Centralized Group Equitability and Individual Envy-Freeness in the Allocation of Indivisible Items

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Abstract

We study the fair allocation of indivisible items for groups of agents from the perspectives of the agents and a centralized allocator. In our setting, the centralized allocator is interested in ensuring the allocation is fair among the groups and between agents. This setting applies to many real-world scenarios, including when a school administrator wants to allocate resources (e.g., office spaces and supplies) to staff members in departments and when a city council allocates limited housing units to various families in need across different communities. To ensure fair allocation between agents, we consider the classical *envy-freeness* (EF) notion. To ensure fairness among the groups, we define the notion of *centralized group equitability* (CGEQ) to capture the fairness for the groups from the allocator's perspective. Because an EF or CGEQ allocation does not always exist in general, we consider their corresponding natural relaxations of *envy-freeness to one item* (EF1) and *centralized group equitability up to one item* (CGEQ1). For different classes of valuation functions of the agents and the centralized allocator, we show that allocations satisfying both EF1 and CGEQ1 always exist and design efficient algorithms to compute these allocations. We also consider the *centralized group maximin share* (CGMMS) from the centralized allocator's perspective as a group-level fairness objective with EF1 for agents and present several results.

1 Introduction

Fair division of indivisible items often deals with fairly allocating a set of (discrete or indivisible) items to a set of agents who have preferences over the items. Due to both practical and theoretical interests, fair division of indivisible items has received considerable attention in various research communities, such as economics, mathematics, and computer science, for much of the past century [Moulin, 2004; Amanatidis *et al.*, 2023]. In practice, fair division of indivisible items has many real-world applications ranging from course allocation (i.e., for allocating schedules of courses to students) to goods division (i.e., dividing artworks or furniture among individuals), in which the Course Match [Budish *et al.*, 2017] mechanism and the Spliddit [Goldman and Procaccia, 2015] online platform have been developed, to provide fair allocations subject to agent preferences and appropriate fairness notions (e.g., envy-freeness and their relaxations) for the respective applications. In theory, fair division of indivisible items has led to the development of numerous notions such as envy-freeness up to one/any item, proportionality up to one/any item, and maximin share fair for quantifying fairness, algorithms (e.g., round-robin or envy-graph procedure [Lipton *et al.*, 2004]) for providing (approximately) fair allocations, and techniques for (partially) characterizing the existence of fair allocation. [Amanatidis *et al.*, 2023].

Allocator's Preference. A main drawback of existing studies of fair division of indivisible items is the lack of consideration from the allocator's perspective, who is responsible for implementing the allocation and has preferences on how the items should be allocated to the set of agents [Bu *et al.*, 2023]. As a result, [Bu *et al.*, 2023] initiated the study of fair division of indivisible items from both the allocator's and agents' perspectives, where each agent has a valuation preference over the items, and the allocator has a separate valuation preference for each agent over the items (specifying their internal values for the agent receiving the items). Moreover, they focus on allocations that satisfy relaxations of envy-freeness between agents under both the allocator's valuation for each agent and agent valuations.

As a motivating example, [Bu *et al.*, 2023] discussed the situations where the government (as the allocator) needs to distribute education resources (e.g., funding and staff members) to different schools (as agents) in which the schools and the government have separate preferences over the education resources based on their needs and macroeconomic policy for schools, respectively. In addition, the work of [Bu *et al.*, 2023] provided examples where a company allocates resources to different departments, an advisor allocates tasks or projects to students, and conference organizers allocate papers to reviewers that require the consideration of both the allocator's and agents' preferences.

Our Study: Centralized Allocator's Preference. Building on the work of [Bu *et al.*, 2023], we introduce a *centralized* allocator who is interested in ensuring fairness at a group level, where each agent naturally belongs to different prede-

Centralized allocator's valuation	Agents' valuations	EF1+CGEQ1
Arbitrary	Identical	✓ (Poly time) (Thm 1)
Ordered		✓ (Poly time) (Thm 2)
Binary	Arbitrary	✓ (Poly time) (Thm 3)

Table 1: Summary of our main results.

fined groups in the fair allocation. For instance, building on the above-mentioned example, a school administrator, tasked with allocating limited resources (e.g., office spaces and supplies) to staff members from departments within the school [Perez, 2022], needs to ensure that the allocation is also fair at the department (group) level. A city council, tasked with allocating limited housing units to various neighborhoods in need across different communities [Gray, 1976], needs to ensure the allocation is fair with respect to different communities. Finally, a government distributing resources to different schools needs to ensure that the allocation is fair with respect to the schools. Therefore, in this paper, our goal is to explore the fair division of indivisible items, which not only provides fairness for the agents but also guarantees fairness for the centralized allocator.

1.1 Our Contribution

We study the fair division of indivisible items for groups of agents from the perspectives of the agents and the centralized allocator. Each agent belongs to one of the groups (e.g., based on their associations) and has an additive valuation function over the items. The centralized allocator has a common additive valuation function indicating their values for the items measured in standardized units (e.g., investment value, monetary amount, and space).

To ensure fair allocation among agents, we consider the classical envy-freeness (EF) notion. To ensure fairness among the groups, we define the notion of centralized group equitability (CGEQ) to capture the fairness for the groups from the centralized allocator's perspective that compares the weighted proportion of values received by each group. Because an EF or CGEQ allocation does not always exist in general, we consider their corresponding natural relaxations of envy-freeness up to one item (EF1) and centralized group equitability up to one item (CGEQ1). Following the idea from [Bu *et al.*, 2023], we strive to answer the following questions.

> *Under which conditions can we guarantee the existence of EF1+CGEQ1 allocations? If so, can we design algorithms to compute them efficiently?*

To address the above questions, we examine different classes of valuation functions of the agents/allocator. The presence of the centralized allocator introduces a fundamental shift in both the fairness notions and the algorithmic challenges involved. The techniques we develop, though sometimes inspired by classic methods such as round-robin, are nontrivial extensions that integrate allocator-aware priorities and group-level proportionality. Specifically, our key contributions are as follows (summarized in Table 1):

- When each agent has an identical valuation function. Even though agents are indistinguishable in terms of preferences, the allocator's independent valuation introduces nontrivial global constraints. Our **DM Algorithm** (Algorithm 1) constructs a temporary allocation satisfying EF1 for both agents and allocator, using a match-based process guided by the allocator's preferences. Then, it reallocates bundles to achieve CGEQ1 while preserving agent-level EF1. This dynamic reassignment highlights the allocator's role in determining group-level equity even under agent homogeneity.
- When all agents and the allocator share the same ordinal ranking of items, we propose the **SPS Algorithm** (Algorithm 2) that simultaneously addresses item distribution across groups and within groups. Despite the aligned ordering, the absolute values may differ significantly under separate allocator's and agents' valuation functions. The allocator's proportional fairness criteria — considering each group's value-to-size ratio — guide item assignment to balance CGEQ1 and agent-level EF1. This synchronization of dual fairness notions introduces dependencies absent in standard ordinal settings.
- When the allocator classifies items into two types. Our **GDRR Algorithm** (Algorithm 3) innovatively extends Round-Robin by introducing a reverse RR phase. This two-phase procedure ensures EF1 for agents while achieving CGEQ1, and cannot be reduced to standard round-robin without losing group fairness.
- We also propose the **Centralized Group Maximin Share** (CGMMS) as an allocator-centric optimization benchmark and seek allocations that satisfy CGMMS for the centralized allocator and EF1 for agents.

The remainder of the paper is organized as follows. In Section 2, we formally define the notations and fairness notions considered in our paper. In Sections 3, 4, and 5, we study EF1+CGEQ1 allocations in identical valuations, ordered valuations, and binary valuations settings, respectively. In Section 6, we discuss EF1+CGMMS allocations. In Section 7, we conclude the paper and provide future research directions. Due to space constraints, we refer readers to the appendix for the missing proofs.

1.2 Related Work

There is an extensive line of work in the fair division of indivisible items. We refer readers to the survey [Amanatidis *et al.*, 2023] for an overview. Below, we review studies focusing on allocations that consider group fairness and fairness from the agents' and allocator's perspectives.

Fairness from the Agents' and Allocator's Perspectives. As discussed earlier, the most relevant work is [Bu *et al.*, 2023], where they initiated the study of fair division of indivisible items from the perspectives of the agents and the allocator. Unlike our setting, their allocator is not centralized

— does not consider groups of agents and aims to ensure (almost) envy-freeness between agents based on the allocator's own valuations of the agents, in addition to the agents' valuations.

Group Fairness. Existing studies have examined group-fair division of indivisible items from only the agent perspective. Some works focus on the predefined group. For example, [Aleksandrov and Walsh, 2018] defined group envy-freeness and group Pareto optimality, and studied the price of group envy-freeness. [Benabbou *et al.*, 2019] considered the fair matching among different groups where each agent can pick at most one item. They studied the fairness criteria named typewise envy-freeness up to one item (TEF1), and showed that when agents have binary valuations, TEF1 allocations can be computed in polynomial time. [Feige and Tahan, 2022] studied the notion of group proportional share fairness and group any price share fairness in different groups that may have different structures like laminar. [Manurangsi and Suksompong, 2024] studied the ordinal maximin share fairness among groups. There are other works that did not consider the predefined group. For example, [Conitzer *et al.*, 2019] studied the group fairness among agents, where they considered any partition of agents, and showed that local optimal Nash welfare allocations satisfy two different relaxations of group fairness that they defined. Later, [Aziz and Rey, 2021] extended it to the setting where items include both goods and chores. The survey of [Amanatidis *et al.*, 2023] offers a comprehensive view of recent progress and open problems in this field.

The most related setting to the proposed study is the work of Scarlett *et al.* [2023], where they studied the compatibility of individual envy-freeness and group envy-freeness from the agent perspective only. Moreover, they did not consider the centralized allocator and defined the group utility based on the agent's valuation function instead of the centralized allocator's valuation.

2 Preliminaries

In this section, we present notations and fairness notions for the considered setting of fair division of indivisible items with a set of agents and a centralized allocator.

2.1 Notations

For $r \in \mathbb{N}$, let $[r] = \{1, 2, \ldots, r\}$. Let $\mathcal{O} = \{o_1, o_2, \ldots, o_m\}$ be a set of m indivisible items, and $\mathcal{N} = [n]$ be a set of n agents. The set of n agents is partitioned into $k \in \mathbb{N}$ groups denoted by $\mathcal{G} = (G_1, \ldots, G_k)$. Each agent belongs to exactly one group, i.e., $G_p \cap G_q = \varnothing$ for any $p, q \in [k]$. Additionally, our setting includes a *centralized* allocator.

Each agent $i \in \mathcal{N}$ has an additive valuation function $v_i : 2^{\mathcal{O}} \to \mathbb{R}_{\geq 0}$, i.e., for any $S \subseteq \mathcal{O}$, $v_i(S) = \sum_{o \in S} v_i(\{o\})$. Specifically, we assume that $v_i(\varnothing) = 0$. The centralized allocator has her own preferences and is endowed with an additive valuation function u: $2^{\mathcal{O}} \to \mathbb{R}_{\geq 0}$, i.e., for any $S \subseteq \mathcal{O}$, $u(S) = \sum_{o \in S} u(\{o\})$ indicating their values for the items measured in standardized units (e.g., investment value, monetary amount, and space). Additionally, we assume that $u(\varnothing) = 0$.

For simplicity, we use $v_i(o)$ and $u(o)$ instead of $v_i(\{o\})$ and $u(\{o\})$, respectively. Let $\Pi(n, \mathcal{O})$ denote all n-partitions of $\mathcal{O}$. An allocation $\mathcal{A} = (A_1, \ldots, A_n) \in \Pi(n, \mathcal{O})$, in which A_i is the bundle allocated to agent i, is an n-partition of $\mathcal{O}$ among n agents, i.e., $\bigcup_{i \in \mathcal{N}} A_i = \mathcal{O}$ and $A_i \cap A_j = \varnothing$ for any two agents $i \neq j$. A fair allocation instance is denoted as $\mathcal{I} = \langle \mathcal{O}, \mathcal{N}, \mathcal{G}, \boldsymbol{v}, u \rangle$, where $\boldsymbol{v} = (v_1, \ldots, v_n)$. Next, we provide the formal definitions of the fairness notions for the agents and the centralized allocator.

2.2 Fairness and Efficiency Notions

To ensure fair allocation between agents, we consider the classical envy-freeness (EF) notion from the agent perspective.

Definition 1 (Envy-Freeness)**.** *An allocation $\mathcal{A}$ is envy-free (EF), if for any two distinct agents $i, j \in \mathcal{N}$, we have $v_i(A_i) \geq v_i(A_j)$.*

However, EF allocations do not always exist. Therefore, we consider a natural and commonly studied relaxation of EF, named envy-free up to one item.

Definition 2 (Envy-Freeness up to One Item)**.** *An allocation $\mathcal{A}$ is envy-free up to one item (EF1) if, for any two agents $i, j \in \mathcal{N}$, $v_i(A_i) \geq v_i(A_j \setminus \{o\})$ holds for some $o \in A_j$.*

Next, we introduce our fairness notion from the centralized allocator's perspective, which is called centralized group equitability (CGEQ).

Definition 3 (Centralized Group Equitability)**.** *An allocation $\mathcal{A}$ is centralized group equitable (CGEQ) if for any two groups $G_p, G_q \in \mathcal{G}$, $\frac{u(\bigcup_{i \in G_p} A_i)}{|G_p|} = \frac{u(\bigcup_{j \in G_q} A_j)}{|G_q|}$ holds.*

This definition reflects an *equitable view from the centralized allocator's perspective*, where the utility function $u(\cdot)$ represents the authority's valuation over bundles allocated to different groups. CGEQ thus requires that each group receives, on average, the same utility as any other group, when judged by the central authority.

Our definition is inspired by the concept of group envy-freeness (GEF) introduced by Conitzer *et al.* [2019], which evaluates group-level fairness using the same valuation function as the one used when computing EF1. However, unlike their setting, where the same function governs both agents' and allocator's evaluations, we follow the framework of Bu *et al.* [2023] to consider separate valuation functions and focus on group-level fairness from the centralized allocator's standpoint. In this way, our model synthesizes the two approaches, applying Conitzer et al.'s group fairness structure to a setting with heterogeneous valuations.

It is well-known that exact fairness conditions (like EQ) are often too strong to satisfy with indivisible items. Similarly, CGEQ allocations do not always exist. For example, consider an instance with only one indivisible item and two groups of equal size: any allocation gives utility to only one group, making CGEQ impossible.

Motivated by the success of EF1 as a relaxation of envy-freeness in the indivisible goods setting Lipton *et al.* [2004], we propose the following relaxation notion.

Definition 4 (Centralized Group Equitability up to One Item)**.** *An allocation $\mathcal{A}$ is said to be centralized group equitable up to one item (CGEQ1) if, for any two groups $G_p, G_q \in \mathcal{G}$, $\frac{u(\bigcup_{i \in G_p} A_i)}{|G_p|} \geq \frac{u(\bigcup_{j \in G_q} A_j \setminus \{o\})}{|G_q|}$ holds for some item $o \in \bigcup_{j \in G_q} A_j$.*

We are interested in allocations that satisfy EF1 from the agent's perspective and CGEQ1 from the centralized allocator's perspective. In particular, we study computing EF1+CGEQ1 allocations in various scenarios, including different classes of valuation functions of the agents and the centralized allocator. If the size of any group is one, i.e., $|G_p| = 1$ for any $G_p \in \mathcal{G}$, CGEQ1 degenerates into EF1. In this case, our setting reduces to a special case in Bu *et al.* [2023], where they showed that an EF1+EF1 allocation can be computed in polynomial time. Therefore, we consider the case where the size of some group is not one in Sections 3, 4, 5 and 6.

3 Identical Valuations

In this section, we first consider the case where the valuations of each agent and the centralized allocator are the same, i.e., $v_1 = \cdots = v_n = u$. In that case, our setting degenerates to that in Scarlett *et al.* [2023], where they study the combination of group and individual fairness and define the group utility based on one set of valuations (i.e., the agents' valuations).

Next, we consider the case where only the valuation of each agent is the same, i.e., $v_1 = \cdots = v_n$, while the centralized allocator's valuation is arbitrary. For simplicity, let v denote the valuation of each agent. In that case, the algorithm in Scarlett *et al.* [2023] does apply directly since its technique heavily depends on the fact that the group utility is computed by using the agents' valuations. However, the centralized allocator's valuation can be different from the agents' valuations in our setting. Therefore, we propose a new algorithm named Draft-and-Match (DM), as described in Algorithm 1. The whole algorithm includes two phases. In Phase 1, the intuition of our algorithm is to partition the items into a temporary allocation $\mathcal{A}'$ by allocating the item that has the highest value to the agent whose bundle has the lowest value in each iteration. This approach ensures that the allocation is envy-free up to one item (EF1) with respect to both the agents' and the centralized allocator's valuation functions. In Phase 2, given the temporary allocation, we then follow specific rules to match and reallocate these bundles to the agents, which ensures the final allocation is CGEQ1 for the centralized allocator. Since we consider identical agents' valuations, this reallocation does not violate the EF1 property of each agent in $\mathcal{A}'$, and the returned allocation is EF1+CGEQ1.

Theorem 1. *Given an instance where the valuation of each agent is the same, the Draft-and-Match Algorithm (Algorithm 1) computes an EF1+CGEQ1 allocation in polynomial time.*

Before proving Theorem 1, we first give the following lemmas. In Lemma 1, we show that the temporary allocation $\mathcal{A}'$ is EF1 with respect to both the agents' and the centralized allocator's valuation functions. Then in Lemma 2, we show that after the reallocation of bundles in $\mathcal{A}'$, the returned allocation $\mathcal{A}$ is CGEQ1.

Algorithm 1: Draft-and-Match (DM)

Input: An instance $\mathcal{I} = \langle \mathcal{O}, \mathcal{N}, \mathcal{G}, \boldsymbol{v}, u \rangle$ with identical agents' valuation functions
Output: An EF1+CGEQ1 allocation $\mathcal{A}$

Let $\mathcal{A}' = (\varnothing, \dots, \varnothing)$ and $\mathcal{A} = (\varnothing, \dots, \varnothing)$;
- - - - - Phase 1: Partition $\mathcal{O}$ into allocation $\mathcal{A}'$ - - - - -
Add $n - (m \mod n)$ dummy items where each agent and the centralized allocator have the valuation of zero to $\mathcal{O}$;
Let $\mathcal{O}_s$ be the array of sorted goods with respect to u in non-increasing order;
Let $\mathcal{N}' = \mathcal{N}$;
while $\mathcal{O}_s \neq \varnothing$ **do**
  Let $\mathcal{O}_n$ be the first n items in $\mathcal{O}_s$ and $\mathcal{O}_s \leftarrow \mathcal{O}_s \setminus \mathcal{O}_n$;
  while $\mathcal{O}_n \neq \varnothing$ **do**
    $A'_{i_{min}} \leftarrow A'_{i_{min}} \cup \{o_{max}\}$, where $i_{min} \in \arg\min_{i \in \mathcal{N}'} v(A'_i)$ and $o_{max} \in \arg\max_{o \in \mathcal{O}_n} v(o)$ (breaking ties arbitrarily);
    $\mathcal{N}' = \mathcal{N} \setminus \{i_{min}\}$ and $\mathcal{O}_n \leftarrow \mathcal{O}_n \setminus \{o_{max}\}$;
- - - - Phase 2: Match the bundles in $\mathcal{A}'$ to agents - - - -
Assume that the groups are in non-decreasing order of size, i.e., $|G_1| \leq \dots \leq |G_k|$;
$t_p \leftarrow 0, \forall p \in [k]$, t_p is the total number of times that G_p has been picked so far;
while $\mathcal{A}' \neq (\varnothing, \dots, \varnothing)$ **do**
  if $\exists t_p = 0$ **then**
    $p^* \leftarrow \min\{p | p \in [k]$ *and* $t_p = 0\}$;
  else
    $p^* \leftarrow \arg\min_{p \in [k]} \frac{t_p}{|G_p|}$ (breaking ties by selecting p that reaches $\min_{p \in [k]} \frac{t_p}{|G_p|}$ the latest);
  Arbitrarily choose one agent $i^* \in G_{p^*}$, where $A_{i^*} = \varnothing$;
  $A_{i^*} \leftarrow A'_{i_{max}}$, where $A'_{i_{max}} \in \arg\max_{A'_i \in \mathcal{A}'} u(A'_i)$;
  $t_{p^*} \leftarrow t_{p^*} + 1$;
  $A'_{i_{max}} \leftarrow \varnothing$;
return $\mathcal{A}$

Lemma 1. *The allocation $\mathcal{A}'$ (computed in Phase 1) in Algorithm 1 is EF1 with respect to both the agents' and the centralized allocator's valuation functions. That is, for any $i, j \in [n]$, we have $v(A'_i) \geq v(A'_j \setminus \{o\})$ and $u(A'_i) \geq u(A'_j \setminus \{o'\})$ for some $o, o' \in A'_j$.*

Since the agents' valuation functions are identical, EF1 allows us to rearrange the bundles without violating fairness at the individual level. Leveraging this flexibility, we aim to rearrange the bundles to meet the group-level fairness criterion (CGEQ1) while preserving EF1 for the agents, as any agent's bundle is already "close" in value to others up to one item.

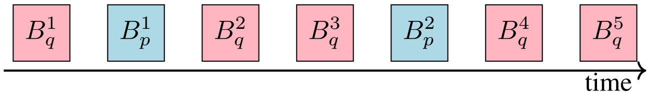

Figure 1: Example where $|G_p| = 2$ and $|G_q| = 5$, with G_q receiving the first bundle during the re-allocation process. We denote B_z^i as the bundle received by G_z in the i-th allocation. The red squares represent bundles received by G_q, and the blue squares represent bundles received by G_p.

Lemma 2. *In Algorithm 1, the returned allocation $\mathcal{A}$ is CGEQ1.*

Proof Sketch. To prove that the allocation is CGEQ1, we need to show that for any two groups G_p and G_q, G_p will not envy G_q. Let $B_p^1, B_p^2, \dots, B_p^{|G_p|}$ and $B_q^1, B_q^2, \dots, B_q^{|G_q|}$ represent the bundles allocated to agents in G_p and G_q, respectively, during the reallocation process (see Figure 1 for an example). We remove the most valuable item in the centralized allocator's perspective from B_q^1, the first bundle received by G_q. After removing this item, B_q^1 becomes the least valuable among all bundles (from both G_p and G_q).

Next, we duplicate each bundle received by G_p $|G_q|$ times and each bundle received by G_q $|G_p|$ times, ordering all duplicated bundles in non-decreasing order of value. In this sequence, $|G_p|$ copies of B_q^1 (with one item removed) now become the least valuable. This duplication transforms the problem of comparing the average values of the bundles into comparing the sum of the values of all duplicated bundles.

We then perform a pairwise comparison between the duplicated bundles from both groups. In this comparison, each duplicated bundle from G_p is at least as valuable as the corresponding duplicated bundle from G_q. Therefore, G_p will not envy G_q, and the allocation satisfies CGEQ1. □

Proof of Theorem 1. By Lemmas 1 and 2, and the fact that each agent has the same valuation, it can be concluded that the final allocation is EF1+CGEQ1. Next, let us consider the time complexity. Without loss of generality, we assume that $m \geq n$. In the first part of our algorithm, it takes $O(m \log m)$ to sort the items, and then for each iteration, selecting an agent takes $O(n)$ time, and choosing their favorite item takes $O(m)$ time. Note that there are $O(\lceil \frac{m}{n} \rceil)$ iterations. In the second part of our algorithm, there are $O(n)$ iterations in the while loop. For each iteration, selecting the target group and agent takes $O(nk)$ time, and choosing the bundle with the highest value from the centralized allocator's perspective takes $O(n)$ time. Therefore, the total running time of our algorithm is $O(m^2 + m \log m + n^2 k)$. □

4 Ordered Valuations

In this section, we consider the instance $\mathcal{I}$ with ordered valuations, where each agent $i \in \mathcal{N}$ and the centralized allocator share the same ranking or preference for all items. Specifically, $v_i(o_1) \geq \cdots \geq v_i(o_m)$ and $u(o_1) \geq \cdots \geq u(o_m)$. The challenge in our setting arises when certain items are valued oppositely by the agents and the centralized allocator. In such cases, ordered valuations may help us circumvent this issue.

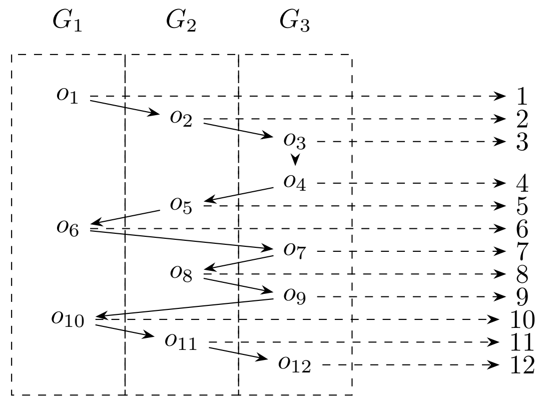

Figure 2: An illustration of the allocation process of the first twelve items. Assume that there are three groups G_1, G_2, and G_3, where each group has 3, 4, and 5 agents respectively. The arrow means the sequence of the allocation of these twelve items. For example, in G_1, agent 1 with $\ell_1 = 0$ picks o_1, agent 6 with $\ell_6 = 2$ picks o_6, and agent 10 with $\ell_{10} = 3$ picks o_{10}. Then, in the following iterations, if G_1 receives some item, the algorithm will follow the order to select the target agent.

Inspired by the algorithm that computes a weighted EF1 allocation in Chakraborty *et al.* [2021], we propose the Synchronous Picking Sequence Algorithm, detailed in Algorithm 2. Our algorithm operates by allocating a set of items in several batches, where each batch has n items. If there are less than n items left, we can add some dummy items that have a zero value for the agents and the centralized allocator. For each item within a batch, there are two phases of allocation. In the first phase, the algorithm assigns the item to a group. In the second phase, the item is allocated to a specific agent within that group. Every agent receives exactly one item per batch. By structuring the allocation in this way, the algorithm mirrors a specialized round-robin algorithm. Figure 2 illustrates the allocation process of the first batch (first twelve items) for an instance where there are three groups that have 3, 4, and 5 agents, respectively, as computed by Algorithm 2.

Theorem 2. *Given an instance with ordered valuations, the Synchronous Picking Sequence (Algorithm 2) computes an EF1+CGEQ1 allocation in polynomial time.*

5 Binary Allocator Valuations

In this section, we consider the instance $\mathcal{I}$ where the centralized allocator has a binary valuation function, i.e., for each item o, either $u(o) = 0$ or $u(o) = 1$ holds. We show that an EF1+CGEQ1 allocation always exists, which can be computed by the Group-Decided Round-Robin Algorithm (GDRR, Algorithm 3) in polynomial time.

The idea behind Algorithm 3 is as follows: First, we figure out a special order for agents. This order is designed so that when we allocate items valued at 1 (from the allocator's perspective) using a round-robin approach, it satisfies the CGEQ1 property. Once the allocation of items with a value of 1 is completed, we proceed to allocate items with a value of 0 from the allocator's perspective. These items do not affect the CGEQ1 property. For this allocation, we use

Algorithm 2: Synchronous Picking Sequence (SPS)

Input: An instance $\mathcal{I} = \langle \mathcal{O}, \mathcal{N}, \mathcal{G}, \boldsymbol{v}, u \rangle$ with ordered valuation functions
Output: An EF1+CGEQ1 allocation $\mathcal{A}$

Let $\mathcal{A} = (\varnothing, \dots, \varnothing)$;
Add $n - (m \mod n)$ dummy items whose value is zero for the agents and the centralized allocator to $\mathcal{O}$;
Assume that the groups are ordered in non-decreasing order of size, i.e., $|G_1| \leq \dots \leq |G_k|$;
Set $t_p \leftarrow 0, \forall p \in [k]$ and $\ell_i \leftarrow 0, \forall i \in \mathcal{N}$;
while $\mathcal{O} \neq \varnothing$ **do**
  Let $\mathcal{O}_n$ be the first n items in $\mathcal{O}$ and $\mathcal{O} \leftarrow \mathcal{O} \setminus \mathcal{O}_n$;
  while $\mathcal{O}_n \neq \varnothing$ **do**
    Let $o_{max} \in \arg\max_{o \in \mathcal{O}_n} u(o)$;
    - Phase 1: Decide which group receives this item -
    if $\exists t_p = 0$ **then**
      $p^* \leftarrow \min\{p | p \in [k] \text{ and } t_p = 0\}$;
    else
      $p^* \leftarrow \arg\min_{p \in [k]} \frac{t_p}{|G_p|}$ (breaking ties by selecting p that reaches $\min_{p \in [k]} \frac{t_p}{|G_p|}$ the latest);
    - Phase 2: Decide which agent picks this item -
    if $\exists i^* \in G_{p^*}$ *such that* $A_{i^*} = \varnothing$ **then**
      $A_{i^*} \leftarrow \{o_{max}\}$ and $\ell_{i^*} \leftarrow t_{p^*}$;
    else
      Find the agent i^* whose label ℓ_{i^*} equals $t_{p^*} \mod |G_p|$, and $A_{i^*} \leftarrow A_{i^*} \cup \{o_{max}\}$;
    $t_{p^*} \leftarrow t_{p^*} + 1$ and $\mathcal{O}_n \leftarrow \mathcal{O}_n \setminus \{o_{max}\}$;
return $\mathcal{A}$

the reverse of the previously determined sequence and perform another round-robin distribution based on the agents' valuation functions.

Theorem 3. *Given an instance where the centralized allocator has the binary valuation function, the Group-Decided Round Robin Algorithm (Algorithm 3) computes an EF1+CGEQ1 allocation in polynomial time.*

We divide the proof of Theorem 3 into two parts, corresponding to the two properties of the allocation computed by Algorithm 3: CGEQ1 and EF1. These properties are established in the following lemmas.

Lemma 3. *The output allocation in Algorithm 3 is CGEQ1.*

Proof. It suffices to show that at any point during the allocation process of the first bundle, the partial allocation is CGEQ1. We prove the statement by induction. Before allocating any item, the allocation is trivially CGEQ1. Fix two groups p and q. Suppose that after allocating k items, G_p and G_q receive c_1 and c_2 items, respectively, and $c_1/|G_p| \leq c_2/|G_q|$. If the $(k+1)$-th item is not allocated to G_p or G_q, then G_p and G_q will not envy each other. Otherwise, the $(k+1)$-th item must go to G_p. Thus, we have $c_1'/|G_p| > c_1/|G_p| \geq (c_2 - 1)/|G_q|$, and $c_2/|G_q| \geq c_1/|G_p| = (c_1' - 1)/|G_p|$. □

Algorithm 3: Group-Decided Round-Robin (GDRR)

Input: An instance $\mathcal{I} = \langle \mathcal{O}, \mathcal{N}, \mathcal{G}, \boldsymbol{v}, u \rangle$ with binary allocator valuation functions
Output: An EF1+CGEQ1 allocation $\mathcal{A}$

Let $\mathcal{A} = (\varnothing, \dots, \varnothing)$;
Denote the collection of items with $u(o) = 1$ as $\mathcal{O}^1$ and the collection of items with $u(o) = 0$ as $\mathcal{O}^2$;
Set $t_p \leftarrow 0, \forall p \in [k]$ and $\ell_i \leftarrow 0, \forall i \in \mathcal{N}$;
Set $t \leftarrow 0$;
while $\mathcal{O}^1 \neq \varnothing$ **do**
  if $\exists i$ *such that* $A_i = \varnothing$ **then**
    if $\exists t_p = 0$ **then**
      $p^* \leftarrow \min\{p | p \in [k] \text{ and } t_p = 0\}$;
    else
      $p^* \leftarrow \arg\min_{p \in [k]} \frac{t_p}{|G_p|}$ (breaking ties by selecting p that reaches $\min_{p \in [k]} \frac{t_p}{|G_p|}$ the latest);
    Find agent $i^* \in G_{p^*}$ with $A_{i^*} = \varnothing$;
    $A_{i^*} \leftarrow \{o_{max}\}$ where $o_{max} \in \arg\max_{o \in \mathcal{O}^1} v_{i^*}(o)$;
    $\ell_{i^*} \leftarrow t$;
    $t_{p^*} \leftarrow t_{p^*} + 1$;
  else
    Find agent i^* whose label ℓ_{i^*} equals t
    $A_{i^*} \leftarrow A_{i^*} \cup \{o_{max}\}$ where $o_{max} \in \arg\max_{o \in \mathcal{O}^1} v_{i^*}(o)$;
  $t \leftarrow (t+1) \mod n$;
  $\mathcal{O}^1 \leftarrow \mathcal{O}^1 \setminus \{o_{max}\}$;
$t \leftarrow n - 1$;
while $\mathcal{O}^2 \neq \varnothing$ **do**
  Find agent i^* with ℓ_{i^*} equals t;
  $A_{i^*} \leftarrow A_{i^*} \cup \{o_{max}\}$ where $o_{max} \in \arg\max_{o \in \mathcal{O}^2} v_{i^*}(o)$;
  $\mathcal{O}^2 \leftarrow \mathcal{O}^2 \setminus \{o_{max}\}$;
  $t \leftarrow (t-1) \mod n$;
return $\mathcal{A}$

Lemma 4. *The allocation computed by Algorithm 3 is EF1.*

Proof. We assume that $|\mathcal{O}^1| = k_1 n$ and $|\mathcal{O}^2| = k_2 n$, as we can always achieve this by adding dummy items with value 0 from all agents' perspectives.

Fix two agents i and j with $i < j$. Denote i's items as $o_{i,1}, \dots, o_{i,k_1+k_2}$ and j's items as $o_{j,1}, \dots, o_{j,k_1+k_2}$. We have $v_i(o_{i,k}) \geq v_i(o_{j,k})$ for $1 \leq k \leq k_1$ and $v_j(o_{j,k}) \geq v_j(o_{i,k+1})$ for $1 \leq k < k_1$. Similarly, we have $v_j(o_{j,k}) \geq v_j(o_{i,k})$ for $k_1 + 1 \leq k \leq k_1 + k_2$ and $v_i(o_{i,k}) \geq v_i(o_{j,k+1})$ for $k_1 + 1 \leq k < k_1 + k_2$.

Thus, we have

$$\begin{aligned}\sum_{k=1}^{k_1+k_2} v_i(o_{i,k}) &\geq \sum_{k=1}^{k_1} v_i(o_{i,k}) + \sum_{k=k_1+1}^{k_1+k_2-1} v_i(o_{i,k}) \\ &\geq \sum_{k=1}^{k_1} v_i(o_{j,k}) + \sum_{k=k_1+2}^{k_1+k_2} v_i(o_{j,k}) \\ &= \left(\sum_{k=1}^{k_1+k_2} v_i(o_{j,k})\right) - v_i(o_{j,k_1+1}),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{k_1+k_2} v_j(o_{j,k}) &\geq \sum_{k=1}^{k_1-1} v_j(o_{j,k}) + \sum_{k=k_1+1}^{k_1+k_2} v_j(o_{j,k}) \\ &\geq \sum_{k=2}^{k_1} v_j(o_{i,k}) + \sum_{k=k_1+1}^{k_1+k_2} v_j(o_{i,k}) \\ &= \left(\sum_{k=1}^{k_1+k_2} v_j(o_{i,k})\right) - v_j(o_{i,1}),\end{aligned}$$

implying that agents i and j will not envy each other up to one item. □

6 Centralized Group Maximin Share

In the previous discussion, the allocator achieves group-level fairness among agents through an additional fairness requirement (CGEQ1). Now, we shift our focus to optimizing group-level fairness objectives directly, aiming to achieve fairness from a centralized perspective while still maintaining EF1 for the agents. This can be understood as the allocator striving to find the "best" fair allocation.

In this case, the utilitarian social welfare ($\sum_{i=1}^{n} u(A_i)$) is not suitable to be the optimization objective since it not only remains invariant regardless of the allocation computed but also fails to reflect group-level fairness. Instead, we focus on the share-based fairness objective from the centralized allocator's perspective, which is called centralized group maximin share (CGMMS). This definition is motivated by the well-study notion – maximin share fairness (MMS) [Budish, 2011]. Our main goal is to find an allocation that satisfies CGMMS and EF1 from the agents' perspective simultaneously.

Definition 5 (Centralized Group Maximin Share)**.** *Let $\mathcal{O}$ be the set of items and $\Pi_n(\mathcal{O})$ be the set of n-partitions of $\mathcal{O}$ (which may be subject to some constraints). The centralized group maximin share* CGMMS *is defined as:*

$$\mathsf{CGMMS} = \max_{\mathcal{A}\in\Pi_n(\mathcal{O})} \min_{G_p\in\mathcal{G}} \frac{u(\cup_{i\in G_p} A_i)}{|G_p|}.$$

An allocation $\mathcal{A}$ is centralized group maximin share fair (CGMMS) if it holds that $\min_{G_p\in\mathcal{G}} \frac{u(\cup_{i\in G_p} A_i)}{|G_p|} = \mathsf{CGMMS}$.

However, for the most general case, computing a CGMMS allocation is strongly NP-hard, which can be reduced from the 3-partition problem. Thus, we directly have the following proposition.

Proposition 1. *Computing a CGMMS allocation subject to EF1 for agents is strongly NP-hard.*

We notice that when agents have identical valuation functions or for instances with ordered valuations, computing an EF1+CGMMS allocation is still strongly NP-hard. However, when the centralized allocator has a binary valuation function, it can be solved efficiently.

Theorem 4. *When the centralized allocator has a binary valuation function, computing a CGMMS allocation subject to EF1 for agents can be achieved in polynomial time.*

Proof. We first notice that CGMMS (without considering EF1) can be computed in polynomial time. Since each item's value (from the allocator's perspective) is either 1 or 0, there are only polynomially many possible values for $u(\cup_{i\in G_p} A_i)/|G_p|$. Denote this set by S. Thus, we can enumerate values $x \in S$ and check whether CGMMS $\geq x$ holds. The constraints are that each group should receive at least a certain number of items valued at 1, and the total number of items with value 1 should be sufficient to meet the requirements of all agents.

Let $r = \lfloor \mathsf{CGMMS} \rfloor$. Define $\mathcal{O}^1$ (resp. $\mathcal{O}^2$) as the set of items valued at 1 (resp. 0) from the allocator's perspective. There exists an allocation where each agent receives r items, and additionally, some agents in each group may receive an extra item to ensure the allocation achieves CGMMS. For these agents receiving extra items, we label them sequentially as $1, 2, \ldots, (|\mathcal{O}^1| - n \cdot r)$. For the remaining agents, we label them as $(|\mathcal{O}^1| - n \cdot r + 1), \ldots, n$.

Next, we show that there is an EF1 allocation that achieves CGMMS. We apply a method similar to Algorithm 3. Using the computed order, we perform a forward round-robin allocation of $\mathcal{O}^1$, followed by a reverse round-robin allocation of $\mathcal{O}^2$. By Lemma 4, the allocation computed is EF1, and it also achieves CGMMS. □

7 Conclusion

In this paper, we study the fair division of indivisible items from the perspectives of agents and a centralized allocator. We propose to use EF1 and CGEQ1 to measure the fairness from the agents' and the centralized allocator's perspectives, respectively, and aim to compute allocations that satisfy EF1 and CGEQ1 simultaneously. We show that EF1+CGEQ1 allocations always exist for different classes of agents' and the centralized allocator's valuation functions, which can be computed in polynomial time. As for optimizing group-level fairness objectives, we show that, in general, finding a CGMMS allocation is hard, but an EF1+CGMMS allocation can be computed within polynomial time when the centralized allocator has a binary valuation function.

For future work, a natural direction is to determine whether an allocation satisfying the above two fairness notions exists in more general settings. *We have searched for a non-existence counterexample with the aid of computer programs, but it seems to be hard to find such an instance.* Further, we can explore the setting where the agents and the centralized allocator have beyond additive valuation functions like submodular or subadditive valuation functions and design algorithms that efficiently return EF1+CGEQ1 allocations. For the group-level fairness objective, we can approximately optimize CGMMS subject to EF1 for agents.

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A Missing proofs

A.1 Proof of Lemma 1

Proof. We use mathematical induction to show that for each agent $i \in \mathcal{N}$ and the centralized allocator, the temporary allocation $\mathcal{A}'$ is EF1 with respect to their valuation functions. Since all agents have the same valuation function, no envy cycle occurs in any allocation. Hence, there is no need to eliminate envy cycles during the iteration. This claim also holds for the centralized allocator.

For the base case, it is clear that the empty bundle is EF1 with respect to the agents' valuation functions and the centralized allocator's valuation function.

Next, for the induction step, we assume that in the kth iteration, any bundle in the partial allocation $\mathcal{A}'^{(k)} = (A_1'^{(k)}, \ldots, A_n'^{(k)})$ is EF1 with respect to the agents' valuation functions and the centralized allocator's valuation function, i.e., for any two bundles $A_i'^{(k)}$ and $A_j'^{(k)}$, both $v(A_i'^{(k)}) \geq v(A_j'^{(k)} \setminus \{o\})$ and $u(A_i'^{(k)}) \geq u(A_j'^{(k)} \setminus \{o\})$ hold for some $o \in A_j'^{(k)}$. We now show that this property continues to hold after the $(k+1)$th iteration.

Fix two arbitrary agents $i, j \in \mathcal{N}$. Let $o_i^{(k+1)}$ and $o_j^{(k+1)}$ denote the items allocated to agent i and j, respectively. Without loss of generality, assume that agent j picks $o_j^{(k+1)}$ first, which implies that $v(A_i'^{(k)}) \geq v(A_j'^{(k)})$, $v(o_j^{(k+1)}) \geq v(o_i^{(k+1)})$, and $u(o_j^{(k+1)}) \geq u(o_i^{(k+1)})$. For agent i, we have

$$v(A_i'^{(k+1)}) \geq v(A_i'^{(k)}) \geq v(A_j'^{(k)}) = v(A_j'^{(k+1)} \setminus \{o_j^{k+1}\}),$$

and for agent j, we have

$$\begin{aligned} v(A_j'^{(k+1)}) &= v(A_j'^{(k)}) + v(o_j^{(k+1)}) \\ &\geq v(A_i'^{(k)} \setminus \{o\}) + v(o_i^{(k+1)}) = v(A_i'^{(k+1)} \setminus \{o\}) \end{aligned}$$

for some $o \in A_i'^{(k)}$. Therefore, after the $(k+1)$th iteration, any bundle in the partial allocation is still EF1 to each agent's valuation function. Since the items are indexed in non-increasing order of the centralized allocator's valuation, and the first n items from the remaining items are allocated to agents in each iteration, we get $u(o_j^{(\ell)}) \geq u(o_i^{(\ell+1)})$ for any two agents $i, j \in \mathcal{N}$, and any $\ell \in [k]$.

Then, for the centralized allocator, it holds that $\forall i, j \in \mathcal{N}$

$$u(A_i'^{(k+1)}) = \sum_{s=1}^{k+1} u(o_i^{(s)}) \geq \sum_{s=2}^{k+1} u(o_j^{(s)}) = u(A_j'^{(k+1)} \setminus \{o_j^{(1)}\})$$

Thus, after the $(k+1)$th iteration, any bundle in $\mathcal{A}'^{(k+1)}$ is still EF1 under the centralized allocator's valuation function. This completes the induction and establishes the correctness of our proof. □

A.2 Proof of Lemma 2

Proof. It suffices to show that for any two groups G_p and G_q, $\frac{u\left(\bigcup_{i\in G_p} A_i\right)}{|G_p|} \geq \frac{u\left(\bigcup_{i\in G_q} A_i \setminus \{o\}\right)}{|G_q|}$ holds for some item $o \in \bigcup_{i\in G_q} A_i$. G_p and G_q will obtain bundles alternatively according to Algorithm 1. Without loss of generality, we assume that $|G_p| \geq |G_q|$. Notice that at any point during the allocation process, the values $\frac{t_p}{|G_p|}$ and $\frac{t_q}{|G_q|}$ differ by at most $\frac{1}{|G_q|}$. This is because when G_q receives a new bundle, its allocation ratio $\frac{t_q}{|G_q|}$ is less than or equal to that of G_p, and receiving one more bundle increases this ratio by $\frac{1}{|G_q|}$. Similarly, if G_p receives a new bundle, its allocation ratio increases by $\frac{1}{|G_p|}$.

Let $r^1, r^2, \ldots, r^{|G_p|}$ represent the first, second, ..., (t_i+1)th pick of some agent in G_p, where the bundle $B^0, B^1, \ldots, B^{|G_p|}$ from $\mathcal{A}'$ is allocated to some agent in G_p, respectively. Let the number of bundles that G_q receives during the time intervals $(r^1, r^2), (r^2, r^3), \ldots, (r^{t_i}, r^{t_i+1})$ be denoted by $f_1, f_2, \ldots, f_{t_i}$, respectively. Define f_0 as the number of bundles received by G_q before r^1, and $f_{|G_p|+1}$ as the number of bundles received after $r^{|G_p|+1}$.

We first prove the desired inequality:

$$\frac{u\left(\bigcup_{i\in G_p} A_i\right)}{|G_p|} \geq \frac{u\left(\bigcup_{i\in G_q} A_j \setminus \{o\}\right)}{|G_q|} \tag{1}$$

for some item $o \in \bigcup_{i\in G_q} A_j$, and the proof of the reverse direction is similar. To prove Inequality (1), it is sufficient to show that

$$u\left(\bigcup_{i\in G_p} A_i\right) \cdot |G_q| \geq u\left(\bigcup_{j\in G_q} A_j \setminus \{o\}\right) \cdot |G_p|.$$

Given the EFI property established previously, for any bundle, removing its most valuable item results in a lower value than any other bundle. Thus, we can prove a stronger condition: we always remove the most valuable item from the first bundle received by G_q and then compare the average values of the received by both groups. Equivalently, we construct the following strategy to compare the two sides of the inequality:

- For each bundle received by G_p, duplicate it $|G_q|$ times.
- For each bundle received by G_q, duplicate it $|G_p|$ times.

This results in a total of $|G_p| \times |G_q|$ bundles for each group. We will now compare the total value of these duplicated bundles.

Note that every time right before G_p receives a new bundle, the following inequality holds:

$$\frac{\sum_{i=0}^{k} f_i}{|G_q|} \leq \frac{k}{|G_p|} + \frac{1}{|G_q|}.$$

This is due to the previously mentioned property that the difference between the allocation ratios of any two groups is bounded by $\frac{1}{|G_q|}$. Rearranging this inequality, we obtain

$$\left(\sum_{i=0}^{k} f_i\right) \cdot |G_p| \leq k \cdot |G_q| + |G_p|. \tag{2}$$

Next, consider the order of the duplicated bundles for both groups. For G_q, the value of the duplicated bundles is arranged in descending order as follows: first

$B_q^2, B_q^3, \ldots, B_q^{|G_q|}$, each repeated $|G_p|$ times. Finally, B_q^1, with its most valuable item removed, is repeated $|G_p|$ times. Similarly, for G_p, the value of its duplicated bundles is arranged in descending order as $B_p^1, B_p^2, \ldots, B_p^{|G_p|}$, each repeated $|G_q|$ times.

We can perform a pairwise comparison of the total values of corresponding duplicated bundles from both groups. Specifically, we compare the ith highest value bundle in G_p with the ith highest value bundle in G_q. For the bundles received by G_p, consider the $((k-1)|G_q|+1)$th to $(k \cdot |G_q|)$th bundles, for $1 \leq k \leq |G_p|$. These bundles have the value of $u(B_p^k)$, which is smaller than the values of bundles received by G_q (in the duplicated scenario) at most

$$\sum_{i=0}^{k-1} f_i \cdot |G_p| - |G_p|.$$

By Inequality (2), $\sum_{i=0}^{k-1} f_i \cdot |G_p| - |G_p|$ is at most $(k-1) \cdot |G_q|$.

Hence, when performing the pairwise comparison between the ith highest value bundle in G_p and the ith highest value bundle in G_q, the value of the bundles in G_p is always greater than or equal to the value of the corresponding bundles in G_q. Therefore, Inequality (1) holds, implying the desired condition

$$\frac{u\left(\bigcup_{i \in G_p} A_i\right)}{|G_p|} \geq \frac{u\left(\bigcup_{j \in G_q} A_j \setminus \{o\}\right)}{|G_q|}$$

for some item $o \in \bigcup_{j \in G_q} A_j$. □

A.3 Proof of Theorem 2

Proof. For the centralized allocator, the notion of CGEQ1 can be seen as the variant of the notion of weighted EF1. In Chakraborty *et al.* [2021], they showed that a weighted EF1 allocation can be computed in polynomial time, so it is not hard to see that the final allocation is CGEQ1. For agents, the whole algorithm has two phases. In the first phase, each agent is relabelled in her group. In the second phase, if one group receives one item, following the rule of the label, the agent who has the smallest number picks this item. If there is a tie, the agent with the lowest index is selected. Besides that, each agent and the centralized allocator have the same preference, which means the item chosen by the centralized allocator in each iteration is also every agent's favorite item. Therefore, the whole allocation process for agents can be regarded as the round-robin protocol, and the final allocation is EF1 to agents. □