One-Bit Matrix Completion With Time-Varying Sampling Thresholds

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***Abstract*—We explore the impact of coarse quantization on matrix completion in the extreme scenario of *generalized one-bit sampling*, where the matrix entries are compared with time-varying threshold levels. In particular, instead of observing a subset of high-resolution entries of a low-rank matrix, we have access to a small number of *one-bit samples*, generated as a result of these comparisons. To recover the low-rank matrix from its highly-quantized known entries, we first formulate the one-bit matrix completion problem with time-varying thresholds as a nuclear norm minimization problem, with one-bit sampled information manifested as linear inequality feasibility constraints. We then modify the popular *singular value thresholding* (SVT) algorithm to accommodate these inequality constraints, resulting in the creation of the *One-Bit SVT* (OB-SVT). Our findings demonstrate that incorporating multiple time-varying sampling threshold sequences in one-bit matrix completion can significantly improve the performance of the matrix completion algorithm. We perform numerical evaluations comparing our proposed algorithm with the maximum likelihood estimation method previously employed for one-bit matrix completion, and demonstrate that our approach can achieve a better recovery performance.**

***Index Terms*—Coarse quantization, matrix completion, one-bit sampling, singular value thresholding, time-varying thresholds.**

I. Introduction

Matrix completion, the recovery of an unknown low-rank matrix from limited information, is a pervasive challenge across various practical domains, including collaborative filtering [1], system identification [2], and sensor localization [3]. As a special case of low-rank matrix sensing, matrix completion poses unique challenges due to the sampling matrices potentially not satisfying the matrix restricted isometry property (RIP) conditions [4, 5].

The literature contains numerous methods for addressing this problem. For instance, one approach is the singular value thresholding (SVT) method proposed in [6, 7], which employs a projected gradient descent technique. Another method, the alternating minimization (AltMin) algorithm, is comprehensively discussed in [4]. Low-rank matrix factorization is suggested in [8] as a means of solving matrix completion problems using gradient descent approaches. The Sketchy Frank-Wolfe algorithm, explored in [9], is another option. Additionally, the OptSpace algorithm [10], one of the earliest nonconvex methods for matrix completion with proven results, utilizes gradient descent on the Grassmann manifold to converge to the optimal solution with high probability.

A prime example of this is the Netflix data matrix consisting of user ratings, which is presumed to be approximately low rank owing to the widely accepted notion that only a handful of factors significantly influence a person's taste or preference [5, 6]. Another emerging application of matrix completion is in waveform design for multiple-input and multiple-output (MIMO) radars [11, 12]. MIMO radars equipped with sparse sensing and matrix completion techniques have the potential to drastically decrease the amount of data necessary for precise target detection and estimation. This reduction in data volume can lead to a more efficient and accurate MIMO radar system [11, 13, 14].

Quantization is a crucial step in digital signal processing that converts continuous signals into discrete representations. However, to achieve high-resolution quantization, a large number of quantization levels are required, which can result in increased power consumption, manufacturing cost, and a reduced sampling rate in analog-to-digital converters (ADCs). To mitigate these issues, researchers have explored the use of fewer quantization bits, including the extreme case of one-bit quantization where the signals are compared with a fixed threshold at the ADCs, resulting in sign outputs [15, 16]. This approach enables high-rate sampling while reducing implementation cost and energy consumption compared to multi-bit ADCs.

However, one-bit quantization with a fixed threshold can lead to challenges in estimating the signal amplitude. To address this, recent studies have proposed the use of time-varying thresholds, which have been shown to improve signal recovery performance [17–19]. These findings have sparked further investigation into one-bit quantization techniques as a promising alternative to high-resolution signal processing. Several recent studies have explored one-bit quantization using time-varying sampling thresholds and have demonstrated improved estimation of signal characteristics [17, 20, 21].

The reconstruction of low-rank matrices from highly-quantized measurements, where the number of measurements is expected to be much smaller than that of unknowns, is an emerging research area in statistical signal processing and machine learning. This field of study has the potential to facilitate the development of cost-effective and efficient systems.

The theory behind matrix completion typically assumes that observations consist of continuous values within the matrix. However, the Netflix problem involves "quantized" observations that are restricted to integers between 1 and 5. This presents a challenge for high-resolution matrix completion techniques, as the impact of coarse quantization becomes more apparent. This issue is particularly prominent in recommender systems, where ratings are reduced to a single bit indicating a positive or negative rating (e.g., rating music on Pandora, determining the relevance of advertisements on Hulu, or evaluating posts on sites like MathOverflow). In such cases, the assumptions made in existing matrix completion theory do not hold true [22]. MIMO radar systems with one-bit ADC receivers provide another example where traditional matrix completion techniques cannot be used for waveform design due to coarse quantization applied to the received measurements.

In [22, 23], the initial attempt to address one-bit matrix completion involved developing theoretical guarantees under the generalized linear model. The authors derived a maximum likelihood estimate (MLE) based on a probability distribution determined by the real-valued noisy entries of the low-rank matrix. To constrain the MLE problem, the authors employed the nuclear and Frobenius norms,

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drawing inspiration from previous work on one-bit compressed sensing [22]. They utilized projected gradient descent to solve the regularized MLE obtained.

To derive the MLE and take advantage of time-varying thresholds (dithering), authors of [22] considered noisy measurements, where the noise can be seen as time-varying thresholds. However, as demonstrated in [17, 19, 21, 24], the design of time-varying thresholds is a critical aspect of one-bit sampling that can significantly improve signal reconstruction performance. However, by utilizing noise as our dithering, as was demonstrated in [22], we constrain our thresholds to follow the behavior of the noise, which is not under our control.

Dithering is particularly relevant in matrix completion scenarios like recommendation systems, where users may prefer to compare products rather than provide exact ratings. This approach can improve the user interface's usability and, in some cases, enhance the accuracy of ratings [22, 25]. For example, a user may evaluate an anime's quality by comparing it to their favorite show, such as Attack on Titan, and rate it on IMDb accordingly, or simply give it a thumbs-up or thumbs-down as one-bit data.

In this paper, we propose a model for one-bit matrix completion that incorporates time-varying sampling thresholds. We frame this as a nuclear norm minimization problem, constrained by a *linear inequality* system obtained from the one-bit sampling scheme. *Our formulation allows for the freedom to design or select appropriate thresholds to enhance the reconstruction performance.* We propose the *O*ne-*B*it *SVT* (OB-SVT) algorithm, which utilizes an SVT specifically developed to address nuclear norm minimization problems with linear inequality constraints. We numerically compare our proposed scheme with the MLE proposed by [22] and demonstrate that our method achieves superior performance in both noiseless and noisy scenarios.

Notation: Throughout this paper, we use bold lowercase and bold uppercase letters for vectors and matrices, respectively. We represent a vector $\mathbf{x}$ and a matrix $\mathbf{B}$ in terms of their elements as $\mathbf{x}=[x_i]$ and $\mathbf{x}=[B_{i,j}]$, respectively. The sets of complex and real numbers are $\mathbb{C}$ and $\mathbb{R}$, respectively; $(\cdot)^{\top}$, $(\cdot)^{*}$and $(\cdot)^{\mathrm{H}}$ are the vector/matrix transpose, conjugate and the Hermitian transpose, respectively. The function diag$(.)$ returns the diagonal elements of the input matrix. The nuclear norm of a matrix $\mathbf{B}\in\mathbb{C}^{M\times N}$ is denoted $\|\mathbf{B}\|_{\star}=\sum_{i=1}^{r}\sigma_i$ where r and $\{\sigma_i\}$ are the rank and singular values of $\mathbf{B}$, respectively. The Frobenius norm of a matrix $\mathbf{B}\in\mathbb{C}^{M\times N}$ is defined as $\|\mathbf{B}\|_{\mathrm{F}}=\sqrt{\sum_{r=1}^{M}\sum_{s=1}^{N}|b_{rs}|^2}$, where b_{rs} is the (r,s)-th entry of $\mathbf{B}$. We denote the ω-bandlimited Paley-Wiener subspace of the square-integrable function space L^2 by PW_{ω}. The Hadamard (element-wise) products is $\odot$. The vectorized form of a matrix $\mathbf{B}$ is written as $\operatorname{vec}(\mathbf{B})$. Given a scalar x, we define the operator $(x)^{+}$ as $\max\{x,0\}$. For an event $\mathcal{E}$, $\mathbb{1}_{(\mathcal{E})}$ is the indicator function for that event meaning that $\mathbb{1}_{(\mathcal{E})}$ is 1 if $\mathcal{E}$ occurs; otherwise, it is zero. The function $\operatorname{sgn}(\cdot)$ yields the sign of its argument.

II. One-Bit Matrix Completion

In this section, we first present the one-bit sampling approach using multiple time-varying thresholds as a linear inequality feasibility problem. We then formulate the one-bit matrix completion problem as a nuclear norm minimization problem with linear inequality constraints for both noisy and noiseless scenarios. Finally, we develop the SVT algorithm in such a way to handle the linear inequality constraints and recover the low-rank matrix. We call our proposed algorithm, OB-SVT.

A. One-Bit Sampling With Time-Varying Thresholds

Consider a bandlimited continuous-time signal $x\in\mathrm{PW}_{\omega}$ that we represent via Shannon's sampling theorem as [26]

$$0<\mathrm{T}\leqslant\frac{\pi}{\omega},\quad x(t)=\sum_{k=-\infty}^{k=+\infty}x(k\mathrm{T})\operatorname{sinc}\left(\frac{t}{\mathrm{T}}-k\right),\tag{1}$$

where $1/\mathrm{T}$ is the sampling rate, ω is the signal bandwidth, and $\operatorname{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$ is an *ideal* low-pass filter. Denote the uniform samples of $x(t)$ with the sampling rate $1/\mathrm{T}$ by $x_k=x(k\mathrm{T})$.

In practice, the discrete-time samples occupy pre-determined quantized values. We denote the quantization operation on x_k by the function $Q(\cdot)$. This yields the quantized signal as $r_k=Q(x_k)$. In one-bit quantization, compared to zero or constant thresholds, time-varying sampling thresholds yield a better reconstruction performance [17, 27]. These thresholds may be chosen from any distribution. For one-bit quantization with such time-varying sampling thresholds, $r_k=\operatorname{sgn}(x_k-\tau_k)$.

The information gathered through the one-bit sampling with time-varying thresholds may be formulated in terms of an overdetermined linear system of inequalities. We have $r_k=+1$ when $x_k>\tau_k$ and $r_k=-1$ when $x_k<\tau_k$. Collecting all the elements in the vectors as $\mathbf{x}=[x_k]\in\mathbb{R}^n$ and $\mathbf{r}=[r_k]\in\mathbb{R}^n$, therefore, one can formulate the geometric location of the signal as

$$r_k\left(x_k-\tau_k\right)\geq 0.\tag{2}$$

Then, the vectorized representation of (2) is $\mathbf{r}\odot(\mathbf{x}-\boldsymbol{\tau})\geq\mathbf{0}$ or equivalently

$$\boldsymbol{\Omega}\mathbf{x}\succeq\mathbf{r}\odot\boldsymbol{\tau},\tag{3}$$

where $\boldsymbol{\Omega}\triangleq\operatorname{diag}(\mathbf{r})$. Suppose $\mathbf{x},\boldsymbol{\tau}\in\mathbb{R}^n$, and that $\boldsymbol{\tau}^{(\ell)}$ denotes the time-varying sampling threshold in ℓ-th signal sequence, where $\ell\in\mathcal{L}=\{1,\cdots,m\}$.

For the ℓ-th signal sequence, (3) becomes

$$\boldsymbol{\Omega}^{(\ell)}\mathbf{x}\succeq\mathbf{r}^{(\ell)}\odot\boldsymbol{\tau}^{(\ell)},\quad\ell\in\mathcal{L},\tag{4}$$

where $\boldsymbol{\Omega}^{(\ell)}=\operatorname{diag}\left(\mathbf{r}^{(\ell)}\right)$. Denote the concatenation of all m sign matrices as

$$\tilde{\boldsymbol{\Omega}}=\left[\begin{array}{c|c|c}\boldsymbol{\Omega}^{(1)}&\cdots&\boldsymbol{\Omega}^{(m)}\end{array}\right]^{\top},\quad\in\mathbb{R}^{mn\times n}.\tag{5}$$

Rewrite the m linear system of inequalities in (4) as

$$\tilde{\boldsymbol{\Omega}}\mathbf{x}\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}),\tag{6}$$

where $\mathbf{R}$ and $\boldsymbol{\Gamma}$ are matrices, whose columns are the sequences $\left\{\mathbf{r}^{(\ell)}\right\}_{\ell=1}^{m}$ and $\left\{\boldsymbol{\tau}^{(\ell)}\right\}_{\ell=1}^{m}$, respectively.

The linear system of inequalities in (6) associated with the one-bit sampling scheme is overdetermined. We recast (6) into a *one-bit polyhedron* as

$$\mathcal{P}=\left\{\mathbf{x}\mid\tilde{\boldsymbol{\Omega}}\mathbf{x}\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma})\right\}.\tag{7}$$

B. One-Bit Matrix Completion as Nuclear Norm Minimization Problem

Assume we apply the coarse quantization to the observed partial entries of a low-rank matrix $\mathbf{X}\in\mathbb{R}^{n_1\times n_2}$ of rank r. Define $\mathcal{P}_{\Omega}(\mathbf{X})=\left[\widehat{X}_{i,j}\right]$ be the orthogonal projector onto the span of matrices vanishing outside of Ω. These partial entries of $\mathbf{X}$ is obtained in subset Ω as below

$$\widehat{X}_{i,j}=\begin{cases}X_{i,j}&(i,j)\in\Omega,\\0&\text{otherwise.}\end{cases}\tag{8}$$

In one-bit matrix completion, we solely observe the partial matrix through the l-th one-bit data matrix $\boldsymbol{\mathcal{R}}^{(\ell)}=\left[r_{i,j}^{(\ell)}\right]\in\mathbb{R}^{m_1\times m_2}$, where $m_1m_2\ll n_1n_2$. The entries in $\boldsymbol{\mathcal{R}}^{(\ell)}$ are dependent on the comparison between corresponding entries in $\mathcal{P}_{\Omega}(\mathbf{X})$ and l-th dithering matrix $\boldsymbol{\mathcal{T}}^{(\ell)}=\left[\tau_{i,j}^{(\ell)}\right]\in\mathbb{R}^{n_1\times n_2}$ according to the following relationship:

$$r_{i,j}^{(\ell)}=\begin{cases}+1 & X_{i,j}>\tau_{i,j}^{(\ell)},\\ -1 & X_{i,j}<\tau_{i,j}^{(\ell)},\end{cases}\quad (i,j)\in\Omega,\quad \ell\in\mathcal{L}. \tag{9}$$

Define $\mathbf{P}\in\mathbb{R}^{m_1m_2\times n_1n_2}$ be a permutation matrix that only selects subset Ω. For l-th threshold matrix, we formulate the obtained one-bit scheme as the following linear inequality feasibility problem:

$$\boldsymbol{\Omega}^{(\ell)}\mathbf{P}\operatorname{vec}(\mathbf{X})\succeq\operatorname{vec}\left(\boldsymbol{\mathcal{R}}^{(\ell)}\right)\odot\left(\mathbf{P}\operatorname{vec}\left(\boldsymbol{\mathcal{T}}^{(\ell)}\right)\right),\quad \ell\in\mathcal{L}, \tag{10}$$

where $\boldsymbol{\Omega}^{(\ell)}=\operatorname{diag}\left(\operatorname{vec}\left(\boldsymbol{\mathcal{R}}^{(\ell)}\right)\right)$, $\mathbf{P}\operatorname{vec}(\mathbf{X})$ and $\mathbf{P}\operatorname{vec}\left(\boldsymbol{\mathcal{T}}^{(\ell)}\right)$ only return subset Ω from $\operatorname{vec}(\mathbf{X})$ and $\operatorname{vec}\left(\boldsymbol{\mathcal{T}}^{(\ell)}\right)$, respectively. As demonstrated earlier, we rewrite (10) as

$$\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}), \tag{11}$$

where $\mathbf{R}$ and $\boldsymbol{\Gamma}$ are matrices, whose columns are the sequences $\left\{\operatorname{vec}\left(\boldsymbol{\mathcal{R}}^{(\ell)}\right)\right\}_{\ell=1}^{m}$ and $\left\{\mathbf{P}\operatorname{vec}\left(\boldsymbol{\mathcal{T}}^{(\ell)}\right)\right\}_{\ell=1}^{m}$, respectively, and

$$\boldsymbol{\mathcal{B}}=\left[\begin{array}{c|c|c}\boldsymbol{\Omega}^{(1)}\mathbf{P} & \cdots & \boldsymbol{\Omega}^{(m)}\mathbf{P}\end{array}\right]^{\top}. \tag{12}$$

Therefore, to recover the low-rank matrix $\mathbf{X}$ from highly-quantized observed measurements, we must find the optimal solution from the following feasible set:

$$\mathcal{F}=\left\{\mathbf{X}\mid\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}),\ \|\mathbf{X}\|_{\star}\leq\epsilon\right\}. \tag{13}$$

where ϵ is the predefined threshold. The feasible set of one-bit matrix completion is written as a *nuclear norm minimization* problem as below

$$\begin{aligned}\mathcal{F}^{(1)}:\quad &\underset{\mathbf{X}}{\operatorname{minimize}}\quad \tau\|\mathbf{X}\|_{\star}+\frac{1}{2}\|\mathbf{X}\|_{\mathrm{F}}^{2}\\ &\text{subject to}\quad \boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}),\end{aligned} \tag{14}$$

for some fixed $\tau\geq 0$. More than nuclear norm, the Frobenius norm is also considered to control the amplitudes of the unknown data [7]. In Subsection II-D, we will use the SVT algorithm to tackle this problem.

C. One-Bit Matrix Completion With Noisy Entries

Herein, we formulate the noisy version of one-bit matrix completion with time-varying thresholds. Denote $\mathbf{Z}\in\mathbb{R}^{n_1\times n_2}$ as the noise vector. The noisy one-bit samples are generated as

$$r_{i,j}^{(\ell)}=\begin{cases}+1 & X_{i,j}+Z_{i,j}>\tau_{i,j}^{(\ell)},\\ -1 & X_{i,j}+Z_{i,j}<\tau_{i,j}^{(\ell)},\end{cases}\quad (i,j)\in\Omega. \tag{15}$$

Consequently, the linear inequalities of feasible set $\mathcal{F}$ is rewritten as

$$\boldsymbol{\mathcal{B}}(\operatorname{vec}(\mathbf{X})+\operatorname{vec}(\mathbf{Z}))\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}), \tag{16}$$

or equivalently,

$$\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})+\boldsymbol{\nu}\succeq\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma}), \tag{17}$$

where $\boldsymbol{\nu}=\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{Z})$ is the noise of our system. For instance, if we consider $\operatorname{vec}(\mathbf{Z})\sim\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma}_z)$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_z$, the distribution of $\boldsymbol{\nu}$ will be $\mathcal{N}\left(\boldsymbol{\mathcal{B}}\boldsymbol{\mu},\boldsymbol{\mathcal{B}}\boldsymbol{\Sigma}_z\boldsymbol{\mathcal{B}}^{\mathrm{H}}\right)$.

To find the feasible solution from a linear inequality feasibility problem such as $\mathbf{C}\mathbf{x}\succeq\mathbf{b}$, we can rewrite the problem as [28]

$$(\mathbf{b}-\mathbf{C}\mathbf{x})^{+}=0. \tag{18}$$

If our system encounters a noise vector $\mathbf{n}$, we can slightly adjust (18) to minimize the impact of noise using the following formula:

$$\left|(\mathbf{b}-\mathbf{C}\mathbf{x})^{+}\right|\preceq\boldsymbol{\sigma}_n, \tag{19}$$

where $\boldsymbol{\sigma}_n$ is the effect of noise. Thus, if $\mathbf{x}$ is contained within $\mathbf{C}\mathbf{x}\succeq\mathbf{b}$, there is no need to take (19) into account. However, if it is not, we must consider the following:

$$|\mathbf{b}-\mathbf{C}\mathbf{x}|\preceq\boldsymbol{\sigma}_n. \tag{20}$$

Since $\mathbf{C}\mathbf{x}\preceq\mathbf{b}$, (20) is equivalent to

$$\mathbf{b}-\boldsymbol{\sigma}_n\preceq\mathbf{C}\mathbf{x}\preceq\mathbf{b}. \tag{21}$$

By applying the same process to (16), the modified linear inequality feasibility constraint is given by

$$\mathcal{Y}=\left\{\{\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\succeq\mathbf{t}\}\cup\{\mathbf{t}-\boldsymbol{\sigma}_z\preceq\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\preceq\mathbf{t}\}\right\}. \tag{22}$$

where $\mathbf{t}=\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma})$, and $\boldsymbol{\sigma}_z$ is the effect of $\operatorname{vec}(\mathbf{Z})$. Therefore, the nuclear norm minimization problem $\mathcal{F}^{(1)}$ is reformulated as

$$\begin{aligned}\mathcal{F}^{(2)}:\quad &\underset{\mathbf{X}}{\operatorname{minimize}}\quad \tau\|\mathbf{X}\|_{\star}+\frac{1}{2}\|\mathbf{X}\|_{\mathrm{F}}^{2}\\ &\text{subject to}\quad (\mathbf{t}-\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X}))^{+}\preceq\boldsymbol{\sigma}_z.\end{aligned} \tag{23}$$

D. Proposed Algorithm: OB-SVT

To tackle $\mathcal{F}^{(1)}$, we employ the SVT algorithm generalized for linear inequality constraints inspired by [7] as follows: Denote a linear transformation $\mathcal{A}:\mathbb{R}^{n_1\times n_2}\rightarrow\mathbb{R}^{mm_1m_2}$ and $\mathcal{A}^{\star}:\mathbb{R}^{mm_1m_2}\rightarrow\mathbb{R}^{n_1\times n_2}$ as its adjoint operator. In $\mathcal{F}^{(1)}$, we have $\mathcal{A}(\mathbf{X})=\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})$. Then the Lagrangian for this problem is of the form

$$\mathcal{L}(\boldsymbol{X},\boldsymbol{y})=\tau\|\mathbf{X}\|_{\star}+\frac{1}{2}\|\mathbf{X}\|_{\mathrm{F}}^{2}+\boldsymbol{y}^{\top}(\mathbf{t}-\mathcal{A}(\boldsymbol{X})), \tag{24}$$

where $\boldsymbol{y}$ is the Lagrangian multiplier and $\mathbf{t}=\operatorname{vec}(\mathbf{R})\odot\operatorname{vec}(\boldsymbol{\Gamma})$. The Karush-Kuhn-Tucker (KKT) conditions dictate that when dealing with inequality constraints, the Lagrange multiplier must be positive, i.e., $\boldsymbol{y}\succeq 0$. Drawing inspiration from Uzawa's method [7, Section 3.2] , we introduce slight modifications to the SVT algorithm to accommodate the inequality constraint and ensure satisfaction of $\boldsymbol{y}\succeq 0$ at every iteration k. The resulting expression is:

$$\begin{cases}\mathbf{X}^{(k)}=\arg\min\mathcal{L}\left(\boldsymbol{X},\boldsymbol{y}^{(k-1)}\right),\\ \boldsymbol{y}^{(k)}=\left(\boldsymbol{y}^{(k-1)}+\delta_k\left(\mathbf{t}-\mathcal{A}\left(\mathbf{X}^{(k)}\right)\right)\right)^{+},\end{cases} \tag{25}$$

where $\{\delta_k\}$ are positive step sizes. If we consider the singular value decomposition (SVD) of $\mathbf{X}$ as $\mathbf{X}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\top}$ and $\{\sigma_i\}$ as its singular values, the first step can be easily solved by using the singular value shrinkage operator comprehensively investigated in [7, 29] which applied the partial singular value decomposition to achieve the low-rank matrix structure as

$$\mathcal{D}_{\tau}(\mathbf{X})=\mathbf{U}\mathcal{D}_{\tau}(\boldsymbol{\Sigma})\mathbf{V}^{\top},\quad \mathcal{D}_{\tau}(\boldsymbol{\Sigma})=\operatorname{diag}\left((\sigma_i-\tau)^{+}\right), \tag{26}$$

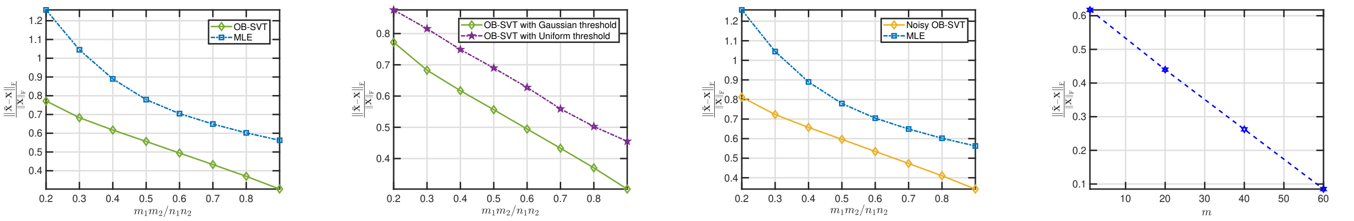
where $\tau\geq 0$ is the predefined threshold. The SVT algorithm is a popular approach for matrix sensing and matrix completion, where the partial SVD can be numerically calculated using Krylov subspace methods (e.g. Lanczos algorithm) [30]. Consequently, (26) is rewritten as

$$\begin{cases}\mathbf{X}^{(k)}=\mathcal{D}_{\tau}\left(\mathcal{A}^{\star}\left(\boldsymbol{y}^{(k-1)}\right)\right),\\ \boldsymbol{y}^{(k)}=\left(\boldsymbol{y}^{(k-1)}+\delta_k\left(\mathbf{t}-\mathcal{A}\left(\mathbf{X}^{(k)}\right)\right)\right)^{+}.\end{cases} \tag{27}$$

For the noisy scenario (21), the Lagrangian problem of $\mathcal{F}^{(2)}$ is given by

$$\mathcal{L}^{n}(\boldsymbol{X},\boldsymbol{y}_1)=\tau\|\mathbf{X}\|_{\star}+\frac{1}{2}\|\mathbf{X}\|_{\mathrm{F}}^{2}+\boldsymbol{y}_1^{\top}\left((\mathbf{t}-\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X}))^{+}-\boldsymbol{\sigma}_z\right), \tag{28}$$

where $\boldsymbol{y}_1$ is the Lagrangian multiplier parameter. As outlined in Subsection II-C, incorporating the noisy inequality constraint (19) can be achieved by only considering $\boldsymbol{\mathcal{B}}\operatorname{vec}\left(\mathbf{X}^{(k)}\right)\preceq\mathbf{t}$ at each iteration k. Otherwise, i.e., $\boldsymbol{\mathcal{B}}\operatorname{vec}(\mathbf{X})\succeq\mathbf{t}$, there is no need to update the Lagrangian multiplier $\boldsymbol{y}_1$. As a result, $(\cdot)^{+}$ can be easily removed



(a) (b) (c) (d)

Figure 1. (a) Comparison between our proposed OB-SVT method and the MLE approach proposed in [22] for noiseless measurements in terms of the relative error when reconstructing a 500×500 matrix $\mathbf{X}$ with the rank $r = 10$. (b) Demonstrating the impact of time-varying sampling threshold selection in the accuracy of input matrix reconstruction by OB-SVT method. (c) Comparison between our proposed noisy OB-SVT method and the MLE approach proposed in [22] for noisy measurements in terms of the relative error when reconstructing a 500×500 matrix $\mathbf{X}$ with the rank $r = 10$. (d) Enhancement of the reconstruction accuracy via the OB-SVT method as the number of time-varying thresholds grows large.

from (28):

$$\mathcal{L}^n(\boldsymbol{X}, \boldsymbol{y}) = \tau \left\|\mathbf{X}\right\|_\star + \frac{1}{2}\left\|\mathbf{X}\right\|_\mathrm{F}^2 + \boldsymbol{y}_1^\top \left(\mathbf{t} - \boldsymbol{\sigma}_z - \boldsymbol{\mathcal{B}}\,\mathrm{vec}\left(\mathbf{X}\right)\right). \tag{29}$$

Denote $\mathbf{t}_1 = \mathbf{t} - \boldsymbol{\sigma}_z$, according to Uzawa's method, the update process proposed to address $\mathcal{F}^{(2)}$ is expressed as:

$$\begin{cases} \mathbf{X}^{(k)} = \mathcal{D}_\tau\left(\mathcal{A}^\star\left(\boldsymbol{y}_1^{(k-1)}\right)\right), \\ \boldsymbol{y}_1^{(k)} = \left(\boldsymbol{y}_1^{(k-1)} + \delta_k \boldsymbol{g}^{(k)}\right)^+, \\ \boldsymbol{g}^{(k)} = \left(\mathbf{t}_1 - \boldsymbol{\mathcal{B}}\,\mathrm{vec}\left(\mathbf{X}^{(k)}\right)\right)^+ \mathbb{1}_{\left(\boldsymbol{\mathcal{B}}\,\mathrm{vec}\left(\mathbf{X}^{(k)}\right) \preceq \mathbf{t}\right)}. \end{cases} \tag{30}$$

III. Numerical Investigations

In this section, we numerically scrutinize the efficacy of our proposed OB-SVT method by comparing its recovery results with the state-of-the-art method (MLE approach) proposed in [22]. In particular, we constructed a random 500×500 matrix $\mathbf{X}$ with rank $r = 10$ by forming $\mathbf{X} = \mathbf{X}_1 \mathbf{X}_2^\top$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are 500×10 matrices with entries drawn i.i.d. from the Gaussian distribution $\mathcal{N}(0, 1)$. We then obtained one-bit observations by comparing the high-resolution values with the generated time-varying sampling thresholds $\boldsymbol{\tau}$ and recording the sign of the resulting value. Herein, we have utilized the Gaussian and Uniform distributed random thresholds. Accordingly, inspired by [31], we have generated the Gaussian time-varying thresholds as $\left\{\boldsymbol{\tau}^{(\ell)} \sim \mathcal{N}\left(\mathbf{0}, \frac{\beta_\mathbf{X}^2}{9}\mathbf{I}\right)\right\}_{\ell=1}^m$, where $\beta_\mathbf{X}$ denotes the dynamic range of the $\mathcal{P}_\Omega(\mathbf{X})$. Similarly, we have generated the Uniform random thresholds as $\left\{\boldsymbol{\tau}^{(\ell)} \sim \mathcal{U}_{[a,b]}\right\}_{\ell=1}^m$, where a and b denote the minimum and maximum entries of $\mathcal{P}_\Omega(\mathbf{X})$, respectively. For simplicity in OB-SVT method, we have utilized step sizes that are independent of the iteration count; i.e. $\delta_k = \delta$ for all k. Inspired from [7], we have set the value of δ to $\delta = 1.2\frac{n_1 n_2}{m_1 m_2}$. The reason behind this selection has been heuristically justified in [7]. The parameter τ is chosen empirically and set to $\tau = \alpha\sqrt{n_1 n_2}$, where the value of α varies for different values of $m_1 m_2$. Define the relative error of the reconstruction as

$$\text{relative error} \triangleq \frac{\left\|\bar{\mathbf{X}} - \mathbf{X}\right\|_\mathrm{F}}{\left\|\mathbf{X}\right\|_\mathrm{F}}, \tag{31}$$

where $\mathbf{X}$ is the real unknown matrix and $\bar{\mathbf{X}}$ denotes the reconstructed version of $\mathbf{X}$ by either OB-SVT or MLE methods.

In Fig. 1a, the accuracy of input matrix reconstruction using OB-SVT and MLE methods is compared in terms of relative error for noiseless measurements, with one sequence of time-varying sampling threshold ($m = 1$). It can be seen that OB-SVT outperforms MLE in reconstructing the input matrix. The experiment used Gaussian random thresholds. Figure 1b shows the effect of time-varying sampling thresholds on input matrix recovery, where Gaussian random thresholds yield better reconstruction accuracy than Uniform random thresholds. This observation can be attributed to the fact that the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ were generated using entries drawn from a Gaussian distribution. Similar to the previous case, we have considered the number of random thresholds $m = 1$. In line with previous work [22], we added Gaussian noise with $\mathbf{\Sigma}_z = 0.09\mathbf{I}$ to the high-resolution measurements to facilitate a numerical comparison between the noisy OB-SVT and MLE methods. Fig. 1c illustrates the reconstruction results associated with the noisy OB-SVT and MLE methods. Our proposed approach outperforms the MLE method in terms of relative error, just like in the noiseless case. Through numerical observations, we found that setting the noise effect $\boldsymbol{\sigma}_z = \gamma_z \mathbf{1}$ to $1.5\sigma_z \leq \gamma_z < 3\sigma_z$ works well for Gaussian noise with standard deviation σ_z. It is worth noting that these results are based on a single time-varying threshold sequence ($m = 1$). Fig. 1d displays the relative error in relation to the number of time-varying threshold sequences. As shown, increasing the number of time-varying thresholds leads to better reconstruction accuracy. This phenomenon occurs because as the number of time-varying sampling thresholds increases, the probability of generating a random threshold that is spatially close to the high-resolution measurements also increases; i.e. increasing the richness of the input information.

IV. Summary

This study aimed to investigate how matrix completion is affected by the use of one-bit sampling with time-varying thresholds. By formulating the problem as a nuclear norm minimization coupled with linear inequality feasibility constraints derived from one-bit samples, we achieved significant performance improvements. We adapted the singular value thresholding algorithm to accommodate these constraints in both noiseless and noisy scenarios. Our numerical comparisons with the MLE method demonstrate that the proposed OB-SVT algorithm achieves a better recovery performance.

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