2024 10th International Conference on Control, Decision and Information Technologies
CoDIT 2024 | Valletta, Malta / July 01-04, 2024 Technically co-sponsored by IEEE & IFAC

2024 10th International Conference on Control, Decision and Information Technologies (CoDIT) | 979-8-3503-7397-4/24/$31.00 ©2024 IEEE | DOI: 10.1109/CoDIT62066.2024.10708136

Stabilization of Input Derivative Positive Systems and its Utilization in Positive Singular Systems

Bahram Shafai[1] , Fatemeh Zarei[2] and Anahita Moradmand[3]

Abstract— **This paper introduces a subclass of positive systems involving input derivatives, which we formally define it as input derivative positive systems. Due to the presence of input derivatives, we provide an algebraic transformation to eliminate the derivative inputs to accommodate the process of stabilization by state feedback. This elimination transfers the input derivative in the output equation, which does not interfere with the design process. Stabilization of input derivative positive systems is performed through its equivalent transformed positive systems in standard form using LMI. To take advantage of this stabilization process, we utilize it for stabilization of positive singular systems. Consequently, we analyze singular systems and its equivalent transformations, which admit derivative input. Thus, algebraic transformation is employed to eliminate these derivative inputs. Finally, we establish the connection between stabilization of positive singular systems and stabilization of input derivative systems by a modified LMI. Numerical examples are included to support the theoretical result.**

I. INTRODUCTION

The class of dynamic systems with derivative input appears naturally in applications such as electric circuits when all capacitor loops and/or inductor cut-sets combine with voltage source and/or current source, respectively. They also appear in electro-mechanical systems when transducers are used in setting the reference inputs in analysis and control design. In robotics, they are used in control design of manipulators, and in active suspension systems when vibration control is required.

The theory of linear systems with derivative inputs originated by few researchers and reported in [1] - [5]. The early research in this direction was based on polynomial matrix representation, which used strict system equivalence and realization theory. This approach was inconvenient due to its lengthy derivation for higher order derivatives. It was also difficult to interpret the results since state variable information is lost when calculating the transfer functions. In fact, it is important to preserve the state space representation of input derivative systems in analysis and control design.

In addition, there are other scenarios in dynamic systems when algebraic constraints are imposed on physical variables leading to the so-called descriptor or singular systems. Singular systems inherently experience the derivative inputs in the process of obtaining an equivalent state space model in standard form. A compilation results on singular systems can be found in [8] - [14]. Singular systems are also tied to the class of positive systems, which have impressive stability and robustness properties [15]-[18]. The response of positive systems to positive initial conditions and inputs remain in the positive orthant of state space. Due to the fact that both systems share the same applications in engineering, economics, social sciences, biology, chemical process, etc. the combined systems, positive singular systems, was investigated by researchers (see [16] and the references therein).

[1]Bahram Shafai, Professor in Electrical and Computer Engineering department at Northeastern University, Boston MA, USA. shafai@ece.neu.edu

[2]Fatemeh Zarei, Ph.D. Student in Electrical and Computer Engineering department at Northeastern University, Boston MA, USA. zarei.fa@northeastern.edu

[3]Anahita Moradmand, Ph.D. Candidate in Electrical and Computer Engineering Department at Northeastern University, Boston MA, USA. moradmand.a@northeastern.edu

Motivated by the above development, we extend the class of input derivative systems and formally introduce the class of input derivative positive systems. Due to the presence of input derivative, we provide an algebraic transformation to eliminate the derivative inputs to accommodate the process of stabilization by state feedback. The stabilization is performed by its equivalent transformed positive systems in standard form using LMI. This is the main contribution of the paper. Subsequently, we utilize the process of stabilization of input derivative systems for positive singular systems and show that it can be performed by a modified LMI.

II. INPUT-DERIVATIVE POSITIVE SYSTEM

Consider the class of Systems with input derivatives described by

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{l} B_i u^{(i)}(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and the derivatives inputs $\dot{u}, \ddot{u}, \ldots$ defined by $u^{(i)}(t) = \frac{d^i u(t)}{dt} \in \mathbb{R}^m$.

Definition 1. The system (1), (2) is called internally positive if for all nonnegative initial conditions $x_0 \in \mathbb{R}^n_+$ and all nonnegative $u(t) \in \mathbb{R}^m_+$ including its derivatives $u^{(i)}(t) \in \mathbb{R}^m_+$ we have $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^p_+$.

Theorem 1. *The input derivative System (1),(2) is internally positive if and only if $A \in M_n$ is a Metzler matrix and $B_i \in \mathbb{R}^{n\times m}$ for all $i = 0, 1, \ldots, l$, $C \in \mathbb{R}^{p\times n}$ are nonnegative matrices provided that $u^{(i)}(t) \in \mathbb{R}^m_+$.*

Proof: The proof of this theorem is a simple generalization of standard positive system by imposing positivity of B_i and $u^{(i)}(t)$ for all $i = 0, 1, \ldots, l$.

The class of systems (1),(2) with one input derivative appeared in a natural way in several applications. In this case $l = 1$ and we have

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 \dot{u}(t) \tag{3}$$

Although traditional approach of polynomial matrix representation and subsequent strict system equivalent of Rosenbrock can transform (3) to standard form, it is not convenient for the general case with l input derivatives [1]. An immediate and more transparent way is to eliminate the derivative input by an algebraic transformation. To apply the process for (3), we define $z(t) = x(t) - B_1 u(t)$ and obtain

$$\dot{z}(t) = Az(t) + (B_0 + AB_1)u(t) \tag{4}$$

$$y(t) = Cz(t) + CB_1 u(t) \tag{5}$$

For $l = 2$, one requires to perform twice the algebraic transformation to obtain

$$\dot{z}(t) = Az(t) + (B_0 + AB_1 + A^2 B_2)u(t) \tag{6}$$

$$y(t) = Cz(t) + C(B_1 + AB_2)u(t) + CB_2 \dot{u}(t) \tag{7}$$

Note that the system (4),(5) remain positive if $AB_1 > 0$. Similarly, the system (6),(7) remain positive if $AB_1 > 0$, $AB_2 > 0$, and $A^2 B_2 > 0$. Continuing the process for $l > 2$, one can arrive at the following closed form expression for l derivative inputs,

$$\dot{z}(t) = Az(t) + \bar{B}_0 u(t) \tag{8}$$

$$y(t) = Cz(t) + C \sum_{j=1}^{l} \bar{B}_j u^{(j-1)}(t) \tag{9}$$

where

$$\bar{B}_j = \sum_{i=j}^{l} A^{i-j} B_i, \quad j = 0, 1, \ldots, l$$

with

$$\bar{B}_0 = \sum_{i=0}^{l} A^i B_i, \quad i = 0, 1, \ldots, l$$

Note that the elimination of input derivatives in state equation shifts the derivative terms in the output equation except for $l = 1$. This does not cause any issue when stabilization by state feedback is applied for (8),(9) in the next section.

Thus, we can state the following result for the equivalent derivative system (8),(9).

Theorem 2. *The input derivative system (1),(2) is internally positive if and only if the equivalent system (8),(9) is positive with $A \in M_n$, $\bar{B}_j \in \mathbb{R}_+^{n\times m}$ and $C \in \mathbb{R}_+^{p\times n}$.*

Remark 1. The requirement $\bar{B}_i \in \mathbb{R}_+^{n\times m}$ can be avoided for the special subset of Metzler matrices $A \in \mathbb{R}_+^{n\times m} \subset \mathbb{M}_n$; since the Metzler matrices are defined by real matrices with nonnegative off-diagonal elements. In this case, the condition $\bar{B}_j \in \mathbb{R}_+^{n\times m}$ in the above theorem can be replaced by $B \in \mathbb{R}_+^{n\times m}$.

III. Stability and Stabilization of Input Derivative Positive Systems

Lemma 1. *Let the input-derivative system (1),(2) or its equivalent form (8),(9) be positive. Then it is asymptotically stable if and only if one of the following conditions is satisfied:*

1) *There exists a positive definite diagonal matrix P such that $A^T P + PA \prec 0$*
2) *There exists a positive vector $v \in \mathbb{R}_+^n$ such that $Av < 0$*

The controllability of the input-derivative system (1),(2) is required for its stabilization using state feedback.

Lemma 2. *The input derivative system (1),(2) or equivalently (8),(9) is controllable if and only if*

$$rank \begin{bmatrix} \bar{B}_0 & A\bar{B}_0 & \cdots & A^{n-1}\bar{B}_0 & \bar{B}_1 & \bar{B}_2 & \cdots & \bar{B}_l \end{bmatrix} = n$$

where $\bar{B}_j = \sum_{i=j}^{l} A^{i-j} B_i$ for all $i, j = 0, 1, \ldots, l$

Proof: The proof of this result for general input derivative system is given in [6]. Consequently, the rank condition remains valid for the class of input derivative positive systems.

Theorem 3. *Let the input derivative system (1),(2) or the equivalent form (8),(9) be positive and controllable. Then the state feedback control law $u = K_z z$ stabilizes (8) and the closed-loop system becomes*

$$\dot{z}(t) = A_z z(t) \tag{10}$$

where

$$A_z = A + \bar{B}_0 K_z, \quad \bar{B}_0 = \sum_{i=0}^{l} A^i B_i, \quad i = 0, 1, \ldots, l \tag{11}$$

and z is related to x by

$$x = \left[I + \sum_{i=j}^{l} \bar{B}_j K_z A^{j-1} \right] z \tag{12}$$

with $u = v + K_x x$ determined by

$$K_x = K_z \left[I + \sum_{i=j}^{l} \bar{B}_j K_z A_z^{j-1} \right]^{-1} \tag{13}$$

if and only if the following LMI has a feasible solution with respect to the variables Y and W

$$WA^T + Y^T \bar{B}_0{}^T + AW + \bar{B}_0 Y \prec 0 \tag{14}$$

$$(AW + \bar{B}_0 Y)_{ij} \geq 0 \quad for \quad i \neq j \tag{15}$$

where $W \succ 0$ is a diagonal positive definite matrix. Furthermore, the gain matrix K_z is obtained from

$$K_z = YW^{-1} \tag{16}$$

and K_x is determined by (13).
Note that (14),(15) guarantee $(AW + \bar{B}_0 Y)_{ij} < 0$

Proof: The proof of (12) and (13) can be established by algebraic manipulations. The LMI (14), (15) are based on the proof of positive stabilization of standard positive systems without input derivatives. Here, one should apply condition 1) of Lemma 1 with respect to $A + \bar{B}_0 K$ in (8).

The above theorem can also be stated in terms of LP using the second equivalent condition of Lemma 1 when replacing A by $A + \bar{B}_0 K$.

IV. Singular Systems and Equivalent Representations

This section concentrates on the class of singular systems, which can be represented by standard input derivative systems (1),(2). Consider the general linear singular system described by

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{17}$$

$$y(t) = Cx(t) \tag{18}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and the triple matrices (A, B, C) have appropriate dimensions. The matrix $E \in \mathbb{R}^{n\times n}$ is singular with rank (E) = r < n. For the unforced singular system represented by the pair $\{E, A\}$, we define the generalized spectral abscissa $\alpha(E, A) = \max \Re(\lambda)$, where $\lambda \in \{s : \det(sE - A) = 0\}$. If $E = I$, then $\alpha(I, A) = \alpha(A)$, which is the usual spectral abscissa for standard systems.

Definition 2. The singular system (17),(18) is called,

1) Regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$.
2) Impulse free if deg $\det(sE - A) = \text{rank}(E) = r$.
3) Stable if all roots of $\det(sE - A)$ have negative real parts.
4) Admissible if it is regular, impulse free, and stable.

Lemma 3. *Consider the singular system (17),(18) and assume that it is regular. Then, there exist two nonsingular matrices L and R such that*

$$\begin{cases} \bar{E} = LER = \textit{block diag}(I_{n_1}, N) \\ \bar{A} = LAR = \textit{block diag}(A_1, I_{n_2}) \\ \bar{B} = LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \bar{C} = CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{cases} \tag{19}$$

where $n_1 = \deg(\det(sE - A)) \leq r$, $n_2 = n - n_1$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with index μ (i.e. $N^\mu = 0$, $N^{\mu-1} \neq 0$).

Furthermore,

1) *The pair $\{E, A\}$ is impulse free if and only if $N = 0$.*
2) *The pair $\{E, A\}$ is stable if and only if $\alpha(A_1) < 0$.*
3) *The pair $\{E, A\}$ is admissible if and only if $N = 0$ and $\alpha(A_1) < 0$.*

The restricted equivalent form of singular system in Lemma 1 can be represented by slow and fast subsystems respectively as

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ y_1(t) &= C_1 x_1(t) \end{aligned} \tag{20}$$

and

$$\begin{aligned} N\dot{x}_2(t) &= x_2(t) + B_2 u(t)) \\ y_2(t) &= C_2 x_2(t) \end{aligned} \tag{21}$$

It is not difficult to derive the transfer function of the system and show that

$$\begin{aligned} G(s) &= C(sE - A)^{-1}B = \bar{C}(sE - \bar{A})^{-1}\bar{B} \\ &= G_1(s) + G_2(s) \end{aligned} \tag{22}$$

where

$$G_1(s) = C_1(sI_{n_1} - A_1)^{-1}B_1, \quad \text{slow subsystem}$$

$$G_2(s) = C_2(sN - I_{n_2})^{-1}B_2, \quad \text{fast subsystem}$$

For the fast subsystem we can write

$$G_2(s) = C_2 \left(-\sum_{i=0}^{\mu-1} s^i N^i \right) B_2 \tag{23}$$

and taking the inverse Laplace transform of $X_2(s)$, we get the solution of (21) as

$$x_2(t) = -\sum_{i=0}^{\mu-1} N^i B_2 u^{(i)}(t), u^{(0)} = u \tag{24}$$

assuming $x_2(0) = 0$ and $\sum_{i=0}^{\mu-1} N^i B_2 u^{(i)}(0) = 0$. Differentiating (24) yields

$$\dot{x}_2(t) = -\sum_{i=0}^{\mu-1} N^i B_2 u^{(i+1)}(t) \tag{25}$$

Combining (25) with the slow subsystem (20) leads to the overall system

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu-1} \tilde{B}_i u^{(i+1)}(t) \tag{26}$$

$$y(t) = \tilde{C}x(t) \tag{27}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ -N^{i-1}B_2 \end{bmatrix}$$

$$\tilde{C} = [C_1 \quad C_2]$$

Thus, the singular system (17), (18) represented by (26), (27) is in the form of input derivative system (1), (2). Consequently, one can apply the process of algebraic transformation to eliminate the derivative inputs.

It is also possible to decompose the singular system to dynamic and static parts by applying singular value decomposition (SVD) to the matrix E. Subsequently, one can employ the so-called Shuffle algorithm [7], [13] to derive an equivalent standard system with input derivatives.

Lemma 4. *Consider the singular system (17), (18) and assume that it is regular. Then, by performing an SVD of the matrix E, it can always be rewritten as*

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}x(t) + \hat{B}u(t) \tag{28}$$

$$y(t) = \hat{C}\hat{x}(t) \tag{29}$$

where

$$\hat{A} = U^T AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B} = U^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\hat{E} = U^T EV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{C} = CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $\Sigma_r = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ defining the nonzero singular values and orthogonal pair of matrices $\{U, V\}$.

Note that one can replace Σ_r by I_r with an additional transformation step. It can be shown that the pair $\{E, A\}$ or equivalently $\{\hat{E}, \hat{A}\}$ is impulse free if and only if A_{22} is nonsingular and in addition, the pair is admissible if and only if $\alpha(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 0$.

It is important to point out that the Canonical form (20), (21) with (19) identifies the slow and fast subsystems of singular system (17), (18) while its SVD form (28), (29) identifies the dynamic and static parts of it. One can conclude that the difference between the number of finite modes in (20),n_1, and the number of dynamic modes in (28),r, is the infinite dynamic modes, which is the rank of N. This means that for a singular system with only finite poles (dynamic modes) we have $n_1 = r = \text{rank } E$.

The equivalent singular system (28), (29) is not in the form of standard system with input derivatives (1), (2). A subsequent Shuffle algorithm can be employed to achieve this goal [7]. In fact, this algorithm can also be performed in a direct fashion to the singular system (17), (18).

However, the transformation to SVD form (28), (29) is numerically reliable for large size system and it's structure prepares the system for initial step of Shuffle algorithm by defining

$$\hat{E} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{30}$$

where $E_1 = \begin{bmatrix} \Sigma_r & 0 \end{bmatrix}, A_1 = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$, and $A_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$. Therefore, starting with (30), we define

$$\begin{aligned} E_1\dot{x} &= A_1x + B_1u \\ 0 &= A_2x + B_2u \end{aligned} \tag{31}$$

Taking the derivative of the second (static) equation and combining it with the first one yields

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \dot{u} \tag{32}$$

If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is nonsingular, multiply both sides by its inverse and stop. Then, define the input derivative system. If $\begin{bmatrix} E_1^T & A_2^T \end{bmatrix}^T$ is singular, then the sequence of row operations continues until a regular pencil after $\mu - 1$ steps is obtained where $\begin{bmatrix} E_{\mu-1}^T & A_{\mu-1}^T \end{bmatrix}^T$ becomes nonsingular. Consequently, (17), (18) represented by SVD form (28), (29) is transformed to the input derivative system (1), (2) with $l = \mu - 1$ as follows

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=0}^{\mu-1} \tilde{B}_i u^{(i)}(t) \tag{33}$$

$$y(t) = \tilde{C}x(t) \tag{34}$$

V. Positive Singular Systems and Stabilization

Consider the singular system represented by (17), (18) and let us analyze its positivity and stability.

Definition 3. The singular system (17), (18) is called weakly positive if and only if $E \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, and $A \in \mathbb{M}_n$ is a Metzler matrix.

This definition is an extension of positivity as defined for standard positive systems. Unfortunately, it does not guarantee the strong positivity of the singular systems. One should apply one of the equivalent representations as discussed in the previous section to determine the positivity of the singular system through its standard form involving derivative inputs.

Obviously, weakly positive singular system should undergo the same process. In fact, even if the singular system does not satisfy any positivity of coefficient matrices, i.e. not being weakly positive, may lead to positivity after transforming it to an input derivative system.

To elaborate on this point, consider the following simple example of a singular system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} u$$

As it can be seen the matrices E and B have negative elements, which disqualifies the singular system from being even weakly positive. So, applying the shuffle algorithm one step, we obtain (32) with

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

Since $\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ is nonsingular, one can easily obtain the input derivative system by (33), (34), where

$$\tilde{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $\widetilde{A} \in \mathbb{M}_n$, $\widetilde{B}_0 \in \mathbb{R}_+^{n\times m}$ and $\widetilde{B}_1 \in \mathbb{R}_+^{n\times m}$, the singular system is positive as we defined it for input derivative system. Thus, we can formally connect the positivity and stability of positive singular systems in terms of its equivalent input derivative system.

Definition 4. The singular system (17), (18) is internally positive if and only if its equivalent input derivative system (33), (34) is internally positive i.e. for every consistent initial condition $x_0 \in \mathbb{R}_+^n$ and every nonnegative input $u(t) \in \mathbb{R}_+^m$ such that $u^{(i)}(t) \in \mathbb{R}_+^m$ for $i = 0, 1, ..., \mu - 1$; $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ where $u^{(i)}(t) = \frac{d^i u(t)}{dt}$.

Theorem 4. *The singular system (17), (18) is internally positive if and only if its equivalent input derivative system (33), (34) satisfies*

$$\widetilde{A} \in \mathbb{M}_n, \quad \widetilde{B}_i \in \mathbb{R}_+^{n\times m}, \quad \widetilde{C} \in \mathbb{R}_+^{p\times n}$$

for all $i = 0, 1, ..., \mu - 1$. Furthermore, it is asymptotically stable if and only if $\widetilde{A}$ is a stable Metzler matrix satisfying any one of the equivalent conditions of Lemma 1.

The development of Section IV showed two different strategies of transforming the singular system to input derivative systems. In this section we only concentrated the positivity of the singular systems in terms of its equivalent input derivative system (33), (34). This allows us to employ the algebraic transformation to transform the input derivative positive system to standard form and apply a modified version of Theorem 3 by proper adjustment of parameter matrices.

Theorem 5. *Let the singular system (17), (18) with its equivalent input derivative system (33), (34) be positive and controllable. Furthermore, assume that its algebraic transformation represented by (8), (9) with the associated parameter matrices $\{A, \bar{B}_0, \bar{B}_j, C\}$ replaced by $\{\widetilde{A}, \tilde{\bar{B}}_0, \tilde{\bar{B}}_j, \widetilde{C}\}$, where $l = \mu - 1$. Then the state feedback control law $u = v + K_x x$ stabilizes the positive singular system with*

$$K_x = K_z \left[I + \sum_{i=j}^{l} \tilde{\bar{B}}_j K_z \widetilde{A}_z{}^{j-1} \right]^{-1} \tag{35}$$

where

$$\widetilde{A}_z = \widetilde{A} + \tilde{\bar{B}}_0 K_z, \quad \tilde{\bar{B}}_0 = \sum_{i=0}^{l} \widetilde{A}^i \widetilde{B}_i$$

if and only if the following LMI has a feasible solution

$$W\widetilde{A}^T + Y^T \tilde{\bar{B}}_0{}^T + \widetilde{A}W + \tilde{\bar{B}}_0 Y \prec 0 \tag{36}$$

$$(\widetilde{A}W + \tilde{\bar{B}}_0 Y)_{ij} \geq 0 \quad i \neq j \tag{37}$$

whereby K_z is obtained by $K_z = YW^{-1}$ and $W \succ 0$ is a diagonal positive definite matrix.

Proof: The proof follows the same line of proof as in Theorem 3 by proper adjustments of design parameters. Thus, Theorem 5 for stabilization of positive singular systems ties directly to the stabilization of input derivative positive systems established in Theorem 3.

VI. Illustrative Examples

Example 1: Consider a simple second order system with one derivative input, which represents an input derivative positive system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_0} u + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B_1} \dot{u}$$

Applying the algebraic transformation $z = x - B_1 u$, we obtain

$$\dot{z} = Az + (B_0 + AB_1)u = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

The state feedback control law $u = K_z z$ results in closed loop system $\dot{z} = A_z z$, $A_z = A + \bar{B}_0 K_z$ with $\bar{B}_0 = B_0 + AB_1$, where K_z can easily be obtained by solving LMI (14), (15) or analytically using Lyapunov formulation as $K_z = \begin{bmatrix} -\frac{1}{3} & -1 \end{bmatrix}$. Finally, K_x is determined by $K_x = K_z(I + B_1 K_z)^{-1} = \begin{bmatrix} 1 & 3 \end{bmatrix}$.

Example 2: Next, we consider a third-order multi-variable system with input derivative

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 3 \end{bmatrix}}_{B_0} u + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}}_{B_1} \dot{u}$$

Applying the algebraic transformation $z = x - B_1 u$, we get

$$\dot{z} = Az + \underbrace{(B_0 + AB_1)}_{\bar{B}_0} u = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u$$

The state feedback control law is $u = K_z z$ results in closed-loop system $\dot{z} = A_z z$, $A_z = A + \bar{B}_0 K_z$ where K_z can be obtained by solving LMI (14), (15) as

$$K_z = YW^{-1} = \begin{bmatrix} -1.2328 & 0.4764 & 0.0007 \\ -0.7672 & -2.4764 & -2.9993 \end{bmatrix}$$

with the pair

$$Y = \begin{bmatrix} -2.9023 & 1.6511 & 0.0040 \\ -1.8061 & -8.5827 & -16.9556 \end{bmatrix}$$

and $W = \text{diag}\{2.3542, 3.4658, 5.6532\}$. The closed-loop system matrix A_z becomes Metzlerian stable,

$$A_z = \begin{bmatrix} -3.2328 & 1.4764 & 0.0007 \\ 0.2328 & -1.4764 & 0.0007 \\ 0 & 0 & -1.9986 \end{bmatrix}$$

Note also that the K_x for the original input derivative representation above can be obtained by $K_x = K_z(I + B_1 K_z)^{-1}$.

Example 3: Consider the third-order positive singular system in Section V, which has been represented by an equivalent regular system with one input derivative using the shuffle algorithm. It is not difficult to determine that the system is not controllable with the aid of Lemma 2, however, it is stabilizable. The eigenvalues of the singular system can be obtained from $\det(sE - A) = 0$ as $\{0, -1\}$, which implies that the system is unstable. Applying the algebraic transformation and using LMI (36) without the constraint (37), a possible stabilizer $u = K_x x$ can be obtained for the original singular system where

$$K_x = \begin{bmatrix} -0.3860 & -1.3271 & -0.2185 \\ -0.7580 & -1.6907 & -1.0901 \end{bmatrix}$$

Thus, $A_x = A + BK_x$ with

$$\det(sE - A_x) = 1.386s^2 + 3.838s + 0.862$$

which has stable roots $(-0.2402, -3.4551)$. Although, a stabilizer can be obtained for this positive singular system, it is not possible to positively stabilize it due to the constraint imposed by Metzlerian structure.

Example 4: This example considers a non-positive singular system. It is possible to show that positive stabilization can be achieved by applying Theorem 5.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -2 & 0 & 2 & -0.1 \\ 0 & -2 & 0 & 3 \\ 1 & 0 & 1 & -0.4 \\ 0 & 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 2 \end{bmatrix} u$$

Since the roots of $\det(sE - A) = 0$ are $\{-4, 1\}$, the singular system is unstable. After the steps of transformation to the equivalent input derivative system and application of algebraic transformation, one can solve the modified LMI (36),(37) to obtain K_x.

The equivalent input derivative system is obtained as

$$\widetilde{A} = \begin{bmatrix} -2 & 0 & 2 & 0.1 \\ 0 & -2 & 0 & 3 \\ 2 & -0.8 & -2 & 1.3 \\ 0 & -2 & 0 & 3 \end{bmatrix},$$

$$\widetilde{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0.4 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.4 & 0.8 \\ 0 & 2 \end{bmatrix}$$

After algebraic transformation, we obtain

$$\dot{z} = \widetilde{A} + \underbrace{\left(\widetilde{B}_0 + \widetilde{A}\widetilde{B}_1\right)}_{\widetilde{\widetilde{B}}_0} u$$

and its stabilizer can be found by applying Theorem 5 with

$$u = K_z z, \quad \dot{z} = \underbrace{\left(\widetilde{A} + \widetilde{\widetilde{B}}_0 K_z\right)}_{A_z} z$$

$$K_z = YW^{-1} = \begin{bmatrix} -0.0833 & 0.0222 & 0.3333 & -0.1111 \\ 0 & 0.1667 & 0 & -0.3333 \end{bmatrix}$$

where

$$Y = \begin{bmatrix} -0.11 & 0.0569 & 1.15 & -0.6267 \\ 0 & 0.4267 & 0 & -1.88 \end{bmatrix}$$

and $W = \text{diag}\{1.32, 2.56, 3.45, 5.64\}$. Finally, we obtain,

$$K_x = K_z \left(I + \widetilde{B}_1 K_z\right)^{-1} = \begin{bmatrix} 0.25 & 0 & -1 & 0.2 \\ 0 & 0.5 & 0 & -1 \end{bmatrix}$$

This feedback gain stabilizes the singular system and the closed-loop system matrix A_x becomes stable and Metzler.

VII. CONCLUSIONS

In this paper we introduced a subclass of positive systems, which we defined as input derivative positive systems. Using an algebraic transformation, we transformed the system to an equivalent form without derivative inputs and applied stabilization by using LMI. The connection of input derivative positive systems to the class of positive singular systems was established, which made it possible to use a modified LMI to stabilize it and maintain positivity.

REFERENCES

[1] Porter, B., and A. Bradshaw. "Eigenvalue assignment in multivariable linear systems with input-derivative control." International Journal of Control 16, no. 1 (1972): 109-113.

[2] Al-Nasr, Nazar, Victor Lovass-Nagy, and David L. Powers. "On transmission zeros and zero directions of multivariable time-invariant linear systems with input-derivative control." International Journal of Control 33, no. 5 (1981): 859-870.

[3] Al-Nasr, Nazar. "On zeros of time-invariant multivariable linear systems containing input derivatives." International Journal of Control 35, no. 4 (1982): 749-753.

[4] Yang, T. C. "Equation of state with input derivative in exploration." Int. Control Suppl 5, no. 20 (1983): 24-25.

[5] DAI, LIYI. "Observer problem for linear systems with input derivatives control." International journal of systems science 20, no. 1 (1989): 55-63.

[6] Qiao, Liang, Qingling Zhang, and Wanquan Liu. "Controllability and dissipativity analysis for linear systems with derivative input." Journal of the Franklin Institute 353, no. 2 (2016): 478-499.

[7] Galvão, Roberto Kawakami Harrop, Karl Heinz Kienitz, and Sillas Hadjiloucas. "Conversion of descriptor representations to state-space form: an extension of the shuffle algorithm." International Journal of Control 91, no. 10 (2018): 2199-2213.

[8] Lewis, Frank L. "A survey of linear singular systems." Circuits, systems and signal processing 5 (1986): 3-36.

[9] DAI, L. "Singular control systems." Lecture notes in control and information sciences 118 (1989).

[10] Xu, Shengyuan, and James Lam. Robust control and filtering of singular systems. Vol. 332. Berlin: Springer, 2006.

[11] Zhang, Qingling, Chao Liu, and Xue Zhang. Complexity, analysis and control of singular biological systems. Vol. 421. Springer Science & Business Media, 2012.

[12] Duan, Guang-Ren. Analysis and design of descriptor linear systems. Vol. 23. Springer Science & Business Media, 2010.

[13] Luenberger, David G. "Time-invariant descriptor systems." Automatica 14, no. 5 (1978): 473-480.

[14] Bender, Douglas, and Alan Laub. "The linear-quadratic optimal regulator for descriptor systems." IEEE Transactions on Automatic Control 32, no. 8 (1987): 672-688.

[15] Shafai, Bahram, Mohammad Naghnaeian, and Jie Chen. "Stability radius formulation of Lσ-gain in positive stabilisation of regular and time-delay systems." IET Control Theory & Applications 13, no. 15 (2019): 2327-2335.

[16] Kaczorek, Tadeusz. Positive 1D and 2D systems. Springer, 2002.

[17] Shafai, Bahram, Jie Chen, and M. Kothandaraman. "Explicit formulas for stability radii of nonnegative and Metzlerian matrices." IEEE transactions on automatic control 42, no. 2 (1997): 265-270.

[18] Rami, M. Ait, and F. Tadeo. "Linear programming approach to impose positiveness in closed-loop and estimated states." In Proc. of the 17th Intern. Symp. on Mathematical Theory of Networks and Systems. 2006.