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Equilibrium games in networks☆

Angsheng Li*, Xiaohui Zhang, Yicheng Pan, Pan Peng

State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, 100190, PR China

HIGHLIGHTS

- We proposed a game approach to resisting cascading failure in networks against a small number of attacks.
- Experiments showed that networks of some classic models have an equilibrium game, but some others fail to have.
- Most real networks fail to have an equilibrium game.

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ABSTRACT

It seems a universal phenomenon of networks that the attacks on a small number of nodes by an adversary player Alice may generate a global cascading failure of the networks. It has been shown (Li et al., 2013) that classic scale-free networks (Barabási and Albert, 1999, Barabási, 2009) are insecure against attacks of as small as $O(\log n)$ many nodes. This poses a natural and fundamental question: Can we introduce a second player Bob to prevent Alice from global cascading failure of the networks? We proposed a game in networks. We say that a network has an equilibrium game if the second player Bob has a strategy to balance the cascading influence of attacks by the adversary player Alice. It was shown that networks of the preferential attachment model (Barabási and Albert, 1999) fail to have equilibrium games, that random graphs of the Erdös–Rényi model (Erdös and Rényi, 1959, Erdös and Rényi, 1960) have, for which randomness is the mechanism, and that homophyly networks (Li et al., 2013) have equilibrium games, for which homophyly and preferential attachment are the underlying mechanisms. We found that some real networks have equilibrium games, but most real networks fail to have. We anticipate that our results lead to an interesting new direction of network theory, that is, equilibrium games in networks.

1. Introduction

Cascading failure of networks generated by a small number of attacks has been a fundamental challenge in network theory. We propose a game theoretical approach to understanding the phenomenon of cascading failure of networks by a small number of attacks. We showed that networks of the ER model have an equilibrium game, that networks of the PA model fail to have, and that some real networks have equilibrium games, but most of them fail to have. Our discoveries here suggest a fundamental issue to characterize the networks that have equilibrium games.

A surprising discovery in modern network theory is that network topology is universal in nature, society and industry [1]. Many real networks follow a power law [2,1,3], and satisfy the small world phenomenon [4–6]. Consequently many real networks are vulnerable to cascading failures against the attacks of an adversary player, Alice say, on a small number of high degree nodes [7].

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* Corresponding author. Tel.: +86 01062661629.
E-mail address: angsheng@ios.ac.cn (A. Li).

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There are different ways to define the cascading failures of attacks. We use a simple definition similar to a linear threshold model [8]. Let $G = (V, E)$ be a graph. Suppose that each node $v \in V$ has a threshold $\phi(v)$. Let $S \subset V$ be a subset of vertices V. We define the *infection set of S in G* recursively: (i) each node $u \in S$ is called infected, and (ii) a node v becomes infected, if $\phi(v)$ fraction of v's neighbors have already been infected. We use $\inf^G(S)$ to denote the set of all infected nodes of S in G defined as above. An adversary player Alice will try to choose an S of small size such that the size of the infection set of S in G is maximized.

In Ref. [7], it has been shown that for a random graph of the Erdös–Rényi model with random thresholds, there is an adversary player Alice to attack $O(\log n)$ many nodes of the graph to generate a global cascading failure of the network, and that for a nontrivial network of the preferential attachment model with random thresholds, there is an adversary player Alice to attack $O(\log n)$ many nodes of the network to generate a global cascading failure of the network. This progress poses a fundamental question: How can we prevent Alice from generating a global cascading failure of the networks by attacking on a small number of nodes?

In this paper, we propose a simple way to solve this problem. We introduce a second player, Bob say, to compete the influence with Alice in a network to prevent the global cascading failure from the attacks of Alice. We say that a network has an *equilibrium game*, if for any attacks S of Alice, Bob can choose a set T of the same size as that of S such that the difference of the cascading failure set of Alice and that of Bob's is negligible. Otherwise, we say that the network fails to have an equilibrium game. Given a network G, we say that *cascading failure in G is resistable*, if for any strategy $\mathcal{A}$ of Alice, Bob has a strategy $\mathcal{B}$ such that the size of the cascading failure set of $\mathcal{A}$ in competing with $\mathcal{B}$ is negligible. Otherwise, we say that cascading failure in G is *non-resistable*.

We will show that there are networks that have equilibrium games, but some others fail to have, and that some networks are resistable, but some others are not. Our approach of equilibrium games partially solves the cascading failures of networks, and equally importantly, it poses a fundamental issue to characterize the networks having equilibrium games, and the networks being resistable.

2. Games in networks

Given a network $G = (V, E)$, suppose that each node $v \in V$ has a threshold $\phi(v)$. Let Z be a set of nodes and $v \in V \setminus Z$. We define the *influence of Z on v*, denoted by $f(Z; v)$, to be $\frac{n(Z,v)}{d(v)}$, where $n(Z, v)$ is the number of v's neighbors that are in Z, and $d(v)$ is the degree of v. We say that Z *attracts* v if $f(Z; v)$ is greater than or equal to $\phi(v)$, the threshold of v.

Suppose that Alice and Bob have strategies $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the game between $\mathcal{A}$ and $\mathcal{B}$ proceeds as follows.

Game $(\mathcal{A}, \mathcal{B})$

Step 0: At first, $\mathcal{A}$ selects a set X by a program α, secondly $\mathcal{B}$ chooses a set Y by a program β such that $X \cap Y = \emptyset$ and $|X| = |Y| = k$ for some k bounded by a polynomial of $\log n$, where n is the number of nodes of G. Set $A^G_{\mathcal{A},\mathcal{B}} \leftarrow X$, and set $B^G_{\mathcal{A},\mathcal{B}} \leftarrow Y$.

Suppose that $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ are the current sets agitated by Alice and Bob respectively.

Step $2i + 1$: Run Program Γ in Section 3.1. If a_i is defined, then go to action phrase to agitate a_i.

Step $2i + 2$: Run Program Δ in Section 3.2. If b_i is defined, then go to action phrase to agitate b_i.

Action phrase (Agitation) Let v be the node a_i or b_i chosen by Γ or Δ respectively. Do the following:

(1) Let l be the number of v's neighbors that are in $A^G_{\mathcal{A},\mathcal{B}}$,
(2) Let r be the number of v's neighbors that are in $B^G_{\mathcal{A},\mathcal{B}}$,
(3) If both $\frac{l}{d}$ and $\frac{r}{d}$ are greater than or equal to $\phi(v)$, then with probability $\frac{l}{l+r}$, v enters $A^G_{\mathcal{A},\mathcal{B}}$, in which case, we say that v is agitated by $\mathcal{A}$, otherwise, then v enters $B^G_{\mathcal{A},\mathcal{B}}$, in which case, we say that v is agitated by $\mathcal{B}$, where d is the degree of v in G, and
(4) Otherwise, then:
   (a) if $\frac{l}{d} \geq \phi(v)$, then v enters $A^G_{\mathcal{A},\mathcal{B}}$, in which case, we say that v is agitated by $\mathcal{A}$, and
   (b) if $\frac{r}{d} \geq \phi(v)$, then v enters $B^G_{\mathcal{A},\mathcal{B}}$, in which case, we say that v is agitated by $\mathcal{B}$.

We use $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ to denote the sets defined at the end of the game.

We notice that in our games, the network G is associated with a threshold function ϕ, and that the initial set X and Y chosen by $\mathcal{A}$ and $\mathcal{B}$ respectively have the same size k. Therefore we will use $A^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ and $B^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ to denote the set of nodes agitated by $\mathcal{A}$ and $\mathcal{B}$ respectively, in the games above. We define $a^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ and $b^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ to be the size of $A^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ and $B^G_{\mathcal{A},\mathcal{B}}(\phi, k)$ respectively.

3. Strategies for the games

3.1. Strategies for Alice

Alice always plays first. We use $\mathcal{A}$ to denote a strategy for Alice. The strategy $\mathcal{A}$ consists of two programs to choose an initial set X and to choose a node a_i at each odd steps $2i + 1$ respectively. We use α and Γ to denote the two programs of $\mathcal{A}$, denoted by $\mathcal{A} = (\alpha, \Gamma)$.

Program α

It chooses an initial set X (of size bounded by a polynomial of $\log n$). Since Alice is the adversary player, it chooses X arbitrarily. However, in our experiments, we assume that Program α of $\mathcal{A}$ chooses the top degree nodes as its initial set X.

Program Γ of $\mathcal{A}$ will choose a node a_i at each odd step of the games. By the same reason as above, Alice may choose a_i arbitrarily for each i. However in our experiments, we assume that $\mathcal{A}$ chooses a_i by the following Program Γ.

Program Γ

Program Γ tries to maximize its payoffs during the games. At an old step $2i+1$, let $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ be the sets defined at the end of step $2i$ by Alice and Bob respectively. Alice will choose a_i by the following program.

(1) (Guaranteed to win) If there is an $x \notin A^G_{\mathcal{A},\mathcal{B}} \cup B^G_{\mathcal{A},\mathcal{B}}$ such that $f(A^G_{\mathcal{A},\mathcal{B}}; x) \geq \phi(x)$, and $f(B^G_{\mathcal{A},\mathcal{B}}; x) < \phi(x)$, then let a_i be the x such that the degree of x is maximized.
(2) (The highest probability to win) Otherwise and there is an $x \notin A^G_{\mathcal{A},\mathcal{B}} \cup B^G_{\mathcal{A},\mathcal{B}}$ such that $f(A^G_{\mathcal{A},\mathcal{B}}; x) \geq \phi(x)$. Then for each such an x, let $l(x)$ and $r(x)$ be the numbers of neighbors of x in $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ respectively. Let $\xi(x) = \frac{l(x)}{l(x)+r(x)}$. Let ξ_0 be the greatest $\xi(x)$ for all such x's. Choose a_i to be the x such that the degree $d(x)$ is maximized, and such that $\xi(x) = \xi_0$.
(3) If a_i is defined, written by $a_i \downarrow$, then go on the game to decide a_i to be in either $A^G_{\mathcal{A},\mathcal{B}}$ or $B^G_{\mathcal{A},\mathcal{B}}$. Otherwise, we say that a_i is undefined, written by $a_i \uparrow$. In this case,
   (a) If $b_{i-1} \uparrow$, then the game is over.
   (b) Otherwise, then go on to step $2(i+1)$.

3.2. Strategies for Bob

We use $\mathcal{B}$ to denote a strategy of Bob.

A strategy $\mathcal{B}$ consists of two programs, the first is to choose the initial set Y, the second is to choose b_i at each even steps $2i+2$. We use β and Δ to denote the two programs of $\mathcal{B}$. In this case, $\mathcal{B} = (\beta, \Delta)$. The program β will choose a set Y with the same size as that of X, the set chosen by Alice. Program Δ will decide the nodes b_i at even steps $2i+2$.

Let $G = (V, E)$ be a network. Suppose that each node $v \in V$ is associated with a threshold $\phi(v)$. Let X be the initial set chosen by Alice. Clearly, there are many algorithms for β. Here we introduce 2 natural programs for β, denoted by β_1 and β_2 respectively.

Program β_1

(1) Suppose that $y_1, y_2, \ldots, y_l$ are all nodes $y \notin X$ having a neighbor in X, listed by decreasing order of degrees, and
(2) Then define $Y = \{y_1, y_2, \ldots, y_k\}$, where $k = |X|$.

Program β_1 is the most simple one which chooses the highest degree neighbors of X.

Program β_2

(1) Let Z be the set of all nodes z satisfying:
   – $z \notin X$, and
   – $f(X; z) \geq \phi(z)$.
(2) Suppose that $y_1, y_2, \ldots, y_l$ is the list of all the nodes $y \notin X$, satisfying:
   – $n(y_j, Z) \geq n(y_{j+1}, Z)$, and
   – $d(y_j) \geq d(y_{j+1})$,
   where $n(y, Z)$ is the number of edges between y and nodes in Z.
(3) Bob chooses $Y = \{y_1, y_2, \ldots, y_k\}$, for $k = |X|$.

β_2 chooses the highest degree nodes that have highest influence on nodes that are probably agitated by the initial set X chosen by $\mathcal{A}$.

At step $2i+2$, Bob chooses b_i as follows. Let $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ be the sets defined at the end of step $2i+1$ by Alice and Bob respectively. Bob will choose b_i by the following program.

Program Δ

(1) (Guaranteed to win) If there is an $x \notin A^G_{\mathcal{A},\mathcal{B}} \cup B^G_{\mathcal{A},\mathcal{B}}$ such that $f(B^G_{\mathcal{A},\mathcal{B}}; x) \geq \phi(x)$, and $f(A^G_{\mathcal{A},\mathcal{B}}; x) < \phi(x)$, then let b_i be the x such that the degree of x is maximized.
   This means that the highest priority is to choose the node for which Bob is guaranteed to win, and that if there are more than one such nodes, then Δ chooses the highest degree node.
(2) (The highest probability to win) Otherwise and there is an $x \notin A^G_{\mathcal{A},\mathcal{B}} \cup B^G_{\mathcal{A},\mathcal{B}}$ such that $f(B^G_{\mathcal{A},\mathcal{B}}; x) \geq \phi(x)$. Then for each such an x, let $l(x)$ and $r(x)$ be the numbers of neighbors of x in $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ respectively. Let $\eta(x) = \frac{r(x)}{l(x)+r(x)}$. Define b_i to be the node x with maximal degree such that $\eta(x)$ is maximized.
   This means that if there is no node for which Bob can be guaranteed to win, then chooses the node for which Bob has the highest probability to win.
(3) If b_i is defined, written by $b_i \downarrow$, then go on the game to decide whether or not b_i is in either $A^G_{\mathcal{A},\mathcal{B}}$ or $B^G_{\mathcal{A},\mathcal{B}}$. Otherwise, then we say that b_i is undefined, written by $b_i \uparrow$, in which case,
   (a) If $a_i \uparrow$, then the game is over.

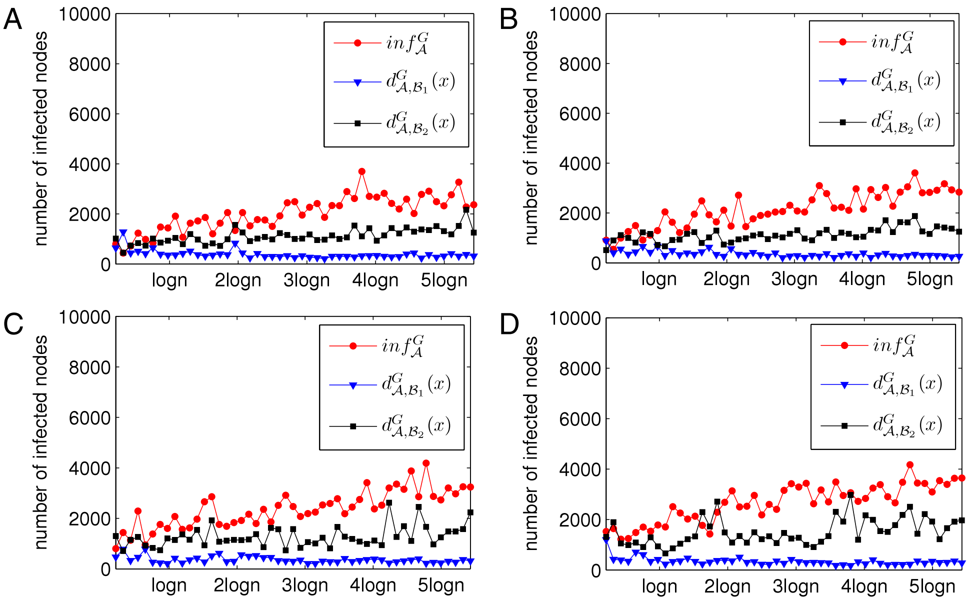

Fig. 1. Random graphs of the Erdös–Rényi model have equilibrium games. A, B, C and D of the figure depict the curves for networks of the ER model for expected average number of edges $d = 8, 9, 10, 11$ respectively. In each of the A–D, the three curves correspond to the sizes of infection sets $\inf_{\mathcal{A}}^{G}$, and the sizes of differences of sizes of sets $A_{\mathcal{A},\mathcal{B}_j}^{G}$ and $B_{\mathcal{A},\mathcal{B}_j}^{G}$, i.e., $d_{\mathcal{A},\mathcal{B}_j}^{G}$ for each $j \in \{1, 2\}$.

(b) Otherwise, then go on to step $2(i + 1) + 1$.
If b_i is undefined, then Bob passes this step. In this case, if Alice passed the last step, then the game is over, else, the game goes on to the next step.

For each $j \in \{1, 2\}$, we define $\mathcal{B}_j = (\beta_j, \Delta)$. We notice that the strategy $\mathcal{B}_1$ depends only on the structure of G and the set X chosen by Alice, and that strategy $\mathcal{B}_2$ depends on the structure of G, the set X chosen by Alice and the thresholds of vertices of V.

Let $G = (V, E)$ be a network. Our games depend on the choices of thresholds $\phi(v)$ for all $v \in V$. In practice, the thresholds may be quite arbitrary. In the paper, we consider the natural choice of random thresholds. In this case, for a node $v \in V$, $\phi(v)$ is defined as $\frac{r}{d}$, where r is uniformly and randomness chosen among the set $\{1, 2, \ldots, d\}$, and d is the degree of v in G.

4. Equilibrium games in networks

Let $G = (V, E)$ be a network. Suppose that ϕ is a threshold function of V. For any k, we define

$$d_{\mathcal{A},\mathcal{B}}^{G}(\phi, k) = a_{\mathcal{A},\mathcal{B}}^{G}(\phi, k) - b_{\mathcal{A},\mathcal{B}}^{G}(\phi, k).$$

We say that G *has an equilibrium game*, if for any strategy $\mathcal{A}$, there is a strategy $\mathcal{B}$ such that for any k bounded by a polynomial of $\log n$, the following property holds:

$$\Pr_{\phi}[d_{\mathcal{A},\mathcal{B}}^{G}(\phi, k) = o(n)] = 1 - o(1),$$

where ϕ is a random threshold function, k is bounded by a polynomial of $\log n$, and $n = |V|$ is sufficiently large.

We will investigate the equilibrium games of the networks of classical models.

4.1. *The ER model*

The first is the Erdös–Rényi model (ER model, for short) [9,10]. In this model, we are given a number n, and a number p, and n nodes $1, 2, \ldots, n$. For every pair $\{i, j\}$ of nodes i and j, we create an edge between i and j with probability p. By definition, every node i randomly and uniformly links to some nodes.

We depict the curves of $d_{\mathcal{A},\mathcal{B}_j}^{G}$ for $\mathcal{A}$ and $\mathcal{B}_j$ for $j = 1, 2$ for random graphs of the ER model in Fig. 1.

From Fig. 1, we observe the following results:

(1) The curve of $\inf_{\mathcal{A}}^{G}$ is higher than that of $d_{\mathcal{A},\mathcal{B}_j}^{G}$ for both $j = 1$ and 2.
(2) The curve of $d_{\mathcal{A},\mathcal{B}_1}^{G}$ is almost flat.
(3) By (1) and (2) above, Bob has a strategy $\mathcal{B} = (\beta_1, \Delta)$ which can balance the influence of Alice in networks of the ER model.

The results above show that networks of the ER model have an equilibrium game.

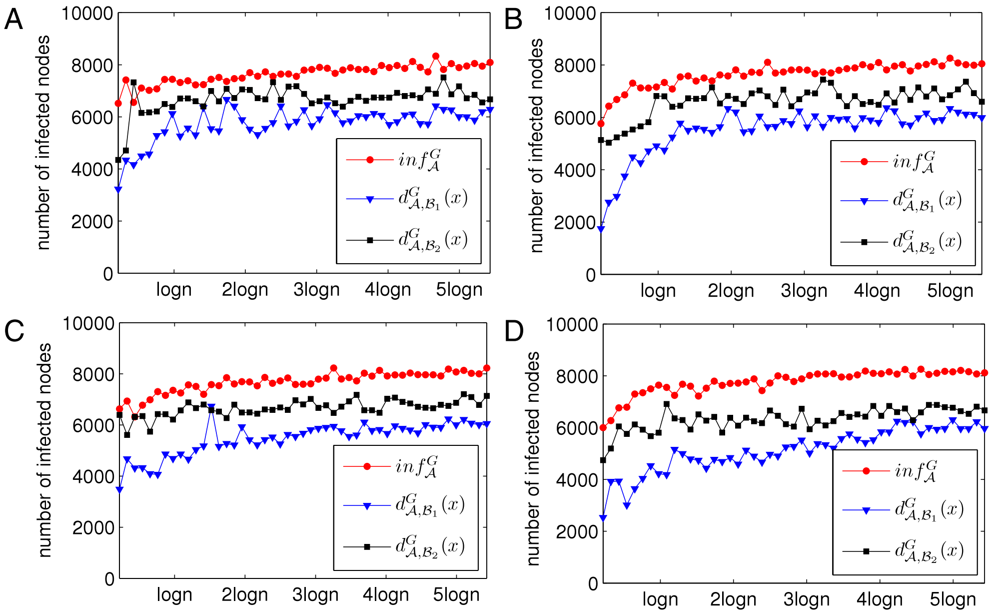

Fig. 2. Networks of the PA mode fail to have an equilibrium game. A, B, C and D of the figure depict the curves for networks of the ER model for expected average number of edges $d = 8, 9, 10, 11$ respectively. In each of the A–D, the three curves correspond to the sizes of infection sets $\inf^G_{\mathcal{A}}$, and the sizes of differences of sizes of sets $A^G_{\mathcal{A},\mathcal{B}_j}$ and $B^G_{\mathcal{A},\mathcal{B}_j}$, i.e., $d^G_{\mathcal{A},\mathcal{B}_j}$ for each $j \in \{1, 2\}$.

4.2. The PA model

The second is the preferential attachment (PA, for short) model [2,1]. Given a natural number d, the model constructs networks by steps: At step 0, we choose G_0 to be an initial graph. At step $t + 1$, create a new node, v say, and create d edges from v to nodes in G_t chosen with probability proportional to the degrees in G_t, where G_t is the network constructed at the end of step t.

We depict the curves of $d^G_{\mathcal{A},\mathcal{B}_j}$ for $\mathcal{A}$ and $\mathcal{B}_j$ for $j = 1, 2$ for networks of the PA model in Fig. 2.

From Fig. 2, we have the following results:

(1) The curve of $\inf^G_{\mathcal{A}}$ is higher than that of $d^G_{\mathcal{A},\mathcal{B}_j}$ for both $j = 1$ and 2.
(2) The curve of $d^G_{\mathcal{A},\mathcal{B}_2}$ is higher than that of $d^G_{\mathcal{A},\mathcal{B}_1}$.
(3) The curve of $d^G_{\mathcal{A},\mathcal{B}_1}$ is slightly lower than that of the $\inf^G_{\mathcal{A}}$.

The results imply that none of the strategies of Bob can balance the influence of Alice, so that non-trivial networks of the PA model fail to have an equilibrium game.

5. Equilibrium games—resisting cascading

Given strategies $\mathcal{A}$ and $\mathcal{B}$ for Alice and Bob respectively, we use $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ to denote the sets agitated by Alice and Bob with strategies $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $a^G_{\mathcal{A},\mathcal{B}}$ and $b^G_{\mathcal{A},\mathcal{B}}$ be the numbers of nodes in $A^G_{\mathcal{A},\mathcal{B}}$ and $B^G_{\mathcal{A},\mathcal{B}}$ respectively.

We say that G is *resistable*, if for any strategy $\mathcal{A}$ of Alice, there is a strategy $\mathcal{B}$ for Bob such that the following property holds:

$$\Pr_{\phi}[a^G_{\mathcal{A},\mathcal{B}}(\phi, k) = o(n)] = 1 - o(1)$$

where ϕ is a random threshold function, k is bounded by a polynomial of $\log n$, $n = |V|$ is sufficiently large.

By definition, if G is resistable, then for any strategy $\mathcal{A}$ of an adversary player Alice, Bob has a strategy $\mathcal{B}$ to resist the global cascading failure against the attacks by the strategy of Alice.

5.1. ER model

To characterize the networks in which cascading failure is resistable, we first analyze the classic models of networks.

In Fig. 3, we depict the curves of $\inf^G_{\mathcal{A}}$, and $a^G_{\mathcal{A},\mathcal{B}_j}$ for $j = 1, 2$ respectively. In A–D of the figure, we depict the curves for random graphs of the Erdös–Rényi model for $d = 8, 9, 10, 11$ respectively, where d is the expected average number of edges.

From Fig. 3, we observe that

(1) The strategy β_1 of Bob is better than β_2 in resisting cascading failure of attacks in networks of the ER model.
(2) Bob has a strategy $\mathcal{B}$ to resist the global cascading failure of a small number of attacks by Alice in networks of the ER model, for which randomness is the mechanism.

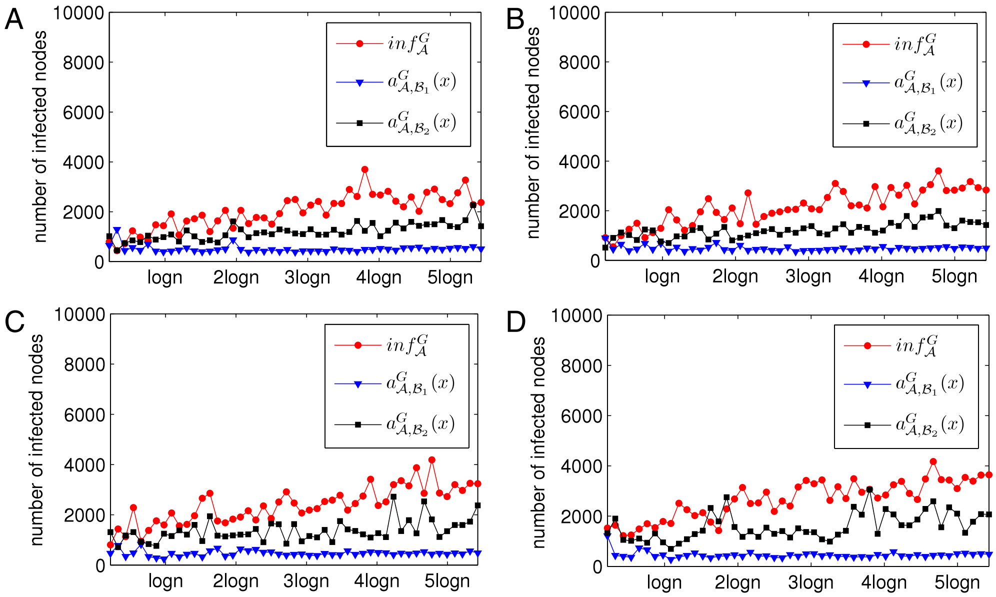

Fig. 3. Random graphs of the Erdös–Rényi model have equilibrium games. A, B, C and D of the figure depict the curves for networks of the ER model for expected average number of edges $d = 8, 9, 10, 11$ respectively. In each of the A–D, the five curves correspond to the sizes of infection sets $\inf_{\mathcal{A}}^{G}$, and the sizes of sets $A_{\mathcal{A},\mathcal{B}_j}^{G}$, i.e., $a_{\mathcal{A},\mathcal{B}_j}^{G}$ for each $j \in \{1, 2\}$.

The results above show that networks of the ER model are resistable, for which randomness is the mechanism.

5.2. The preferential attachment model

In Fig. 4, we depict the curves of the sizes of $\inf_{\mathcal{A}}^{G}$, $A_{\mathcal{A},\mathcal{B}_j}^{G}$ for $j \in \{1, 2\}$ respectively. The networks of the preferential attachment model have average number of edges $d = 8, 9, 10, 11$ respectively.

From Fig. 4, we observe that in networks of the PA model, the first player Alice always has a winning strategy in generating a global cascading failure by a small number of attacks, and that Bob fails to resist cascading failure of an adversary player Alice.

By comparing the results in Sections 4.1, 5.1 and the results in Sections 4.2, 5.2, we have that networks of the ER model have an equilibrium game, and are resistable, and that in networks of the PA model fail to have equilibrium games.

This poses a natural question: Are there dynamic and scale-free networks which have equilibrium games? To answer this question, we have to explore new mechanisms of networks. We examine the homophyly model of networks proposed in Ref. [11].

6. Homophyly networks

Given a homophyly exponent a (a positive number), and a natural number d, the homophyly model, written $\mathcal{H}$, constructs a network by steps as follows. Initially, let G_d be an initial d-regular graph such that each node is associated with a distinct color, and called a seed node. For $i + 1 > d$, suppose that G_i has been defined, let $p_i = 1/(\log i)^a$. We create a new node, v say. With probability p_i, v chooses a new color, in which case, we call v a seed node, and create d edges from v to nodes in G_i choosing with probability proportional to the degrees of nodes in G_i. Otherwise, v chooses an old color, in which case, v chooses randomly and uniformly an old color as its own color, and create d edges from v to nodes of the same color as that of v, chosen with probability proportional to the degrees of nodes in G_i.

In Ref. [11], it has been shown that for a network of n nodes, G say, constructed from the homophyly model, the following properties hold: (1) (power law) G follows a power law, (2) (small diameter) the diameter of G is $O(\log^2 n)$, (3) (small community phenomenon) nodes of the same color form a quality community of size bounded by a polynomial of $\log n$, (4) (internal centrality) the induced subgraph of a community follows a power law, (5) (external centrality) the external links of a community is bounded by $O(\log\log n)$, and (6) (local communication law) diameters of communities are bounded by $O(\log\log n)$.

These properties may play some roles in equilibrium games in the networks.

7. Games in homophyly networks

Let $G = (V, E)$ be a homophyly network. For a homochromatic set Z, the induced subgraph G_Z of Z in G is a natural community of size bounded by $\log^{O(1)} n$. By the construction, a community G_Z has a seed node, z_0 say. Since G_Z is small,

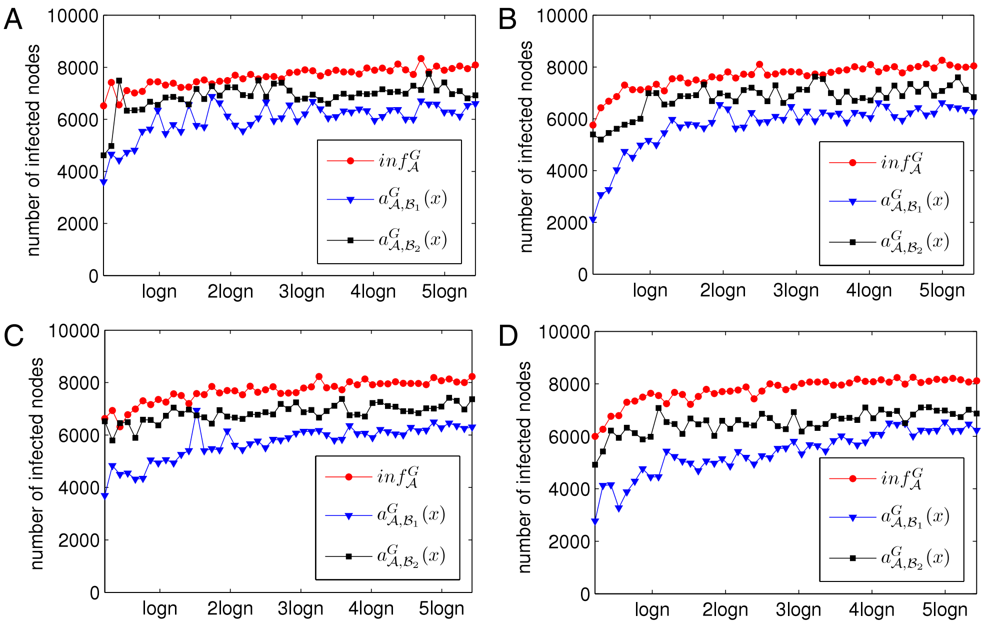

Fig. 4. Networks of the PA mode fail to have an equilibrium game. A, B, C and D of the figure depict the curves for networks of the ER model for expected average number of edges $d = 8, 9, 10, 11$ respectively. In each of the A–D, the five curves correspond to the sizes of infection sets $\inf_{\mathcal{A}}^{G}$, and the sizes of sets $A_{\mathcal{A},\mathcal{B}_j}^{G}$, i.e., $a_{\mathcal{A},\mathcal{B}_j}^{G}$ for each $j \in \{1, 2\}$.

the main problem for $\mathcal{B}$ is to resist the agitations of nodes in the neighbor communities of Z. By the construction, for a community G_Z, we have that there are many seed nodes which link to the seed node z_0 of Z, that for each non-seed node $z \in Z$, there are few edges from z to seed nodes outside of Z, and that there are some seed nodes linking to the non-seed nodes of Z.

To understand the structure of G, we analyze an example. Suppose that X, Y and Z are three communities of G. Let x_0, y_0 and z_0 be the seed nodes of X, Y and Z respectively. Suppose that $\mathcal{A}$ chooses x_0. In this case, $\mathcal{A}$ may have more chances to agitate elements in the community X. However even if there is an edge between x_0 and y_0, since y_0 is the seed of community Y, so that $\mathcal{A}$ is hard to agitate y_0. The problem is if there is an edge (x_0, y) for some non-seed node $y \in Y$, then $\mathcal{A}$ is easy to agitate y. If y has been agitated by $\mathcal{A}$ through x_0, then y_0 would be easily agitated by $\mathcal{A}$ through y.

The intuition above suggests that the strategy $\mathcal{B}$ for Bob should choose the seed nodes of the communities in which there are nodes that are easily agitated by the initial choices X chosen by Program α of $\mathcal{A}$.

Therefore, the community structure of the homophyly networks allows us to design new programs for $\mathcal{B}$ to choose its initial set Y. We will give one such program.

7.1. Strategies

For a homophyly network $G = (V, E)$, we introduce one more strategy for Bob to choose Y.

Program β_3

(1) Let $X = \{x_1, x_2, \ldots, x_k\}$ be the set chosen by Program α of strategy $\mathcal{A}$.
(2) Let C_j be the community of x_j for each $j \in \{1, 2, \ldots, k\}$. Let $C = \cup_j C_j$ be the union of all the communities C_j. Let $N(C)$ be the set of all neighbors of nodes in C. Define $Z = C \cup N(C)$.
The motivation of this step is that the choices X of $\mathcal{A}$ have more influences on nodes in the union C of communities of all $x \in X$ and the neighbors of C.
(3) Suppose that $y_1, y_2, \ldots, y_l$ are all the seed nodes y satisfying:
(a) $y \notin X$,
(b) there is a $z \in Z$ such that y and z share the same color, and
(c) $d(y_i) \geq d(y_{i+1})$ for all $i \in \{1, 2, \ldots, l\}$, where $d(y)$ is the degree of y.
This steps means that $\mathcal{B}$ resists $\mathcal{A}$ by choosing the highest degree seed nodes of communities which contains elements in C or the neighbors of C.
(4) Bob defines $Y = \{y_1, y_2, \ldots, y_k\}$.

β_3 chooses the highest degree seed nodes of communities that either contain some x_j or some neighbors of nodes in the union C of all the communities C_j. The motivation of β_3 is that if Program α of $\mathcal{A}$ chooses X as its initial set, then $\mathcal{A}$ has more power to agitate nodes in the communities C_j for each $j \in \{1, 2, \ldots, k\}$. In this case, the best choice for $\mathcal{B}$ is to select the most

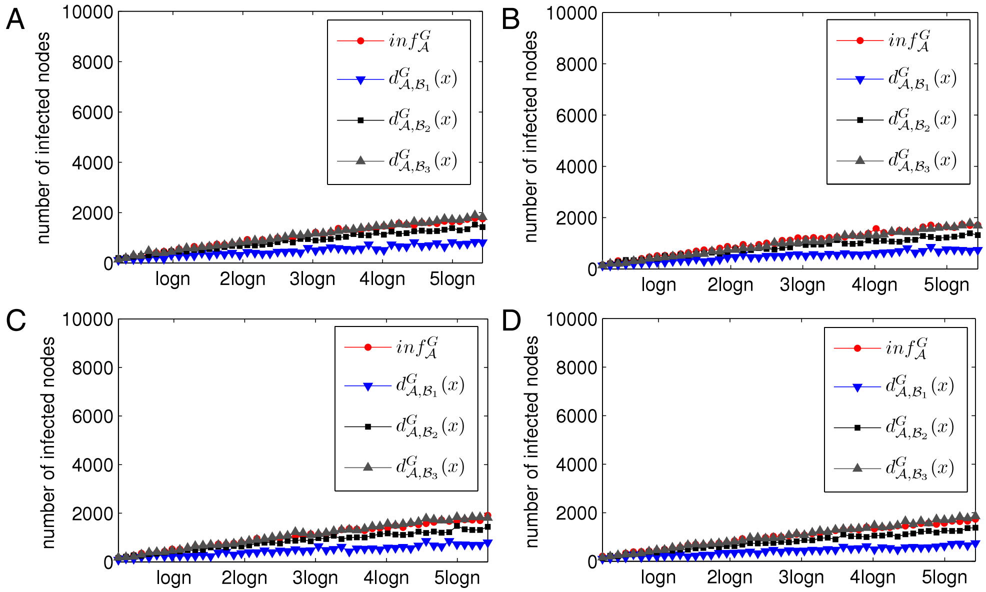

Fig. 5. Homophyly networks have abstract equilibrium games. The curves in A, B, C, D correspond to homophyly networks with $n = 10,000$, $a = 1.5$ and $d = 8, 9, 10, 11$ respectively. In each of the A–D, the three curves correspond to the sizes of infection sets $\inf_{\mathcal{A}}^{G}$, and the sizes of differences of sizes of sets $A_{\mathcal{A},\mathcal{B}_j}^{G}$ and $B_{\mathcal{A},\mathcal{B}_j}^{G}$, i.e., $d_{\mathcal{A},\mathcal{B}_j}^{G}$ for $j \in \{1, 2, 3\}$ respectively.

influential seed nodes whose communities link to the communities C_j for $j \in \{1, 2, \ldots, k\}$. By the construction of G, there is a large number of seed nodes whose communities link to nodes in C, the union of all the communities C_j, $j \in \{1, 2, \ldots, k\}$. Therefore $\mathcal{B}$ has chance to select high degree seed nodes which on one hand resist the cascading by attacks of $\mathcal{A}$, and on the other hand, $\mathcal{B}$ opens chances to agitate more nodes. Clearly, β_3 is designed based on the community structure of the homophyly networks.

Define $\mathcal{B}_3 = (\beta_3, \Delta)$. Then $\mathcal{B}_3$, is a strategy for Bob which assumed and used the community structures of the homophyly networks.

7.2. Equilibrium games in the homophyly networks

We depict the curves of $\inf_{\mathcal{A}}^{G}$, and $d_{\mathcal{A},\mathcal{B}_j}^{G}$ for $\mathcal{A}$ and $\mathcal{B}_j$ for $j = 1, 2, 3$ for homophyly networks in Fig. 5.

From Fig. 5, we observe the following:

(1) The curve of $d_{\mathcal{A},\mathcal{B}_1}^{G}$ is the lowest among all 4 curves, which is almost flat.

(2) The curve of $\inf_{\mathcal{A}}^{G}$ increases much more slower than that of networks of the PA model in Fig. 2.

(1) implies that homophyly networks have equilibrium games, and (2) implies that homophyly does hinder cascading failure in the homophyly networks.

7.3. The equilibrium games: resisting cascading in homophyly networks

Fig. 6 depicts the curves of $\inf_{\mathcal{A}}^{G}$ and $a_{\mathcal{A},\mathcal{B}_j}^{G}$, for $j \in \{1, 2, 3\}$ for homophyly networks.

From Fig. 6, we observe that all the curves of $\inf_{\mathcal{A}}^{G}$, $a_{\mathcal{A},\mathcal{B}_j}^{G}$, for $j \in \{1, 2, 3\}$ are the same. Therefore homophyly networks are non-resistable.

7.4. Theoretical analysis of homophyly networks

Why do homophyly networks have an equilibrium game as shown in Fig. 5? We analyze the experiments as follows. In a homophyly network $G = (V, E)$, for a homochromatic set $Z \subset V$, the induced subgraph G_Z is a natural community of G. Clearly, all the natural communities have sizes bounded by $\log^{O(1)} n$.

Let $x \in X$ be a node chosen by Alice. Then we have:

(1) If x is the seed node of a community Z, then there are quite a few communities whose seeds link to x or nodes in its community Z, in which case, Bob has chance to select seed node y_0 say of a neighbor community of x or nodes in Z such that the degree of y_0 is large, so that y_0 has strong influence.

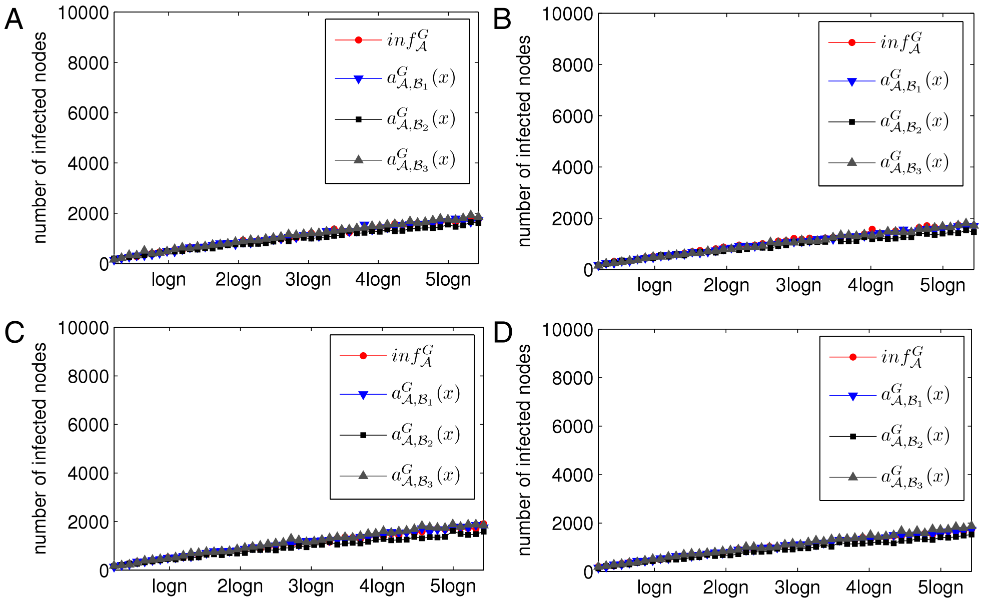

Fig. 6. Homophyly networks have abstract equilibrium games. The curves in A, B, C, D correspond to homophyly networks with $n = 10{,}000$, $a = 1.5$ and $d = 8, 9, 10, 11$ respectively. In each of the A–D, the curves represent the sizes of sets $\inf^G$, $A^G_{\mathcal{A},\mathcal{B}_j}$, i.e., $a^G_{\mathcal{A},\mathcal{B}_j}$, for $j \in \{1, 2, 3\}$ respectively.

(2) If x is a non-seed node of a community Z, then x has influence mainly within the community Z, and the degree of x cannot be large. In this case, Bob may simply select the seed node x_0 of Z to resist the influence of x on nodes outside of the community Z.

In either case, the choice of Bob is not significantly less powerful than the choice of Alice. This intuitively explains the reason why homophyly networks have an equilibrium game.

The experiments in Fig. 5 seems to imply that for appropriately large network, $G = (V, E)$ say, of the homophyly model, almost surely (that is, with probability $1 - o(1)$), the following occurs:

For any strategy $\mathcal{A}$ of Alice, there is a strategy $\mathcal{B}$ for Bob such that $d^G_{\mathcal{A},\mathcal{B}} = o(n)$.

However it is a grand challenge to theoretically prove the results, for which a proof calls for a complete understanding of the experiments, and the homophyly model.

8. Games in real networks

In this section, we implement experiments on some real network data. We depict the curves of both $d^G_{\mathcal{A},\mathcal{B}_j}$ and $a^G_{\mathcal{A},\mathcal{B}_j}$ for the games on 9 real networks.

8.1. Financial network

Our first example is a financial network given in Ref. [12]. We depict the curves of infection sets $\inf^G_{\mathcal{A}}$, $d^G_{\mathcal{A},\mathcal{B}_j}$ and $a^G_{\mathcal{A},\mathcal{B}_j}$ for $j = 1, 2$ of the financial network in Fig. 7.

By observing Fig. 7, we know that Bob has no effective strategy to either balance the influence of Alice or to resist the cascading failure by attacks of Alice. Therefore the financial network fails to have an equilibrium game.

8.2. Equilibrium games in real networks

In this part, we depict the curves of $\inf^G_{\mathcal{A}}$ and $d^G_{\mathcal{A},\mathcal{B}_j}$ for $j = 1, 2$ on more real network data.

Fig. 8 depicts the experiments of games in 4 Amazon networks, and Fig. 9 depicts the experiments of the game in 4 social networks. All these networks are from Stanford Large Network Dataset Collection.

The Amazon network is product co-purchasing network, specifically, which is based on Customers Who Bought This Item Also Bought feature of the Amazon website. Each node represents a product, and if a product i is frequently co-purchased with product j, the graph contains a directed edge from i to j. The four networks contain 262 111 nodes and 1 234 877 edges, 400 727 nodes and 3 200 440 edges, 410 236 nodes and 3 356 824 edges, and 403 394 nodes and 3 387 388 edges respectively. The data of the four networks A, B, C, D were collected in March 02 2003, March 12 2003, May 05 2003, June 01 2003 respectively.

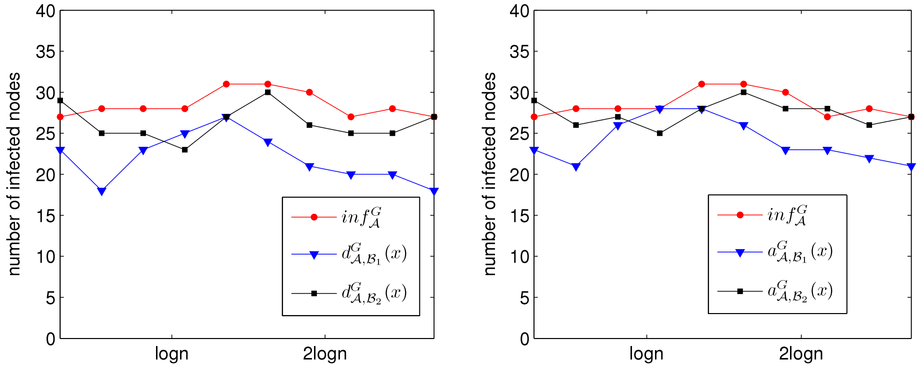

Fig. 7. Two-party games on the financial network.

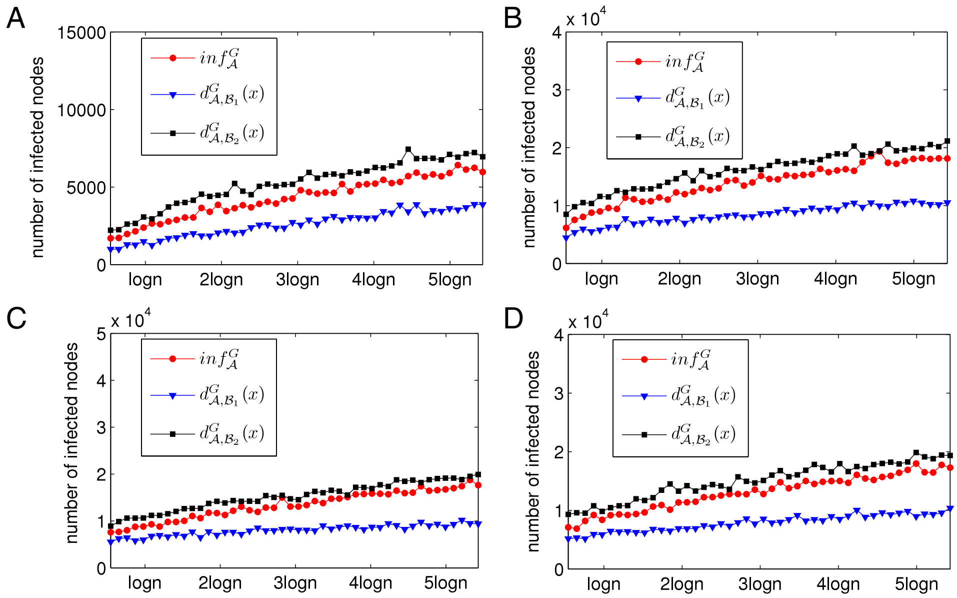

Fig. 8. Games on the Amazon networks.

By observing Fig. 8, we have that, in each of the networks, Bob has a strategy to significantly balance the influence of Alice, and that the curves are similar to that in Fig. 5 for the homophyly networks.

The 4 social networks in Fig. 9 are: The networks in A and B are from Slashdot, on which users can tag each other as friends or foes. Nodes of these networks represent users and edges represent friends or foes relationship between the users of Slashdot. Network A contains 77 360 nodes and 905 468 edges, and B has 82 168 nodes and 948 464 edges, which were collected in November 2008 and February 2009, respectively. C is a who-trust-whom online social network of a general consumer review site Epinions.com. Nodes of this network represent members of this site and a directed edge from *i* to *j* represents user *i* trusts user *j*. The numbers of nodes and edges are 75 879 and 508 837 respectively. Network D contains all the Wikipedia voting data from the inception of Wikipedia till January 2008. Nodes in the network represent Wikipedia users and a directed edge from node *i* to node *j* represents that user *i* voted on *j*. There are 7115 nodes and 103 689 edges in it.

By observing Fig. 9, we know that Bob fails to balance the influence of Alice in each of the networks, and that the curves in 9 are similar to that of networks of the PA model in Fig. 2.

Therefore, there are real networks which have equilibrium games, and at the same time there are real networks which fail to have equilibrium games.

8.3. *Resisting cascading failure in real networks*

In Fig. 10, we depict the curves of $\inf_{\mathcal{A}}^{G}$ and $a_{\mathcal{A},\mathcal{B}_j}^{G}$ for $j = 1, 2$ for the games in the 4 Amazon networks.

In Fig. 11, we depict the curves of $\inf_{\mathcal{A}}^{G}$ and $a_{\mathcal{A},\mathcal{B}_j}^{G}$ for $j = 1, 2$ for the games in the 4 social networks.

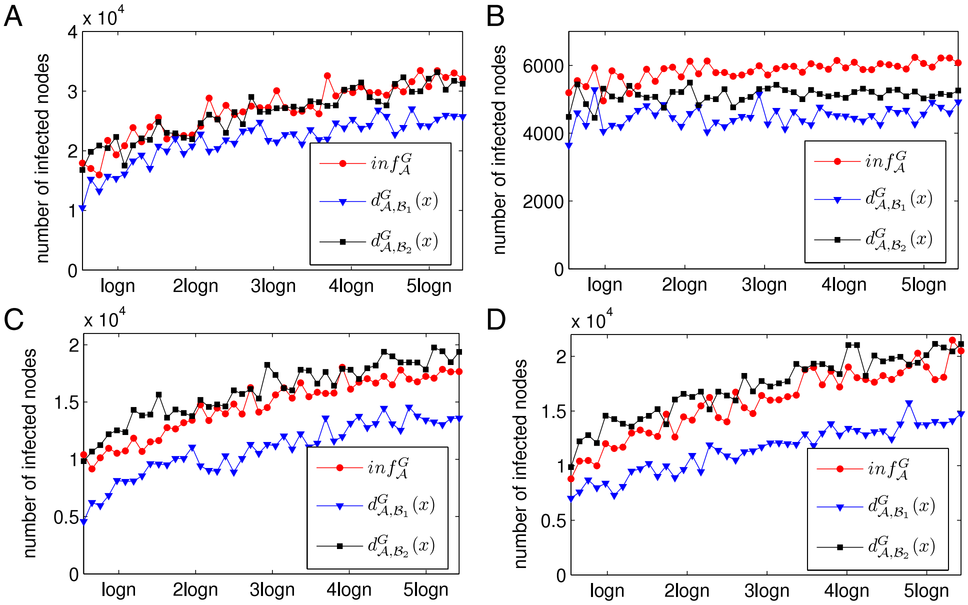

Fig. 9. Games on the social networks.

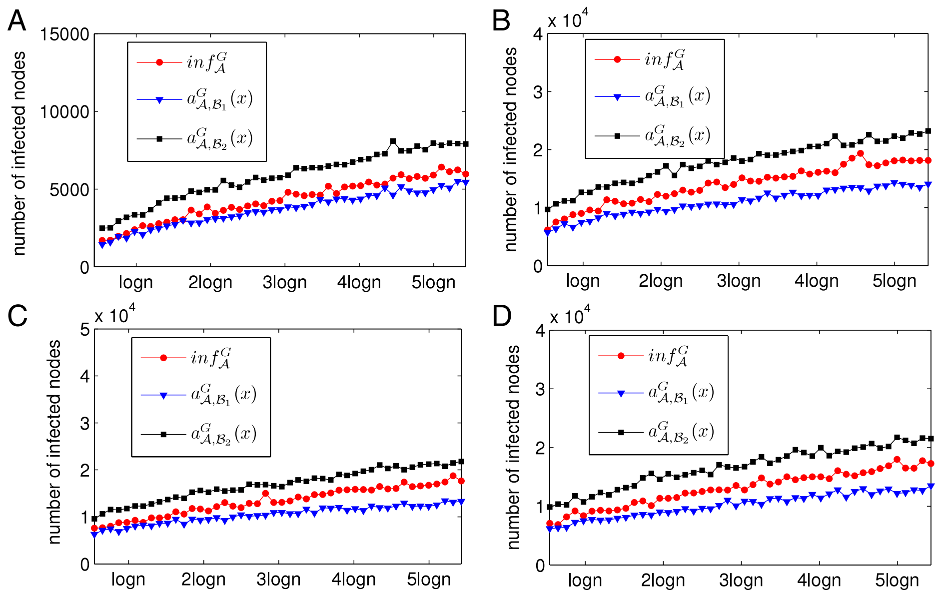

Fig. 10. Games on the Amazon networks.

By observing Figs. 10 and 11, we have that in each of the real networks, Bob fails to have a strategy to resist cascading failures by attacks of Alice. This result poses a fundamental question: How can we resist global cascading failures by a small number of attacks by an adversary player Alice in a power law or real world network?

9. Conclusions

We proposed a two-party game by introducing a second player Bob to balance the influence or to resist global cascading failure of attacks by an adversary player Alice. We showed that random graphs of the Erdös–Rényi model have an equilibrium game and have a strategy for Bob to resist cascading failures by attacks of an adversary player Alice, for which randomness is the mechanism. We showed that networks of the PA model fail to have an equilibrium game, that homophyly networks have an equilibrium game for which homophyly and preferential attachment are the underlying mechanisms, and that for power law networks of either the PA model or the homophyly model, there is no strategy for Bob to resist cascading failures in

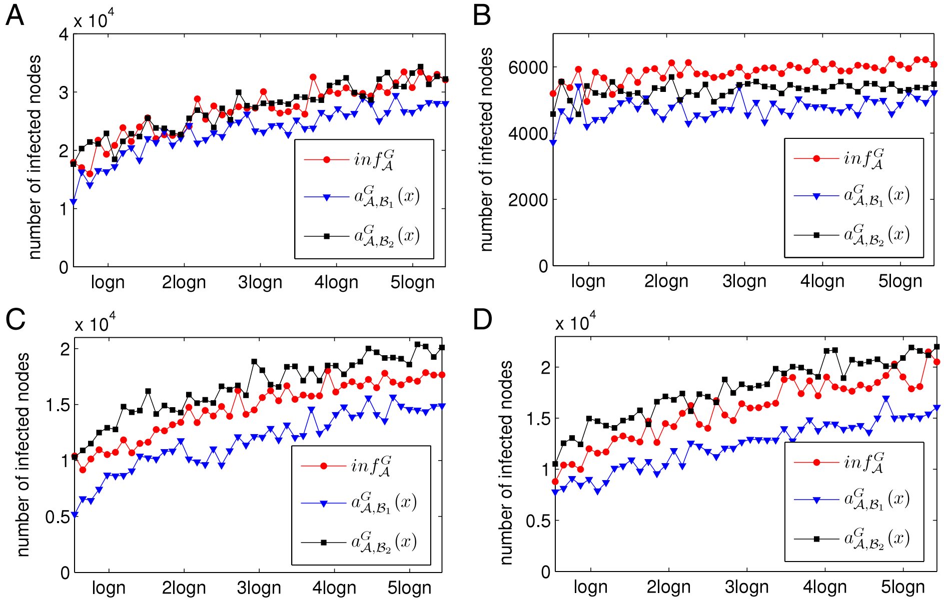

Fig. 11. Games on the social networks.

the networks by attacks of the adversary player Alice. We also showed that there are real networks which have equilibrium games, but some others fail to have, and that for most real networks, there is no any strategy for Bob to resist cascading failures by attacks on the networks by an adversary player Alice. Our results pose new fundamental open questions such as: What mechanisms and mathematical properties of networks guarantee the existence of equilibrium games in the networks? How can we resist cascading failures by attacks of a small number of nodes in a power law network or in a real network data?

Methods. In all the experiments in each of Figs. 1–11, the curves correspond to the greatest values of $\inf_{\mathcal{A}}^{G}$, $d_{\mathcal{A},\mathcal{B}_j}^{G}$ and $a_{\mathcal{A},\mathcal{B}_j}^{G}$ for $j = 1, 2, 3$ among 100 times of the games each of which depends on different choices of the random thresholds.

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