Time Encoding of Sparse Signals with Flexible Filters

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***Abstract*—In this work, we consider the problem of recovering a sparse signal consisting of a sum of filtered spikes from the output of a time encoding machine (TEM). This problem was addressed before with recovery methods designed for filters with specific shapes, mostly relying on Prony's method for recovery. Here we propose a new recovery method for sparse inputs from TEM samples. Compared to existing approaches, the new method relaxes significantly the assumption on the filters. The method is associated by a theoretically guaranteed algorithm. We provide numerical examples to evaluate the new method, including an example with filters that are not compatible with previous methods.**

I. Introduction

A model encountered in many problems in basic and applied sciences, the signal model represents a sum of filtered spikes with different amplitudes. In the frequency domain, this is equivalent to resolving parameters of complex-exponentials. This representation is typically used when measurements are directly performed in the Fourier domain. Mathematically, the input is modelled as

$$g(t) = \sum\nolimits_{k=1}^{K} a_k \varphi(t-\tau_k), \quad t \in [0, t_{\mathsf{max}}], \tag{1}$$

where $\{\tau_k\}_{k=1}^K$ are positive real numbers representing time-domain shifts of $\varphi(t)$, and $\{a_k\}_{k=1}^K$ are real coefficients. The recovery problem is to compute $\{\tau_k, a_k\}_{k=1}^K$ using measurements of $g(t)$. Given its various different applications, this problem was considered under various names, including (a) Tauberian Approximation [1], (b) Time-Delay Estimation [2], (c) Sparse Deconvolution [3], [4], (d) Super-resolution [5] and (e) Finite-rate-of-Innovation Sampling [6]. More recent applications have shown that (1) represents a core concept also for areas such as ultra-fast time-of-flight imaging [7], the Unlimited Sensing Framework [8]–[10] or analytical imaging of paintings [11].

Motivation. Despite the extensive work in the topic, see [1]–[6] and follow-up literature, there are remaining fundamental research gaps due to the mathematical tractability of (1) in the case of non-conventional sampling approaches. One such approach is time encoding, which converts a continuous-time input $g(t)$ into an increasing sequence of time events t_k. The sampling model is known as a Time Encoding Machine (TEM), which is inspired by the information processing in the brain, and is characterized by low power consumption [12]. A TEM with input $g(t)$ is an operator $\mathcal{T}$ defined as $\mathcal{T}g = \{t_k\}_{k\in\mathbb{Z}}$. The TEM thus generates a strictly increasing sequence of time samples $\{t_k\}_{k\in\mathbb{Z}}$, known as spikes or trigger times. The input recovery approach, known as a Time Decoding Machine (TDM) has been realised for the case of bandlimited inputs [12]–[14], inputs belonging to shift-invariant spaces [14], [15], and also inputs with jump discontinuities [16]–[18].

In this work, we consider the problem of recovering sparse inputs of the form (1) for a wide class of filteres $\varphi(t)$ from their corresponding TEM samples. This was considered before for the case where $\varphi(t)$ is a polynomial or exponential B-spline [19], hyperbolic secant kernel [20], [21] or a biologically inspired alpha synaptic model [22]. In [23], the problem of sparse recovery for TEMs was considered for a filter comprising a sum of complex exponentials, whose output (TEM input) is a periodic bandlimited function. The recovery conditions combine TEM Nyquist rate conditions with classical sparse input recovery. Moreover, as before, the constraints on the acquisition setup may be restrictive, e.g., in applications where the TEM input is not bandlimited. This line of work was further extended in [24], [25].

We note that the existing methods for sparse input recovery for TEMs are dependent on specific expressions of $\varphi(t)$. This is a strong assumption, as a filter in real applications may deviate from a specific shape. Moreover, the existing methods are mostly based on Prony's method for recovery. In this work, we generalize the problem of sparse input recovery from TEM samples for filters of arbitrary shape satisfying some mild assumptions. Our contributions are as follows.

Contributions. We introduce theoretical guarantees under which a sequence of filters of flexible shape can be recovered from the TEM observations. We complement the theory with a theoretically guaranteed recursive recovery algorithm that is evaluated numerically for different filter shapes.

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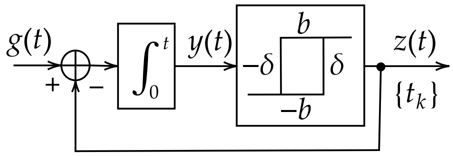

Figure 1: The asynchronous sigma-delta modulator (ASDM) diagram.

II. The Time Encoding Machine

Here we consider the case of an Asynchronous Sigma-Delta Modulator (ASDM) TEM, which is characterised by low power consumption [26] and modular design [27]. The ASDM comprises a loop with an adder, integrator, and a noninverting Schmitt trigger, as depicted in Fig. 1. At $t=0$, the Schmitt trigger is initialised as $z(0)=-b$, where b is a constant denoting the amplitude of the ASDM output. The first ASDM output sample is $t_0=0$. Subsequently, the output sequence $\{t_n\}_{n\geqslant 1}$ satisfies the *t-transform* equations [12]

$$\mathcal{L}_n g = (-1)^n \left[2\delta - b(t_{n+1}-t_n)\right], \quad n \in \mathbb{Z}_+^*, \tag{2}$$

where $\mathcal{L}_n g \triangleq \int_{t_n}^{t_{n+1}} g(s)$, δ is the threshold and b is the amplitude of the Schmitt trigger output. We assume that $|g(t)| \leqslant c < b$, $c>0$, which yields the following bounds on the density of the ASDM samples $\Delta t_n \triangleq t_{n+1}-t_n$ [12]

$$\frac{2\delta}{b+c} \leqslant \Delta t_n \leqslant \frac{2\delta}{b-c}. \tag{3}$$

When the input $g(t)$ is bandlimited to Ω rad/s, it was shown that exact input recovery is possible provided that $|\Delta t_n| < \frac{\pi}{\Omega}$.

III. Sampling Sparse Signals with a TEM

A. Proposed Sampling Pipeline

We consider that $g(t)$ belongs to the input space spanned by (1), where $\{a_k, \tau_k\}_{k=1}^K$ are unknown real numbers satisfying $\varepsilon_a < |a_k| < c, 0 < \tau_k < \tau_{k+1} < t_{\max}, \tau_{k+1}-\tau_k > \varepsilon_\tau$, $\varepsilon_a, \varepsilon_\tau, c, t_{\max} > 0$ and K is the unknown number of pulses with shape $\varphi(t)$. We assume that $\varphi(t)$ is known and has finite support $[-L/2, L/2]$. Without reducing the generality, we assume that the pulses are normalized such that $\max_t |\varphi(t)| = 1$. Furthermore, we assume $\max_t |g(t)| < c$, $\varepsilon_\tau < L/2$, $\tau_1 \geqslant L/2$ and $\varphi(t)$ is second order differentiable on $(-L/2, L/2)$ and continuous on $\mathbb{R}$. We also assume that $\varphi'(t) > 0, t \in [-L/2, 0)$. Besides the normalization condition, the conditions on $\varphi(t)$ are defining a space of functions that are relatively common, including the previously considered cases of polynomial and exponential splines [19], hyperbolic secant kernel [20] and alpha synaptic activation function [22].

We assume that $g(t)$ is sampled with an ASDM TEM with parameters δ, b over the finite time interval $[0, t_{\max}]$ to yield output time encoded samples $\{t_n\}_{n=1}^N$. We assume that $\varepsilon_\tau > \frac{6\delta}{b-c}$, which, based on the TEM properties in Section II, ensures that there are at least 3 TEM output samples in between each two consecutive pulses. The problem we propose is to recover $\{a_k, \tau_k\}_{k=1}^K$ from $\{t_n\}_{n=1}^N$.

B. Proposed Recovery Approach

The idea behind the proposed recovery is that one can use a number of TEM samples to uniquely determine the time and amplitude of each pulse, under significantly reduced assumptions on the filter. Let $n_k \in \mathbb{Z}$ be the index of the TEM output located right before the onset of the kth filter

$$n_k = \max_{n \in \{1,\dots,N\}} \left\{ n \;\middle|\; t_n \leqslant \tau_k - L/2 \right\}. \tag{4}$$

Using the ASDM equations it follows that, given that $g(t)=0$ for $t \in [t_n, t_{n+1}], n < n_1$ then $\mathcal{L}_n g = 0, \forall n \in \{1, \dots, n_1 - 1\}$. Furthermore, using the pulse separation assumption $\varepsilon_\tau > \frac{6\delta}{b-c}$, we have that $\mathcal{L}_{n_1} g \neq 0$ and $\mathcal{L}_{n_1+1} g \neq 0$, since $|a_1| > \varepsilon_a > 0$ and

$$\mathcal{L}_n g = \int_{t_n}^{t_{n+1}} a_1 \varphi(t-\tau_1)\, dt, \quad n \in \{n_1, n_1+1\}. \tag{5}$$

Thus, using (2), one can find n_1 as

$$\widetilde{n}_1 = \min_{n \in \{1,\dots,N\}} \left\{ n \;\middle|\; |2\delta - b(t_{n+1}-t_n)| > 0 \right\}. \tag{6}$$

In practice, due to numerical errors, one would compute $|2\delta - b(t_{n+1}-t_n)| > tol$, where tol is a small tolerance set by the user, modelling the estimation error of the previous filters at time t_n. Using $\mathcal{L}_{n_1} g$ and $\mathcal{L}_{n_1+1} g$, the first objective is to compute τ_1, a_1. We first denote $I_n(\tau) \triangleq \int_{t_n}^{t_{n+1}} \varphi(t-\tau) dt$, which implies that $\mathcal{L}_{n_1} g = a_1 I_{n_1}(\tau_1)$ and $\mathcal{L}_{n_1+1} g = a_1 I_{n_1+1}(\tau_1)$ (5). We note that $\frac{\mathcal{L}_{n_1+1}}{\mathcal{L}_{n_1}} = \frac{I_{n_1+1}(\tau_1)}{I_{n_1}(\tau_1)}$ is not a function of a_1. Therefore, if $\frac{I_{n_1+1}(\tau)}{I_{n_1}(\tau)}$ is strictly monotonic as a function of τ, then τ_1 can be uniquely estimated. Subsequently, a_1 can be estimated as $\frac{\mathcal{L}_{n_1} g}{I_{n_1}(\tau_1)}$. We first give a theorem analysing the monotonicity of $\frac{I_{n_1+1}(\tau)}{I_{n_1}(\tau)}$. Subsequently, we show how to recover recursively the remaining values $\{\tau_k, a_k\}_{k=2}^K$.

Theorem 1. *Let* $\{t_{n+i}\}_{i=-1}^2$ *be a set satisfying*

$$0 < t_{n-1} \leqslant \tau - L/2 < t_n < t_{n+1} < t_{n+2}.$$

Furthermore, assume that $\frac{2\delta}{b+c} \leqslant \Delta t_{n+i} \leqslant \frac{2\delta}{b-c}$ *for some* $\delta, b, c > 0$ *and* $i \in \{-1, 0, 1\}$. *Let* $I_n(\tau) = \int_{t_n}^{t_{n+1}} \varphi(t-\tau) dt$, *for* $\tau \in (t_{n-1} + L/2, t_n + L/2)$. *Then* $\frac{I_{n+1}(\tau)}{I_n(\tau)}$ *is well-defined, differentiable, and strictly increasing if the following holds*

$$\frac{b-c}{b+c} \geqslant \frac{{\varphi'_{\max,\delta}}^2}{{\varphi'_{\min,\delta}}^2} + \frac{2\varphi\left(-\frac{L}{2} + \frac{2\delta}{b-c}\right)}{{\varphi'_{\min,\delta}}^2} \cdot \|\varphi''\|_\infty \cdot \left(1 + \frac{b+c}{b-c}\right) - 2, \tag{7}$$

where $\|\varphi''\|_\infty$ *denotes the absolute norm on* $(-L/2, L/2)$ *and*

$$\varphi'_{\max,\delta} \triangleq \max_{t \in \mathbb{S}_\delta} \varphi'(t), \quad \varphi'_{\min,\delta} \triangleq \min_{t \in \mathbb{S}_\delta} \varphi'(t), \tag{8}$$

where $\mathbb{S}_\delta \triangleq \left[-\frac{L}{2}, -\frac{L}{2} + \frac{6\delta}{b-c}\right]$.

Proof. The filter satisfies $\varphi(t) = 0, t < -L/2$ and, given that $\varphi'(t) > 0, t \in [-L/2, 0)$, it follows that $\varphi(t) > 0, t \in (-L/2, 0)$ and thus $\varphi(t-\tau) > 0, t \in (-L/2+\tau, \tau)$. It follows that $I_n(\tau) > 0$ and thus $\frac{I_{n+1}(\tau)}{I_n(\tau)}$ is well-defined. Moreover, $I_n(\tau)$ is the composition of differentiable functions, therefore it is itself differentiable.

For simplicity, let $f(t) \triangleq \varphi(t-\tau), t \in [\tau - L/2, \tau], f_l = f(t_l), l \in \{n, n+1, n+2\}$, and $\Delta f_l = f_{l+1} - f_l$, $\Delta t_l = t_{l+1} - t_l, l \in \{n, n+1\}$. The following holds

$$I_n'(\tau) = \varphi(t_n - \tau) - \varphi(t_{n+1} - \tau) = -\Delta f_n,$$
$$I_{n+1}'(\tau) = \varphi(t_{n+1} - \tau) - \varphi(t_{n+2} - \tau) = -\Delta f_{n+1}.$$

$$\begin{aligned}\left(\frac{I_{n+1}(\tau)}{I_n(\tau)}\right)' &= \frac{I_{n+1}'(\tau) I_n(\tau) - I_{n+1}(\tau) I_n'(\tau)}{I_n^2(\tau)} \\ &= \frac{\Delta f_n \cdot I_{n+1}(\tau) - \Delta f_{n+1} \cdot I_n(\tau)}{I_n^2(\tau)}.\end{aligned} \tag{9}$$

The sign of the derivative is dictated by the numerator in (9), which is strictly positive if and only if

$$\frac{1}{\Delta f_n} \int_{t_n}^{t_{n+1}} f(t) dt < \frac{1}{\Delta f_{n+1}} \int_{t_{n+1}}^{t_{n+2}} f(t) dt. \tag{10}$$

Function $f(t)$ inherits the properties of being positive, differentiable and strictly increasing from $\varphi(t)$. We expand $f(t)$ in Taylor series with anchor points t_n and t_{n+1}, respectively,

$$f(t) = f_n + f'(\xi_n)(t - t_n) \leqslant f_n + f'_{\mathsf{max}}(t - t_n), t \in [t_n, t_{n+1}]$$

$$f(t) = f_{n+1} + f'(\xi_{n+1})(t - t_{n+1}) \geqslant f_{n+1} + f'_{\mathsf{min}}(t - t_{n+1}),$$

for $t \in [t_{n+1}, t_{n+2}]$, such that

$$t_n \leqslant \xi_n \leqslant t \leqslant t_{n+1}, \quad f'_{\mathsf{max}} = \max_{t \in [t_n, t_{n+2}]} f'(t)$$
$$t_{n+1} \leqslant \xi_{n+1} \leqslant t \leqslant t_{n+2}, \quad f'_{\mathsf{min}} = \min_{t \in [t_n, t_{n+2}]} f'(t).$$

Then we can bound the corresponding integrals of $f(t)$ as

$$\int_{t_n}^{t_{n+1}} f(t) dt \leqslant f_n \cdot \Delta t_n + f'_{\mathsf{max}} \frac{\Delta t_n^2}{2}, \tag{11}$$

$$\int_{t_{n+1}}^{t_{n+2}} f(t) dt \geqslant f_{n+1} \cdot \Delta t_{n+1} + f'_{\mathsf{min}} \frac{\Delta t_{n+1}^2}{2}. \tag{12}$$

Then a sufficient condition for (10) is

$$f_n \frac{\Delta t_n}{\Delta f_n} + \frac{f'_{\mathsf{max}}}{2} \frac{\Delta t_n^2}{\Delta f_n} < f_{n+1} \frac{\Delta t_{n+1}}{\Delta f_{n+1}} + \frac{f'_{\mathsf{min}}}{2} \frac{\Delta t_{n+1}^2}{\Delta f_{n+1}}. \tag{13}$$

We then use that $f_n, f_{n+1} > 0$, and thus

$$\begin{aligned} f_n \frac{\Delta t_n}{\Delta f_n} + \frac{f'_{\mathsf{max}}}{2} \frac{\Delta t_n^2}{\Delta f_n} &\leqslant \frac{f_n}{f'_{\mathsf{min}}} + \frac{f'_{\mathsf{max}} \Delta t_n}{2 f'_{\mathsf{min}}}, \\ f_{n+1} \frac{\Delta t_{n+1}}{\Delta f_{n+1}} + \frac{f'_{\mathsf{min}}}{2} \frac{\Delta t_{n+1}^2}{\Delta f_{n+1}} &\geqslant \frac{f_{n+1}}{f'_{\mathsf{max}}} + \frac{f'_{\mathsf{min}} \Delta t_{n+1}}{2 f'_{\mathsf{max}}}. \end{aligned} \tag{14}$$

As before, we use (14) to get a sufficient condition for (13), by rearranging the terms as

$$\begin{aligned} \frac{\Delta t_{n+1}}{\Delta t_n} &\geqslant \frac{2 f'_{\mathsf{max}}}{f'_{\mathsf{min}}} \left[\frac{f'_{\mathsf{max}}}{2 f'_{\mathsf{min}}} - \frac{f_{n+1}/f'_{\mathsf{max}} - f_n/f'_{\mathsf{min}}}{\Delta t_n} \right] \\ &= \frac{{f'_{\mathsf{max}}}^2}{{f'_{\mathsf{min}}}^2} - 2 \frac{f_{n+1} \cdot f'_{\mathsf{min}} - f_n \cdot f'_{\mathsf{max}}}{{f'_{\mathsf{min}}}^2 \Delta t_n}. \end{aligned} \tag{15}$$

We note that, if $f'(t)$ is constant for $t \in [t_n, t_{n+2}]$ then $f'_{\mathsf{min}} = f'_{\mathsf{max}}$ and the condition above amounts to $\frac{\Delta t_{n+1}}{\Delta t_n} \geqslant 1 - 2 \frac{\Delta f_n}{\Delta t_n f'_{\mathsf{min}}}$, sufficiently guaranteed by $\frac{b-c}{b+c} > -1$, which is always true given the LHS is strictly positive. This gives us a margin that can be exploited for the case when $f'(t)$ is not constant, as shown next. By rearranging the terms,

$$\frac{\Delta t_{n+1}}{\Delta t_n} \geqslant \frac{{f'_{\mathsf{max}}}^2}{{f'_{\mathsf{min}}}^2} - 2 \frac{f_{n+1} - f_n}{f'_{\mathsf{min}} \Delta t_n} + 2 f_n \frac{f'_{\mathsf{max}} - f'_{\mathsf{min}}}{{f'_{\mathsf{min}}}^2 \Delta t_n}. \tag{16}$$

We continue the derivation of a sufficient condition by finding a lower bound for the LHS as $\frac{\Delta t_{n+1}}{\Delta t_n} \geqslant \frac{b-c}{b+c}$. Furthermore, we get an upper bound for the second term on the RHS as

$$-2 \frac{f_{n+1} - f_n}{f'_{\mathsf{min}} \Delta t_n} = -\frac{\Delta f_n}{\Delta t_n} \frac{2}{f'_{\mathsf{min}}} = -\frac{2 f'(\bar{\xi}_n)}{f'_{\mathsf{min}}} \leqslant -2, \tag{17}$$

where $\bar{\xi}_n \in [t_n, t_{n+1}]$. Lastly, we get an upper bound for the third term in the RHS of (16) as

$$\begin{aligned} 2 f_n \frac{f'_{\mathsf{max}} - f'_{\mathsf{min}}}{{f'_{\mathsf{min}}}^2 \Delta t_n} &= 2 f_n \frac{f'(\zeta_M) - f'(\zeta_m)}{{f'_{\mathsf{min}}}^2 \Delta t_n} \\ &= \frac{2 f_n}{{f'_{\mathsf{min}}}^2} \frac{f'(\zeta_M) - f'(\zeta_m)}{\zeta_M - \zeta_m} \cdot \frac{\zeta_M - \zeta_m}{t_{n+1} - t_n} \\ &= \frac{2 f_n}{{f'_{\mathsf{min}}}^2} \cdot \left| f''(\bar{\zeta}_n) \right| \cdot \frac{\zeta_M - \zeta_m}{t_{n+1} - t_n} \\ &\leqslant \frac{2 f\left(\frac{2\delta}{b-c} + \tau - \frac{L}{2}\right)}{{f'_{\mathsf{min}}}^2} \cdot f''_{\mathsf{max}} \cdot \frac{t_{n+2} - t_n}{t_{n+1} - t_n} \\ &\leqslant \frac{2 \varphi\left(-\frac{L}{2} + \frac{2\delta}{b-c}\right)}{{f'_{\mathsf{min}}}^2} \cdot f''_{\mathsf{max}} \cdot \left(1 + \frac{b+c}{b-c}\right), \end{aligned} \tag{18}$$

where $\zeta_m, \zeta_M \in [t_n, t_{n+2}]$ s.t. $f'(\zeta_m) = f'_{\mathsf{min}}$ and $f'(\zeta_M) = f'_{\mathsf{max}}$, $\bar{\zeta}_n \in [t_n, t_{n+2}]$ s.t. $\left| f''(\bar{\zeta}_n) \right| = \frac{f'(\zeta_M) - f'(\zeta_m)}{\zeta_M - \zeta_m}$ and $f''_{\mathsf{max}} = \max_{t \in [t_n, t_{n+2}]} |f''(t)|$. Furthermore, in the last inequality, we use that $t_{n-1} \leqslant \tau - L/2 < t_n$ to compute a bound for f_n. Lastly, we have that

$$\varphi'_{\mathsf{max},\delta} = \max_{t \in \mathbb{S}_\delta^\tau} f'(t), \quad \varphi'_{\mathsf{min},\delta} = \min_{t \in \mathbb{S}_\delta^\tau} f'(t), \tag{19}$$

where $\mathbb{S}_\delta^\tau = \left[\tau - L/2, \tau - L/2 + \frac{6\delta}{b-c}\right]$. Using $[t_n, t_{n+1}] \subseteq \mathbb{S}_\delta^\tau$, we get that $f'_{\mathsf{max}} \leqslant \varphi'_{\mathsf{max},\delta}$ and $\varphi'_{\mathsf{min},\delta} \leqslant f'_{\mathsf{min}}$. By plugging this in (16) together with (18) and (17), we get the final sufficient condition. □

Remark 1. *We note that the sufficient condition* (7) *in Theorem 1 is achievable. In fact, if we assume that* $\varphi'(-L/2) \neq 0$, *then*

$$\lim_{\delta \to 0} \frac{{\varphi'_{\mathsf{max},\delta}}^2}{{\varphi'_{\mathsf{min},\delta}}^2} = 1, \quad \lim_{\delta \to 0} \frac{2 \varphi\left(-\frac{L}{2} + \frac{2\delta}{b-c}\right)}{{\varphi'_{\mathsf{min},\delta}}^2} = 0. \tag{20}$$

The latter equality holds because $\varphi(t) = 0, t < -L/2$, *and using continuity on* $\mathbb{R}$ *we get* $\varphi(-L/2) = 0$. *It follows that, for* $\delta \to 0$, (7) *becomes*

$$\frac{b-c}{b+c} \geqslant -1,$$

Algorithm 1: Recovery Algorithm.

Data: $\{t_n\}_{n=1}^N, \delta, b, \varphi(t), L, \varepsilon_\tau, tol.$

Result: $K,\ \{\widetilde{\tau}_k, \widetilde{a}_k\}_{k=1}^K$

1) Compute $\mathcal{L}_n^1 g = (-1)^n \left[2\delta - b\Delta t_n\right], n \in \{1, \dots, N\}$, set $k = 1$.
2) While $\exists n \in \{1, \dots, N\}$ s.t. $\left|\mathcal{L}_n^k g\right| > tol$
   - $2a)$ Compute $\widetilde{n}_k = \min_{n \in \{1,\dots,N\}} \left\{n \;\middle|\; \left|\mathcal{L}_n^k g\right| > tol\right\}$
   - $2b)$ Compute $I_n(\tau) = \int_{t_n}^{t_{n+1}} \varphi(t-\tau) dt$ for $n \in \{\widetilde{n}_k, \widetilde{n}_k + 1\}$ and $\tau \in (t_{n-1} + L/2, t_n + L/2)$.
   - $2c)$ Compute $\widetilde{\tau}_k$ by solving $\frac{I_{\widetilde{n}_k+1}(\tau)}{I_{\widetilde{n}_k}(\tau)} = \frac{\mathcal{L}_{\widetilde{n}_k+1}}{\mathcal{L}_{\widetilde{n}_k}}$ via line search.
   - $2d)$ Compute $\widetilde{a}_k = \frac{\mathcal{L}_{\widetilde{n}_k}^k g}{I_{\widetilde{n}_k}(\widetilde{\tau}_k)}$.
   - $2e)$ Compute $\mathcal{L}_n^{k+1} g \triangleq \mathcal{L}_n^k g - \int_{t_n}^{t_{n+1}} \widetilde{a}_k \varphi\left(t - \widetilde{\tau}_k\right) dt$, $k = k + 1$.
3) Compute $K = k - 1$.

which is always true. Therefore, if $\varphi'(-L/2) \neq 0$ then (7) *is true for δ small enough. We show in the numerical study section that the recovery works even when $\varphi'(-L/2) = 0$.*

Using Theorem 1, $\frac{I_{n_1+1}(\tau)}{I_{n_1}(\tau)} = \frac{\mathcal{L}_{n_1+1}}{\mathcal{L}_{n_1}}$ has a unique solution τ_1. We estimate it as $\widetilde{\tau}_1$ via a line search algorithm. Thereon, the amplitude of the first pulse is estimated as $\widetilde{a}_1 = \frac{\mathcal{L}_{n_1} g}{I_{n_1}(\widetilde{\tau}_1)}$. Note that computing τ_k, a_k can be de-coupled and each can be computed by a different method. For the next pulse, we remove the contribution of the first pulse from the measurements via

$$\mathcal{L}_n^2 g \triangleq \mathcal{L}_n g - \int_{t_n}^{t_{n+1}} a_1 \varphi\left(t - \tau_1\right) dt. \tag{21}$$

The process continues recursively. The proposed recovery approach is summarized in Algorithm 1.

IV. Numerical Study

We test our recovery algorithm for two different pulse shapes $\varphi(t)$. The input is $g(t) = \sum_{k=1}^K a_k \varphi\left(t - \tau_k\right)$, where $K = 6$, amplitudes a_k are randomly generated in $[-4, -2] \cup [2, 4]$, values τ_k are randomly generated in $[0, 10]$ with minimal distance $\varepsilon_\tau = 1$. For the first example $\varphi(t) = \beta^3(t)$ is the B-spline of order 3 with support $[-1, 1]$ and amplitude 1. Without any additional pre-filtering, the input is then encoded with an ASDM TEM with parameters $\delta = 1, b = 12$, which leads to a sequence of TEM samples $\{t_n\}_{n=1}^{59}$. We then implement Algorithm 1 for $L = 2, \varepsilon_\tau = 1, tol = 4.5 \cdot 10^{-2}$ and estimate $\widetilde{K} = K = 6$, and sparse signal parameters $\{\widetilde{\tau}_k, \widetilde{a}_k\}_{k=1}^K$ with the following average percentage errors $\mathsf{Err}_\tau = 0.05\%, \mathsf{Err}_a = 0.42\%$, defined as

$$\mathsf{Err}_\tau = \frac{1}{K} \sum_{k=1}^K 100 \cdot \frac{\left|\widetilde{\tau}_k - \tau_k\right|}{\left|\tau_k\right|}, \quad \mathsf{Err}_a = \frac{1}{K} \sum_{k=1}^K 100 \cdot \frac{\left|\widetilde{a}_k - a_k\right|}{\left|a_k\right|}. \tag{22}$$

The input, filter output TEM output and recovered input are depicted in Fig. 2(a). We note that our algorithm works well for this input, despite the fact that $\varphi'(-L/2) = 0$ (see Remark 1). We repeat the experiment above with a new pulse shape generated using the hyperbola curve $1/t$

$$\varphi(t) = \begin{cases} -\frac{1}{t-1} - 0.5, & t \in (-1, 0], \\ \frac{1}{t+1} - 0.5, & t \in (0, 1), \\ 0, & t \in (-\infty, -1] \cup [1, \infty). \end{cases} \tag{23}$$

We note that this pulse is generated in a fundamentally differently way from the pulses in the existing literature, that are generated using complex exponentials [24] or real exponentials [19], [20], [22]. This means the existing methods are not directly applicable. We use Algorithm 1 with a new set of 6 randomly generated values $\{\tau_k, a_k\}$. In this case $\varphi'(-L/2) \neq 0$, and thus Remark 1 via Theorem 1 guarantees recovery for δ small enough. We use the same parameters as in the previous example. The results are depicted in Fig. 2(b), and the errors are $\mathsf{Err}_\tau = 0.05\%, \mathsf{Err}_a = 0.57\%$.

V. Conclusions

In this work we presented a new method for recovering a sparse input from the output of a TEM. The new method is part of a new class of approaches not relying on Prony's method and does not assume a specific filter shape. We introduced recovery guarantees and validated the method via numerical simulations. This approach opens the path for working with real world pulses that are known, but rarely have a mathematically precise shape. In the future, we will also consider extending this sparse input recovery problem for TEM architectures for inputs with high dynamic range [16].

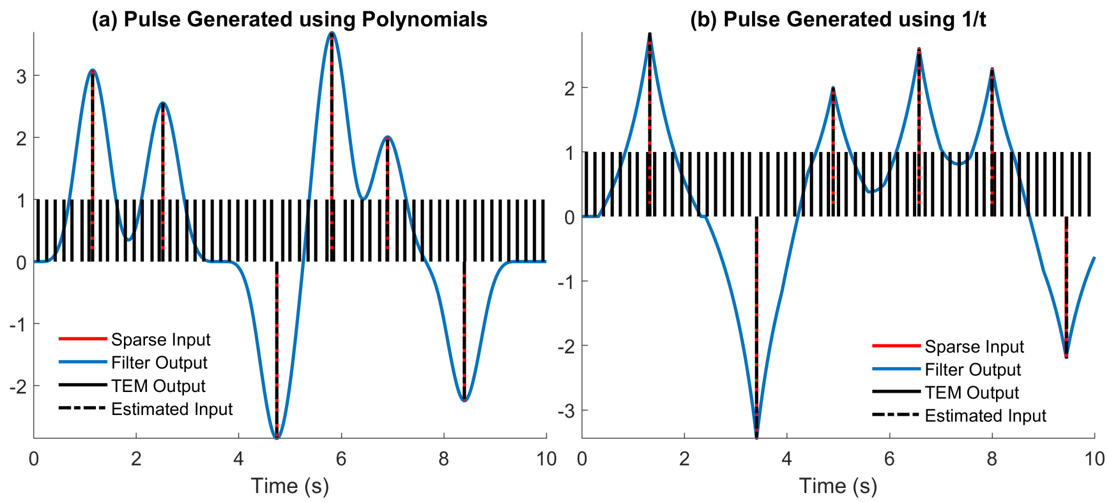

Figure 2: Recovery of a sparse input from TEM samples with the proposed method for two clases of pulses generated using: (a) polynomials and (b) function $1/t$.

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