Novel Spectral Algorithms for the Partial Credit Model

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Abstract

The Partial Credit Model (PCM) of Andrich (1978) and Masters (1982) is a fundamental model within the psychometric literature with wide-ranging modern applications. It models the integer-valued response that a subject gives to an item where there is a natural notion of monotonic progress between consecutive response values, such as partial scores on a test and customer ratings of a product. In this paper, we introduce a novel, time-efficient and accurate statistical spectral algorithm for inference under the PCM model. We complement our algorithmic contribution with in-depth non-asymptotic statistical analysis, the first of its kind in the literature. We show that the spectral algorithm enjoys the optimal error guarantee under three different metrics, all under reasonable sampling assumptions. We leverage the efficiency of the spectral algorithm to propose a novel EM-based algorithm for learning mixtures of PCMs. We perform comprehensive experiments on synthetic and real-life datasets covering education testing, recommendation systems, and financial investment applications. We show that the proposed spectral algorithm is competitive with previously introduced algorithms in terms of accuracy while being orders of magnitude faster.

1. Introduction

Item Response Theory (IRT) is the study of how people make choices in the presence of uncertainty. Since its popularization in the psychometric literature in the 1960s with the Rasch model for binary response data (Rasch, 1960), IRT has been utilized in many applications such as education testing, recommendation systems, evaluation of machine learning models, among others. Our work focuses on a fundamental IRT model known as the Partial Credit Model (PCM) (Andrich, 1978; Masters, 1982). The PCM is a generalization of the Rasch model to multi-level response data. It models the *ordered discrete response* of an individual l to an object i. Examples of ordered discrete responses include product ratings and test scores where there is a natural notion of monotonic progress from one level of response to the next (e.g., a product receiving 3 stars out of 5 stars review, a student scoring 6 out of 10 in a test). Let $X_{li} \in \{0, 1, 2, \ldots, K\}$, where K is a fixed integer, denote the response of individual l with parameter $\theta_l^* \in \mathbb{R}$ to item i with parameter $\boldsymbol{\beta}_i^* = \left(\beta_i^{*(1)}, \ldots, \beta_i^{*(K)}\right)^\top \in \mathbb{R}^K$. The PCM assumes that

$$\Pr\left(X_{li} = k\right) = \begin{cases} \frac{{Q_{li}}^{(1)} \ldots {Q_{li}}^{(K)}}{N_{li}} & \text{if } k = 0 \\ \frac{{P_{li}}^{(1)} \ldots {P_{li}}^{(K)}}{N_{li}} & \text{if } k = K \\ \frac{P_{li}^{(1)} \ldots P_{li}^{(k)} Q_{li}^{(k+1)} \ldots Q_{li}^{(K)}}{N_{li}} & \text{otherwise} \end{cases} \tag{1}$$

where $P_{li}^{(k)} = \frac{e^{\theta_l^*}}{e^{\theta_l^*} + e^{\beta_i^{*(k)}}}$, $Q_{li}^{(k)} = 1 - P_{li}^{(k)}$ and $N_{li} = \prod_{k'=1}^K Q_{li}^{(k')} + \sum_{k'=1}^{K-1} P_{li}^{(1)} \ldots P_{li}^{(k')} Q_{li}^{(k'+1)} \ldots Q_{li}^{(K)} + \prod_{k'=1}^K P_{li}^{(k')}$ is a normalization factor. To build an intution behind (1), consider the context of education testing, $P_{li}^{(k)}$ is the probability that a student solves the k-th step of a problem. The probability that a student receives partial grade $k \in [K]$ is proportional to the probability that the student simultaneously solves all the steps $1, \ldots, k$ while simulatenously fails to solve steps $k+1, \ldots, K$ where solving each step is independent of one another. In this sense, we can loosely interpret θ_l^* as the knowledge of student l and $\beta_i^{*(k)}$ as the difficulty of step k of problem i.

In this work and many classic papers in the PCM literature, the focus is on the *one-sided item estimation problem* where the goal is to estimate $\boldsymbol{\beta}^*$ accurately. Knowledge of β^* can be used to calibrate the scores/difficulty of different versions of a standardized test, or to characterize items for recommendation systems applications. The user parameters can also be easily recovered from the data given an estimate of $\boldsymbol{\beta}^*$ (Andrich, 1978; Masters, 1982). Previous inference approaches in the PCM literature include the Maximum Marginal Likelihood Estimate (MMLE) (Johnson, 2007) and Joint Maximum Likelihood Estimate (JMLE) (Andrich, 1978; Masters, 1982). In JMLE, both the user parameters

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θ^* and the item parameters $\boldsymbol{\beta}^*$ are jointly estimated by maximizing the full log likelihood function.

$$\theta_{\text{JMLE}}, \boldsymbol{\beta}_{\text{JMLE}} = \underset{\theta, \boldsymbol{\beta}}{\text{argmax}} \sum_{l \in [n], i \in [m]} \log \Pr\left(X_{li} \,\middle|\, \theta_l, \boldsymbol{\beta}_i \right) .$$

On the other hand, in MMLE, the statistician first specifies a prior distribution over the user parameters $\mathcal{D}_{\theta^*}$. The marginal likelihood is a function over $\boldsymbol{\beta}$ which is obtained by integrating the joint likelihood function with respect to the measure $\mathcal{D}_{\theta^*}$. The item estimate is obtained by maximizing the marginal likelihood.

$$\boldsymbol{\beta}_{\text{MMLE}} = \underset{\boldsymbol{\beta}}{\text{argmax}} \sum_{l \in [n], i \in [m]} \log \mathbb{E}_{\theta_l \sim \mathcal{D}_{\theta^*}} \Pr\left(X_{li} \,\middle|\, \theta_l, \boldsymbol{\beta}_i \right) .$$

While widely used, these popular approaches have their own limitations. It is well known that JMLE may produce inconsistent estimate for $\boldsymbol{\beta}^*$ when the number of users grows much more quickly than the number of items (Neyman & Scott, 1948; Andersen, 1973; Haberman, 1977; Ghosh, 1995). This behaviour is caused by the presence of many more 'nuisance' user parameters when the objective is to estimate $\boldsymbol{\beta}$ and is observed in experiments as well (cf. Figure 1). MMLE, on the other hand, tends to be more accurate and faster than JMLE but depends on the accuracy of the prior distribution $\mathcal{D}_{\theta^*}$, which may not be readily available in practice. This means that the algorithm requires careful hyperparameter selection which may add a layer of complexity to inference. For datasets with a large number of items and users, we also observe that JMLE and MMLE incur significant numerical issues related to optimization instability and slow convergence. In recent years, spectral estimators for IRT models (Choppin, 1982; Saaty, 1987; Garner Jr, 2002; Nguyen & Zhang, 2022; 2023b) have received increasing attention as they are competitive to the classical inference methods in terms of accuracy while often being significantly faster, more numerically stable and requiring little hyper-parameter optimization.

Our Contribution. We propose a novel spectral algorithm for inference under the PCM. We show, *for the first time* in the literature of the PCM, an inference algorithm that comes with a finite-sample error guarantee. We show that under reasonable sampling conditions, the spectral algorithm in fact achieves the optimal *average estimation error*, almost optimal *entrywise estimation error* and optimal sample complexity for *top-L ranking*. As an algorithmic extension, we exploit the efficiency of the spectral algorithm and propose a weighted generalization of the spectral algorithm. The weighted spectral algorithm is a key ingredient in a novel EM algorithm for learning a mixture of PCMs. We perform detailed experiments to show that the spectral algorithm is competitive in terms accuracy with the previously proposed estimators while being significantly faster. Our experiments span multiple different datasets from small scale education testing datasets to large recommendation systems datasets with hundreds of thousands of users and items. We also experiment with a novel application of the PCM to financial investment where we show that the spectral algorithm can be used as an accurate estimator within a meta trading strategy that achieves meaningful returns that outperform the aggregate market baseline.

Related Works. IRT was popularized by the celebrated works of George Rasch (Rasch, 1960) in the 1960s under the context of psychological testings. The namesake model, the Rasch model, is a probabilistic model over *binary response data*. Over the years, many variants of the Rasch model as well as more complicated models have been proposed including the Partial Credit Model (Andrich, 1978; Masters, 1982) and the Graded Response Model (Samejima, 1969). In recent years, IRT has been studied for applications beyond psychometrics including recommendation systems (Chen et al., 2004; 2005), evaluation of machine learning algorithms (Chen et al., 2019c; Chen & Ahn, 2020), active learning for computerized adaptive testing (Fries et al., 2014; Zhuang et al., 2023). Despite these wide ranging applications of IRT, statistical understandings of fundamental IRT models such as the Rasch model and the PCM model under realistic finite sample settings are only recently studied (Chen et al., 2019b; 2021; Nguyen & Zhang, 2022; 2023b). Recently, mixture models of PCMs have been applied to model response population heterogeneity in psychometric tests (Eid & Rauber, 2000; Kim et al., 2017). However, previous models assume availability of user features (i.e., demographic information) that help distinguish between the subgroups. Our approach, however, requires no user covariates.

2. Model Description

Let n be the number of users and m the number of items. To allow for missing data, let $A_{li} \in \{0, 1\}$ denote whether user l is shown item i. Let $\boldsymbol{\beta}^* = \left(\beta^{*(1)}, \ldots, \beta^{*(K)} \right) \in \mathbb{R}^{m \times K}$ where $\beta^{*(k)} = \left(\beta_1^{*(k)}, \ldots, \beta_m^{*(k)} \right)^\top \in \mathbb{R}^m$. Let $p \in (0, 1]$ and $A_{li} \sim \text{Bernoulli}(p)$ independently for every user-item pair. The user-item response is $X_{li} \in \{*, 0, 1, \ldots, K\}$ where $X_{li} = *$ if $A_{li} = 0$. Each response, conditioned on $A_{li} = 1$, is independently distributed according to (1). Let $X_l = (X_{l1}, \ldots, X_{lm})^\top$ denote the response vector of user l. Note that there is an inherent identifiability issue associated with translation of the parameters $\boldsymbol{\beta}^*$ and θ^*. That is, two PCMs parametrized by $\{\boldsymbol{\beta}^*, \theta^*\}$ and $\{\boldsymbol{\beta}^* + \alpha \mathbb{1}_m \mathbb{1}_K^\top, \theta^* + \alpha \mathbb{1}_n\}$ for any $\alpha \in \mathbb{R}$ are distributionally indistinguishable. Without loss of generality, one may then assume that $\langle \beta^{*(1)}, \mathbb{1}_m \rangle = 0$ where $\langle ., . \rangle$ is the vector inner product. We further assume that the parameters of the model are bounded, $\theta^*_{\min} \leq \theta^*_l \leq \theta^*_{\max} \, \forall l \in [n]$ and $\beta^*_{\min} \leq \beta_i^{*(k)} \leq \beta^*_{\max} \, \forall i \in [m], k \in [K]$ for some constants $\theta^*_{\min}, \theta^*_{\max}, \beta^*_{\min}, \beta^*_{\max}$. We use $\|.\|_F$ to mean the Frobenius norm, $\|.\|_\infty$ to mean the ℓ_∞ norm and $\mathbb{I}\{.\}$ to mean the indicator function.

3. Algorithm

In this section, we describe our spectral algorithm for the PCM, detailed in Algorithm 1. The algorithm takes in as input a response matrix $X \in \{*, 0, \ldots, K\}^{n \times m}$ and returns parameter estimate $\boldsymbol{\beta} = (\beta^{(1)}, \ldots, \beta^{(K)}) \in \mathbb{R}^{m \times K}$ where $\langle \beta^{(1)}, \mathbb{1}_m \rangle = 0$. We refer to the final output of the algorithm $\boldsymbol{\beta}$ as an *anchored estimate*. At a high level, the algorithm can be divided into two phases. In the first phase, we obtain *normalized* estimate $\hat{\beta}^{(k)}$ that satisfies $\langle \hat{\beta}^{(k)}, \mathbb{1}_m \rangle = 0$ and is *level-wise* close to $\beta^{*(k)}$ by a shift. That is $\hat{\beta}^{(k)} + \delta^{(k)} \mathbb{1}_m \approx \beta^{*(k)}$ for some $\delta^{(k)} \in \mathbb{R}$. The second phase of the algorithm estimates this shift $\delta^{(k)}$.

In the first phase, we iterate through K levels. In each level, we construct a m-state Markov chain $M^{(k)}$. A key quantity used in the algorithm is the following measurement. For $i \neq j$ and $k, k' \in [K]$, define

$$Y_{ij}^{(k,k')} := \sum_{l=1}^{n} A_{li} \, A_{lj} \, \mathbb{I}\{X_{li} = k, X_{lj} = k'\} \, . \quad (2)$$

The pairwise transition probability is defined in (3) where the factor d is any sufficiently large normalization factor to ensure that no entries of ${M_{ij}}^{(k)}$ are negative. For consistency, we set $d = \max_{i \in [m]} \left\{ \sum_{j' \neq i} B_{ij'} \right\}$. The stationary distribution of $M^{(k)}$, denoted as $\pi^{(k)}$, can be obtained using any eigenvector method (e.g., power iteration). We recover the normalized estimate $\hat{\beta}^{(k)}$ using an entrywise log transformation and normalization. The intuition for the Markov chain construction is explained in the next section.

It is worth pointing out that when $K = 1$, the *first phase* of our proposed spectral algorithm reduces to the inference algorithm for the Rasch model for *binary response data* in Nguyen & Zhang (2022). That algorithm, however, only produces a normalized estimate. In the PCM model, for $k > 1$, $\langle \beta^{*(k)}, \mathbb{1}_m \rangle \neq 0$ in general. We therefore develop a novel procedure in the second phase of the algorithm. Specifically, for each level $k \in \{2, \ldots, K\}$, we estimate an appropriate scalar shift $\delta^{(k)}$ from the response data such that $\beta^{(k)} = \hat{\beta}^{(k)} + \delta^{(k)} \approx \beta^{*(k)}$. Recall that by our identifiability ensuring assumption, $\delta^{(1)} = \delta^{*(1)} = 0$.

Note that in both phases of the algorithm, each iteration (each level k) of the for loop can be performed in *parallel*. A thoughtful implementation of the spectral algorithm can therefore be significantly accelerated.

Algorithm 1 Spectral Algorithm

Input: Response matrix $X \in \{*, 0, \ldots, K\}^{n \times m}$.
Output: Parameter estimates $\boldsymbol{\beta} = \left(\beta^{(1)}, \ldots, \beta^{(K)}\right)$.
for $k = 1, \ldots, K$ **do**
    Construct a Markov chain $M^{(k)}$ whose entries satisfy

$$M_{ij}^{(k)} = \begin{cases} \frac{1}{d} \, Y_{ij}^{(k,k-1)} & \text{if } i \neq j \\ 1 - \sum_{j' \neq i} M_{ij'}^{(k)} & \text{if } i = j \end{cases} , \quad (3)$$

    where $d = \max_{i \in [m]} \left\{ \sum_{j' \neq i} B_{ij'} \right\}$.
    Compute the stationary distribution $\pi^{(k)}$ of $M^{(k)}$.
    Obtain the normalized estimate ${\hat{\beta}_i}^{(k)} = \log({\pi_i}^{(k)}) - \frac{1}{m} \sum_{j=1}^{m} \log({\pi_j}^{(k)})$ for $i \in [m]$.
end for
Set $\beta^{(1)} = \hat{\beta}^{(1)}$ and $\delta^{(1)} = 0$.
for $k = 2, \ldots, K$ **do**
    Estimate the shift $\delta^{(k)}$ as

$$\delta^{(k)} = \log \left(\frac{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}_j^{(1)}} Y_{ij}^{(k-1,1)}}{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}_i^{(k)}} Y_{ij}^{(k,0)}} \right) . \quad (4)$$

    Set $\beta^{(k)} = \hat{\beta}^{(k)} + \delta^{(k)} \mathbb{1}_m$.
end for

4. Analysis

In this section, we provide an intuition for the principles underlying Algorithm 1 and the key theoretical guarantee enjoyed by the algorithm.

The Normalized Estimate $\hat{\beta}^{(k)}$. By carefully inspecting (1), one could verify that for any $k \in \{1, \ldots, K\}$ and any l, i, the following equation holds.

$$\frac{\mathbb{E}\left[\mathbb{I}\{X_{li} = k-1\}\right]}{\mathbb{E}\left[\mathbb{I}\{X_{li} = k\}\right]} = \frac{\Pr\left(X_{li} = k-1\right)}{\Pr\left(X_{li} = k\right)} = \frac{e^{\beta_i^{*(k)}}}{e^{\theta_l^*}} \, .$$

We can remove the user-specific parameter θ^*_l by noting that

$$\frac{\mathbb{E}\left[\mathbb{I}\{X_{li} = k-1\}\right]}{\mathbb{E}\left[\mathbb{I}\{X_{li} = k\}\right]} \, \frac{\mathbb{E}\left[\mathbb{I}\{X_{lj} = k\}\right]}{\mathbb{E}\left[\mathbb{I}\{X_{lj} = k-1\}\right]} = \frac{e^{\beta_i^{*(k)}}}{e^{\beta_j^{*(k)}}} .$$

We emphasize that by design, the spectral algorithm is *agnostic of any assumptions about the distribution of user parameters*. Rearranging the terms of the display above and summing over all item pairs $i \neq j$ and user $l \in [n]$, we get

$$e^{\beta_j^{*(k)}} \sum_{l=1}^{n} \mathbb{E}\left[\mathbb{I}\{X_{li} = k-1, X_{lj} = k\}\right] = e^{\beta_i^{*(k)}} \sum_{l=1}^{n} \mathbb{E}\left[\mathbb{I}\{X_{li} = k, X_{lj} = k-1\}\right] \, .$$

By defining $\pi_i^{*(k)} := \frac{e^{\beta_i^{*(k)}}}{\sum_{i'=1}^m e^{\beta_{i'}^{*(k)}}}$, we rewrite the above as

$$\pi_j^{*(k)} \mathbb{E}\left[Y_{ji}{}^{(k,k-1)}\right] = \pi_i^{*(k)} \mathbb{E}\left[Y_{ij}{}^{(k,k-1)}\right] .$$

This condition is also known as *detailed balance* in Markov chain analysis. Consider an idealized Markov chain $M^{*(k)}$ whose transition probabilities are defined as

$$M_{ij}^{*}{}^{(k)} = \begin{cases} \frac{1}{d}\mathbb{E}\left[Y_{ij}{}^{(k,k-1)}\right] & \text{if } i \neq j \\ 1 - \sum_{j' \neq i} M_{ij'}^{*}{}^{(k)} & \text{if } i = j \end{cases} ,$$

The stationary distribution of the above Markov chain is precisely $\pi^{*(k)}$ (cf. Lemma A.18). The Markov chain constructed in (3) is an empirical estimate of this Markov chain. Given a sufficiently large sample size n, the stationary distribution $\pi^{(k)}$ will be close to the idealized stationary distribution $\pi^{*(k)}$. Let $\hat{\beta}^{*(k)} = \beta^{*(k)} - \mathbb{1}_m \delta^{*(k)}$, where $\delta^{*(k)} = \frac{1}{m}\langle \beta^{*(k)}, \mathbb{1}_m \rangle$, to be the normalized version of $\beta^{*(k)}$. One could verify that $\pi_i^{*(k)} = \frac{e^{\hat{\beta}_i^{*(k)}}}{\sum_{j \in [m]} e^{\hat{\beta}_j^{*(k)}}}$. Naturally, if $\pi^{(k)} \approx \pi^{*(k)}$ then $\hat{\beta}^{(k)} \approx \hat{\beta}^{*(k)}$ (cf. Theorem A.10).

Recovering the shift $\delta^{(k)}$. To build an intuition for the second phase of the algorithm, again inspect (1). For any two levels $k, k' \in \{1, \ldots, K\}$, the following holds.

$$\frac{\mathbb{E}\left[\mathbb{I}\left\{X_{li} = k\right\}\right]}{\mathbb{E}\left[\mathbb{I}\left\{X_{li} = k-1\right\}\right]} = \frac{\Pr\left(X_{li} = k\right)}{\Pr\left(X_{li} = k-1\right)} = \frac{e^{\theta_l^*}}{e^{\beta_i^{*(k)}}} ,$$
$$\frac{\mathbb{E}\left[\mathbb{I}\left\{X_{lj} = k'-1\right\}\right]}{\mathbb{E}\left[\mathbb{I}\left\{X_{lj} = k'\right\}\right]} = \frac{\Pr\left(X_{lj} = k'-1\right)}{\Pr\left(X_{lj} = k'\right)} = \frac{e^{\beta_j^{*(k')}}}{e^{\theta_l^*}} .$$

We remove the user-dependent factor θ_l^* by multiplying the above two equations and rearranging the terms. We have

$$\mathbb{E}\left[\mathbb{I}\left\{X_{li} = k, X_{lj} = k'-1\right\}\right] e^{\beta_i^{*(k)}} = \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k-1, X_{lj} = k'\right\}\right] e^{\beta_j^{*(k')}} .$$

Sum both sides of the above equality over all pairs i, j and over all users. Recall the definition of $\hat{\beta}^{*(k)} = \beta^{*(k)} - \mathbb{1}_m \, \delta^{*(k)}$. We have

$$\sum_{i \neq j} e^{\hat{\beta}_i^{*(k)} + \delta^{*(k)}} \sum_{l \in [n]} \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k, X_{lj} = k'-1\right\}\right]$$
$$= \sum_{i \neq j} e^{\hat{\beta}_j^{*(k')} + \delta^{*(k')}} \sum_{l \in [n]} \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k-1, X_{lj} = k'\right\}\right] .$$

Pick $k' = 1$ and recall that $\delta^{*(1)} = 0$. Rearranging the above display, we have, for any $k \in \{2, \ldots, K\}$

$$\delta^{*(k)} = \log\left(\frac{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}_j^{*(1)}} \mathbb{E}[Y_{ij}^{(k-1,1)}]}{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}_i^{*(k)}} \mathbb{E}[Y_{ij}^{(k,0)}]}\right) .$$

The reader can see that Line 10 of Algorithm 1 replaces the above idealized quantity with its empirical estimate. We will show that, given a sufficiently large sample size n, $\delta^{(k)} \approx \delta^{*(k)}$ (cf. Lemma A.11).

The Final Estimate. The accuracy of the anchored estimate can be expressed in terms of the accuracy of the normalized estimate and the shift via triangle's inequality.

$$\begin{aligned} \left\|\beta^{(k)} - \beta^{*(k)}\right\|_2 &= \left\|\hat{\beta}^{(k)} - \hat{\beta}^{*(k)} + \left(\delta^{(k)} - \delta^{*(k)}\right) \mathbb{1}_m\right\|_2 \\ &\leq \left\|\hat{\beta}^{(k)} - \hat{\beta}^{*(k)}\right\|_2 + \sqrt{m}\left|\delta^{(k)} - \delta^{*(k)}\right| . \end{aligned} \tag{5}$$

By upper-bounding the two error terms on the RHS of the above display across all levels, we obtain an aggregate Frobenius norm error bound for Algorithm 1.

Theorem 4.1. *Consider the uniform sampling model in Section 2. There exist constants C_0, C_0', C_1, C_1' that only depend on K such that if $np^2 \geq C_0 \log m$ and $mp \geq C_0' \log n$ and $np \geq C_1$ then with probability at least $1 - O(n^{-10}) - O(K \exp\left(-O(np^2)\right)) - O(K(m)^{-10})$,*

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}^*\|_F^2 \leq \frac{C_1' m}{np} .$$

To complete our analysis, we have the following matching lower bound which asserts the optimality, up to a constant factor, of the spectral method. As mentioned earlier, finite sample analysis of the PCM has not been established in the literature and we are the first to present a matching upper and lower bound.

Theorem 4.2. *Consider the uniform sampling model in Section 2. There exist constants c, c' that only depend on K such that if $np \geq c'$ then there exists a class of Partial Credit Models $\mathcal{B}$ such that for any statistical estimator $\boldsymbol{\beta}'$, the minimax risk is lower bounded as*

$$\inf_{\boldsymbol{\beta}'} \sup_{\boldsymbol{\beta}^* \in \mathcal{B}} \mathbb{E}\left[\|\boldsymbol{\beta}' - \boldsymbol{\beta}^*\|_F^2\right] \geq \frac{cm}{np} .$$

Entrywise Error and Top-L Ranking Guarantee. In certain applications (e.g., stock ranking in our experiments), we wish to identify the 'best' items from user response data. To the best of our knowledge, learning to rank from graded response data has never been studied in the IRT literature. Towards this end, we first propose the following *novel* definition of ranking score under the PCM model.

$$s_i^* := \frac{1}{n} \sum_{l=1}^{n} \frac{\Pr(X_{li} = K | \theta_l^*, \boldsymbol{\beta}_i^*)}{\Pr(X_{li} = 0 | \theta_l^*, \boldsymbol{\beta}_i^*)} . \tag{6}$$

The ranking score can be interpreted as the likelihood ratio that a user gives an item the highest possible rating over the lowest rating. In the top-L ranking problem, we want to find the L items with the *highest* scores, denoted as $\mathcal{S}_L^*$. It can be shown that $\mathcal{S}_L^*$ is also the set of items with the *highest negative parameter sums*. That is, $\mathcal{S}_L^* = \mathcal{S}_L(\boldsymbol{\beta}^*)$ where

$$\mathcal{S}_L(\boldsymbol{\beta}) := \underset{\mathcal{S}\subseteq[m]:|\mathcal{S}|=L}{\operatorname{argmax}} -\sum_{i\in\mathcal{S}}\sum_{k=1}^{K}\beta_i^{(k)} \,. \tag{7}$$

This equivalency means that a PCM ranking algorithm would run any PCM inference algorithm on user data to obtain estimate $\boldsymbol{\beta}$ and return $\mathcal{S}_L(\boldsymbol{\beta})$. We will show that the fundamental difficulty of correctly finding the top-L set is controlled by the gap between the L-th and $L+1$-th item.

$$\Delta_L^* := \sum_{k=1}^{K} {\beta^*_{[L+1]}}^{(k)} - {\beta^*_{[L]}}^{(k)} \,, \tag{8}$$

where $\boldsymbol{\beta}^*_{[t]}$ denotes the parameter of the item with the t-th largest item score per (6). Via a refined analysis of the spectral estimate, we obtain the following *entrywise* error bound. Subsequently, we obtain a sample complexity bound for top-L ranking using the spectral algorithm.

Theorem 4.3. *Consider the setting of Theorem 4.1. There exists a constant C_1'' that only depends on K such that, with probability at least $1-O(n^{-10})-O(K\exp\left(-O(np^2)\right))-O(K(m)^{-10})$,*

$$\max_{k\in[K]}\left\|\beta^{(k)}-\beta^{*(k)}\right\|_\infty \leq \frac{C_1''\sqrt{\log m}}{\sqrt{np}} \,.$$

Additionally, there exists a constant C_L such that if $np \geq \frac{C_L \log m}{{\Delta_L^*}^2}$ then $\mathcal{S}_L(\boldsymbol{\beta}) = \mathcal{S}_L^*$.

Due to space constraint, we defer matching lower bound results to the supplementary materials. The key take-away is that the spectral algorithm achieves the *optimal entrywise estimation error* (up to a log factor) and the *optimal sample complexity* for top-L ranking.

5. Algorithmic Extension

In this section, we discuss an algorithmic extension of our spectral algorithm to enhance its usefulness to real life applications where the data exhibits user heterogeneity. We propose a novel EM-based algorithm to learn *mixtures of PCM models*, summarized in Algorithm 3.

A Mixture Model of PCMs. Fix an integer C, our proposed mixture model is parametrized by C set of item parameters $\boldsymbol{\beta}_1^*, \ldots, \boldsymbol{\beta}_C^* \in \mathbb{R}^{m\times K}$. A subtle yet important distinction between the single model setting and the mixture setting is that in the former we can assume without loss of generality that $\langle \beta^{*(1)}, \mathbb{1}_m\rangle = 0$ due to the shift invariance property of the PCM. However, in the mixture setting, assuming that $\langle \beta_{c,\cdot}^{*(1)}, \mathbb{1}_m\rangle = 0\,\forall c\in[C]$ would substantially restrict the parameter space. Instead, we posit that the user parameters θ^* are independently drawn from the same distribution with *zero mean* such as the standard Gaussian distribution. For a user l with a latent class membership z_l and parameter θ_l^*, her item response probabilities follow a PCM parametrized by $\boldsymbol{\beta}_{z_l}^* = \left({\beta_{z_l,\cdot}^*}^{(1)}, \ldots, {\beta_{z_l,\cdot}^*}^{(K)}\right)$ and θ_l^*. As a simplifying assumption, we assume a uniform prior class membership probability, $\Pr(z_l = c) = \frac{1}{C}\,\forall c\in[C]$. While we do not know C in general, in our experiments, we determine the most appropriate number of mixture components using validation data. A mixture of PCMs is a more expressive model than a single PCM and might be preferred in applications where the statistician has strong reasons to believe that the user population exhibits heterogeneity. As an example, one may suspect that there are different groups of movie watchers who tend to like different genres of movies and different groups of students with varying academic interests who tend to have different performance across different subjects.

A Clustering Initialization. An EM algorithm requires an accurate initial estimate in order to efficiently converge to a good local optimum. To this end, we propose using the C-means clustering algorithm to cluster users into different groups based on their response patterns. One natural idea is to cluster the response vectors $\{X_l\}_{l=1}^m$. However, if a user has a highly positive θ_l^* parameter or a highly negative θ_l^* parameter, her responses tend to be high or low respectively, regardless of the underlying class membership z_l. As an analogy, a very strong student with high ability parameter θ_l^* will do well across all tests. Hence, by focusing on the *differences in responses* of the same user towards different items, we can more accurately cluster the users. This motivates us to embed the user response X_l as a pairwise difference vector $Z_l = \left(Z_{l,ij}\right)_{i,j\in[m]:i\neq j} \in \mathbb{R}^{\binom{m}{2}}$ where

$$Z_{l,ij} := A_{li}A_{lj}\left(X_{li}-X_{lj}\right) \,. \tag{9}$$

Let $Z = \left(Z_1,\ldots,Z_n\right)^\top \in \mathbb{R}^{n\times\binom{m}{2}}$ denote the pairwise difference matrix. Lines 1-2 of Algorithm 3 construct Z and obtain C clusters of the *rows* of Z. For each cluster, we apply Algorithm 1 on the corresponding rows of the matrix X (Line 5).

The Spectral-EM Algorithm. Building on the Expectation-Maximization framework, the second half of Algorithm 3 alternates between the E-step (Line 9) and the M-step (Lines 10-18) starting from the initial estimate until convergence. The E-step computes the posterior distribution over the class membership of each user. Let $q_{lc} = \Pr\left(z_l = c \,\middle|\, \boldsymbol{\beta}, X_l\right) = \mathbb{E}_{\theta_l\sim\mathcal{N}(0,1)}\Pr\left(z_l = c\,\middle|\,\boldsymbol{\beta}, X_l, \theta_l\right)$ which can be efficiently evaluated using single-variable numerical integration.

Fixing the posterior probabilities $Q = (q_{lc})_{l\in[n],c\in[C]}$, the M-step consists of solving C Markov chain problems and is parallelizable. For component $c \in [C]$, define the weighted generalization of ${Y_{ij}}^{(k,k')}$ as follows.

$$ {Y_{ij}}^{(k,k')}(Q,c) := \sum_{l=1}^{n} q_{lc} A_{li} A_{lj} \mathbb{I}\left\{X_{li} = k, X_{lj} = k'\right\}. \tag{10} $$

Comparing to the definition of ${Y_{ij}}^{(k,k')}$ in (2), the weighted generalization weighs each user response with the corresponding posterior probability. The weighted Markov chain is defined analogously to (3). For $c \in [C]$ and $k \in [K]$,

$$ {M_{ij}}^{(k)}(Q,c) = \begin{cases} \frac{1}{d} {Y_{ij}}^{(k,k-1)}(Q,c) & \text{if } i \neq j \\ 1 - \sum_{j' \neq i} {M_{ij'}}^{(k)}(Q,c) & \text{if } i = j \end{cases} \tag{11} $$

Lines 9-10 compute the normalized estimates from the stationary distribution of the weighted Markov chains. Similar to Lines 10-11 of Algorithm 1, Line 15 of Algorithm 3 produces an anchored estimate $\tilde{\boldsymbol{\beta}}_c$ where $\langle \tilde{\beta}^{(1)}_{c,\cdot}, \mathbb{1}_m \rangle = 0$. The intuition for the weighted generalization of the spectral algorithm can be developed from a similar angle as the vanilla spectral algorithm. Due to space constraint, we defer the detailed description to the supplementary materials.

Estimating the Component-wise Parameter Shift. Step 5 and 15 of Algorithm 3 produce anchored estimate $\tilde{\boldsymbol{\beta}}_c$ where $\langle \tilde{\beta}^{(1)}_{c,\cdot}, \mathbb{1}_m \rangle = 0$. Note that in the *single model setting*, because of the assumption that $\langle \beta^{(1)}, \mathbb{1}_m \rangle = 0$, the first level shift $\delta^{(1)} = 0$. On the other hand, in the mixture setting, there is a component-wise shift ${\delta_c}^{(1)}$ that needs to be estimated from the data such that $\tilde{\beta}^{(k)}_{c,\cdot} + {\delta_c}^{(1)} \approx {\beta^*_{c,\cdot}}^{(k)}$ for all $c \in [C]$. Recall the assumption that the user parameter distribution is zero-meaned. We can then estimate the component-wise shift from the data by maximizing the log marginal likelihood as follows.

$$ {\delta_c}^{(1)} = \operatorname*{argmax}_{\delta} \sum_{l=1}^{n} \log \mathbb{E}_{\theta_l \sim \mathcal{N}(0,1)} \left[\Pr\left(X_l, c_l = c \,\middle|\, \theta_l, \tilde{\boldsymbol{\beta}}_c + \delta \mathbb{1}_m \mathbb{1}_K^\top \right) \right]. \tag{12} $$

The above optimization problem has a single variable δ and can thus be implemented using many off-the-shelf integral optimization or approximate solvers. Lastly, we obtain the final estimate from the shift as ${\beta_{c,\cdot}}^{(k)} = \tilde{\beta}^{(k)}_{c,\cdot} + {\delta_c}^{(1)} \mathbb{1}_m$ for $k = 1, \ldots, K$. This subroutine is summarized in Algorithm 2 and is used in Line 7 and Line 18 of Algorithm 3.

Algorithm 2 Parameter Shift Estimation Algorithm

1: **Input:** Response matrix $X \in \{*, 0, \ldots, K\}^{n \times m}$, posterior weights $Q \in \mathbb{R}^{n \times C}$, estimates $\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_C$.
2: **Output:** Refined estimate $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_C$.
3: **for** $c = 1, \ldots, C$ **do**
4: Compute ${\delta_c}^{(1)}$ per (12) and set ${\beta_{c,\cdot}}^{(k)} = \tilde{\beta}^{(k)}_{c,\cdot} + {\delta_c}^{(1)} \mathbb{1}_m$ for $k \in [K]$.
5: **end for**

Algorithm 3 The Spectral-EM Algorithm

1: **Input:** Response matrix $X \in \{*, 0, \ldots, K\}^{n \times m}$, number of mixture components C.
2: **Output:** Component-wise parameter estimates $\boldsymbol{\beta}_c \in \mathbb{R}^{m \times K}$ for $c \in [C]$.
3: Embed the user responses per (9). Run the C-means clustering algorithm on the rows of the matrix Z. Let $Q_{\text{init}} \in \{0,1\}^{n \times C}$ denote the estimated membership matrix.
4: **for** cluster $c \in [C]$ **do**
5: Run Algorithm 1 on the corresponding submatrix X to obtain anchored estimate $\tilde{\boldsymbol{\beta}}_c$.
6: **end for**
7: Run Algorithm 2 on X, Q_{init} and $\{\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_C\}$ to obtain initial estimate $\boldsymbol{\beta}_{\text{init}}$.
8: **for** $t = 0, 1 \ldots$ until convergence **do**
9: Estimate the posterior distribution for $l \in [n], c \in [C]$, $q_{lc} = \Pr\left(z_l = c \,\middle|\, \boldsymbol{\beta}_{\text{current}}, X_l \right)$.
10: **for** $c \in [C]$ **do**
11: **for** $k = 1, \ldots, K$ **do**
12: Construct weighted Markov chains $M^{(k)}(Q,c)$ per (11) and compute its stationary distribution, ${\pi_{c,\cdot}}^{(k)}$. Recover the normalized estimate $\hat{\beta}^{(k)}_{c,\cdot} = \log\left({\pi_{c,\cdot}}^{(k)}\right) - \frac{1}{m} \langle \log {\pi_{c,\cdot}}^{(k)}, \mathbb{1}_m \rangle \mathbb{1}_m$.
13: **end for**
14: **for** $k = 2, \ldots, K$ **do**
15: Estimate the shift

$$ {\delta_c}^{(k)} = \log \left(\frac{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}^{(1)}_{c,j}} Y_{ij}^{(k-1,1)}(Q,c)}{\sum_{i,j \in [m]: i \neq j} e^{\hat{\beta}^{(k)}_{c,i}} Y_{ij}^{(k,0)}(Q,c)} \right). \tag{13} $$

and anchored estimate $\tilde{\beta}^{(k)}_{c,\cdot} = \hat{\beta}^{(k)}_{c,\cdot} + {\delta_c}^{(k)} \mathbb{1}_m$.
16: **end for**
17: **end for**
18: For $c \in [C]$, set $\tilde{\boldsymbol{\beta}}_c = \left(\hat{\beta}^{(1)}_{c,\cdot}, \tilde{\beta}^{(2)}_{c,\cdot}, \ldots, \tilde{\beta}^{(K)}_{c,\cdot} \right)$. Update the current estimate $\boldsymbol{\beta}$ by running Algorithm 2 on X, Q and $\{\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_C\}$.
19: **end for**

6. Experiments

In this section, we present the empirical findings to complement our theoretical contribution and showcase the usefulness of the spectral algorithm. The detailed description of the experiment is deferred to the supplementary materials. We compare the spectral algorithm against two well known and popular PCM inference algorithms – the marginal likelihood estimate (MMLE) (Basu, 2011) and the joint maximum likelihood estimate (JMLE) (Andersen, 1973; Fischer, 1981; Haberman, 1977). The open source implementation of MMLE and JMLE can be found in Sanchez (2021). The implementation of our algorithms can be found at `https://github.com/dnguyen1196/spectral-algos-discrete-data`.

Synthetic Data. We first focus on synthetic single-model data where have knowledge of the true model parameters $\theta^*, \boldsymbol{\beta}^*$. Figure 1 shows that when the prior distribution over the user parameter θ^* is correctly specified, both MMLE and the spectral algorithm perform similarly. On the other hand, JMLE produces inaccurate estimate of $\boldsymbol{\beta}$ as n grows much bigger than m. This is expected from prior theoretical analysis that when the number of items is much smaller than the number of users, JMLE produces inconsistent $\boldsymbol{\beta}$ estimate (Ghosh, 1995; Haberman, 1977). As for speed, Figure 2 shows that the spectral algorithm is magnitude of orders faster than JMLE and MMLE in terms of inference time. Figure 3 shows that the spectral algorithm is the only algorithm that is robust to misspecification of the user distribution. While MMLE's performance crucially depends on an accurate specification of $\mathcal{D}_{\theta^*}$, the spectral algorithm is agnostic of this assumption and produces an accurate estimate. All lines show the average error over 100 trials.

Learning Mixtures of PCMs. Figure 4 shows the average error over 50 trials of the Spectral-EM algorithm and *the initialization error* of the cluster-then-learn estimate (Line 7) in Algorithm 3 on synthetic data drawn from mixtures of 4 PCMs with 50 items and 5 response levels ($K = 4$). In our set up, the number of mixture components is assumed to be known. We measure the average Frobenius norm error, $\min_{\alpha \in \Pi_C} \sqrt{\frac{\sum_{c=1}^{C} \|\boldsymbol{\beta}_{\alpha_c} - \boldsymbol{\beta}_c^*\|_F^2}{C}}$, where Π_C is the set of all permutations of $\{1, \ldots, C\}$. The figure shows that as the sample size increases, estimation error decreases. The reader can also see that the iterative EM algorithm significantly refines the initial estimate, leading to a more accurate final estimate.

Real Data. In addition to synthetic datasets, we compare the spectral algorithm against MMLE and JMLE on real life datasets reflecting the PCM's application in education testing and recommendations systems. Table 1 summarizes the performance of the three algorithms on 9 datasets in terms of predictive performance on respective heldout test set as well as inference time. Our experiments cover small scale dataset with fewer than 100 users and items to large recommendation systems datasets with tens of thousands of users and items. The reader can see that the spectral algorithm performs consistently well and is competitive with both MMLE and JMLE. Most notably, the spectral algorithm is much faster than the two competitors. This significant advantage of the spectral algorithm together with its competitive accuracy highlights the usefulness of the spectral algorithm. The Spectral-EM algorithm generally produces more accurate estimate at the expense of worse time complexity than the spectral algorithm. Therefore, it is reasonable to deploy the EM algorithm when there is a strong reason to suspect

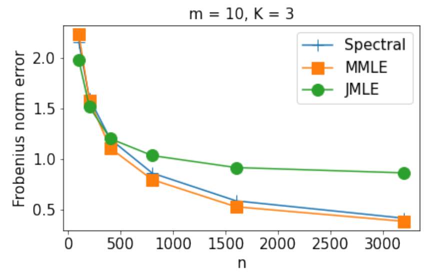

Figure 1: When the prior is correctly specified, MMLE performs as well as the spectral algorithm. JMLE is inconsistent when the number of tests is small.

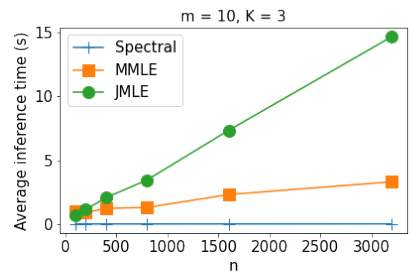

Figure 2: The spectral algorithm is much more efficient than both JMLE and MMLE.

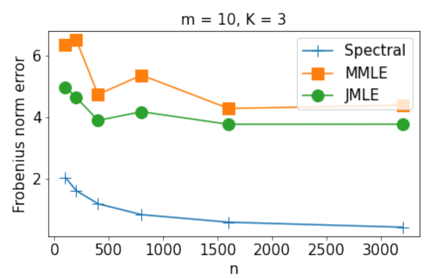

Figure 3: When the prior is mis-specified, MMLE produces inaccurate estimate.

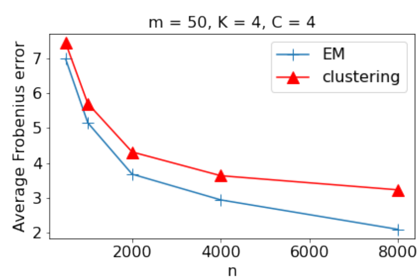

Figure 4: The Spectral-EM iterative algorithm refines the cluster-then-learn initial estimate.

that there is substantial heterogeneity in the data and that accuracy is the priority of the statistician.

Stock Ranking Applications. Besides applications in recommendation systems and education testing, we find that finance is a real-life application where IRT approaches can be valuable. To the best of our knowledge, we are the first to apply an IRT framework to finance by mining datasets of stock ratings given by financial analysts. The dataset (u/nobjos) consists of ratings given by 500 financial analysts to over 200 stocks over a 10 year period from 2011-01 to 2021-02. The recommendations are divided into three groups (sell, hold, buy) and are augmented with daily stock price data extracted from Yahoo Finance.

We build a meta trading strategy that makes use of IRT inference algorithms in formulating trading decisions. Specifically, we aggregate rating data in a moving window of size W (measured in months). We use *single model* PCM inference algorithms to infer the item (stock) parameters and induce a *ranking over the stocks*. The meta trading strategy starts with \$10000 and, at the end of each month, liquidates the current position and buys the current top 10 stocks defined per Equation (7) in equal dollar amounts. The performance of the trading strategy depends on the top-10 ranking accuracy of the underlying inference algorithm. The performance of the overall trading strategy is measured using the *total profit and loss* (PNL) over time, which is the total amount of money the algorithm manages minus the initial investment. We set 2011-01 to 2018-01 as the training period where we find the model with the appropriate hyper-parameters that maximizes the PNL within the training period. The testing period is left untouched until the final evaluation.

Table 2 shows the performance of the three inference algorithms and hyperparameters during the training historical period and the subsequent heldout test period from 2019-2021. Among all the models, the meta trading strategy that uses the spectral algorithm trained on ratings data aggregated from a 18-month moving window performs the best in both the training data as well as the heldout data. One can also see that the performance of the spectral algorithm has less variance and is more predictable compared to that of MMLE and JMLE.

Figure 5 shows the PNL of the best meta trading strategy that uses the spectral algorithm versus those using MMLE and JMLE during both the training period and the testing period. From the training data, we select the hyperparameters for each method that yields the highest PNL. For the testing period, we initialize from \$10000 seed capital again and compute the cummulative PNL. One can see that the strategy leveraging the spectral algorithm significantly outperforms those using MMLE and JMLE as well as a passive S&P500 benchmark. The spectral algorithm is also much faster than MMLE and JMLE, making it the best overall performer and corroborating the theoretical optimality of the spectral algorithm as shown in our analysis.

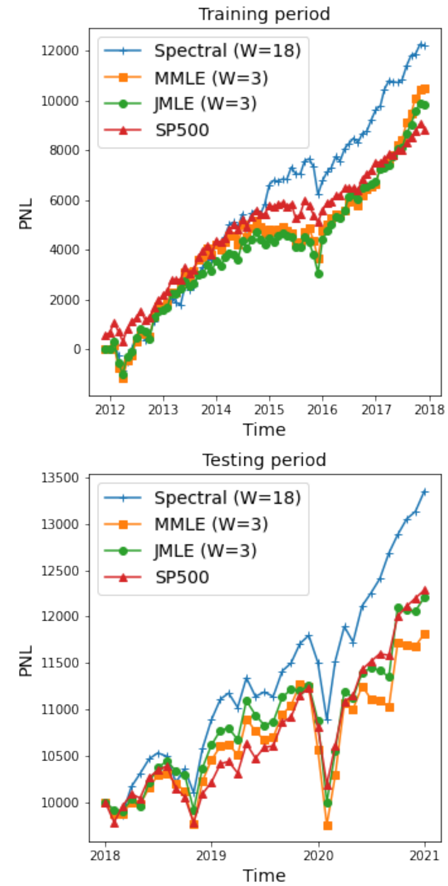

Figure 5: Cummulative PNL of trading strategies using PCM inference algorithms.

7. Conclusion

In conclusion, we introduce a novel statistical spectral algorithm for inference under the Partial Credit Model. The algorithm is intuitive and simple to implement. To augment our algorithmic contribution, we prove that the spectral algorithm achieves the optimal finite sample error guarantee up to a constant factor. To support our theoretical findings, we perform comprehensive experiments. The algorithm achieves accuracy on par with previously proposed methods while enjoying much faster inference time. The practical significance of the spectral algorithm shows that it is valuable in diverse applications from small scale education testing, large scale recommendation systems and financial investments.

Dataset	Heldout MAE				Heldout LLH				Inference Time (s)			
	Spectral	Mixture	MMLE	JMLE	Spectral	Mixture	MMLE	JMLE	Spectral	Mixture	MMLE	JMLE
LSAT	0.29	0.25	0.27	0.29	-1.356	-0.918	-1.397	-1.359	0.002	2.0495	0.334	0.698
UCI	0.21	0.23	0.18	0.21	-0.923	-0.617	-0.923	-0.893	0.002	0.3727	0.160	0.339
Grades3	0.10	0.12	0.10	0.10	-0.693	-0.607	-0.710	-0.716	0.002	1.096	0.141	0.084
Book Genome	0.64	0.66	0.72	0.63	-1.207	-1.19	-1.269	-1.238	46	2.7K	3.2K	27K
EachMovie	1.08	0.99	1.16	0.93	-1.618	-1.354	-1.869	-1.652	32	287	660	6.8K
Hetrec-2k	0.66	0.64	0.74	0.61	-1.274	-1.230	-1.468	-1.398	19	240	650	6.1K
ml-1M	0.71	0.68	0.80	0.70	-1.288	-1.239	-1.396	-1.481	4.5	412	484	5350
ml-10M	0.68	0.65	0.79	0.65	-1.290	-1.243	-1.406	-1.382	105	7.2K	14K	135K
ml-20M	0.67	0.66	0.78	0.65	-1.284	-1.248	-1.412	-1.383	170	7.8K	18K	200K

Table 1: The spectral algorithm is competitive in terms of accuracy while being orders of magnitude faster than the other algorithms.

	Train PNL	Test PNL
Spectral (W = 3)	10288	12323
Spectral (W = 6)	11673	13081
Spectral (W = 12)	12200	13139
Spectral (W = 18)	12309	14003
MMLE (W = 3)	10465	12559
MMLE (W = 6)	9511	12404
MMLE (W = 12)	4987	13545
MMLE (W = 18)	9097	11928
JMLE (W = 3)	9813	12006
JMLE (W = 6)	9526	12538
JMLE (W = 12)	5195	13177
JMLE (W = 18)	7700	12262

Table 2: PNL of trading algorithms with different PCM inference algorithms in the training period and the test period.

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Impact Statement

This paper falls within the literature on item response theory. The models within this literature have often been used to model real-life human responses. In certain applications such as psychological testing and recommendation systems, the data may contain sensitive and private information. When deployed in such settings, it is imperative for the statistician to carefully weigh and consider any potential human impact that the inference and reporting of the model parameters may bring. In a similar vein, we believe that fairness study and privacy analysis are understudied area within the item response theory literature.

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A. Proof

In this section, we present the proofs for all the theoretical results in our paper. For the sake of clarity and conciseness, we will defer proofs of some intermediate helper lemmas to a later section and only go into detailed proofs of important theorems and lemmas here.

A.1. Preliminaries

We use $a \vee b$ to mean $\max\{a, b\}$ and $a \wedge b$ to mean $\min\{a, b\}$. Define, for any $k \in [K]$, $i, j \in [m]$,

$$\beta^*{}_{\max} := \max_{i\in[m],k\in[K]} \left\{\beta_i^{*(k)}\right\}, \beta^*{}_{\min} := \min_{i\in[m],k\in[K]} \left\{\beta_i^{*(k)}\right\} .$$

$$\theta^*{}_{\min} := \min_{l\in[n]} \{\theta_l^*\}, \theta^*{}_{\max} := \max_{l\in[n]} \{\theta_l^*\} .$$

$$\kappa := \max_{k\in[K]} \{\beta^*{}_{\max} - \beta^*{}_{\min}\} .$$

$$\gamma := \min_{l\in[n],i,j\in[m],k,k'\in[K]} \mathbb{E}\left[\mathbb{I}\left\{X_{ni} = k, X_{nj} = k'\right\}\right] .$$

We have the following useful properties of the theoretical quantities of the PCM. The key takeaway is that $\pi_i^{*(k)} = O\left(\frac{1}{m}\right)$ and γ is lower bounded by a constant that only depends on $K, \beta^*_{\max}, \beta^*_{\min}, \theta^*_{\max}, \theta^*_{\min}$.

Proposition A.1.

$$\frac{1}{me^{\kappa}} \leq \pi_i^{*(k)} \leq \frac{e^{\kappa}}{m} \quad \forall i \in [m], k \in [K].$$

$$\gamma \geq \left(\frac{\exp(\Delta^*) - 1}{\exp(K\Delta^*) - 1}\right)^2 ,$$

where $\Delta^* = |\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|$.

Proof. See proof. □

The following lemma establishes that under the random sampling model, $B_{ij} := \sum_{l=1}^n A_{li}A_{lj}$ is concentrated. Conditioned on the event $\mathcal{A}$ defined below, the analysis greatly simplifies.

Lemma A.2. *Consider the uniformly random sampling model described in Section 2. There exist constants C_0, C_0' such that if $np^2 \geq C_0 \log m$ and $mp \geq C_0' \log n$, then the condition*

$$\mathcal{A} := \left\{\frac{np^2}{2} \leq B_{ij} \leq \frac{3np^2}{2} \;\forall i \neq j \in [m]\right\} \cup \left\{\frac{1}{2}mp \leq \sum_{i=1}^m A_{li} \leq \frac{3}{2}mp \;\forall l \in [n]\right\}$$

holds with probability at least $1 - 2\exp\left(-\frac{np^2}{24}\right) - \frac{2}{n^{10}}$. ***Additionally,*** *when $\mathcal{A}$ holds setting $d = \frac{3}{2}mnp^2$ is a valid choice of the normalizing factor for the Markov chain $M^{(k)}$ in Algorithm 1 (i.e., all entries of $M^{(k)}$ are positive).*

Proof. See proof. □

The following lemma establishes that so long as the normalizing factor d is sufficiently large to ensure that all of the entries of the Markov chain is positive, the output of Algorithm 1 is the unchanged.

Lemma A.3. *For a fixed scoring matrix X, so long as the normalization factor in Algorithm 1 d satisfies $d \geq \max_i \sum_{j\neq i} Y_{ij}$, then the output $\pi^{(k)}$ of step 1.b of Algorithm 1 does not change for all levels $k \in [K]$.*

Proof. See proof. □

Choice of normalization factor d under $\mathcal{A}$. The above lemma allows us to simplify the analysis by setting $d = \frac{3}{2}mnp^2$ which is a valid normalization factor when $\mathcal{A}$ holds per Lemma A.2.

A.2. Overview of Upper Bound Proofs

The parameters $\boldsymbol{\beta}^*$ consists of K sets of vectors ${\beta^*}^{(1)}, \ldots, {\beta^*}^{(K)}$. The first half of Algorithm 1 produces the *normalized estimates* $\hat{\beta}^{(1)}, \ldots \hat{\beta}^{(K)}$. The second half of Algorithm 1 produces the shift estimates $\delta^{(1)}, \ldots, \delta^{(K)}$ such that

$$\hat{\beta}^{(k)} + \delta^{(k)} \mathbb{1}_m \approx {\beta^*}^{(k)} \; \forall k \in [K].$$

This decomposition allows us to break our proof into two main components. The first is to obtain an upper bound on

$$\left\| \hat{\beta}^{(k)} - {\hat{\beta}^*}{}^{(k)} \right\|_2,$$

where ${\hat{\beta}^*}{}^{(k)}$ is the *normalized* true parameter defined as ${\hat{\beta}^*}{}^{(k)} := {\beta^*}^{(k)} - \frac{\mathbb{1}_m^\top {\beta^*}^{(k)}}{m} \mathbb{1}_m$. Section A.3 focuses on this component of the proof and culminates in Theorem A.10. The second component of the proof centers around proving that

$$\left| \delta^{(k)} - {\delta^*}^{(k)} \right|$$

is small for all $k \in [K]$. This second set of result is deferred to Section A.4 and summarized in Lemma A.11. The reader can immediately see that by the triangle's inequality,

$$\left\| \beta^{(k)} - {\beta^*}^{(k)} \right\|_2 \leq \left\| \hat{\beta}^{(k)} - {\hat{\beta}^*}{}^{(k)} \right\|_2 + \sqrt{m} \left| \delta^{(k)} - {\delta^*}^{(k)} \right|,$$

as summarized in Theorem 4.1 which is built up in Section A.5.

A.3. Guarantee for Normalized Estimate $\hat{\beta}^{(k)}$

In this section, we work towards proving an upper bound on $\left\| \hat{\beta}^{(k)} - {\hat{\beta}^*}{}^{(k)} \right\|$. In order to get there, we will obtain an upper bound on $\left\| \pi^{(k)} - {\pi^*}^{(k)} \right\|$ and $\left\| \pi^{(k)} - {\pi^*}^{(k)} \right\|_\infty$. The starting point is the following result.

Lemma A.4 (Lemma A.3 of Nguyen & Zhang (2022) and Theorem 8 of Chen et al. (2019a))**.** *Consider two Markov chains M, M^* defined on the same set of states with stationary distribution π and π^*, respectively. Suppose that M^* is reversible. Then,*

$$\|\pi - \pi^*\| \leq \frac{\left\| {\pi^*}^\top (M - M^*) \right\|_2}{\mu^*(M^*) - \|M - M^*\|_{\text{op}}} \tag{14}$$

where $\mu^*(M^*)$ is the spectral gap of M^*.

We first introduce an intermediate result which will be useful for the remaining of the proof.

Proposition A.5. *Suppose that Condition $\mathcal{A}$ holds. Fix a level $k \in [K]$. There exists a constant C_1 such that if $np^2 \geq \frac{C_1 e^{4\kappa}}{\gamma^3} \log m$ then, with probability at least $1 - 2\exp\left(-\frac{\gamma^3 np^2}{12^3 2 e^{4\kappa}}\right)$,*

$$\left| {M_{ij}}^{(k)} - {M_{ij}^*}^{(k)} \right| \leq \frac{\gamma}{12 e^{2\kappa}} {M_{ij}^*}^{(k)} \quad \forall i \neq j. \tag{15}$$

where ${M_{ij}^*}^{(k)} := \mathbb{E}\left[{M_{ij}}^{(k)} \right]$.

Proof. See proof. □

We have the following lower bound on the denominator of (14).

Lemma A.6. *Fix a $k \in [K]$. Suppose that Condition $\mathcal{A}$ and Inequality (15) hold. Then the following holds deterministically.*

$$\mu^*\left({M^*}^{(k)} \right) - \left\| M^{(k)} - {M^*}^{(k)} \right\|_{\text{op}} \geq \frac{\gamma}{6 e^{2\kappa}}.$$

Proof. The proof directly uses Lemma A.21 and Lemma A.22. □

By obtaining an upper bound on the numerator of (14), we obtain the following result.

Lemma A.7. *Fix a level $k \in [K]$. Suppose that Condition $\mathcal{A}$ and Inequality (15) hold. There exists a constant C_2 such that*

$$\left\|\pi^{(k)} - \pi^{*(k)}\right\| \leq \frac{C_2 e^{3\kappa}}{\gamma} \frac{1}{\sqrt{mnp}} \,. \tag{16}$$

with probability at least $1 - 2\exp\left(-10m\right)$.

Proof. We first obtain a linear bound on $\left\|{\pi^{*(k)}}^\top \left(M^{(k)} - M^{*(k)}\right)\right\|$. We follow the argument in the proof of Lemma 8.4 of Chen et al. (2022).

$$\begin{aligned}
&\left\|{\pi^{*(k)}}^\top \left(M^{(k)} - M^{*(k)}\right)\right\| \\
&= \sqrt{\sum_{i\in[m]} \left(\sum_{j\in[m]} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right)\right)^2} \\
&= \sqrt{\sum_{i\in[m]} \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left(M_{ii}^{(k)} - {M_{ii}^*}^{(k)}\right)\right)^2} \\
&= \sqrt{\sum_{i\in[m]} \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left(\sum_{j'\neq i} {M_{ij'}^*}^{(k)} - {M_{ij'}}^{(k)}\right)\right)^2} \\
&= \sqrt{\sum_{i\in[m]} \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right)^2}
\end{aligned}$$

Let $\mathcal{B}$ denote the unit norm ball in $\mathbb{R}^m$ and $\mathcal{V}$ be a $\frac{1}{2}$-net of $\mathcal{B}$. That is, for every $u \in \mathcal{B}$, there exists a $v \in \mathcal{V}$ such that $\|u - v\| \leq \frac{1}{2}$. Consider any $u \in \mathcal{B}$ and any corresponding v, we have

$$\begin{aligned}
&\sum_{i\in[m]} u_i \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right) \\
&= \sum_{i\in[m]} v_i \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right) \\
&+ \sum_{i\in[m]} (u_i - v_i) \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right) \\
&\leq \sum_{i\in[m]} v_i \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right) \\
&+ \frac{1}{2}\sqrt{\sum_{i\in[m]} \left(\sum_{j\neq i} \pi_j^{*(k)} \left(M_{ji}^{(k)} - {M_{ji}^*}^{(k)}\right) + \pi_i^{*(k)} \left({M_{ij}^*}^{(k)} - {M_{ij}}^{(k)}\right)\right)^2} \,.
\end{aligned}$$

The above inequality must hold uniformly for all $u \in \mathcal{B}$. Then, maximizing the LHS with respect to u and re-arranging the terms gives

$$\begin{aligned}
&\left\|\pi^{*(k)^\top}\left(M^{(k)}-M^{*(k)}\right)\right\| \\
&=\sqrt{\sum_{i\in[m]}\left(\sum_{j\neq i}\pi_j^{*(k)}\left(M_{ji}^{(k)}-M_{ji}^{*(k)}\right)+\pi_i^{*(k)}\left(M_{ij}^{*(k)}-M_{ij}^{(k)}\right)\right)^2} \\
&\leq 2\max_{v\in\mathcal{V}}\sum_{i\in[m]}v_i\left(\sum_{j\neq i}\pi_j^{*(k)}\left(M_{ji}^{(k)}-M_{ji}^{*(k)}\right)+\pi_i^{*(k)}\left(M_{ij}^{*(k)}-M_{ij}^{(k)}\right)\right)\,.
\end{aligned}$$

We expand on the RHS as follows.

$$\begin{aligned}
&\sum_{i\in[m]}v_i\left(\sum_{j\neq i}\pi_j^{*(k)}\left(M_{ji}^{(k)}-M_{ji}^{*(k)}\right)+\pi_i^{*(k)}\left(M_{ij}^{*(k)}-M_{ij}^{(k)}\right)\right) \\
&=\frac{1}{d}\sum_{i\in[m]}v_i\left(\sum_{j\neq i}\pi_j^{*(k)}\left(Y_{ji}^{(k)}-Y_{ji}^{*(k)}\right)+\pi_i^{*(k)}\left(Y_{ij}^{*(k)}-Y_{ij}^{(k)}\right)\right) \\
&=\frac{1}{d}\sum_{i\in[m]}v_i\left(\sum_{j\neq i}\pi_j^{*(k)}\left(\sum_{l=1}^{n}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}\right]\right)\right)+\right. \\
&\left.\pi_i^{*(k)}\left(\sum_{l=1}A_{li}A_{lj}\left(\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}\right]-\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}\right)\right)\right) \\
&=\frac{1}{d}\sum_{i\in[m]}v_i\left(\sum_{j\neq i}\pi_j^{*(k)}\left(\sum_{l=1}^{n}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}\right]\right)\right)-\right. \\
&\left.\pi_i^{*(k)}\left(\sum_{l=1}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}\right]\right)\right)\right) \\
&=\frac{1}{d}\sum_{l=1}^{n}\sum_{i,j\in[m]:i\neq j}v_i\pi_j^{*(k)}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}\right]\right)- \\
&\frac{1}{d}\sum_{l=1}^{n}\sum_{i,j\in[m]:i\neq j}v_i\pi_i^{*(k)}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k,X_{lj}=k-1\right\}\right]\right) \\
&=\frac{1}{d}\sum_{l=1}^{n}\sum_{i,j\in[m]:i\neq j}(v_i-v_j)\pi_j^{*(k)}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}\right]\right)
\end{aligned}$$

The rest of the proof will focus on obtaining an upper bound on

$$f^{(k)}(X):=\frac{1}{d}\sum_{l=1}^{n}\sum_{i,j\in[m]:i\neq j}(v_i-v_j)\pi_j^{*(k)}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k-1,X_{lj}=k\right\}\right]\right)\,. \tag{17}$$

The function $f^{(k)}$ defined above is one over $n\times m$ independent Bernoulli random variables and $\mathbb{E}_X[f^{(k)}(X)]=0$. Let us use the method of bounded difference to obtain a concentration bound on (17). The argument using the method of bounded difference proceeds as follows. Consider X and X' that are identical except for one entry $l\in[n], i\in[m]$ where $X_{li}\neq X'_{li}$. The maximum deviation between $f^{(k)}(X)$ and $f^{(k)}(X')$ is given as follows.

$$
\begin{aligned}
\left|f^{(k)}(X) - f^{(k)}(X')\right| &\leq \frac{1}{d}\left|\sum_{j\in[m]:j\neq i} \left(A_{li}A_{lj}(v_i - v_j)\right)\, A_{lj}\,\pi_j^{*(k)}\left(1 - 2\,\mathbb{I}\left\{X_{li} = k-1\right\}\right)\right| \\
&\leq \frac{1}{d}\sqrt{\sum_{j\in[m]:j\neq i} A_{li}A_{lj}(v_i - v_j)^2}\cdot\sqrt{\sum_{j\in[m],j\neq i} A_{lj}^2}\,\max_{j\in[m]}\,\pi_j^{*(k)} \\
&\leq \frac{1}{d}\sqrt{\sum_{j\in[m]:j\neq i} A_{li}A_{lj}(v_i - v_j)^2}\cdot\sqrt{\frac{3}{2}mp}\,\max_{j\in[m]}\,\pi_j^{*(k)} \\
&\leq \frac{1}{d}\sqrt{\sum_{j\in[m]:j\neq i} A_{li}A_{lj}(v_i - v_j)^2}\cdot\frac{e^\kappa\sqrt{3p}}{\sqrt{2m}}\,,
\end{aligned}
$$

where the second inequality comes from Cauchy-Schwarz inequality and the third inequality comes from Condition $\mathcal{A}$ which states that $\sum_{j=1}^m A_{lj} \leq \frac{3mp}{2}$ and that $\max_i \pi_i^{*(k)} \leq \frac{e^\kappa}{m}$. Fix a $v \in \mathcal{V}$,

$$
\begin{aligned}
\Pr\left(\left|f^{(k)}(X)\right| > t\middle| v, \mathcal{A}\right) &\leq 2\,\exp\left(-\frac{2t^2}{\sum_{l\in[n]}\sum_{i\in[m]}\left(\frac{1}{d}\sqrt{\sum_{j\in[m]:j\neq i} A_{li}A_{lj}(v_i - v_j)^2}\cdot\frac{e^\kappa\sqrt{3p}}{\sqrt{2m}}\right)^2}\right) \\
&= 2\,\exp\left(-\frac{2t^2}{\sum_{l\in[n]}\sum_{i\in[m]}\frac{3pe^{2\kappa}}{2md^2}\sum_{j\in[m]:j\neq i} A_{li}A_{lj}(v_i - v_j)^2}\right) \\
&= 2\,\exp\left(-\frac{2t^2}{\frac{3pe^{2\kappa}}{2md^2}\sum_{l\in[n]}\sum_{i,j\in[m]:i\neq j} A_{li}A_{lj}(v_i - v_j)^2}\right) \\
&= 2\,\exp\left(-\frac{2t^2}{\frac{3pe^{2\kappa}}{2md^2}\sum_{i,j\in[m]:i\neq j} B_{ij}(v_i - v_j)^2}\right) \\
&\leq 2\,\exp\left(-\frac{2t^2}{\frac{3pe^{2\kappa}}{2md^2}\frac{3mnp^2}{2}}\right) \\
&= 2\,\exp\left(-\frac{2m^2npt^2}{e^{2\kappa}}\right)\,.
\end{aligned}
\tag{18}
$$

Note that the last inequality uses the fact that

$$
\sum_{i,j\in[m]:i\neq j} B_{ij}(v_i - v_j)^2 \leq \frac{3np^2}{2}\sum_{i,j\in[m]:i\neq j}(v_i - v_j)^2 \leq \frac{3np^2}{2}\max_{v\in\mathcal{V}} v^\top(I_m - \mathbb{1}_m\mathbb{1}_m^\top)v \leq \frac{3mnp^2}{2}\,.
$$

Applying union bound over all $v \in \mathcal{V}$ and noting that $\mathcal{V}$ has cardinality 5^m (Vershynin, 2018), we have

$$
\begin{aligned}
\Pr\left(\left|f^{(k)}(X)\right| > t\middle|\mathcal{A}\right) &\leq 5^m\cdot 2\,\exp\left(-\frac{2m^2npt^2}{e^{2\kappa}}\right) \\
&\leq 2\exp\left(-\frac{2m^2npt^2}{e^{2\kappa}} + 2m\right) \\
&\leq 2\exp\left(-\frac{m^2npt^2}{e^{2\kappa}}\right)\,.
\end{aligned}
\tag{19}
$$

The last inequality holds so long as $\frac{2m^2npt^2}{e^{2\kappa}} > 4m$ or $t > \frac{e^\kappa}{\sqrt{2}}\frac{1}{\sqrt{mnp}}$. Set

$$
t = \frac{e^\kappa\sqrt{10}}{\sqrt{mnp}}\,.
$$

We conclude that

$$\left\|{\pi^*}^{(k)\top}\left(M^{(k)} - {M^*}^{(k)}\right)\right\| \le 2 \max_{v\in\mathcal{V}} \sum_{i\in[m]} v_i \left(\sum_{j\neq i} \pi_j^{*(k)}\left(M_{ji}^{(k)} - M_{ji}^{*(k)}\right) + \pi_i^{*(k)}\left(M_{ij}^{*(k)} - {M_{ij}}^{(k)}\right)\right)$$
$$\le \frac{e^\kappa\sqrt{10}}{\sqrt{mnp}}$$

with probability at least $1-\exp(-10m)$. Substitute the above inequality and the lower bound in Lemma A.6 into (14) completes the proof. ☐

Entrywise error bound for $\pi^{(k)}$. Pick an index $i\in[m]$. We have the following deterministic decomposition for any level $k\in[K]$.

$$\begin{aligned}
{\pi_i}^{(k)} - \pi_i^{*(k)} &= (\pi^\top M^{(k)})_i - ({\pi^*}^\top {M^*}^{(k)})_i \\
&= \sum_j \pi_j^{(k)} {M_{ji}}^{(k)} - \sum_j \pi_j^* M_{ji}^{*(k)} \\
&= \sum_j \pi_j^{(k)} {M_{ji}}^{(k)} - \sum_j \pi_j^{*(k)}(M_{ji}^{*(k)} - {M_{ji}}^{(k)} + M_{ji}^{(k)}) \\
&= \sum_j (\pi_j^{(k)} - \pi_j^{(k)*}){M_{ji}}^{(k)} - \sum_j \pi_j^{*(k)}(M_{ji}^{*(k)} - M_{ji}^{(k)}) \\
&= \sum_j (\pi_j^{(k)} - \pi_j^{(k)*}){M_{ji}}^{(k)} + \sum_j \pi_j^{*(k)}({M_{ji}}^{(k)} - M_{ji}^{*(k)})\,. \\
&= \sum_{j\neq i} (\pi_j^{(k)} - \pi_j^{*(k)})M_{ji} + ({\pi_i}^{(k)} - \pi_i^{*(k)})M_{ii} + \sum_{j\neq i}\pi_j^{*(k)}({M_{ji}}^{(k)} - M_{ji}^{*(k)}) + \pi_i^{*(k)}(M_{ii}^{(k)} - M_{ii}^{*(k)})\,.
\end{aligned}$$

Rearranging the terms gives

$${\pi_i}^{(k)} - \pi_i^{*(k)} = \frac{1}{1-M_{ii}^{(k)}}\cdot\left(\sum_{j\neq i}(\pi_j^{(k)} - \pi_j^{*(k)})M_{ji}^{(k)} + \sum_{j\neq i}\pi_j^{*(k)}({M_{ji}}^{(k)} - M_{ji}^{*(k)}) + \pi_i^{*(k)}(M_{ii}^{(k)} - M_{ii}^{*(k)})\right). \quad (20)$$

Lemma A.8. *Fix a level $k\in[K]$. Suppose that Condition $\mathcal{A}$ holds and Inequality (15) holds. Then*

$$1 - {M_{ii}}^{(k)} \ge \frac{\gamma}{2}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right),$$
$$\left|\sum_{j\neq i}(\pi_j^{(k)} - \pi_j^{*(k)})M_{ji}^{(k)}\right| \le \frac{1}{\sqrt{m}}\left(1+\frac{\gamma}{12e^{2\kappa}}\right)\left\|\pi^{(k)} - {\pi^*}^{(k)}\right\|.$$

Additionally, with probability at least $1-4m^{-10}$,

$$\left|\sum_{j\neq i}\pi_j^{*(k)}\left({M_{ji}}^{(k)} - M_{ji}^{*(k)}\right)\right| \vee \left|\pi_i^{*(k)}\left({M_{ii}}^{(k)} - M_{ii}^{*(k)}\right)\right| \le \frac{4\sqrt{10}\,e^\kappa}{3m}\sqrt{\frac{\log m}{np}} \quad (21)$$

Proof. See proof. ☐

Lemma A.9. *Fix a level $k\in[K]$. Suppose that Inequality (15), Inequality (16) and Inequality (21) hold. Then*

$$\left\|\pi^{(k)} - {\pi^*}^{(k)}\right\|_\infty \le \frac{1}{\frac{\gamma}{2}\left(1-\frac{\gamma}{12e^{2\kappa}}\right)}\cdot\left(\frac{8\sqrt{10}}{3}\sqrt{\frac{\log m}{np}} + \frac{2C_2e^{3\kappa}}{\gamma}\left(1+\frac{\gamma}{12e^{2\kappa}}\right)\frac{1}{\sqrt{np}}\right)\cdot\frac{1}{m}.$$

If we further assume that $np \ge \frac{C_3e^{8\kappa}}{\gamma^4} \vee \frac{C_3'e^{2\kappa}}{\gamma^2}\log m$ then

$$\left\|\pi^{(k)} - {\pi^*}^{(k)}\right\|_\infty \le \frac{1}{2me^\kappa}, \quad (22)$$

where C_3, C_3' are absolute constants.

Proof. We substitute the upper bounds in Lemma A.8 into (20) and note that the inequality holds for all $i \in [m]$. Therefore,

$$\left\|\pi^{(k)} - \pi^{*(k)}\right\|_\infty \leq \frac{1}{1 - M_{ii}^{(k)}} \cdot \left(\left|\sum_{j \neq i} (\pi_j^{(k)} - \pi_j^{*(k)}) M_{ji}^{(k)}\right| + \left|\sum_{j \neq i} \pi_j^{*(k)} (M_{ji}{}^{(k)} - M_{ji}^{*}{}^{(k)})\right| + \left|\pi_i^{*(k)} (M_{ii}^{(k)} - M_{ii}^{*(k)})\right| \right)$$

$$\leq \frac{1}{\frac{\gamma}{2}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right)} \left(\frac{8\sqrt{10}}{3m} \sqrt{\frac{\log m}{np}} + \frac{1}{\sqrt{m}} \left(1 + \frac{\gamma}{12e^{2\kappa}}\right) \left\|\pi^{(k)} - \pi^{*(k)}\right\| \right)$$

$$\leq \frac{1}{\frac{\gamma}{2}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right)} \cdot \left(\frac{8\sqrt{10}}{3} \sqrt{\frac{\log m}{np}} + \frac{2C_2 e^{3\kappa}}{\gamma} \left(1 + \frac{\gamma}{12e^{2\kappa}}\right) \frac{1}{\sqrt{np}} \right) \cdot \frac{1}{m}$$

If both of the following inequalites hold

$$np \geq \left(\frac{64\sqrt{10} e^\kappa}{3\gamma\left(1 - \frac{\gamma}{12e^{2\kappa}}\right)} \right)^2 \log m$$

$$np \geq \left(\frac{16 C_2 e^{4\kappa} \left(1 + \frac{\gamma}{12e^{2\kappa}}\right)}{\gamma^2 \left(1 - \frac{\gamma}{12e^{2\kappa}}\right)} \right)^2$$

then

$$\left\|\pi^{(k)} - \pi^{*(k)}\right\|_\infty \leq \frac{1}{2me^\kappa}.$$

This completes the proof. □

With the upper bounds on $\left\|\pi^{(k)} - \pi^{*(k)}\right\|$ and $\left\|\pi^{(k)} - \pi^{*(k)}\right\|_\infty$, we can now provide an upper bound on the error of the normalized estimate $\hat{\beta}^{(k)}$.

Theorem A.10. *Fix a level $k \in [K]$. Suppose that Condition $\mathcal{A}$, Inequality (15), Inequality (16) and Inequality (21) hold. Suppose further that $np \geq \frac{C_3 e^{8\kappa}}{\gamma^4} \vee \frac{C_3' e^{2\kappa}}{\gamma^2} \log m$ where C_3, C_3' are the same absolute constants in Lemma A.9. Then the normalized estimate $\hat{\beta}^{(k)}$ obtained by the first half of Algorithm (1) satisfies*

$$\left\|\hat{\beta}^{(k)} - \hat{\beta}^{*(k)}\right\|_2 \leq \frac{4C_2 e^{4\kappa}}{\gamma} \frac{\sqrt{m}}{\sqrt{np}}, \tag{23}$$

and

$$\left\|\hat{\beta}^{(k)} - \hat{\beta}^{*(k)}\right\|_\infty \leq \frac{e^\kappa C_2'}{\gamma} \sqrt{\frac{\log m}{np}} + \frac{e^{4\kappa} C_2''}{\gamma^2} \frac{1}{\sqrt{np}} \tag{24}$$

where C_2 is the same absolute constant as in Lemma A.7 and C_2', C_2'' are absolute constants.

Proof. Suppose that Condition $\mathcal{A}$, Inequality (15) and Inequality (16) hold. By Lemma A.9, assume that

$$np \geq \frac{C_3 e^{8\kappa}}{\gamma^4} \vee \frac{C_3' e^{2\kappa}}{\gamma^2} \log m$$

then Inequality (22) holds, or

$$\left\|\pi^{(k)} - \pi^{*(k)}\right\|_\infty \leq \frac{1}{2me^\kappa}.$$

Recall that $\frac{e^\kappa}{m} \geq \pi_i^{*(k)} \geq \frac{1}{me^\kappa}$ for any $i \in [m]$. Hence, conditioned on (22), $\frac{e^\kappa}{m} + \frac{1}{2me^\kappa} \geq \pi_i^{(k)} \geq \frac{1}{2me^\kappa}$. We have

$$\left|\log(\pi_i{}^{(k)}) - \log(\pi_i^{*(k)})\right| \leq 2me^\kappa \left|\pi_i{}^{(k)} - \pi_i^{*(k)}\right|$$

where we use the fact that within the range $\left[\frac{1}{2me^{\kappa}}, \frac{e^{\kappa}}{m} + \frac{1}{2me^{\kappa}}\right]$, the absolute value of the gradient of the log function is upper bounded by $2me^{\kappa}$.

$$
\begin{aligned}
&\left\|\hat{\beta}^{(k)} - \hat{\beta^*}^{(k)}\right\|_2^2 = \sum_{i=1}^{m}\left(\hat{\beta}_i^{(m)} - \hat{\beta}_i^{*(k)}\right)^2 \\
&= \sum_{i=1}^{m}\left(\log({\pi_i}^{(k)}) - \log({\pi_i^*}^{(k)}) + \frac{1}{m}\sum_{j=1}^{m}\left(\log({\pi_j^*}^{(k)}) - \log({\pi_j}^{(k)})\right)\right)^2 \\
&\leq 2\sum_{i=1}^{m}\left(\log({\pi_i}^{(k)}) - \log({\pi_i^*}^{(k)})\right)^2 + \frac{1}{m^2}\left(\sum_{j=1}^{m}\log({\pi_j^*}^{(k)}) - \log({\pi_j}^{(k)})\right)^2 \\
&\leq 2\sum_{i=1}^{m}\left(\log({\pi_i}^{(k)}) - \log({\pi_i^*}^{(k)})\right)^2 + 2\sum_{i=1}^{m}\frac{1}{m^2}\left(\sum_{j=1}^{m}\log({\pi_j^*}^{(k)}) - \log({\pi_j}^{(k)})\right)^2 \\
&= 2\sum_{i=1}^{m}\left(\log({\pi_i}^{(k)}) - \log({\pi_i^*}^{(k)})\right)^2 + \frac{2}{m}\left(\sum_{j=1}^{m}\log({\pi_j^*}^{(k)}) - \log({\pi_j}^{(k)})\right)^2 \\
&\leq 4\sum_{i=1}^{m}\left(\log({\pi_i}^{(k)}) - \log({\pi_i^*}^{(k)})\right)^2 \\
&\leq 16m^2e^{2\kappa}\left\|\pi^{(k)} - {\pi^*}^{(k)}\right\|_2^2.
\end{aligned}
$$

Taking the square root of both sides and invoking Lemma A.7 gives

$$
\left\|\hat{\beta}^{(k)} - \hat{\beta^*}^{(k)}\right\| \leq \frac{4C_2e^{4\kappa}}{\gamma}\frac{\sqrt{m}}{\sqrt{np}}.
$$

To obtain an ℓ_∞ error bound, we invoke Lemma A.9.

$$
\begin{aligned}
&\left\|\beta^{(k)} - {\beta^*}^{(k)}\right\|_\infty = \max_{i\in[m]}\left|\log\left({\pi_i}^{(k)}\right) - \log\left({\pi_i^*}^{(k)}\right) - \frac{1}{m}\left(\sum_{j=1}^{m}\log\left({\pi_j}^{(k)}\right) - \log\left({\pi_j^*}^{(k)}\right)\right)\right| \\
&\leq 4me^{\kappa}\left\|\pi^{(k)} - {\pi^*}^{(k)}\right\|_\infty \\
&\leq \frac{4e^{\kappa}}{\frac{\gamma}{2}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right)}\cdot\left(\frac{8\sqrt{10}}{3}\sqrt{\frac{\log m}{np}} + \frac{2C_2e^{3\kappa}}{\gamma}\left(1 + \frac{\gamma}{12e^{2\kappa}}\right)\frac{1}{\sqrt{np}}\right).
\end{aligned}
$$

This completes the proof. □

A.4. Guarantee for Shift Estimate $\delta^{(k)}$

In this section, we work towards providing an upper bound on $\left|\delta^{(k)} - \delta^{*(k)}\right|$, where

$$\delta^{*(k)} = \log\left(\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_j^{*(1)}}\,\mathbb{E}\left[Y_{ij}^{(k-1,1)}\right]}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{*(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)$$

and

$$\delta^{(k)} = \log\left(\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_j{}^{(1)}}\,Y_{ij}^{(k-1,1)}}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i{}^{(k)}}\,Y_{ij}^{(k,0)}}\right).$$

Note that $\delta^{(1)} = \delta^{*(k)} = 0$.

Lemma A.11. *Fix a $k \in [2,\ldots,K]$. Suppose that Condition $\mathcal{A}$, Inequality (15), Inequality (16) and Inequality (22) hold. Suppose further that $np \geq \frac{C_4 e^{6\kappa}}{\gamma^4}$ then there exists a constant C_5 such that*

$$\left|\delta^{(k)} - \delta^{*(k)}\right| \leq \frac{C_5 e^{4\kappa}}{\gamma^2}\frac{1}{\sqrt{np}} \tag{25}$$

with probability at least $1 - 4\exp(-10m)$.

f

Proof. We have

$$\left|\delta^{(k)} - \delta^{*(k)}\right| \leq \left|\log\left(\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{(k)}} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{*(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)\right| + \left|\log\left(\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_j^{(1)}} Y_{ij}^{(k-1,1)}}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_j^{*(1)}}\,\mathbb{E}\left[Y_{ij}^{(k-1,1)}\right]}\right)\right|. \tag{26}$$

The two absolute log terms in (26) can be bounded in the same manner. We therefore focus our attention on the first term. By noting that $\hat{\beta}_i^{(k)} = \log\left(\pi_i{}^{(k)}\right) - \frac{1}{m}\sum_{i'=1}^m \log\left(\pi_{i'}{}^{(k)}\right)$ and that $\hat{\beta}_i^{*(k)} = \log\left(\pi_i^{*(k)}\right) - \frac{1}{m}\sum_{i'=1}^m \log\left(\pi_{i'}^{*(k)}\right)$, we can write the ratio inside the log as follows.

$$\begin{aligned}\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{(k)}} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{*(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]} &= \frac{\sum_{i,j\in[m]:i\neq j} \pi_i{}^{(k)} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)}\mathbb{E}\left[Y_{ij}^{(k,0)}\right]} \cdot \frac{\exp\left(\frac{1}{m}\sum_{i'=1}^m \log \pi_{i'}^{*(k)}\right)}{\exp\left(\frac{1}{m}\sum_{i'=1}^m \log \pi_i{}^{(k)}\right)}\\ &= \frac{\sum_{i,j\in[m]:i\neq j} \pi_i{}^{(k)} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)}\mathbb{E}\left[Y_{ij}^{(k,0)}\right]} \cdot \exp\left(\frac{1}{m}\sum_{i'=1}^m \log \pi_{i'}^{*(k)} - \log \pi_{i'}{}^{(k)}\right).\end{aligned} \tag{27}$$

Taking the log of both sides gives

$$\log\left(\frac{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{(k)}} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} e^{\hat{\beta}_i^{*(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right) = \log\left(\frac{\sum_{i,j\in[m]:i\neq j} \pi_i{}^{(k)} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)}\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right) + \frac{1}{m}\sum_{i'=1}^m \log \pi_{i'}^{*(k)} - \log \pi_{i'}{}^{(k)}.$$

The ratio inside the log term can be broken down into a sum of 1 and an error term as

$$
\begin{aligned}
&\frac{\sum_{i,j\in[m]:i\neq j} \pi_i{}^{(k)} Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \mathbb{E}\left[Y_{ij}^{(k,0)}\right]} \\
&= \frac{\sum_{i,j\in[m]:i\neq j}\left(\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)+\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]+\pi_i^{*(k)}\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)+\pi_i^{*(k)}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]\right)}{\sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]} \\
&= 1+\frac{\sum_{i,j\in[m]:i\neq j}\left(\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)+\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]+\pi_i^{*(k)}\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)\right)}{\sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\,.
\end{aligned}
\tag{28}
$$

Bounding the denominator of (28). We can lower bound the denominator as follows.

$$
\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)}\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right] \geq \sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)}\gamma B_{ij} \geq \frac{\gamma n p^2}{2} \sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \geq \frac{\gamma m n p^2}{2}\,, \tag{29}
$$

where the second inequality comes from $\mathcal{A}$.

Bounding the numerator of (28). The first term in the numerator can be bounded as follows. Conditioned on (15),

$$
\begin{aligned}
\left|\sum_{i,j\in[m]:i\neq j}\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)\right| &\leq \sum_{i,j\in[m]:i\neq j}\left|\pi_i{}^{(k)}-\pi_i^{*(k)}\right|\left|Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right| \\
&\leq \sum_{i,j\in[m]:i\neq j}\left|\pi_i{}^{(k)}-\pi_i^{*(k)}\right|\cdot\frac{\gamma}{12e^{2\kappa}}B_{ij} \\
&\leq \frac{\gamma}{8e^{2\kappa}}\,mnp^2\sum_{i\in[m]}\left|\pi_i{}^{(k)}-\pi_i^{*(k)}\right| \\
&\leq \frac{\gamma}{8e^{2\kappa}}\,mnp^2\sqrt{m}\,\left\|\pi^{(k)}-\pi^{*(k)}\right\|_2\,.
\end{aligned}
\tag{30}
$$

The second term can be bounded similarly to the first.

$$
\begin{aligned}
\left|\sum_{i,j\in[m]:i\neq j}\left(\pi_i{}^{(k)}-\pi_i^{*(k)}\right)\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right| &\leq \sum_{i,j\in[m]:i\neq j}\left|\pi_i{}^{(k)}-\pi_i^{*(k)}\right|\,B_{ij} \\
&\leq \frac{3mnp^2}{2}\sum_{i\in[m]}\left|\pi_i{}^{(k)}-\pi_i^{*(k)}\right|. \\
&\leq \frac{3mnp^2}{2}\sqrt{m}\,\left\|\pi^{(k)}-\pi^{*(k)}\right\|_2
\end{aligned}
\tag{31}
$$

We now proceed to upper bound the third term in the numerator in (28). We first decompose it as follows.

$$
\begin{aligned}
&\left|\sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\left(Y_{ij}{}^{(k,0)}-\mathbb{E}\left[Y_{ij}{}^{(k,0)}\right]\right)\right| \\
&= \sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\sum_{l=1}^{n}A_{li}A_{lj}\left(\mathbb{I}\left\{X_{li}=k,X_{lj}=0\right\}-\mathbb{E}\left[\mathbb{I}\left\{X_{li}=k,X_{lj}=0\right\}\right]\right) =: g^{(k)}(X)\,.
\end{aligned}
$$

We use the method of bounded difference, similarly to the proof of Theorem A.10. Consider two sets of responses X and X' that differs in exactly a single entry $X_{li} \neq X'_{li}$ for some $l \in [n], i \in [m]$. By Cauchy-Schwarz inequality,

$$\begin{aligned} \left| g^{(k)}(X) - g^{(k)}(X') \right| &\leq \pi_i^{*(k)} \sum_{j \neq i} A_{lj} A_{li} A_{lj} \\ &\leq \pi_i^{*(k)} \sqrt{\sum_{j\neq i} A_{lj}} \sqrt{\sum_{j \neq i} A_{li} A_{lj}} \\ &\leq \pi_i^{*(k)} \sqrt{\frac{3mp}{2}} \sqrt{\sum_{j\neq i} A_{li} A_{lj}} \\ &\leq \frac{\sqrt{3p}e^{\kappa}}{\sqrt{2m}} \sqrt{\sum_{j\neq i} A_{li} A_{lj}} \,. \end{aligned} \tag{32}$$

Therefore, by the method of bounded difference,

$$\begin{aligned} &\Pr\left(\left| \sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right) \right| > t \,\middle|\, \mathcal{A} \right) \\ &= \Pr\left(\left| \sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \sum_{l=1}^{n} \left(\mathbb{I}\left\{ X_{li} = k, X_{lj} = 0 \right\} - \mathbb{E}\left[\mathbb{I}\left\{ X_{li} = k, X_{lj} = 0 \right\} \right] \right) \right| > t \,\middle|\, \mathcal{A} \right) \\ &\leq 2\exp\left(- \frac{2t^2}{\sum_{l=1}^{n} \sum_{i=1}^{m} \pi_i^{*(k)} m \sum_{j\neq i} A_{li} A_{lj}} \right) \\ &\leq 2\exp\left(- \frac{2t^2}{\frac{3pe^{2\kappa}}{2m} \sum_{i,j\in[m]:i\neq j} B_{ij}} \right) \\ &\leq 2\exp\left(- \frac{8t^2}{9e^{2\kappa} mnp^3} \right). \end{aligned}$$

Then, with probability at least $1 - 2\exp(-10m)$,

$$\left| \sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right) \right| \leq \frac{\sqrt{9\cdot 10}\, e^{\kappa} \sqrt{m} \sqrt{np^3} \sqrt{m}}{\sqrt{8}} \,. \tag{33}$$

We now divide the upper bounds on the numerator terms (30), (31), (33) by the lower bound on the denominator term in (29). We have have the following upper bound hold with probability at least $1 - 2\exp(-10m)$.

$$\begin{aligned} &\left| \frac{\sum_{i,j\in[m]:i\neq j} \left(\left({\pi_i}^{(k)} - \pi_i^{*(k)} \right) \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right) + \left({\pi_i}^{(k)} - \pi_i^{*(k)} \right) \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] + \pi_i^{*(k)} \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right) \right)}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \, \mathbb{E}\left[Y_{ij}^{(k,0)} \right]} \right| \\ &\leq \left| \frac{\sum_{i,j\in[m]:i\neq j} \left({\pi_i}^{(k)} - \pi_i^{*(k)} \right) \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right)}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \, \mathbb{E}\left[Y_{ij}^{(k,0)} \right]} \right| + \left| \frac{\left({\pi_i}^{(k)} - \pi_i^{*(k)} \right) \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right]}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \, \mathbb{E}\left[Y_{ij}^{(k,0)} \right]} \right| + \left| \frac{\pi_i^{*(k)} \left({Y_{ij}}^{(k,0)} - \mathbb{E}\left[{Y_{ij}}^{(k,0)} \right] \right)}{\sum_{i,j\in[m]:i\neq j} \pi_i^{*(k)} \, \mathbb{E}\left[Y_{ij}^{(k,0)} \right]} \right| \\ &\leq \frac{\sqrt{m}}{4e^{2\kappa}} \left\| \pi^{(k)} - \pi^{*(k)} \right\| + \frac{3\sqrt{m}}{\gamma} \left\| \pi^{(k)} - \pi^{*(k)} \right\| + \frac{\sqrt{9\cdot 10}\, e^{\kappa} \sqrt{m} \sqrt{np^3} \sqrt{m}}{\sqrt{8}} \frac{2}{\gamma mnp^2} \\ &\leq \frac{C_2 e^{\kappa}}{4\gamma} \frac{1}{\sqrt{np}} + \frac{3C_2 e^{3\kappa}}{\gamma^2} \frac{1}{\sqrt{np}} + \frac{\sqrt{45}\, e^{\kappa}}{\gamma\sqrt{np}} \\ &\leq \left(\frac{C_2 e^{\kappa}}{4\gamma} + \frac{3C_2 e^{3\kappa}}{\gamma^2} + \frac{\sqrt{45}\, e^{\kappa}}{\gamma} \right) \frac{1}{\sqrt{np}} \,. \end{aligned}$$

Putting things together. Conditioned on (22)

$$\begin{aligned}\left|\frac{1}{m}\sum_{i'=1}^{m}\log\pi_{i'}^{*\,(k)}-\log\pi_{i'}{}^{(k)}\right| &\leq \frac{1}{m}\sqrt{m}\left\|\log\pi^{*\,(k)}-\log\pi^{(k)}\right\| \\ &\leq \frac{1}{\sqrt{m}}\left\|\log\pi^{*\,(k)}-\log\pi^{(k)}\right\| \\ &\leq \sqrt{m}2e^{\kappa}\left\|\pi^{*\,(k)}-\pi^{(k)}\right\| \\ &\leq \frac{2C_2e^{4\kappa}}{\gamma}\frac{1}{\sqrt{np}}\,.\end{aligned} \tag{34}$$

Recall that we ultimately wish to provide an upper bound on the following.

$$\begin{aligned}&\log\left(\frac{\sum_{i,j\in[m]:i\neq j}e^{\hat{\beta}_i^{(k)}}Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j}e^{\hat{\beta}_i^{*\,(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)\\ &=\log\left(\frac{\sum_{i,j\in[m]:i\neq j}\pi_i{}^{(k)}Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)+\left(\frac{1}{m}\sum_{i'=1}^{m}\log\pi_{i'}^{*\,(k)}-\log\pi_{i'}{}^{(k)}\right)\,.\end{aligned}$$

One could verify that

$$\log(1+x)\leq x\quad\forall x\geq 0\,,$$

and

$$\log(1+x)\geq 2x\quad\forall x\in(-0.78,0)\,.$$

Hence, so long as

$$\left(\frac{C_2e^{\kappa}}{4\gamma}+\frac{3C_2e^{3\kappa}}{\gamma^2}+\frac{\sqrt{30}\,e^{\kappa}}{\gamma}\right)\frac{1}{\sqrt{np}}<0.78 \tag{35}$$

then we obtain the following upper bound when we take the log of (28)

$$\left|\log\left(\frac{\sum_{i,j\in[m]:i\neq j}\pi_i{}^{(k)}Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j}\pi_i^{*(k)}\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)\right|\leq 2\left(\frac{C_2e^{\kappa}}{4\gamma}+\frac{3C_2e^{3\kappa}}{\gamma^2}+\frac{\sqrt{45}\,e^{\kappa}}{\gamma}\right)\frac{1}{\sqrt{np}}\,.$$

Combine with (34), we have

$$\begin{aligned}&\left|\log\left(\frac{\sum_{i,j\in[m]:i\neq j}e^{\hat{\beta}_i^{(k)}}Y_{ij}^{(k,0)}}{\sum_{i,j\in[m]:i\neq j}e^{\hat{\beta}_i^{*\,(k)}}\,\mathbb{E}\left[Y_{ij}^{(k,0)}\right]}\right)\right|\\ &\leq 2\left(\frac{C_2e^{\kappa}}{4\gamma}+\frac{3C_2e^{3\kappa}}{\gamma^2}+\frac{\sqrt{45}\,e^{\kappa}}{\gamma}\right)\frac{1}{\sqrt{np}}+\frac{12C_2e^{4\kappa}}{\gamma}\frac{1}{\sqrt{np}}\\ &\leq \frac{c_5e^{4\kappa}}{\gamma^2}\frac{1}{\sqrt{np}}\,,\end{aligned}$$

where c_5 is an absolute constant. The extra condition on np comes from (35). Lastly, the upper bound on the second absolute log term can be shown in the same way. □

A.5. Final Error Bound

In this section, we combine the upper bounds on $\left\|\hat{\beta}^{(k)} - \hat{\beta}^{*(k)}\right\|_2$ and $\left|\delta^{(k)} - \delta^{*(k)}\right|$ to obtain the following bound on $\left\|\beta^{(k)} - \beta^{*(k)}\right\|_2$. Note that in the following proof of Theorem 4.1, we also present the entrywise error guarantee of Theorem 4.3.

Theorem 4.1. Consider the uniform sampling model described in Section 2. Suppose that $np^2 \geq \left(C_0 \vee \frac{C_1 e^{4\kappa}}{\gamma^3}\right)\log m$, $mp \geq C_0' \log n$ and that $np \geq \left(\frac{C_4 e^{6\kappa}}{\gamma^4} \vee \frac{C_3 e^{8\kappa}}{\gamma^4} \vee \frac{C_3' e^{2\kappa}}{\gamma^2} \log m\right)$ where $C_0, C_0', C_1, C_3, C_3', C_4$ are some absolute constants. There exist absolute constants C_6, C_6' such that

$$\left\|\beta^{(k)} - \beta^{*(k)}\right\|_2 \leq \frac{C_6 e^{4\kappa}}{\gamma^2} \frac{\sqrt{m}}{\sqrt{np}},$$
$$\left\|\beta^{(k)} - \beta^{*(k)}\right\|_\infty \leq \frac{e^{4\kappa} C_6'}{\gamma^2} \sqrt{\frac{\log m}{np}}$$

simultaneously for $k = 1, \ldots, K$ with probability at least $1 - 6K\exp(-10m) - 4Km^{-10} - 2K\exp\left(-\frac{\gamma^3 np^2}{12^3 2e^{4\kappa}}\right) - 2\exp\left(-\frac{np^2}{24}\right) - 2n^{-10}$.

Proof. We have by Proposition A.2,

$$\Pr(\neg\mathcal{A}) \leq 2\exp\left(-\frac{np^2}{24}\right) + 2n^{-10}.$$

Invoking Proposition A.5 and applying union bound over all levels $k \in [K]$, we have

$$\Pr\left((15), (16) \text{ do } \textit{not} \text{ hold for all } k \in [K] \,\middle|\, \mathcal{A}\right) \leq 2K\exp\left(-\frac{\gamma^3 np^2}{12^3 2e^{4\kappa}}\right) + 2K\exp(-10m).$$

By Lemma A.9,

$$\Pr\left((21) \text{ does } \textit{not} \text{ hold} \,\middle|\, \mathcal{A}, (15) \text{ hold for all } k \in [K]\right) \leq 4Km^{-10}.$$

By Theorem A.10

$$\Pr\left((23), (24) \text{ do } \textit{not} \text{ hold} \,\middle|\, \mathcal{A}, (15), (16), (21) \text{ hold for all } k \in [K]\right) = 0.$$

By Lemma A.11,

$$\Pr\left((25) \text{ does } \textit{not} \text{ hold} \,\middle|\, \mathcal{A}, (15) \text{ hold for all } k \in [K]\right) \leq 4K\exp(-10m).$$

We have, by union bound,

$$
\begin{aligned}
&\Pr\left((25) \text{ or } (24) \text{ or } (23) \text{ do } \textit{not} \text{ hold for all } k \in [K]\right) \\
&\leq \Pr\left((25) \text{ or } (24) \text{ or } (23) \text{ do } \textit{not} \text{ hold for all } k \in [K] \,\middle|\, \mathcal{A} \right) \Pr(\mathcal{A}) + \Pr(\neg \mathcal{A}) \\
&\leq \Pr\left((25) \text{ or } (24) \text{ or } (23) \text{ do } \textit{not} \text{ hold for all } k \in [K] \,\middle|\, \mathcal{A} \right) \Pr(\mathcal{A}) + 2\ \exp\left(-\frac{np^2}{24}\right) \\
&\leq \Pr\left((25) \text{ or } (24) \text{ or } (23) \text{ do } \textit{not} \text{ hold } \,\middle|\, (15), (16) \text{ hold } \forall k \in [K], \mathcal{A} \right) \Pr\left((15), (16) \text{ hold } \forall k \in [K] \,\middle|\, \mathcal{A} \right) \Pr(\mathcal{A}) \\
&+ \Pr\left((15), (16) \text{ do } \textit{not} \text{ hold for all } k \in [K] \,\middle|\, \mathcal{A} \right) \Pr(\mathcal{A}) + 2\ \exp\left(-\frac{np^2}{24}\right) + 2n^{-10} \\
&\leq \Pr\left((25) \text{ or } (24) \text{ or } (23) \text{ do } \textit{not} \text{ hold for all } k \in [K] \,\middle|\, (15), (16) \text{ hold } \forall k \in [K], \mathcal{A} \right) \\
&+ 2K \exp\left(-\frac{\gamma^3 np^2}{12^3 2 e^{4\kappa}}\right) + 2K \exp\left(-10m\right) + 2\ \exp\left(-\frac{np^2}{24}\right) + 2n^{-10} \\
&\leq 4Km^{-10} + 4K \exp\left(-10m\right) + 2K \exp\left(-\frac{\gamma^3 np^2}{12^3 2 e^{4\kappa}}\right) + 2K \exp\left(-10m\right) + 2\ \exp\left(-\frac{np^2}{24}\right) + 2n^{-10}\,.
\end{aligned}
$$

We then have

$$
\Pr\left((25), (23), (24) \text{ hold for all } k \in [K]\right) \geq 1 - 4Km^{-10} - 6K \exp\left(-10m\right) - 2K \exp\left(-\frac{\gamma^3 np^2}{12^3 2 e^{4\kappa}}\right) - 2\ \exp\left(-\frac{np^2}{24}\right) - 2n^{-10}.
$$

Now suppose that (23), (24), (25) hold. Then the following bounds all hold deterministically. Substitute (25) and (23) into (5). We have

$$
\begin{aligned}
\left\| \beta^{(k)} - \beta^{*(k)} \right\| &\leq \left\| \hat{\beta}^{(k)} - \hat{\beta}^{*(k)} \right\| + \sqrt{m} \left| \delta^{(k)} - \delta^{*(k)} \right| \\
&\leq \frac{24 C_2 e^{4\kappa}}{\gamma} \frac{\sqrt{m}}{\sqrt{np}} + \frac{C_5 e^{4\kappa}}{\gamma^2} \frac{\sqrt{m}}{\sqrt{np}} \\
&\leq \frac{C_6 e^{4\kappa}}{\gamma^2} \frac{\sqrt{m}}{\sqrt{np}},
\end{aligned}
$$

where C_6 is an absolute constant.

On the other hand, we have

$$
\begin{aligned}
&\left\| \beta^{(k)} - \beta^{*(k)} \right\|_\infty \leq \left\| \hat{\beta}^{(k)} - \hat{\beta}^{*(k)} \right\|_\infty + \left| \delta^{(k)} - \delta^{*(k)} \right| \\
&\leq \frac{e^{\kappa} C_2'}{\gamma} \sqrt{\frac{\log m}{np}} + \frac{e^{4\kappa} C_2''}{\gamma^2} \frac{1}{\sqrt{np}} + \frac{C_5 e^{4\kappa}}{\gamma^2} \frac{1}{\sqrt{np}} \\
&\leq \frac{e^{4\kappa} C_6'}{\gamma^2} \sqrt{\frac{\log m}{np}}
\end{aligned}
$$

for some constant C_6'. □

A.6. Proofs of Top-L Ranking Guarantee

In the previous section, we show the entrywise error of the spectral estimate together with the Frobenius norm error in the proof of Theorem 4.1. We therefore focus on proving the sample complexity bound for top L ranking. We first show the equivalency between the top-L set item score (6) and top-L set by parameter sum (7).

Theorem A.12. *Consider a PCM model as described in Section 2. For any two items $i \neq j$, and for any user with parameter θ_l^**

$$\frac{\frac{\Pr(X_{li}=K|\boldsymbol{\beta}_i^*,\theta_l^*)}{\Pr(X_{li}=0|\boldsymbol{\beta}_i^*,\theta_l^*)}}{\frac{\Pr(X_{lj}=K|\boldsymbol{\beta}_i^*,\theta_l^*)}{\Pr(X_{lj}=0|\boldsymbol{\beta}_i^*,\theta_l^*)}} = \exp\left(\sum_{k=1}^{K} \beta_j^{*(k)} - \beta_i^{*(k)}\right) .$$

Proof. We first state an alternative formulation of the PCM likelihood function. It can be shown (Andrich, 1978) that (1) is equivalent to the following. For $k > 0$, and any user l with parameter θ_l^*,

$$\Pr(X_{li} = k|\boldsymbol{\beta}_i^*, \theta_l^*) = \frac{\exp\left(k\theta_l^* - \sum_{s=1}^{k} \beta_i^{*(s)}\right)}{1 + \sum_{k'=1}^{K} \exp\left(k'\theta_l^* - \sum_{s=1}^{k'} \beta_i^{*(s)}\right)} \tag{36}$$

and

$$\Pr(X_{li} = 0|\boldsymbol{\beta}_i^*, \theta_l^*) = \frac{1}{1 + \sum_{k'=1}^{K} \exp\left(k'\theta_l^* - \sum_{s=1}^{k'} \beta_i^{*(s)}\right)} .$$

One can then verify that

$$\frac{\Pr(X_{li} = K|\boldsymbol{\beta}_i^*, \theta_l^*)}{\Pr(X_{li} = 0|\boldsymbol{\beta}_i^*, \theta_l^*)} = \exp\left(K\theta_l^* - \sum_{s=1}^{K} \beta_i^{*(s)}\right) .$$

Substitute the above display into the numerator and denominator of the LHS of the display inside the theorem statement completes the proof. ☐

It's not hard to see that from the above theorem statement, $s_i^* > s_j^*$ if and only if $-\sum_{k=1}^{K} \beta_i^{*(k)} > -\sum_{k=1}^{K} \beta_j^{*(k)}$. The following theorem formally states the top-L ranking guarantee of the spectral algorithm.

Theorem 4.3. Consider the setting of Theorem 4.1. There exists a constant C_L that only depends on K such that if $np \geq \frac{C_L \log m}{{\Delta_L^*}^2}$ then $\mathcal{S}_L(\beta) = \mathcal{S}_L^*$.

Proof. Invoking the entrywise error bound obtained in Theorem 4.1, we have

$$\max_{k\in[K]} \max_{i\in[m]} \left|\beta_i^{(k)} - \beta_i^{*(k)}\right| \leq \frac{e^{4\kappa} C_6'}{\gamma^2} \sqrt{\frac{\log m}{np}} .$$

Define $f_i = \sum_{k=1}^{K} \beta_i^{(k)}$ and $f_i^* := \sum_{k=1}^{K} \beta_i^{*(k)}$. Recall that $\Delta_L^* = \sum_{k=1}^{K} \beta_{[L+1]}^{*}{}^{(k)} - \beta_{[L]}^{*}{}^{(k)} = f_{[L]}^* - f_{[L+1]}^*$. Furthermore, we can also mathematically define the top L items as

$$\mathcal{S}_L^* = \arg \min_{\mathcal{S}\in[m]:|\mathcal{S}|=L} \sum_{i\in\mathcal{S}} f_i^* .$$

One can see that if the entrywise error satisfies $\max_{k\in[K]} \max_{i\in[m]} \left|\beta_i^{(k)} - \beta_i^{*(k)}\right| \leq \frac{\Delta_L^*}{2K}$ then

$$|f_i - f_i^*| \leq \frac{\Delta_L^*}{2}$$

and $\mathcal{S}_L(\beta) = \mathcal{S}_L^*$. The sufficient sample size is therefore determined as

$$\frac{e^{4\kappa} C_6'}{\gamma^2} \sqrt{\frac{\log m}{np}} \leq \frac{\Delta_L^*}{2K} .$$

Solving for np and $\log m$ gives the condition stated in the theorem statement. This completes the proof. ☐

A.7. Proofs of Lower Bounds

We first establish the pairwise Fano's inequality (Thomas & Joy, 2006).

Lemma A.13. *(Pairwise Fano minimax lower bound) Suppose that we can construct a set $\mathcal{W} = \{\beta^{*1}, \ldots, \beta^{*M}\}$ with cardinality M such that*

$$\max_{a \neq b \in [M]} \|\beta^{*a} - \beta^{*b}\|_2^2 \geq \delta^2 \quad \textit{and} \quad \max_{a \neq b \in [M]} KL(Pr^a(X) \| Pr^b(X)) \leq \xi\,,$$

where $\Pr^a(X)$ denotes the distribution over X under a model parametrized by β^{*a}. Then the minimax risk is lower bounded as

$$\inf_{\hat{\beta}} \sup_{\beta^* \in \mathcal{B}} \mathbb{E}[\|\hat{\beta} - \beta^*\|_2^2] \geq \frac{\delta^2}{2}\left(1 - \frac{\xi + \log 2}{\log M}\right),$$

where $\hat{\beta}$ is the output of any statistical estimator.

Intuitively, the above theorem states that if we can somehow construct a set of models, where every pair of models are sufficiently different (in ℓ_2^2 distance sense) yet they parametrize 'similar' distributions (in KL divergence sense), then any statistical estimator will fail to identify the correct model with a constant probability and thus suffer from a minimum expected estimation error.

Next, to construct the set of models, we follow the construction similar to that in the proof of Theorem 1 of (Shah et al., 2015). We first restate a coding theoretic due to Shah et al. (2015).

Lemma A.14. *(Lemma 7 of Shah et al. (2015)) For any $\alpha \in (0, \frac{1}{4})$, there exists a set of $M(\alpha)$ binary vectors $\{z^1, \ldots, z^{M(\alpha)}\} \subset \{0,1\}^d$ such that $|M(\alpha)| \asymp \exp(d)$ and*

$$\alpha d \leq \|z^a - z^b\|_2^2 \leq d \quad \forall a \neq b \in [1, \ldots, M(\alpha)]\,.$$

Lemma A.15 (Reverse Pinsker's inequality)**.** *Consider two probability measures P and Q defined on the same probability space, the KL divergence between P and Q can be bounded as*

$$KL(P\|Q) \leq \left(\frac{\log_2 e}{Q_{\min}}\right) \cdot \|P - Q\|_{TV}^2\,,$$

where $\|P - Q\|_{TV}$ is the total variation distance between the two distributions and $Q_{\min} := \min_{x \in \mathbb{A}} Q(x)$.

Theorem 4.2. Consider the sampling model described in Section 2 and further assume that $np \geq c' K^4$ for some constant c'. There exists a class of Partial Credit Models $\mathcal{B}$ such that for any statistical estimator, the minimax risk is lower bounded as

$$\inf_{\hat{\boldsymbol{\beta}}} \sup_{\boldsymbol{\beta}^* \in \mathcal{B}} \mathbb{E}[\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_F^2] \geq \frac{cKm}{np}\,,$$

where c is an absolute constant.

Proof. Consider a set $\{z^1, \ldots, z^{M(\alpha)}\} \subset \{0,1\}^{m \times K}$ of $m \times K$-dimensional binary vectors (matrices) given by Lemma A.14. Let

$$\boldsymbol{\beta}^{*a} = \frac{\delta}{\sqrt{mK}} \cdot z^a \quad \forall a \in [M(\alpha)]\,,$$

where δ is to be detemined later. It is easy to see that for $a \neq b \in [M(\alpha)]$

$$\|\boldsymbol{\beta}^{*a} - \boldsymbol{\beta}^{*b}\|_2^2 = \frac{\delta^2}{mK} \cdot \|z^{*a} - z^{*b}\|_2^2 \geq \alpha\delta^2\,.$$

Set $\theta_l = 0$ for all $l \in [n]$. Let P denote the PCM parametrized by $\boldsymbol{\beta}^{*a}$ and Q denote the PCM parametrized by $\boldsymbol{\beta}^{*b}$. The key to obtaining the lower bound is to provide an upper bound on the following.

$$\begin{aligned} \mathrm{KL}(P(X \circ A) \| Q(X \circ A)) &= \sum_{l=1}^{n} \sum_{i=1}^{m} \mathrm{KL}(P(A_{li} X_{li}) \| Q(A_{li} X_{li})) \\ &= np \sum_{i=1}^{m} \mathrm{KL}(P(X_{li}) \| Q(X_{li})). \end{aligned}$$

By the reverse Pinsker's inequality,

$$\mathrm{KL}(P(X_{li})\|Q(X_{li})) \leq \left(\frac{\log_2 e}{Q_{\min}}\right) \|P(X_{li}) - Q(X_{li})\|_{TV}^2 \,. \tag{37}$$

To obtain an upper bound on the RHS of (37), we will use a Lipschitz-type argument. In order to do so, we first obtain an upper bound on the absolute value of the derivative of $P(X_{li})$ with respect to the individual entries of $\boldsymbol{\beta}^*$. Define $\tau_{ik} := \sum_{t=1}^{k} \beta_i^{*(t)}$. One can verify the following alternative formula for the likelihood of the PCM (Andrich, 1978; Masters, 1982) for the case when $\theta_l = 0 \, \forall l \in [n]$.

$$P(X_{li} = k) = \frac{\exp(-\tau_{ik})}{1 + \sum_{s=1}^{K} \exp(-\tau_{is})}.$$

We can compute exactly the derivative of $P(X_{li} = k)$ with respect to $\beta_i^{*(t)}$ for every pair of $i \in [m], t \in [K]$.

$$\begin{aligned}\frac{\partial P(X_{li} = k)}{\partial \beta_i^{*(t)}} &= \sum_{k'=1}^{K} \frac{\partial P(X_{li} = k)}{\partial \tau_{ik'}} \frac{\partial \tau_{ik'}}{\partial \beta_i^{*(t)}} = \sum_{k'=1}^{K} \frac{\partial P(X_{li} = k)}{\partial \tau_{ik'}} \mathbb{I}\{t \leq k'\} \\ &= \sum_{k'=1}^{K} \left(\frac{\exp(-\tau_{ik} - \tau_{ik'})}{\left(1 + \sum_{s=1}^{K} \exp(-\tau_{is})\right)^2} - \frac{\exp(-\tau_{ik})}{1 + \sum_{s=1}^{K} \exp(-\tau_{is})} \mathbb{I}\{k' = k\} \right) \mathbb{I}\{t \leq k'\}.\end{aligned} \tag{38}$$

Note that we can decompose first term inside (38) as

$$\frac{\exp(-\tau_{ik})}{1 + \sum_{s=1}^{K} \exp(-\tau_{is})} \frac{\exp(-\tau_{ik'})}{1 + \sum_{s=1}^{K} \exp(-\tau_{is})} \,.$$

We have by construction, $\tau_{ik} \geq 0$ hence, $\exp(-\tau_{ik}) \leq 1$.

$$\begin{aligned}1 + \sum_{s=1}^{K} \exp(-\tau_{is}) &\geq 1 + K - \sum_{s=1}^{k} \tau_{is} = 1 + K - \left(K\beta{^*}_i^{(1)} + (K-1)\beta{^*}_i^{(2)} + \ldots + \beta{^*}_i^{(K)}\right) \\ &\geq 1 + K - \frac{\delta}{\sqrt{mK}} (1 + \ldots + K) \\ &\geq \frac{K}{2}\end{aligned}$$

so long as $\frac{\delta\sqrt{K}}{\sqrt{m}} \leq 1$. That is,

$$\frac{\exp(-\tau_{ik})}{1 + \sum_{s=1}^{K} \exp(-\tau_{is})} \leq \frac{2}{K}.$$

We can then bound the absolute value of (38) as

$$\left| \frac{\partial P(X_{li} = k)}{\partial \beta_i^{*(t)}} \right| \leq \left| \sum_{k'=1}^{K} \left(\frac{4}{K^2} - \frac{2}{K} \mathbb{I}\{k' = k\} \right) \mathbb{I}\{t \leq k'\} \right| \leq \frac{4}{K} \,.$$

We have

$$\begin{aligned}\|P(X_{li}) - Q(X_{li})\|_{TV}^2 &\leq K \cdot \sum_{k=1}^{K} \left(P(X_{li} = k) - Q(X_{li} = k)\right)^2 \\ &\leq \frac{16}{K} \cdot \sum_{k=1}^{K} \left(\beta_i^{*(k)} - \beta_i^{*\prime(k)}\right)^2 \,.\end{aligned}$$

Following the proof of Proposition A.1, we can show that

$$\min_{l,i,k} \Pr(X_{li} = k) \geq \frac{\exp(\frac{\delta}{\sqrt{mK}}) - 1}{\exp(\frac{K\delta}{\sqrt{mK}}) - 1}.$$

Therefore,

$$\begin{aligned}
\mathrm{KL}(P(X\circ A)\|Q(X\circ A)) &\leq np\sum_{i=1}^{m}\mathrm{KL}(P(X_{li})\|Q(X_{li}))\\
&\leq \left(\frac{\log_2 e}{Q_{\min}}\right) np\sum_{i=1}^{m}\|P(X_{1i})-Q(X_{1i})\|_{TV}^2\\
&\leq \left(\frac{\log_2 e}{Q_{\min}}\right)\frac{16np}{K}\left\|\beta^*-\beta^{*\prime}\right\|_F^2\\
&\leq \frac{16\log_2 e\left(\exp\left(\frac{\delta}{\sqrt{mK}}\right)^K-1\right)}{\exp(\frac{\delta}{\sqrt{mK}})-1}\frac{np}{K}\delta^2\,.
\end{aligned}\tag{39}$$

Invoking Lemma A.13,

$$\inf_{\hat\beta}\sup_{\beta^*\in\mathcal{B}}\mathbb{E}[\|\hat\beta-\beta^*\|_2^2]\geq\frac{\delta^2}{2}\left(1-\frac{16\log_2 e\,\frac{\exp\left(\frac{\delta}{\sqrt{mK}}\right)^K-1}{\exp(\frac{\delta}{\sqrt{mK}})-1}\,\frac{np}{K}\delta^2+\log 2}{mK}\right).\tag{40}$$

Using the two inequalities $e^x\leq 1+x+x^2$ if $x<1.79$ and $e^x\geq 1+x$. Note that

$$\frac{\exp\left(\frac{\delta}{\sqrt{mK}}\right)^K-1}{\exp(\frac{\delta}{\sqrt{mK}})-1}\leq\frac{\frac{\delta\sqrt{K}}{\sqrt{m}}+\left(\frac{\delta\sqrt{K}}{\sqrt{m}}\right)^2}{\frac{\delta}{\sqrt{mK}}}\leq 2K$$

so long as $\frac{\delta K\sqrt{K}}{\sqrt{m}}\leq 1$. Set

$$\delta^2=\frac{mK}{64\log_2(e)\,np}.$$

The RHS of Inequality (40) reduces to

$$\inf_{\hat\beta}\sup_{\beta^*\in\mathcal{B}}\mathbb{E}[\|\hat\beta-\beta^*\|_2^2]\geq\frac{mK}{64\log(2)\,np}\left(1-\frac{1}{2}-\frac{\log 2}{mK}\right)$$

so long as $np\geq 64^2\log_2(e)^2\,K^4$.

□

Near Optimality of Entrywise Error Guarantee. The above theorem asserts the optimality of the spectral algorithm in terms of Frobenius norm estimation error. It is not hard to see that the above lower bound also implies a near optimality (modulo a log factor) for the entrywise estimation guarantee of the spectral algorithm. Specifically, we make use of the property that $\|v\|_\infty\geq\frac{1}{m}\|v\|_2$ for any $v\in\mathbb{R}^m$.

Lower Bounds for Top-L Ranking We next show a lower bound for the sample complexity of top-L ranking. We first restate a different version of Fano's inequality (Thomas & Joy, 2006) which will be useful to our proof.

Lemma A.16 (Fano's inequality)**.** *Consider a set of N distributions $\{\Pr^1,\dots,\Pr^N\}$. Suppose that we observe a random variable (or a set of random variables) Y that was generated by first picking an index $A\in\{1,\dots,N\}$ uniformly at random and then $Y\sim\Pr_A$. Fano's inequality states that any hypothesis test ϕ for this problem has an error probability lower bounded as*

$$\Pr[\phi(Y)\neq A]\geq 1-\frac{\max_{a,b\in[N],a\neq b}\mathit{KL}(\Pr_a(Y)\|\Pr_b(Y))+\log 2}{\log N}\,.$$

Model Construction. Let us consider the following constructions for $m-L+1$ models $\boldsymbol{\beta}^{*L},\dots,\boldsymbol{\beta}^{*m}$. For simplicity, let us consider the unnormalized parameter space for now. This is valid because we are only concerned about the KL divergence between any pair of models, which is not affected by parameter shifting. For model $a \in [L, L+1, \ldots, m]$ and for all $k \in [K]$, set

$$\beta_i^{*a(k)} = \begin{cases} \frac{\delta}{\sqrt{K}} & \text{if } i \leq L-1 \\ \frac{\delta}{\sqrt{K}} & \text{if } i - L = a \\ 0 & \text{otherwise} \end{cases}.$$

In other words, the $m - L + 1$ models differ exactly by the identity of the L-th best item. For the user parameters, we consider the case where $\theta_1 = \ldots = \theta_n = \frac{1}{2}$.

Theorem A.17. *Consider the uniform sampling model described in Section 2. Suppose that $np \geq cK^3$ for some constant c. There exists a constant c_L that only depends on K such that if $np \leq \frac{c_L \log m}{\Delta_L^{*2}}$ then any statistical ranking algorithm will fail to identify the top L items from user response data with probability at least $1/2$.*

Proof. We follow a similar derivations as Inequality (39). For any two models $a \neq b$, we have

$$\begin{aligned}
\mathrm{KL}(\Pr_a(X) \| \Pr_b(X)) &= \sum_{l=1}^{n} \sum_{i=1}^{m} \mathrm{KL}(\Pr_a(X_{li}) \| \Pr_b(X_{li})) \\
&\leq \frac{16 \log_2 e \left(\exp\left(\frac{\delta}{\sqrt{mK}}\right)^K - 1 \right)}{\exp(\frac{\delta}{\sqrt{mK}}) - 1} \frac{np}{K} \left\| \boldsymbol{\beta}^a - \boldsymbol{\beta}^b \right\|_F^2 \\
&= \frac{32 \log_2 e \left(\exp\left(\frac{\delta}{\sqrt{mK}}\right)^K - 1 \right)}{\exp(\frac{\delta}{\sqrt{mK}}) - 1} \frac{np\delta^2}{K} \\
&\leq 64 \log_2(e) np\delta^2
\end{aligned}$$

where the last inequality holds so long as $\frac{\delta K \sqrt{K}}{m} \leq 1$. Invoke Lemma A.16, we have the probability that any statistical estimator failing to identify the top L items correctly being lower bounded as

$$\begin{aligned}
&\geq 1 - \frac{\max_{a,b \in [m-L+1], a \neq b} \mathrm{KL}(\Pr_a(X) \| \Pr_b(X)) + \log 2}{\log(m - L + 1)} \\
&\geq 1 - \frac{64 \log_2(e)\, np\delta^2}{\log(m - L + 1)} - \frac{\log 2}{\log(m - L + 1)} \\
&\approx 1 - \frac{64 \log_2(e)\, np\delta^2}{\log(m)} \geq \frac{1}{2}
\end{aligned}$$

for a sufficiently large m and for

$$\delta \leq \frac{8\sqrt{\log_2(e)\, \log m}}{\sqrt{2np}}.$$

The extra condition on np in the theorem statement comes from $\frac{\delta K \sqrt{K}}{m} \leq 1$. Now one can verify that, by construction,

$$\Delta_L^* = 2\sqrt{K}\delta,$$

We conclude that so long as $np \geq cK^3$ for some sufficiently large constant c and that if

$$np \leq \frac{32 \log_2(e)\, \log m}{\delta^2} = \frac{144K \log_2(e)\, \log m}{\Delta_L^{*2}},$$

then any estimator fails to correctly identify all of the top L items with probability at least $\frac{1}{2}$. This completes the proof. □

A.8. Additional Proofs

Lemma A.18. *Consider a discrete state Markov chain with transition probabilities M^* and a vector π^* such that $\langle \pi^*, \mathbb{1}_m \rangle = 1$ that satisfy*

$$\pi_i^* M_{ij}^* = \pi_j^* M_{ji}^* \quad \forall i \neq j$$

and $M_{ii}^* = 1 - \sum_{j \neq i} M_{ij}^*$. Then π^* is the stationary distribution of the Markov chain. That is

$$(\pi^*)^\top M^* = (\pi^*)^\top .$$

Proof. One can verify that

$$\left((\pi^*)^\top M^*\right)_i = \sum_{j=1}^m \pi_j^* M_{ji}^* = \sum_{j=1}^m \pi_i^* M_{ij}^* = \pi_i^* .$$

This equality holds for all $i \in [m]$, which is the definition of stationary distribution. ☐

Proof of Proposition A.1.

$$\pi_i^{*(k)} = \frac{e^{\beta_i^{*(k)}}}{\sum_{k'=1}^K e^{\beta_i^{*(k')}}} \leq \frac{e^{\beta^*_{\max}}}{m e^{\beta^*_{\min}}} = \frac{e^\kappa}{m}.$$

$$\pi_i^{*(k)} = \frac{e^{\beta_i^{*(k)}}}{\sum_{k'=1}^K e^{\beta_i^{*(k')}}} \geq \frac{e^{\beta^*_{\min}}}{m e^{\beta^*_{\max}}} = \frac{1}{m e^\kappa}.$$

To prove the lower bound on γ, we make use of the following property of the PCM.

$$\frac{\Pr(X_{li} = k)}{\Pr(X_{li} = k+1)} = \exp\left(\beta_i^{*(k)} - \theta_l^*\right) \leq \exp\left(|\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|\right).$$

The above bound implies that for $k \neq k'$,

$$\frac{\Pr(X_{li} = k')}{\Pr(X_{li} = k)} \leq \exp\left(|k' - k|\, |\beta^*_{\max} - \theta^*_{\min}|\right)$$

By definition, $\sum_{k=0}^K \Pr(X_{li} = k) = 1$. Hence, for any $k \in [K]$,

$$\begin{aligned}
\Pr(X_{li} = k) &\geq \frac{1}{1 + \sum_{k' \neq k} \exp\left(|k' - k| \ |\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|\right)} \\
&\geq \frac{1}{1 + \sum_{k'=1}^K \exp\left(k' \left(|\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|\right)\right)} \\
&\geq \frac{\exp\left(|\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|\right) - 1}{\exp\left(K \ \left(|\beta^*_{\max} - \theta^*_{\min}| \vee |\beta^*_{\min} - \theta^*_{\max}|\right)\right) - 1}.
\end{aligned}$$

And $\gamma \geq \min_{l,i,k} \Pr(X_{li} = k)^2$. ☐

Proof of Lemma A.2. By Chernoff bound, we have

$$\begin{aligned}
\Pr\left(|B_{ij} - \mathbb{E}\left[B_{ij}\right]| > t\mathbb{E}[B_{ij}]\right) &= \Pr\left(\left|\sum_{l=1}^n A_{li}A_{lj} - \mathbb{E}\left[A_{li}A_{lj}\right]\right| > tnp^2\right) \\
&\leq 2\exp\left(-\frac{t^2 np^2}{3}\right).
\end{aligned}$$

Set $t = \frac{1}{2}$. Take union bound over all pairs $i \neq j$ we get

$$\Pr\left(|B_{ij} - \mathbb{E}\left[B_{ij}\right]| > t\mathbb{E}[B_{ij}]\, \forall i \neq j\right) \leq 2\exp\left(-\frac{np^2}{12} + 2\log m\right).$$

We obtain the bound in the lemma when $np^2 \geq 48 \log m$. To prove the probability of Condition $\mathcal{A}^+$, we also use Chernoff's bound. Fix an $i \in [m]$.

$$\Pr\left(\left|\sum_{i=1}^m A_{li} - \mathbb{E}[A_{li}]\right| > t\mathbb{E}[A_{li}]\right) \leq 2\exp\left(-\frac{t^2 mp}{3}\right)$$

Pick $t = \frac{1}{2}$. Apply union bound over all $l \in [n]$. We have

$$\Pr\left(\left|\sum_{i=1}^m A_{li} - \mathbb{E}[A_{li}]\right| > \frac{1}{2}\mathbb{E}[A_{li}] \ \forall l \in [n]\right) \leq 2\exp\left(-\frac{mp}{12} + \log n\right) \leq \frac{2}{n^{-10}}.$$

The last inequality holds so long as $mp \geq 132 \log n$. The probability that $\mathcal{A}^+$ holds can be obtained by union bounding the probability that $\mathcal{A}$ doesn't hold and the obtained probability bound above. $\square$

Proof of Lemma A.3. Consider two Markov chains M and M' that are constructed per Algorithm 1 on the same data X but using normalization factors d and d', respectively, where $d \neq d'$ and $d, d' \geq \max_i \sum_{j \neq i} Y_{ij}$. By definition, the stationary distributions π and π' of M and M' must satisfy the self-consistent equations for all state $i \in [m]$,

$$\sum_{j \neq i} \pi_j M_{ji} = \sum_{j \neq i} \pi_i M_{ij},$$
$$\sum_{j \neq i} \pi'_j M'_{ji} = \sum_{j \neq i} \pi_i M'_{ij}.$$

Recall the definition of the transition probabilities, we have

$$\sum_{j \neq i} \pi_j \frac{Y_{ji}}{d} = \sum_{j \neq i} \pi_i \frac{M_{ij}}{d},$$
$$\sum_{j \neq i} \pi'_j \frac{Y_{ji}}{d'} = \sum_{j \neq i} \pi_i \frac{Y_{ij}}{d'}.$$

One can verify that any stationary distribution π of M must also be the stationary distribution of M' and vice versa. Under assumptions of connectedness and ergodicity (the Markov chain has no sink and source nodes), M and M' must have the same and unique stationary distribution. $\square$

Proof of Proposition A.5. We first prove (15). Note that by construction,

$$M_{ij} - M^*_{ij} = \frac{1}{d}\left(Y_{ij} - Y^*_{ij}\right) \ .$$

By Chernoff bound, we have

$$\begin{aligned}\Pr\left(\left|M^{(k)}_{ij} - M^{*\,(k)}_{ij}\right| \geq \epsilon M^{*\,(k)}_{ij} \middle| \mathcal{A}\right) &= \Pr\left(\left|Y^{(k)}_{ij} - Y^{*\,(k)}_{ij}\right| \geq \epsilon Y^{*\,(k)}_{ij} \middle| \mathcal{A}\right) \\ &\leq 2\exp\left(-\frac{\epsilon^2 Y^{*\,(k)}_{ij}}{3}\right) \\ &\leq 2\exp\left(-\frac{\epsilon^2 \gamma B_{ij}}{3}\right) \leq 2\exp\left(-\frac{\epsilon^2 \gamma np^2}{6}\right) \ .\end{aligned}$$

Set $\epsilon = \frac{\gamma}{12e^{2\kappa}}$ and apply union bound over all pairs $i \neq j$. We have,

$$\max_{i \neq j}\left|M^{(k)}_{ij} - M^{*\,(k)}_{ij}\right| \leq \frac{\gamma}{12e^{2\kappa}} M^{*\,(k)}_{ij} \quad \forall i \neq j$$

with probability at least

$$\begin{aligned} &1 - 2m^2 \exp\left(-\frac{\gamma^3 np^2}{12^2\, 6\, e^{4\kappa}}\right) \\ &= 1 - 2\exp\left(-\frac{\gamma^3 np^2}{12^2\, 6\, e^{4\kappa}} + 2\log m\right) \\ &\geq 1 - 2\exp\left(-\frac{\gamma^3 np^2}{12^3\, 2\, e^{4\kappa}}\right), \end{aligned}$$

where the last inequality holds so long as $\frac{\gamma^3 np^2}{12^2 6 e^{4\kappa}} \geq 4\log m$. □

The following Lemma A.21 and Lemma A.22 are used to prove Lemma A.6. Note that the proof of Lemma A.21 uses Lemma A.19 and Lemma A.20.

Lemma A.19. *Consider two reversible Markov chains M^*, Q with stationary distributions π^* and λ, respectively, that are defined on the same graph $G = (V, E)$ of m states. Define $\alpha := \min_{i,j\in E} \frac{\pi_i^* M_{ij}^*}{\lambda_i Q_{ij}}$ and $\tau := \max_{i\in[m]} \frac{\pi_i^*}{\lambda_i}$. We have*

$$\frac{\mu^*(M^*)}{\mu^*(Q)} \geq \frac{\alpha}{\tau}$$

where $\mu^*(.)$ is the spectral gap function.

Consider the following Markov chain Q

$$Q_{ij} = \begin{cases} \frac{B_{ij}}{d} & \text{if } i \neq j \\ 1 - \frac{1}{d}\sum_{k\neq j} B_{ik} & \text{if } i = j \end{cases}, \tag{41}$$

where $d = \frac{3}{2} mnp^2$. It is not difficult to check that, by construction, the stationary distribution of Q is $\lambda = \frac{\mathbb{1}_m}{m}$. λ is a left eigenvector that corresponds to an eigvenvalue of 1. In order to bound the spectral gap of Q, we need to find the second largest eigenvalue of Q.

Lemma A.20. *Suppose that condition $\mathcal{A}$ holds. The Markov chain Q defined in (41) satisfies*

$$\mu^*(Q) \geq \frac{1}{3}.$$

Proof. We can write Q as $Q = \frac{1}{d}B + I_m - \frac{1}{d}\text{diag}\,(B\mathbb{1}_m)$. Let $s_{\max,\perp\!\!\!\perp}$ denote the largest eigenvalue of the subspace orthogonal to $\mathbb{1}_m$.

$$\begin{aligned} s_{\max,\perp\!\!\!\perp}(Q) &= s_{\max,\perp\!\!\!\perp}\left(\frac{1}{d}B + I_m - \frac{1}{d}\text{diag}\,(B\mathbb{1}_m)\right) \\ &= 1 - \frac{1}{d}\, s_{\min,\perp\!\!\!\perp}\left(\text{diag}\,(B\mathbb{1}_m) - B\right) \\ &= 1 - \frac{1}{d} \min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} v^\top \left(\text{diag}\,(B\mathbb{1}_m) - B\right) v \\ &= 1 - \frac{1}{2d} \min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} \sum_{i\neq j} B_{ij}(v_i - v_j)^2 \,. \end{aligned}$$

Rearranging the terms, we recognize that

$$\mu^*(Q) = 1 - s_{\max,\perp\!\!\!\perp}(Q) = \frac{1}{2d} \min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} \sum_{i\neq j} B_{ij}(v_i - v_j)^2 \,.$$

Conditioned on $\mathcal{A}$, $B_{ij} \geq \frac{1}{2}np^2$. We therefore have

$$\begin{aligned}
&\min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} \sum_{i\neq j} B_{ij}(v_i - v_j)^2 \\
&\geq \frac{np^2}{2} \min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} \sum_{i\neq j} (v_i - v_j)^2 \\
&= np^2 \min_{v:\|v\|_2=1, v\perp\!\!\!\perp \mathbb{1}_m} v^\top \left(mI_m - \mathbb{1}_m \mathbb{1}_m^\top\right) v \\
&\geq np^2\, s_{\min,\perp\!\!\!\perp} \left(mI_m - \mathbb{1}_m \mathbb{1}_m^\top\right) \\
&\geq mnp^2\,.
\end{aligned}$$

Therefore, $\mu^*(Q) \geq \frac{1}{2d}\, mnp^2 = \frac{1}{3}$, where we have substituted $d = \frac{3}{2}mnp^2$. □

Lemma A.21. *Suppose that condition $\mathcal{A}$ holds. For all levels $k \in [K]$,*

$$\mu^*(M^{*(k)}) \geq \frac{\gamma}{3e^{2\kappa}}$$

Proof. For each level $k \in [K]$, we obtain a lower bound for the spectral gap of $M^{*(k)}$ by applying Lemma A.19 with the two Markov chains $M^{*(k)}$ and the reference Markov chain Q. Let

$$\begin{aligned}
\alpha^{(k)} &= \min_{i,j\in[m]} \frac{\pi_i^{*(k)} M_{ij}^{*(k)}}{\lambda_i Q_{ij}} \\
&\geq \min_{i,j\in[m]} \frac{\pi_i^{*(k)} M_{ij}^{*(k)}}{\lambda_i Q_{ij}}\,.
\end{aligned}$$

We have

$$\lambda_i Q_{ij} = \frac{B_{ij}}{md}\,;$$

$$\pi_i^{*(k)} M_{ij}^{*(k)} \geq \frac{1}{me^\kappa} \frac{1}{d} \sum_{l=1}^{n} \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k, X_{lj} = k-1, A_{ij} = 1\right\}\right] \geq \frac{\gamma B_{ij}}{mde^\kappa}\,.$$

Then,

$$\alpha^{(k)} \geq \frac{\gamma}{e^\kappa}\,.$$

On the other hand,

$$\tau^{(k)} = \max_{i\in[m]} \frac{\pi_i^{*(k)}}{\lambda_i} = \frac{\max_{i\in[m]} \pi_i^{*(k)}}{\frac{1}{m}} \leq e^\kappa\,.$$

Invoking Lemma A.19, we have

$$\mu^*(M^{*(k)}) \geq \mu^*(Q)\, \frac{\alpha^{(k)}}{\tau^{(k)}} \geq \frac{\gamma}{3e^{2\kappa}}\,.$$

□

Lemma A.22. *Fix a level $k \in [K]$. Suppose that Condition $\mathcal{A}$ and Inequality (15) hold. Then*

$$\left\| M^{(k)} - M^{*(k)} \right\|_{\text{op}} \leq \frac{\gamma}{6e^{2\kappa}}\,.$$

Proof.

$$\left\|M^{(k)} - M^{*(k)}\right\|_{\text{op}} \leq \left\|\text{diag}(M^{(k)}) - \text{diag}(M^{*(k)})\right\|_{\text{op}} + \left\|\left[M^{(k)} - M^{*(k)}\right]_{i\neq j}\right\|_{\text{op}}$$
$$= \max_{i\in[m]}\left|M_{ii}^{(k)} - M^{*}{}_{ii}^{(k)}\right| + \left\|\left[M^{(k)} - M^{*(k)}\right]_{i\neq j}\right\|_{\text{op}}.$$

One can bound the first term as follows.

$$\max_{i\in[m]}\left|M_{ii}^{(k)} - M^{*}{}_{ii}^{(k)}\right| = \max_{i\in[m]}\left|\sum_{j'\neq i} M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right|$$
$$\leq (m-1)\max_{i\neq j}\cdot\left|M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right|.$$

For the second term we have

$$\left\|\left[M^{(k)} - M^{*(k)}\right]_{i\neq j}\right\|_{\text{op}} = \max_{u,v:\|u\|=\|v\|=1} u^\top\left[M^{(k)} - M^{*(k)}\right]_{i\neq j} v$$
$$= \max_{u,v:\|u\|=\|v\|=1}\sum_{i\neq j} u_i v_j\left(M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right)$$
$$\leq \max_{u,v:\|u\|=\|v\|=1}\sum_{i\neq j} u_i v_j\left|M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right|$$
$$\leq \max_{u,v:\|u\|=\|v\|=1}\sum_{i\neq j} u_i v_j\cdot\max_{i\neq j}\left|M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right|$$
$$\leq m\cdot\max_{i\neq j}\left|M_{ij}^{(k)} - M^{*}{}_{ij}^{(k)}\right|.$$

Combining the two displays above gives

$$\left\|M^{(k)} - M^{*(k)}\right\|_{\text{op}} \leq 2m\cdot\max_{i\neq j}\left|M_{ij}^{(k)} - M_{ij}^{*}{}^{(k)}\right|.$$

Substitute bound the maximum entrywise deviation of $M^{(k)} - M^{*(k)}$ in Inequality (15). We have

$$\left\|M^{(k)} - M^{*(k)}\right\|_{\text{op}} \leq 2m\cdot\max_{i\neq j}\left|M_{ij}^{(k)} - M_{ij}^{*}{}^{(k)}\right| \leq \frac{\gamma}{3e^{2\kappa}}\cdot\max_{i\neq j} M_{ij}^{*}{}^{(k)} \leq \frac{\gamma}{3me^{2\kappa}},$$

where in the last inequality we have used the fact that

$$\max_{i\neq j} M_{ij}^{*}{}^{(k)} \leq \max_{i\neq j}\frac{B_{ij}}{d} \leq \frac{1}{m}.$$

□

Proof of Lemma A.8. Recall the entrywise decomposition

$$\pi_i{}^{(k)} - \pi_i^{*}{}^{(k)} = \frac{1}{1 - M_{ii}^{(k)}}\cdot\left(\sum_{j\neq i}(\pi_j^{(k)} - \pi_j^{*}{}^{(k)})M_{ji}^{(k)} + \sum_{j\neq i}\pi_j^{*}{}^{(k)}(M_{ji}{}^{(k)} - M_{ji}^{*}{}^{(k)}) + \pi_i^{*}{}^{(k)}(M_{ii}^{(k)} - M_{ii}^{*}{}^{(k)})\right).$$

The first term can be bounded as follows.

$$1 - M_{ii}{}^{(k)} = \sum_{j\neq i} M_{ji} \geq \sum_{j\neq i}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right)M_{ij}^{*}$$
$$\geq \left(1 - \frac{\gamma}{12e^{2\kappa}}\right)\sum_{j\neq i}\frac{\gamma B_{ij}}{d}$$
$$\geq \frac{\gamma}{2}\left(1 - \frac{\gamma}{12e^{2\kappa}}\right).$$

The second term satisfies

$$\begin{aligned}
&\sum_{j\neq i}(\pi_j^{(k)} - \pi_j^{*(k)})M_{ji}^{(k)} \\
&\leq \left\|\pi^{(k)} - \pi^{*(k)}\right\| \sqrt{\sum_{k\neq i} M_{ji}^{*\,(k)}} \leq \left(1+\frac{\gamma}{6e^{2\kappa}}\right)\left\|\pi^{(k)} - \pi^{*(k)}\right\| \sqrt{\sum_{k\neq i} M_{ji}^{*\,(k)}} \\
&\leq \frac{1}{\sqrt{m}}\left(1+\frac{\gamma}{12e^{2\kappa}}\right)\left\|\pi^{(k)} - \pi^{*(k)}\right\| ,
\end{aligned}$$

where the last inequality comes from the observation that $M_{ji}^{*\,(k)} \leq \frac{B_{ji}}{d} \leq \frac{1}{m}$.

The third term, we use the method of bounded difference. Firstly, we write

$$\begin{aligned}
&\sum_{j\neq i}\pi_j^{*(k)}\left({M_{ji}}^{(k)} - M_{ji}^{*\,(k)}\right) \\
&= \frac{1}{d}\sum_{j\neq i}\pi_j^{*(k)}\sum_{l=1}^{n} A_{li}A_{lj}\left(\mathbb{I}\left\{X_{lj}=k\right\}\mathbb{I}\left\{X_{li}=k-1\right\} - \mathbb{E}\left[\mathbb{I}\left\{X_{lj}=k\right\}\mathbb{I}\left\{X_{li}=k-1\right\}\right]\right) .
\end{aligned}$$

One could verify that the above display is a function of the random variable X. If the value of X_{li} changes for any l, the maximum possible absolute difference is bounded by

$$\frac{1}{d}\sum_{j\neq i}\pi_j^{*(k)}A_{li}A_{lj} \leq \frac{e^{\kappa}}{md}A_{li}mp = \frac{2e^{\kappa}}{3mnp}A_{li} .$$

On the other hand, if the value of X_{lj} changes for any $j \neq i$, the maximum possible absolute difference is bounded by $\frac{1}{d}\pi_j^{*(k)}A_{li}A_{lj} \leq \frac{2e^{\kappa}}{3m^2np^2}A_{li}A_{lj}$. Condition on $\mathcal{A}$, note that $\sum_{l=1}^{n}A_{li} \leq \frac{3np}{2}$ and $\sum_{j=1}^{m}A_{lj} \leq \frac{3mp}{2}$. The method of bounded difference gives

$$\begin{aligned}
&\Pr\left(\left|\sum_{j\neq i}\pi_j^{*(k)}\left({M_{ji}}^{(k)} - M_{ji}^{*\,(k)}\right)\right| \geq t\right) \\
&\leq 2\exp\left(-\frac{\frac{1}{2}t^2}{\sum_{l=1}^{n}\left(\left(\frac{2e^{\kappa}}{3mnp}A_{li}\right)^2 + \sum_{j\neq i}\left(\frac{2e^{\kappa}}{3m^2np^2}A_{li}A_{lj}\right)^2\right)}\right) \\
&= 2\exp\left(-\frac{\frac{1}{2}t^2}{\frac{4e^{2\kappa}}{9}\left(\sum_{l=1}^{n}\frac{1}{m^2n^2p^2}A_{li} + \sum_{l=1}^{n}\sum_{j\neq i}\frac{1}{m^4n^2p^4}A_{li}A_{lj}\right)}\right) \\
&\leq 2\exp\left(-\frac{\frac{1}{2}t^2}{\frac{4e^{2\kappa}}{9}\left(\frac{np}{m^2n^2p^2} + \frac{mnp^2}{m^4n^2p^4}\right)}\right) \leq 2\exp\left(-\frac{\frac{1}{2}t^2}{\frac{4e^{2\kappa}}{9}\left(\frac{1}{m^2np} + \frac{1}{m^3np^2}\right)}\right) \\
&\leq 2\exp\left(-\frac{\frac{1}{2}t^2}{\frac{4\sqrt{10}e^{2\kappa}}{9}\frac{2}{m^2np}}\right) .
\end{aligned}$$

Then, with probability at least $1-2m^{-10}$,

$$\left|\sum_{j\neq i}\pi_j^{*(k)}\left({M_{ji}}^{(k)} - M_{ji}^{*\,(k)}\right)\right| \leq \frac{4e^{\kappa}}{3}\frac{\sqrt{\log m}}{m\sqrt{np}} .$$

Similarly, we also use the method of bounded difference to bound the fourth term

$$\pi_i^{*(k)} \left(M_{ii}{}^{(k)} - M_{ii}^{*(k)} \right) = \pi_i^{*(k)} \left(\sum_{j \neq i} M_{ij}^{*(k)} - M_{ij}{}^{(k)} \right)$$
$$\leq \frac{1}{d} \pi_i^{*(k)} \sum_{l=1}^{n} \sum_{j \neq i} \left(\mathbb{I}\left\{X_{li} = k\right\} \mathbb{I}\left\{X_{lj} = k-1\right\} - \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k\right\} \mathbb{I}\left\{X_{lj} = k-1\right\}\right]\right) .$$

One can thus see that the above display is very similar to the expansion of the third term. We follow the same argument as shown for the third term. The final probability bound is obtained using union bound argument. This completes the proof □

B. Intuition for the Spectral-EM Algorithm

The cluster than learn initialization procedure is inspired by the EM algorithm proposed in Nguyen & Zhang (2023a) for learning a mixture of Plackett-Luce models. The Plackett-Luce model is a model over permutations and in that paper the authors cluster mixtures of permutations by embedding each permutation as pairwise comparison vectors where each entry corresponds to a paired comparison between items. In our setting, when the number of items is large embedding each response vector as a $\binom{m}{2}$ pairwise difference vector is space inefficient. To perform efficient clustering, we restrict to the set of items to the 50 items with the highest variance in ratings.

The E-step of Algorithm 3 computes the posterior distribution, similar to the general E-step in the EM framework. Generally speaking, the M-step maximizes the joint log likelihood, fixing the current posterior distribution. The M-step can be divided into C subproblems which can be solved independently of one another. For each subproblem c, we solve for

$$\boldsymbol{\beta}_{c,\text{next}} = \underset{\boldsymbol{\beta}_c}{\text{argmax}} \sum_{l=1}^{n} q_{lc} \log \Pr\left(X_l | \boldsymbol{\beta}_c, z_l = c\right) .$$

Algorithm 3 does not solve this optimization problem exactly and instead produces an approximate solution. Let $q^*_{lc} = \Pr\left(z_l = c \,\middle|\, X_l, \boldsymbol{\beta}^*_c\right)$ denote the true posterior probability. One can verify that for any pair of items i, j,

$$\frac{\Pr\left(X_{li} = k, X_{lj} = k-1, z_l = c\right)}{\Pr\left(X_{li} = k-1, X_{lj} = k, z_l = c\right)} = \frac{e^{\beta_j^{*(k)}}}{e^{\beta_i^{*(k)}}} .$$

Following similar intuition as the spectral algorithm, rearranging the above display and summing over all users l gives

$$e^{\beta_i^{*(k)}} \cdot \sum_{l=1}^{n} q^*_{lc} \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k, X_{lj} = k-1\right\}\right] = e^{\beta_j^{*(k)}} \cdot \sum_{l=1}^{n} q^*_{lc} \mathbb{E}\left[\mathbb{I}\left\{X_{li} = k-1, X_{lj} = k\right\}\right] .$$

The above equation is the weighted analogy of the reversible condition as shown for Algorithm 1 in the single model setting. More importantly, the stationary distribution of the weighted Markov chain remains the true parameter while the pairwise transition probabilities are weighted modification of the pairwise transition probabilities in (3). The reader can see that the weighted Markov chain constructed per (10) is an empirical approximation of the true weighted Markov chain. Given sufficiently large sample size and when $Q \approx Q^*$, the estimate produced by the M step of the Algorithm 3 will be accurate.

Similarly to Algorithm 1, in Line 15 of Algorithm 3, we also have to estimate the level-wise shift. This estimate is the weighted generalization of the shift estimate in Algorithm 1. It is not hard to follow the same derivation as shown above to arrive at (13).

C. Additional Experiment Descriptions

In this section we provide additional descriptions of the experimental setup.

Synthetic Data. The item parameter $\beta_i^{*(k)}$ is independently drawn from $\mathcal{N}(0, 1)$. This parameter is then fixed across all trials. For each value of n, we repeat for 100 trials. In each trial, we generate $\theta_l \sim \mathcal{N}(0, \sigma_0)$ for either $\sigma_0 = 1$ (Figure 1) or $\sigma_0 = 2$ (Figure 3). In all experiments, the prior distribution used in MMLE is set to be the standard normal distribution.

Real Data. Table 3 summarizes the metadata for all the real-life datasets used in our experiments. For the recommendation systems datasets, after processing the data, we remove users with fewer than 100 ratings and items with fewer than 100 ratings. For each of the remaining users, we leave out one rating from each user as part of the heldout test dataset and one rating as part of a validation dataset (for MMLE and Spectral-EM). Inference algorithms (spectral, MMLE and JMLE) are run on the training data and we obtain $\boldsymbol{\beta}$. We then use the learned $\boldsymbol{\beta}$ to estimate θ (in the case of JMLE, we directly use the estimated θ.) We use the estimated parameters to predict the left out rating (maximum likelihood estimate) and compute log likelihood on the heldout data. The validation dataset is used by MMLE to select the best user parameter distribution $\mathcal{D}_{\theta^*}$ and by Spectral-EM to select the number of mixture components. For MMLE and Spectral-EM, the error reported in Table 1 shows the performance on the heldout test set of the best model selected using validation data. The time reported shows the average inference time across all model runs with different hyper-parameter choice.

Dataset	m	n	K	Reference
LSAT	5	1000	2	(McDonald, 2013)
UCI	4	131	2	(Hussain et al., 2018)
3 GRADES	3	649	2	(Cortez & Silva, 2008)
HETREC	6590	1617	5	(Cantador et al., 2011)
EACH MOVIE	7217	1156	5	(Harper & Konstan, 2015)
ML-1M	2499	4247	5	(Harper & Konstan, 2015)
ML-10M	7214	42971	5	(Harper & Konstan, 2015)
ML-20M	8532	51869	5	(Harper & Konstan, 2015)
BOOK-GENOME	9374	9102	5	(Kotkov et al., 2022)

Table 3: Datasets metadata and references.

Financial Investment Application. The dataset was made publicly available on a Reddit post (u/nobjos). We encode sell recommendations as 0, hold recommendations as 1 and buy recommendations as 2. We will also publish the cleaned version of the dataset upon publication of our paper to ensure anonymity. The cleaning procedure includes mapping different rating description (buy, overweight, etc) to the same numerical scores, linking different versions of the same analyst (Nomura Inv Bank, Nomura Investment). The daily stock price can be extracted from Yahoo finance.

The dataset consists of ratings given between 2012-01 and 2021-01. There are 502 stocks in total, and about 300 stocks receive more than 100 ratings. There are 263 rating firms , 72 with at least 100 ratings, 52 firms with more than 200 ratings, only a handful of stocks receive less than 10 ratings. The weekly price is taken as the average of the lowest and highest daily prices within the corresponding week. Trading is performed on the last week of each month. Capital is divided equally among the top 10 stocks. To obtain a ranking using the estimate $\boldsymbol{\beta}$, define for each stock (item) a score

$$f_i := \sum_{k=1}^{K} -\beta_i{}^{(k)} \,.$$

We have shown that sorting the items by f_i is equivalent to sorting the items by $\frac{\Pr(X_{li}=K)}{\Pr(X_{li}=0)}$ for any l. For simplicity, we ignore trading costs.