A modified Casteljau algorithm to solve interpolation problems on Stiefel manifolds ☆

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Abstract

The main objective of this paper is to propose a new method to generate smooth interpolating curves on Stiefel manifolds. This method is obtained from a modification of the geometric Casteljau algorithm on manifolds and is based on successive quasi-geodesic interpolation. The quasi-geodesics introduced here for Stiefel manifolds have constant speed, constant covariant acceleration and constant geodesic curvature, and in some particular circumstances they are true geodesics.

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1. Introduction

Stiefel and Graßmann manifolds arise naturally in several vision applications, such as machine learning and pattern recognition, since features and patterns that describe visual objects may be represented as elements in those spaces. These geometric representations facilitate the analysis of the underlying geometry of the data. The Graßmann manifold is the space of k-dimensional subspaces

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in $\mathbb{R}^n$ and the Stiefel manifold is the space of k orthonormal vectors in $\mathbb{R}^n$. While a point in the Graßmann manifold represents a subspace, a point in the Stiefel manifold identifies exactly what frame (basis of vectors) is used to specify that subspace.

Although these two manifolds are related, the geometry of the Graßmann manifold is much simpler than that of the Stiefel manifold. This reflects on solutions of simple formulated problems, such as the case of geodesics that join two given points. A formula for the geodesic that joins two points on Graßmann manifolds and depends explicitly only on those points was recently presented in Batzies et al. [1]. Knowing such explicit formulas is also a crucial step to solve other important problems such as, averaging, fitting and interpolation of data. Results about geodesics on Stiefel manifolds are not so easy to obtain. Even the simpler problem of finding a geodesic that starts at a given point with a prescribed velocity is not so straightforward, as can be seen for instance in the work of Edelman et al. [2]. In the present paper we solve a slightly different but related problem, which consists of joining two points on the Stiefel manifold by quasi-geodesics. These curves have constant speed, constant covariant acceleration and, therefore, constant geodesic curvature. Moreover, they are defined explicitly in terms of the points they join. In some cases, depending on those points, the quasi-geodesics are true geodesics. Interestingly enough, these special curves can be used successfully to generate smooth interpolating curves on the Stiefel manifold, as will be explained later. These results may have a great impact in computer vision, since a curve that interpolates a set of time-labeled points on the Stiefel manifold may correspond, for instance, to the temporal evolution of an event or dynamic scene from which only a limited number of observations was captured, as nicely explained in Su et al. [3].

The organization of this paper is the following. After this introduction that motivates the reader to the importance of the problems studied here in the context of applications, we introduce in Section 2 the manifolds that play a major role throughout the paper: Graßmann and Stiefel manifolds. This section also includes known results about geodesics and, in particular, a closed formula for the geodesic in the Graßmann manifold that joins two given points. In Section 3 we present quasi-geodesics in the Stiefel manifold. These curves have some interesting properties, such as constant speed, constant covariant acceleration and constant geodesic curvature. We provide an explicit formula for quasi-geodesics that join two arbitrary points in the Stiefel manifold and show that in two particular circumstances they are true geodesics. Interpolation problems are formulated and solved in the next three sections of the paper. First, we review in Section 4 the Casteljau algorithm on manifolds and then implement this algorithm to generate a $\mathcal{C}^1$-smooth interpolating curve on the Graßmann manifold satisfying some prescribed boundary conditions. This algorithm, first introduced in Park and Ravani [4] and later explored for instance in Crouch et al. [5], Popiel and Noakes [6], and Nava-Yazdani and Polthier [7], is a generalisation to manifolds of the classical algorithm to generate Bézier curves in $\mathbb{R}^n$, which was derived independently by De Casteljau [8] and Bézier [9]. The algorithm is based on successive geodesic interpolation, so its implementation requires that explicit formulas for the geodesic joining two points is known. Due to the work in Batzies et al. [1], the implementation of the Casteljau algorithm is possible on Graßmann manifolds, but not on Stiefel manifolds. To overcome this difficulty, we explain in Section 5 how one can modify the Casteljau algorithm in a way that will prove to be very important in Section 6, where an interpolation problem on the Stiefel manifold is solved intrinsically, that is, without resorting to other spaces. This is done using a convenient modification of the Casteljau algorithm, which consists of replacing successive geodesic interpolation by successive quasi-geodesic interpolation. This overcomes the difficulties that arise from not knowing explicit formulas for geodesics that join two arbitrary points on the Stiefel manifold and justifies the introduction of quasi-geodesics. In Section 7 we include the results of two experiments to illustrate the loss of smoothness when instead of the interpolating method proposed in Section 6 for data on the Stiefel manifold one uses interpolating methods performed on the Graßmann manifold. The paper ends with some concluding remarks.

2. Preliminaries

In this section we recall the main definitions associated to Graßmann and Stiefel manifolds and several properties that will be used throughout this paper. Due to the important role that these manifolds play in applied areas, they have been studied in the context of numerical algorithms, for instance in Edelman et al. [2], Absil et al. [10] and Helmke et al. [11], and in a more abstract form in Kobayashi and Nomizu [12]. Recently, Batzies et al. [1] found a closed form expression for a geodesic in the Graßmann manifold that joins two given points. This formula turns out to be very important for the developments throughout the whole paper. If not stated otherwise, the definitions and concepts in this section are taken from references Edelman et al. [2] and Batzies et al. [1].

2.1. The Graßmann manifold

Let $\mathfrak{s}(n)$ and $\mathfrak{so}(n)$ denote the set of all $n \times n$ real symmetric matrices and the set of all $n \times n$ real skew-symmetric matrices respectively.

The (real) Graßmann manifold $\mathscr{G}_{n,k}$ is the set of all k-dimensional linear subspaces in $\mathbb{R}^n$, where $n \geq k \geq 1$. This manifold has a matrix representation

$$\mathscr{G}_{n,k} := \left\{ P \in \mathfrak{s}(n) \ : \ P^2 = P \text{ and } \operatorname{rank}(P) = k \right\}$$

so that it is considered a submanifold of $\mathbb{R}^{n \times n}$ with dimension $k(n-k)$. Graßmann manifold $\mathscr{G}_{n,k}$ can also be viewed as a homogeneous space

$$\mathscr{G}_{n,k} \cong \mathbb{O}(n)/(\mathbb{O}(k) \times \mathbb{O}(n-k)),$$

where $\mathbb{O}(n)$ is the orthogonal Lie group.

Given a point $P \in \mathscr{G}_{n,k}$ define the following sets

$$\begin{aligned}\mathfrak{gl}_P(n) &:= \{ X \in \mathfrak{gl}(n) \ : \ X = PX + XP \}, \\ \mathfrak{s}_P(n) &:= \mathfrak{s}(n) \cap \mathfrak{gl}_P(n) \quad \text{and} \\ \mathfrak{so}_P(n) &:= \mathfrak{so}(n) \cap \mathfrak{gl}_P(n).\end{aligned}$$

We will need the following properties. Here and in the sequel $[A, B] = AB - BA$ is the matrix commutator and $\mathbf{I}_k$ denotes the $k \times k$ identity matrix.

Proposition 1 (Batzies et al. [1]). *Let $P \in \mathscr{G}_{n,k}$ and $X \in \mathfrak{gl}_P(n)$ then*

1. $PX^{2i-1}P = 0$*, for any* $i \geq 1$*,*
2. $PX^{2i} = PX^{2i}P = X^{2i}P$*, for any* $i \geq 0$*,*
3. $[P, [P, X]] = X$*.*

The tangent space to a point $P \in \mathscr{G}_{n,k}$ is given by

$$\mathbf{T}_P \mathscr{G}_{n,k} = \{ [X, P] \ : \ X \in \mathfrak{so}_P(n) \}.$$

The Graßmann manifold will be equipped with the metric inherited from the Euclidean space $\mathbb{R}^{n \times n}$, *cf.* [11],

$$\langle [X_1, P], [X_2, P] \rangle = \mathrm{tr}(X_1^{\mathrm{T}} X_2).$$

If $P \in \mathscr{G}_{n,k}$ then $\Theta P \Theta^{\mathrm{T}} \in \mathscr{G}_{n,k}$, for any $\Theta \in \mathbb{O}(n)$. Thus $\gamma \colon (-\varepsilon, \varepsilon) \to \mathscr{G}_{n,k}$ given by $\gamma(t) = \Theta(t) P \Theta^{\mathrm{T}}(t)$, where Θ is a curve in $\mathbb{O}(n)$ satisfying $\Theta(0) = \mathbf{I}$, is a curve in the Graßmann manifold passing through P at $t = 0$.

Minimal geodesics in the Graßmann manifold satisfy the following second order differential equation

$$\ddot{\gamma} + [\dot{\gamma}, [\dot{\gamma}, \gamma]] = 0.$$

A geodesic γ in $\mathscr{G}_{n,k}$, starting from P with initial velocity $\dot{\gamma}(0) = [X, P]$, is given by

$$\gamma(t) = e^{tX} P e^{-tX}.$$

An explicit formula for a geodesic $\gamma \colon [0, 1] \to \mathscr{G}_{n,k}$ joining a point P to a point Q was derived in Batzies et al. [1] and gives the initial velocity vector in terms of the initial and final points only. This geodesic is given by

$$\gamma(t) = e^{tX} P e^{-tX}, \quad \text{where} \quad X = \frac{1}{2} \log((\mathbf{I} - 2Q)(\mathbf{I} - 2P)). \tag{1}$$

Here 'log' stands for the *principal logarithm* of a matrix.

We recall that, a nonsingular matrix Y without negative eigenvalues always has a unique logarithm whose spectrum lies in the horizontal strip defined by $\{ z \in \mathbb{C} \ : \ -\pi < \mathrm{Im}(z) < \pi \}$. This unique matrix is called the principal logarithm of Y and is denoted by $\log(Y)$ (Horn and Johnson [13], Higham [14]).

It follows from here that if the orthogonal matrix $(\mathbf{I} - 2Q)(\mathbf{I} - 2P)$ has no negative real eigenvalues then (1) defines *a unique geodesic*. The existence and uniqueness of such minimising geodesics joining two points is guaranteed in the open ball of radius π, which is the injectivity radius of the orthogonal group Krakowski et al. [15].

The following property of the velocity vector field along a geodesic will be used later.

Proposition 2. *If* $\gamma(t) = e^{tX} P e^{-tX}$ *is a geodesic in* $\boldsymbol{\mathscr{G}}_{n,k}$*, then* $X \in \mathfrak{so}_{\gamma(t)}(n)$.

Proof. One needs to show that

$$X = \gamma(t)X + X\gamma(t). \tag{2}$$

By the hypothesis, equality (2) holds for $t = 0$. From the commuting properties of the matrix exponential with its argument,

$$\begin{aligned}\gamma(t)X + X\gamma(t) &= e^{tX}Pe^{-tX}X + Xe^{tX}Pe^{-tX} \\ &= e^{tX}(PX + XP)e^{-tX} = e^{tX}Xe^{-tX} = X.\end{aligned}$$

□

2.2. The Stiefel manifold

The Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$ has the following matrix representation:

$$\boldsymbol{\mathscr{S}}_{n,k} := \left\{ S \in \mathbb{R}^{n\times k} \;:\; S^{\mathrm{T}}S = \mathbf{I}_k \right\}.$$

This is a submanifold of $\mathbb{R}^{n\times k}$, having dimension $nk - (k+1)k/2$. Stiefel manifolds $\boldsymbol{\mathscr{S}}_{n,k}$ are homogeneous spaces

$$\boldsymbol{\mathscr{S}}_{n,k} \cong \mathbb{O}(n)/\mathbb{O}(n-k).$$

The tangent space to $\boldsymbol{\mathscr{S}}_{n,k}$ at a point $S \in \boldsymbol{\mathscr{S}}_{n,k}$ can be parametrized as

$$\mathbf{T}_S\boldsymbol{\mathscr{S}}_{n,k} = \left\{ V \in \mathbb{R}^{n\times k} \;:\; V^{\mathrm{T}}S + S^{\mathrm{T}}V = 0 \right\}.$$

The Stiefel manifold will be equipped with the *canonical metric*, given by

$$\langle V_1, V_2\rangle = \mathrm{tr}\big(V_1^{\mathrm{T}}(\mathbf{I} - \tfrac{1}{2}SS^{\mathrm{T}})V_2\big), \quad \text{where} \quad V_1, V_2 \in \mathbf{T}_S\boldsymbol{\mathscr{S}}_{n,k}. \tag{3}$$

The projection onto the tangent space $\pi_{\top} \colon \mathbb{R}^{n\times k} \to \mathbf{T}_S\boldsymbol{\mathscr{S}}_{n,k}$ is given by

$$\pi_{\top}(X) := S\,\mathbf{skew}(S^{\mathrm{T}}X) + (\mathbf{I} - SS^{\mathrm{T}})X,$$

where, for a square matrix A, $\mathbf{skew}(A)$ denotes $(A - A^{\mathrm{T}})/2$.

Minimal geodesics in the Stiefel manifold satisfy the following second order differential equation, *cf.* [2],

$$\ddot\gamma + \dot\gamma\dot\gamma^{\mathrm{T}}\gamma + \gamma\Big((\gamma^{\mathrm{T}}\dot\gamma)^2 + \dot\gamma^{\mathrm{T}}\dot\gamma\Big) = 0.$$

A geodesic γ in $\boldsymbol{\mathscr{S}}_{n,k}$ starting from a point $S = \Theta\Delta$, where $\Theta \in \mathbb{O}(n)$ and $\Delta = \left[\begin{array}{c}\mathbf{I}_k \\ \hline 0\end{array}\right]_{n\times k}$, is given by

$$\gamma(t) = \Theta e^{tX}\Delta,$$

where $X \in \mathfrak{so}(n)$ has the following structure:

$$X = \left[\begin{array}{c|c} A & -B^{\mathrm{T}} \\ \hline B & 0 \end{array}\right].$$

Compared with what happens for the Graßmann manifolds, solving the geodesic equation for Stiefel manifolds is quite hard. Nevertheless, Edelman et al. [2] have included formulas for geodesics on Stiefel manifolds that start at a given point with a prescribed velocity. Also, according to Theorem 3 in Harms and Mennucci [16], any two points in the Stiefel manifold $\boldsymbol{\mathscr{S}}_{n,k}$, when $n \geq 2k$, can be connected by a minimal geodesic. However, as far as we know, there are no explicit formulas for the geodesic joining two arbitrary points that depends on these points only.

2.3. Relationships

There are some intimate relationships between $\mathbb{SO}(n)$, $\boldsymbol{\mathscr{G}}_{n,k}$ and $\boldsymbol{\mathscr{S}}_{n,k}$ that can be expressed in terms of the following surjective mappings, where Δ is the matrix defined in the previous section and $\Lambda = \Delta\Delta^{\mathrm{T}}$.

- The projection $\pi\colon \mathbb{SO}(n) \to \boldsymbol{\mathscr{S}}_{n,k}$ defined by $\pi(\Theta) := \Theta\Delta$;
- The mapping $\varphi\colon \mathbb{SO}(n) \to \boldsymbol{\mathscr{G}}_{n,k}$ defined by $\varphi(\Theta) := \Theta\Lambda\Theta^{\mathrm{T}}$;
- The mapping $\psi\colon \boldsymbol{\mathscr{S}}_{n,k} \to \boldsymbol{\mathscr{G}}_{n,k}$ defined by $\psi(S) := SS^{\mathrm{T}}$.

Clearly, $(\Theta\Delta)^{\mathrm{T}}(\Theta\Delta) = \Delta^{\mathrm{T}}\Delta = \mathbf{I}_k \quad \text{and} \quad (\Theta\Delta)(\Theta\Delta)^{\mathrm{T}} = \Theta\Lambda\Theta^{\mathrm{T}}$.

The following commutative diagram summarises these relationships.

$$\begin{tikzcd}
\mathbb{SO}(n) \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\
\boldsymbol{\mathscr{S}}_{n,k} \arrow[r, "\psi"] & \boldsymbol{\mathscr{G}}_{n,k}
\end{tikzcd}$$

Geodesics in $\boldsymbol{\mathscr{G}}_{n,k}$ and geodesics in $\boldsymbol{\mathscr{S}}_{n,k}$ are projections of special geodesics on $\mathbb{SO}(n)$, as explained next.

Let $\gamma\colon [0,1] \to \mathbb{SO}(n)$ be a geodesic given by $\gamma(t) = e^{tX}\Theta$. Define a curve $\sigma\colon [0,1] \to \boldsymbol{\mathscr{G}}_{n,k}$ by

$$\sigma(t) := (\varphi \circ \gamma)(t) = e^{tX}\Theta\Lambda\Theta^{\mathrm{T}}e^{-tX}.$$

Denote $P = \Theta\Lambda\Theta^{\mathrm{T}} \in \boldsymbol{\mathscr{G}}_{n,k}$ and suppose that $X \in \mathfrak{so}_P(n)$. Then, σ is a geodesic in $\boldsymbol{\mathscr{G}}_{n,k}$ starting from P, with initial velocity equal to $[X,P]$, *cf.* [1].

A simple calculation shows that the condition $X \in \mathfrak{so}_P(n)$ is equivalent to $\Theta^{\mathrm{T}} X \Theta \in \mathfrak{so}_\Lambda(n)$, and the latter implies a particular matrix structure for $\Theta^{\mathrm{T}} X \Theta$, namely

$$\Theta^{\mathrm{T}} X \Theta = \left[\begin{array}{c|c} 0 & -B^{\mathrm{T}} \\ \hline B & 0 \end{array}\right].$$

In a similar way, one may analyse which geodesics in $\mathbb{SO}(n)$ project to geodesics in the Stiefel manifold. Edelman et al. [2] proved that the curve defined by

$$\alpha(t) = e^{tZ} S = e^{tZ} \Theta \Delta = \pi(e^{tZ}\Theta)$$

is a geodesic in $\mathscr{S}_{n,k}$ starting at S when the skew-symmetric matrix Z satisfies the following block structure

$$\Theta^{\mathrm{T}} Z \Theta = \left[\begin{array}{c|c} A & -B^{\mathrm{T}} \\ \hline B & 0 \end{array}\right].$$

But a closed form expression for Z, given the initial and final points, is not yet known.

To overcome this difficulty and being able to propose a solution for an interpolating problem on Stiefel manifolds, we are going to introduce, in the next section, other interesting curves in $\mathscr{S}_{n,k}$ that will play an important role.

We end this section with two properties that are immediate consequences of those in Proposition 1 and will be useful later on.

Proposition 3. *Let $S \in \mathscr{S}_{n,k}$ and $X \in \mathfrak{gl}_{SS^{\mathrm{T}}}(n)$. Then*

1. $S^{\mathrm{T}} X S = 0$;
2. $SS^{\mathrm{T}} X^2 S = X^2 SS^{\mathrm{T}} S = X^2 S$.

3. Quasi-Geodesics in Stiefel Manifolds

In this section we define certain smooth curves in the Stiefel manifold $\mathscr{S}_{n,k}$ that join two arbitrary points S_1 and S_2 and have many interesting properties. In some cases these curves are geodesics, but in general their velocity vector field may fail to have zero covariant derivative. The generic term for these curves will be quasi-geodesics because they have constant geodesic curvature and are associated to certain retractions on $\mathscr{S}_{n,k}$. We use here the notion of a retraction introduced in [17] for general manifolds.

Definition 4. *A retraction R on the Stiefel manifold $\mathscr{S}_{n,k}$ is a smooth mapping from the tangent bundle $\mathbf{T}\mathscr{S}_{n,k}$ to $\mathscr{S}_{n,k}$ that, when restricted to each tangent space at a point $S \in \mathscr{S}_{n,k}$ (restriction denoted by R_S), satisfies the following properties:*

(i) $R_S(0) = S$.
(ii) $dR_S(0) = id$.

If $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, one can define a smooth curve $\beta_V : t \mapsto R_S(tV)$ associated to the retraction R. The curve β_V which satisfies $\beta_V(0) = S$ and $\dot{\beta}_V(0) = V$ is called a *quasi-geodesic*. In the sequel we will present a quasi-geodesic on the Stiefel manifold, which is different from the example included in Absil et al. [17] or in Nishimori and Akaho [18], but has other very interesting properties. Before that, we present a representation of the tangent space to $\mathscr{S}_{n,k}$ at a point S, which differs from that considered in Edelman et al. [2] but will prove to be very important for further derivations.

Proposition 5. *Let $S \in \mathscr{S}_{n,k}$, so that $P = SS^{\mathrm{T}} \in \mathscr{G}_{n,k}$. Then,*

$$\mathbf{T}_S\mathscr{S}_{n,k} = \{\, XS + S\Omega,\ \textit{where } X \in \mathfrak{so}_P(n) \textit{ and } \Omega \in \mathfrak{so}(k) \,\}. \tag{4}$$

Moreover, if $V = XS + S\Omega \in \mathbf{T}_S\mathscr{S}_{n,k}$, then

$$X = VS^{\mathrm{T}} - SV^{\mathrm{T}} + 2SV^{\mathrm{T}}SS^{\mathrm{T}} \quad \textit{and} \quad \Omega = S^{\mathrm{T}}V. \tag{5}$$

Proof. Let $M := \{\, XS + S\Omega,$ where $X \in \mathfrak{so}_P(n)$ and $\Omega \in \mathfrak{so}(k) \,\}$. Notice that the dimensions of M and $\mathbf{T}_S\mathscr{S}_{n,k}$ match. Indeed,

$$\dim(\mathfrak{s}_P(n)) = \dim(\mathbf{T}_P\mathscr{G}_{n,k}) = k(n-k), \quad \dim(\mathfrak{so}(k)) = k(k-1)/2,$$

and so,

$$\dim(M) = k(n-k) + k(k-1)/2 = nk - k(k+1)/2 = \dim(\mathbf{T}_S\mathscr{S}_{n,k}).$$

To show that (4) is a good parametrisation of $\mathbf{T}_S\mathscr{S}_{n,k}$, we must prove that $M \subset \mathbf{T}_S\mathscr{S}_{n,k}$ and $\mathbf{T}_S\mathscr{S}_{n,k} \subset M$. For the first part, a trivial calculation shows that if $V = XS + S\Omega \in M$ then, since X and Ω are skew symmetric, V satisfies the equation $V^{\mathrm{T}}S + S^{\mathrm{T}}V = 0$, that is, $V \in \mathbf{T}_S\mathscr{S}_{n,k}$. For the second part, we show that if $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, there exists $\Omega \in \mathfrak{so}(k)$ and $X \in \mathfrak{so}_P(n)$ such that $V = XS + S\Omega$. This is done by construction:

$$\Omega := S^{\mathrm{T}}V \text{ and } X := VS^{\mathrm{T}} - SV^{\mathrm{T}} + 2SV^{\mathrm{T}}SS^{\mathrm{T}}.$$

It is just a matter of simple calculations, using the fact that $V \in \mathbf{T}_S\mathscr{S}_{n,k}$, to check that indeed $V = XS + S\Omega$, $\Omega \in \mathfrak{so}(k)$, $X \in \mathfrak{so}(n)$, and moreover $X = XSS^{\mathrm{T}} + SS^{\mathrm{T}}X$, that is $X \in \mathfrak{so}_P(n)$.

The last statement in the proposition follows from the previous considerations. □

Proposition 6. *Let S, X and Ω be as in the Proposition 5. Then, the mapping $R\colon \mathbf{T}\mathscr{S}_{n,k} \to \mathscr{S}_{n,k}$ whose restriction to $\mathbf{T}_S\mathscr{S}_{n,k}$ is defined by $R_S(V) = e^X S e^\Omega$ is a retraction on the Stiefel manifold, and $\beta\colon t \mapsto e^{tX} S e^{t\Omega}$ is a quasi-geodesic in $\mathscr{S}_{n,k}$ that satisfies*

1. $\beta(0) = S$;
2. $\dot{\beta}(t) = e^{tX}(XS + S\Omega)e^{t\Omega}$;
3. $\ddot{\beta}(t) = e^{tX}(X^2S + 2XS\Omega + S\Omega^2)e^{t\Omega}$.

Proof. This statement is true because the mapping R satisfies both conditions of the Definition 4 and β is the quasi-geodesic associated to the retraction. The formulas for the derivatives of β are also straightforward. □

3.1. Joining points in the Stiefel manifold by quasi-geodesics

Given two distinct points S_1 and S_2 in the Stiefel manifold, our objective now is to choose $X \in \mathfrak{so}_{P_1}(n)$ ($P_1 = S_1 S_1^{\mathrm{T}}$) and $\Omega \in \mathfrak{so}(k)$ so that the quasi-geodesic defined by $\beta(t) = e^{tX} S_1 e^{t\Omega}$ joins the point S_1 (at $t = 0$) to the point S_2 (at $t = 1$).

Theorem 7. *Let S_1 and S_2 be two distinct points in $\mathscr{S}_{n,k}$ so that, for $i = 1, 2$, $P_i = S_i S_i^{\mathrm{T}} \in \mathscr{G}_{n,k}$. Then, if*

$$X = \frac{1}{2} \log\big((\mathbf{I} - 2S_2 S_2^{\mathrm{T}})(\mathbf{I} - 2S_1 S_1^{\mathrm{T}})\big) \quad \textit{and} \quad \Omega = \log\big(S_1^{\mathrm{T}} e^{-X} S_2\big), \tag{6}$$

the quasi-geodesic $\beta\colon [0,1] \to \mathscr{S}_{n,k}$ defined by

$$\beta(t) := e^{tX} S_1 e^{t\Omega}, \tag{7}$$

has the following properties:

1. $\beta(0) = S_1$;
2. $\beta(1) = S_2$;
3. $\|\dot\beta(t)\|^2 = -\operatorname{tr}\big(S_1^{\mathrm{T}} X^2 S_1 + \frac{1}{2}\Omega^2\big)$ *(constant speed);*
4. $L_0^1(\beta) = \sqrt{-\operatorname{tr}\big(S_1^{\mathrm{T}} X^2 S_1 + \frac{1}{2}\Omega^2\big)}$ *(length of β);*
5. $D_t\dot\beta(t) = X\beta(t)\Omega$;
6. $\|D_t\dot\beta(t)\|^2 = \operatorname{tr}\big(\Omega^2 S_1^{\mathrm{T}} X^2 S_1\big)$ *(constant covariant acceleration).*

Proof. Before starting the proof, we show that X and Ω agree with the parametrization of the tangent space given in Proposition 5. According to (1), $[X, P_1]$ is the initial velocity vector of the geodesic in the Graßmann manifold that joins the point P_1 (at $t = 0$) to P_2 (at $t = 1$), so $X \in \mathfrak{so}_{P_1}(n)$. Moreover, $S_1^{\mathrm{T}} e^{-X} S_2$ is orthogonal as can be easily checked. Indeed, from the expression for X we immediately get

$$\big(S_1^{\mathrm{T}} e^{-X} S_2\big)\big(S_1^{\mathrm{T}} e^{-X} S_2\big)^{\mathrm{T}} = S_1^{\mathrm{T}} e^{-X} S_2 S_2^{\mathrm{T}} e^{X} S_1 = S_1^{\mathrm{T}} S_1 S_1^{\mathrm{T}} S_1 = \mathbf{I}_k;$$

$$\big(S_1^{\mathrm{T}} e^{-X} S_2\big)^{\mathrm{T}}\big(S_1^{\mathrm{T}} e^{-X} S_2\big) = S_2^{\mathrm{T}} e^{X} S_1 S_1^{\mathrm{T}} e^{-X} S_2 = S_2^{\mathrm{T}} S_2 S_2^{\mathrm{T}} S_2 = \mathbf{I}_k.$$

Now, the first two properties follow from the definition of the curve β. To simplify notations we omit the dependency on t. With the canonical metric on the Stiefel manifold, we can write

$$\|\dot\beta\|^2 = \langle\dot\beta, \dot\beta\rangle = \operatorname{tr}\Big(\dot\beta^{\mathrm{T}}(\mathbf{I} - \tfrac{1}{2}\beta\beta^{\mathrm{T}})\dot\beta\Big) = \operatorname{tr}\Big(\dot\beta^{\mathrm{T}}\dot\beta - \tfrac{1}{2}\dot\beta^{\mathrm{T}}\beta\beta^{\mathrm{T}}\dot\beta\Big)$$

But

$$\dot\beta^{\mathrm{T}}\dot\beta = -e^{-t\Omega} S_1^{\mathrm{T}} X^2 S_1 e^{t\Omega} - \Omega^2,$$
$$\beta^{\mathrm{T}}\dot\beta = \Omega, \quad \text{hence} \quad \dot\beta^{\mathrm{T}}\beta = -\Omega.$$

So

$$\begin{aligned}\|\dot{\beta}\|^2 &= -\operatorname{tr}\big(S_1^{\mathrm{T}} X^2 S_1\big) - \operatorname{tr}\big(\Omega^2\big) + \tfrac{1}{2}\operatorname{tr}\big(\Omega^2\big) \\ &= -\operatorname{tr}\big(S_1^{\mathrm{T}} X^2 S_1 + \tfrac{1}{2}\Omega^2\big).\end{aligned}$$

This concludes the proof of 3.. The property 4. follows immediately from the formula $L_0^1(\beta) = \int_0^1 \|\dot{\beta}(t)\|\, dt$ and the property 3.. To prove property 5., we take into consideration the formulas for $\dot{\beta}$ and $\ddot{\beta}$ in Proposition 6 and the following formula for the covariant derivative of $\dot{\beta}$ along β, given in [2]:

$$D_t\dot{\beta} = \ddot{\beta} + \dot{\beta}\dot{\beta}^{\mathrm{T}}\beta + \beta\Big((\beta^{\mathrm{T}}\dot{\beta})^2 + \dot{\beta}^{\mathrm{T}}\dot{\beta}\Big).$$

Using the properties in Proposition 3, this can be simplified to obtain $D_t\dot{\beta}(t) = X\beta(t)\Omega$. Finally, we show that the acceleration vector field along β has constant length. This requires some tedious calculations that are partially omitted because simplifications only require properties that have already been used before.

$$\begin{aligned}\|D_t\dot{\beta}\|^2 &= \operatorname{tr}\Big((D_t\dot{\beta})^{\mathrm{T}}(\mathbf{I} - \tfrac{1}{2}\beta\beta^{\mathrm{T}})(D_t\dot{\beta})\Big) \\ &= \operatorname{tr}\big(\Omega\beta^{\mathrm{T}} X (X\beta\Omega - \tfrac{1}{2}\beta\beta^{\mathrm{T}} X\beta\Omega)\big) \\ &= \operatorname{tr}\big(\Omega S_1^{\mathrm{T}} X^2 S_1 \Omega\big) = \operatorname{tr}\big(\Omega^2 S_1^{\mathrm{T}} X^2 S_1\big).\end{aligned}$$

This completes the proof. □

Remark 8. *It is immediate to check that if $\beta(t) := e^{tX} S_1 e^{t\Omega}$ is the quasi-geodesic given in Theorem 7, which joins S_1 (at $t = 0$) to S_2 (at $t = 1$), then $\overline{\beta}(t) := \beta(1-t) = e^{-tX} S_2 e^{-t\Omega}$ is the reversed quasi-geodesic that joins S_2 (at $t = 0$) to S_1 (at $t = 1$). This quasi-geodesic is associated with the retraction $R_{S_2}(-V)$, where $V = XS_2 - S_2\Omega$.*

Remark 9. *Some restrictions on S_1 and S_2 must be imposed so that the principal logarithm of the orthogonal matrices $(\mathbf{I} - 2S_2S_2^{\mathrm{T}})(\mathbf{I} - 2S_1S_1^{\mathrm{T}})$ and $S_1^{\mathrm{T}} e^{-X} S_2$ exists and is unique. As already mentioned in Subsection 2.1, if these matrices have no eigenvalues in $\mathbb{R}^-$, the quasi-geodesic joining S_1 to S_2 exists and is unique.*

The numerical computation of a quasi-geodesic only requires the computation of matrix logarithms of orthogonal matrices and exponentials of skew-symmetric matrices. An overview of numerical methods to compute these matrix functions is available, for instance, in Higham [14], where numerical stability and computational costs are also analysed. The problem of computing geodesics in the Stiefel manifold is also briefly addressed in Rentmeesters [19], but only for the case of a minimal geodesic joining S_1 to a point in the orbit of S_2, that is, to a frame that generates the same subspace as the frame S_2, not necessarily S_2. Using our approach, this corresponds to the case when $\Omega = 0$ and the quasi-geodesic reduces to a geodesic, as will be better explained after this remark. Our results open the door to further research on finding algorithms to compute efficiently geodesics and quasi-geodesics associated to the canonical metric, but this is not the goal of the present paper.

There are two situations when the quasi-geodesic defined in (7) is a true geodesic, as will be detailed in the next Corollary. Figures 1 and 2 illustrate these situations. For the situation in Figure 3, no explicit form for the geodesic joining the frames S_1 and S_2 is known. In this case, what can be easily exhibited is a two-piece broken geodesic joining those frames, each piece being a geodesic of one of the two types illustrated in Figures 1 and 2.

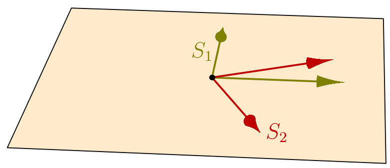

Figure 1: The frames S_1 and S_2 span the same subspace

Corollary 10.

1. *If the frames S_1 and S_2 generate the same subspace, then the curve defined in* (7) *is a geodesic in $\mathscr{S}_{n,k}$ joining S_1 to S_2. This geodesic, denoted by β_1, has length equal to $\sqrt{-\frac{1}{2}\operatorname{tr}(\Omega^2)}$.*
2. *If the frames S_1 and S_2 do not generate the same subspace but the frame $e^X S_1$, where $X = \frac{1}{2}\log\big((\mathbf{I} - 2S_2S_2^{\mathrm{T}})(\mathbf{I} - 2S_1S_1^{\mathrm{T}})\big)$, coincides with S_2, then the curve defined in* (7) *is also a geodesic in $\mathscr{S}_{n,k}$ joining S_1 to S_2. This geodesic, denoted by β_2, has length equal to $\sqrt{-\operatorname{tr}(S_1^{\mathrm{T}}X^2S_1)}$.*

Proof. In the first case $X = 0$ and $\beta_1(t) = S_1e^{t\Omega}$. In the second case $\Omega = 0$ and $\beta_2(t) = e^{tX}S_1$. Clearly both curves are geodesics because, according to Theorem 7, item 5., $D_t\dot{\beta}_i(t) \equiv 0$, for $i = 1, 2$. The formulas for the length of these geodesics follow from Theorem 7, item 4.. $\square$

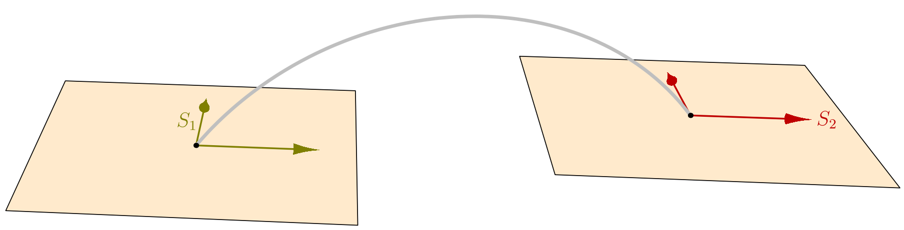

Figure 2: The frames S_1 and S_2 span two different subspaces, but $e^X S_1 = S_2$

Remark 11. *For the two extreme cases when $k = 1$ and $k = n$, the Stiefel manifold becomes respectively a sphere and an orthogonal group. The quasi-geodesics are then true geodesics since both cases fit into the two exceptions in Corollary 10. Indeed, $\Omega = 0$ if $k = 1$ and $X = 0$ if $k = n$. In the later case, $\Omega = \log(S_1^{\mathrm{T}}S_2) \in \mathfrak{so}(n)$.*

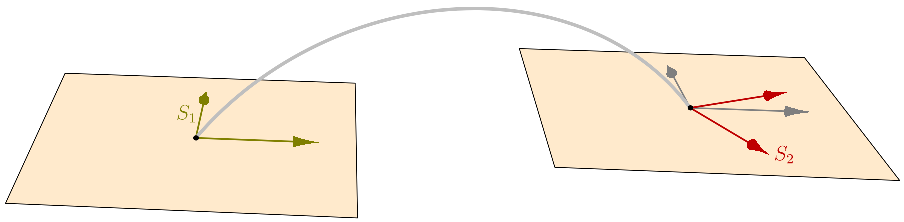

Figure 3: The frames S_1 and S_2 span two different subspaces, but $e^X S_1 \neq S_2$

Proposition 12. *The geodesic curvature κ of the quasi-geodesic defined in* (7) *is constant and given by*

$$\kappa = -\frac{\sqrt{\operatorname{tr}\left(\Omega^2 S_1^{\mathrm{T}} X^2 S_1\right)}}{\operatorname{tr}\left(S_1^{\mathrm{T}} X^2 S_1 + \frac{1}{2}\Omega^2\right)}.$$

Moreover, $0 \leq \kappa < 1$.

Proof. We use the following formula, obtained from Lee [20, p. 137], for computing the geodesic curvature

$$\kappa = \frac{\|D_t \dot{\beta}\|}{\|\dot{\beta}\|^2} - \frac{\langle D_t \dot{\beta}, \dot{\beta} \rangle}{\|\dot{\beta}\|^3}. \tag{8}$$

Since β has constant speed, the second term in (8) vanishes and the expression for the geodesic curvature reduces to the first term that is immediately obtained from the formulas in Theorem 7.

To show that $0 \leq \kappa < 1$, we use a trace inequality due to von Neumann (see, for instance, von Neumann [21] or Marques de Sá [22]), which states that for any $k \times k$ complex matrices A and B with singular values $a_1 \geq a_2 \geq \cdots \geq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_k$ respectively, $|\operatorname{tr}(AB)| \leq \sum_i a_i b_i$. If we consider $A = -S_1^{\mathrm{T}} X^2 S_1$ and $B = -\Omega^2$ which are real symmetric and nonnegative definite, their singular values coincide with their eigenvalues and so $\operatorname{tr}(AB) \leq \sum_i a_i b_i$. Consequently,

$$\begin{aligned}
\kappa^2 = \frac{\operatorname{tr}(AB)}{\operatorname{tr}^2(A + \frac{1}{2}B)} &= \frac{\operatorname{tr}(AB)}{\operatorname{tr}^2(A) + \frac{1}{4}\operatorname{tr}^2(B) + \operatorname{tr}(A)\operatorname{tr}(B)} \\
&\leq \frac{\sum_i (a_i b_i)}{(\sum_i a_i)^2 + \frac{1}{4}(\sum_i b_i)^2 + \sum_{i \neq j}(a_i b_j) + \sum_i (a_i b_i)}.
\end{aligned}$$

Since the eigenvalues of A and B are nonnegative and not simultaneously equal to zero and κ is nonnegative, the geodesic curvature has the required bounds. □

Note that the geodesic curvature is zero whenever A or B is zero, that is, when $X = 0$ or $\Omega = 0$. As expected, this result is consistent with the statements in Corollary 10.

3.2. Comparison results

In this subsection we make some comparison between geodesics and the quasi-geodesics defined by (7).

1. The quasi-geodesic $\beta(t) = e^{tX} S_1 e^{t\Omega}$ is not a geodesic unless $X = 0$ or $\Omega = 0$.

   This follows from Theorem 7, item 5..

2. If $X \neq 0$ and $\Omega \neq 0$, the points S_1 and S_2 can also be joined by a broken geodesic, which is the concatenation of two geodesics, the first one defined by $\beta_1(t) = e^{tX} S_1$ and the second one defined by $\beta_2(t) = e^X S_1 e^{t\Omega}$. Moreover, these two geodesics intersect at a right angle, i.e., $\langle \dot\beta_1(1), \dot\beta_2(0)\rangle = 0$.

   This can be easily checked, since $\beta_1(0) = S_1$, $\beta_1(1) = \beta_2(0) = e^X S_1$, and $\beta_2(1) = e^X S_1 e^{\Omega} = S_2$. Also, $\dot\beta_1^{\mathrm{T}}(1) = -s_1^{\mathrm{T}} X e^{-X}$ and $\dot\beta_2(0) = e^X S_1 \Omega$. So, using formula (3) and Proposition 3, it follows after simplification

$$
\begin{aligned}
\langle \dot\beta_1(1), \dot\beta_2(0)\rangle &= \operatorname{tr}\left(\dot\beta_1^{\mathrm{T}}(1)\dot\beta_2(0)\right) - \frac{1}{2}\operatorname{tr}\left(\dot\beta_1^{\mathrm{T}}(1) e^X S_1 S_1^{\mathrm{T}} e^{-X} \dot\beta_2(0)\right) \\
&= \frac{1}{2}\operatorname{tr}(-S_1^{\mathrm{T}} X S_1) = 0.
\end{aligned}
$$

3. In the previous situation, i.e., when $X \neq 0$ and $\Omega \neq 0$, we have the following relationship between the length of the quasi-geodesic β joining the points S_1 and S_2 and the length of the broken geodesic joining the same points:

$$
\left(L_0^1(\beta)\right)^2 = \left(L_0^1(\beta_1)\right)^2 + \left(L_0^1(\beta_2)\right)^2 .
$$

   So, we can conclude that the quasi geodesic is shorter than the broken geodesic.

   This follows immediately from the formulas for the lengths in Theorem 7, item 4. and in Corollary 10. (Note that the previous equality in Euclidean geometry is the Pythagorean theorem and in this geometry the quasi-geodesic would have to be a geodesic.)

4. The projection ψ of the quasi-geodesic $\beta(t) = e^{tX} S_1 e^{t\Omega}$ on the Graßmann manifold is a geodesic joining $P_1 = S_1 S_1^{\mathrm{T}}$ to $P_2 = S_2 S_2^{\mathrm{T}}$.

   This follows immediately from $\beta(t)\beta^{\mathrm{T}}(t) = e^{tX} S_1 S_1^{\mathrm{T}} e^{t\Omega} = e^{tX} P_1^{\mathrm{T}} e^{t\Omega}$ and the fact that $X = \frac{1}{2}\log((\mathbf{I} - 2P_2)(\mathbf{I} - 2P_1))$.

5. The projection ψ on the Graßmann manifold of a geodesic in the Stiefel manifold is not necessarily a geodesic.

We justify this statement with an example. According to Subsection 2.2, the curve defined by

$$\gamma(t) = e^{tX}\Delta, \quad \text{with} \quad X = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{and} \quad A \in \mathfrak{so}(k)$$

is a geodesic in the Stiefel manifold starting at Δ. However, its projection on the Graßmann manifold is the curve $\gamma(t)\gamma^{\mathrm{T}}(t) = e^{tX}\Delta\Delta^{\mathrm{T}}e^{-tX} = e^{tX}\Lambda e^{-tX}$, which is not a geodesic since $X\Lambda + \Lambda X = 2X$, that is, X doesn't belong to $\mathfrak{so}_\Lambda(n)$.

4. The Casteljau Algorithm on manifolds

The classical Casteljau algorithm, introduced independently by De Casteljau [8] and Bézier [9], is a geometric construction to generate polynomial curves in $\mathbb{R}^n$ based on successive linear interpolation techniques. After the basic idea of Park and Ravani [4] of replacing linear interpolation by geodesic interpolation, the Casteljau algorithm was generalized to accommodate geometric polynomial curves and also interpolating splines on Riemannian manifolds (see, for instance, the work of Crouch et al. [5], Popiel and Noakes [6], Nava-Yazdani and Polthier [7], and Nava-Yazdani [23]). We next give a succinct description of this algorithm for generating polynomials of degree m on a complete Riemannian manifold $\mathbf{M}$, where for the sake of simplicity we parametrize the curves on the $[0,1]$ interval.

If $x_0, \ldots, x_m$ are distinct points in $\mathbf{M}$ and $\sigma_1(t, x_i, x_{i+1})$ is the geodesic arc joining x_i (at $t=0$) to x_{i+1} (at $t=1$), a smooth curve $t \mapsto \sigma_m(t)$, joining x_0 (at $t=0$) to x_m (at $t=1$), may be constructed by recursive geodesic interpolation and depends on the given points. The curves produced by this recursive process which involves m steps are defined by

$$\sigma_k(t, x_i, \ldots, x_{i+k}) = \sigma_1\left(t, \sigma_{k-1}(t, x_i, \ldots, x_{i+k-1}), \sigma_{k-1}(t, x_{i+1}, \ldots, x_{i+k})\right), \qquad k = 2, \ldots, m; \quad i = 0, \ldots, m-k.$$

The curve σ_m obtained in the last step, that is $\sigma_m(t) := \sigma_m(t, x_0, \ldots, x_m)$, generalizes the Euclidean polynomials of degree m and is called *geometric polynomial* in $\mathbf{M}$. This curve doesn't interpolate the points $x_1, \ldots, x_{m-1}$. They are only used to generate the curve that joins x_0 to x_m but, of course, influence the shape of the curve. For that reason, they are called *control points*. Alternatively, one can prescribe other boundary conditions, such as $m-1$ initial conditions $D_t^k\dot{\sigma}_m(0)$, $k = 0, \ldots, m-2$, and compute from them the control points needed for the algorithm. This is theoretically possible, but the complexity of the computations increases significantly with m. This algorithm can also be used to generate *geometric polynomial splines*, which are interpolating curves obtained by piecing together several geometric polynomials in a smooth manner.

Figure 4 illustrates the idea behind the Casteljau algorithm in the 2-sphere. It shows how to generate several points (points in red) of a quadratic curve.

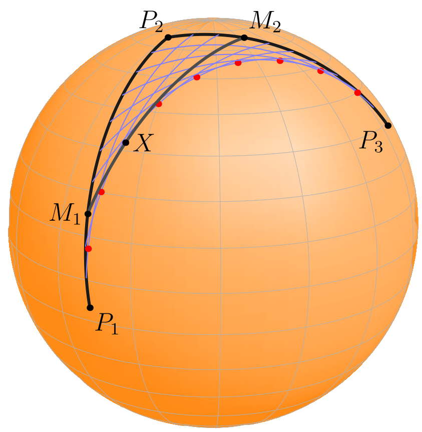

Figure 4: Illustration of the Casteljau algorithm to generate a quadratic polynomial in S^2

4.1. The Casteljau algorithm on Graßmann manifolds

The generalisation of the classical Casteljau algorithm will be used to solve the following problem on the Graßmann manifolds.

Problem 1. *Given a set of points $P_0, \ldots, P_m$ in the Graßmann manifold $\mathcal{G}_{n,k}$ and $\Omega_0 \in \mathfrak{so}_{P_0}(n)$, find a $\mathcal{C}^1$-smooth curve σ that interpolates the points P_i at time i and has initial velocity equal to $[\Omega_0, P_0]$, that is*

$$\sigma(i) = P_i, \quad \textit{for} \quad i = 0, 1, \ldots, m \quad \textit{and} \quad \dot{\sigma}(0) = [\Omega_0, P_0].$$

4.1.1. Solving Problem 1 using the Casteljau algorithm

Since the Casteljau algorithm is based on geodesic interpolation, its implementation on a specific Riemannian manifold requires that an explicit formula for the geodesic that joins two points is available. Although this is not the case in general, it happens that for the Graßmann manifold such a formula has been derived recently in Batzies et al. [1]. This is the explicit formula (1) included in Section 2.

The interpolating curve σ may be generated by piecing together quadratic polynomials defined on each subinterval $[i, i+1]$, and joining P_i to P_{i+1} with control point C_i, that is

$$\sigma(t)|_{[i,i+1]} = \sigma_2(t - i, P_i, C_i, P_{i+1}).$$

The first control point C_0 is computed from P_0, P_1 and Ω_0, Figure 5. In order to ensure that σ is $\mathcal{C}^1$-smooth, the initial velocity of each subsequent spline segment must equal the final velocity of the previous segment.

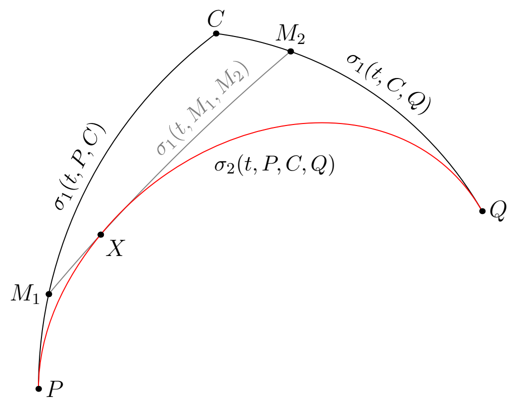

Figure 5: The generalised Casteljau algorithm; segments $\sigma_1(t, P, C)$ and $\sigma_1(t, C, Q)$ form the first step of the algorithm. Then, given $t_0 \in [0, 1]$, $X = \sigma_1(t_0, M_1, M_2)$ is a point in the spline, where $M_1 = \sigma_1(t_0, P, C)$ and $M_2 = \sigma_1(t_0, C, Q)$

4.1.2. Generating a second order spline

Given a sequence of data points $P_0, P_1, \ldots, P_m$ and an initial Ω_0, the algorithm produces a second order spline $\sigma\colon [0, m] \to \boldsymbol{\mathscr{G}}_{n,k}$, passing through the data points, such that

$$\sigma(i) = P_i, \quad \text{for} \quad 0 \le i \le m \quad \text{and} \quad \dot\sigma(0) = [\Omega_0, P_0].$$

Each segment $\sigma([i, i+1])$ joins P_i to P_{i+1}. The algorithm is based on a general version of the Casteljau algorithm described at the beginning of the section.

To find a point $\sigma(t)$ of the spline, first iterate Algorithm 2 to find the component of initial velocity vector Ω_i, for the segment $\sigma([i, i+1])$, where $t \in [i, i+1]$. Then with the triple Ω_i, P_i and P_{i+1}, apply Algorithm 1 to get the desired point.

5. A modification of the Casteljau algorithm

Although the Casteljau algorithm appeared as a geometric tool to construct polynomials of any order by successive linear interpolation, it can be modified to accommodate curves with other properties. This has been done, for instance, in Jakubiak et al. [24] and Rodrigues et al. [25]. On manifolds where explicit formulas for geodesics are not available, this is particularly useful and will also be used to generate a $\mathcal{C}^1$ interpolating curve on the Stiefel manifold in the next section. For now we proceed with some generic results for curves obtained with only one control point and two steps and first prove a result involving the initial and final velocity of a curve generated with two steps, but not necessarily a quadratic polynomial.

Given a set of three points $\{\, x_i \,\}_{i=0}^{2}$ in a manifold $\mathbf{M}$, let $t \mapsto \sigma_1(t, x_i, x_{i+1})$ be curves joining x_i to x_{i+1}, for $i = 0, 1$. Define a family of curves $\gamma\colon [0, 1] \times [0, 1] \to$

Algorithm 1: calculate a point $\sigma(t)$, for $t \in [0,1]$, such that: $\sigma(0) = P$, $\sigma(1) = Q$, and $\dot{\sigma}(0) = 2[\Omega, P]$ *(note that C and Ω do not depend on t and can be pre-computed to improve the efficiency)*

Input: $t \in [0,1]$, $P, Q \in \mathscr{G}_{n,k}$, $\Omega \in \mathfrak{so}_P(n)$
Output: $X = \sigma(t)$

1 Calculate control point C:

$$C = \exp(\Omega) \cdot P \cdot \exp(-\Omega)$$

Calculate first step end points M_1 and M_2:

$$M_1 = \exp(t\Omega) \cdot P \cdot \exp(-t\Omega)$$
$$\Theta_0 = \frac{1}{2} \log\big((\mathbf{I} - 2\,Q) \cdot (\mathbf{I} - 2\,C)\big)$$
$$M_2 = \exp(t\Theta_0) \cdot C \cdot \exp(-t\Theta_0)$$

Compute the point on the geodesic from M_1 to M_2 at t:

$$\Theta_1 = \frac{1}{2} \log\big((\mathbf{I} - 2\,M_2) \cdot (\mathbf{I} - 2\,M_1)\big)$$
$$X = \exp(t\Theta_1) \cdot M_1 \cdot \exp(-t\Theta_1)$$

return X

Algorithm 2: calculate $\widetilde{\Omega}$ so that the final velocity $\dot{\sigma}_1(1, C, Q) = [\widetilde{\Omega}, Q]$, where σ satisfies: $\sigma(0) = P$, $\sigma(1) = Q$, and $\dot{\sigma}_1(0, P, C) = [\Omega, P]$ *(note that to improve the efficiency, all quantities can be computed in advance, once this data is known)*

Input: $P, Q \in \mathscr{G}_{n,k}$, $\Omega \in \mathfrak{so}_P(n)$
Output: $\widetilde{\Omega}$ such that $\dot{\sigma}(1) = [\widetilde{\Omega}, Q]$

1 Calculate control point C:

$$C = \exp(\Omega) \cdot P \cdot \exp(-\Omega)$$

Calculate a component of the initial velocity vector for a geodesic from C to Q:

$$\widetilde{\Omega} = \frac{1}{2} \log\big((\mathbf{I} - 2\,Q) \cdot (\mathbf{I} - 2\,C)\big)$$

return $\widetilde{\Omega}$

M as follows. For a fixed $t_0 \in [0,1]$, the map $t \mapsto \gamma(t, t_0)$ is a curve joining $\sigma_1(t_0, x_0, x_1)$ to $\sigma_1(t_0, x_1, x_2)$, as illustrated in Figure 6. Then $\sigma_2 \colon [0,1] \to \mathbf{M}$ given by $\sigma_2(t) = \gamma(t,t)$ is a curve joining x_0 to x_2. If the curves considered above are geodesics then σ_2 is a second order polynomial.

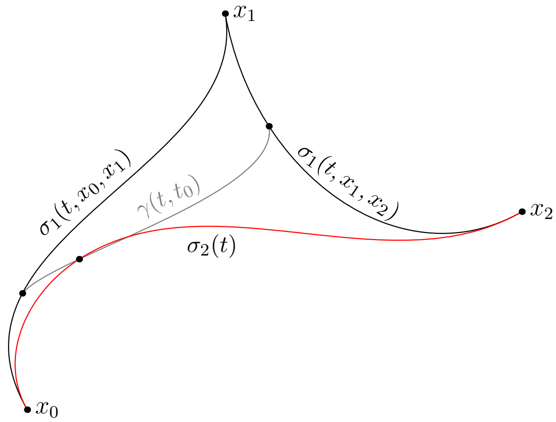

Figure 6: A modification of the two-step Casteljau algorithm

We are interested in finding out how the velocities of σ_2 at the end points are related with the velocities of the curves used along the algorithm steps. The answer is given in the following proposition.

Proposition 13. *Suppose that the curves σ_1 are differentiable and that*

$$\gamma(t,0) = \sigma_1(t, x_0, x_1) \quad \textit{and} \quad \gamma(t,1) = \sigma_1(t, x_1, x_2). \tag{9}$$

Then

$$\dot\sigma_2(0) = 2\dot\sigma_1(0, x_0, x_1) \quad \textit{and} \quad \dot\sigma_2(1) = 2\dot\sigma_1(1, x_1, x_2).$$

Proof. The following identities follow from the definition of the curve γ:

$$\gamma(0,t) = \sigma_1(t, x_0, x_1) \quad \text{and} \quad \gamma(1,t) = \sigma_1(t, x_1, x_2). \tag{10}$$

Note that

$$\dot\sigma_2(t) = \left.\frac{\partial}{\partial s}\right|_{s=t} \gamma(s,t) + \left.\frac{\partial}{\partial s}\right|_{s=t} \gamma(t,s).$$

Therefore, from the hypothesis (9) and by identities (10) it follows that

$$\dot\sigma_2(0) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(s,0) + \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(0,s) = 2\dot\sigma_1(0, x_0, x_1).$$

Similarly

$$\dot\sigma_2(1) = \left.\frac{\partial}{\partial s}\right|_{s=1} \gamma(s,1) + \left.\frac{\partial}{\partial s}\right|_{s=1} \gamma(1,s) = 2\dot\sigma_1(1, x_1, x_2).$$

$\square$

This result easily generalizes to higher order curves.

6. Solving Interpolation problems on Stiefel manifolds

For interpolation problems where orientation is irrelevant, one could simply interpolate points on the Graßmann using the Casteljau algorithm described in Section 4. This would be the case if a frame and a rotated version of that same frame produce equivalent visual scenes. However, when orientation matters, if the interpolating data on the Stiefel manifold is projected on the Graßmann manifold and the interpolating problem is solved at the level of subspaces, there is no guarantee that lifting the solution to the level of frames will produce a reasonable solution. Most likely, the smoothness conditions will not be met in the final curve. This main drawback is due to the fact that all frames $S\Theta$, with $\Theta \in \mathbb{SO}(k)$, generate the same subspace as the frame S, that is, the projection from Stiefel to Graßmann is many to one. Yet another situation in which orientation is irrelevant arises in certain optimisation problems on the Stiefel manifold where the objective function is invariant with respect to right multiplication by elements in $\mathbb{SO}(k)$.

The objective of this section is to present an alternative to generate a smooth interpolation curve on the Stiefel manifold which is intrinsic to this manifold and results from replacing geodesics in the Casteljau algorithm by the quasi-geodesics introduced in Section 3. This procedure may be extended to other more general interpolation problems with the appropriate adaptations.

Problem 2. *Given a set of points* $\{ S_i \}_{i=0}^m$ *belonging to the Stiefel manifold* $\mathscr{S}_{n,k}$*, and a vector* $V_0 \in T_{S_0}\mathscr{S}_{n,k}$*, find a* $\mathcal{C}^1$ *interpolating curve passing through these points and having initial velocity equal to* V_0*.*

6.1. Solving the interpolation problem using quasi-geodesics

Since we know how to join two points on the Stiefel manifold by a quasi-geodesic, we use these curves to perform a modified Casteljau algorithm where successive linear interpolation is replaced by successive quasi-linear interpolation.

The crucial procedure is the generation of the first curve segment, joining S_0 to S_1 and having prescribed initial velocity equal to V_0. Without loss of generality, we assume that all segments are parameterised in the $[0, 1]$ time interval.

6.2. Generating the first spline segment

First we need to find a control point C_0, which is the end point of the quasi-geodesic that starts at the point S_0 with initial velocity equal to $\frac{1}{2}V_0$. This quasi-geodesic is given by

$$\beta_0(t) = e^{tX_0} S_0 e^{t\Omega_0},$$

where, according to (5) in Proposition 5,

$$X_0 = \frac{1}{2}V_0S_0^{\mathrm{T}} - \frac{1}{2}S_0V_0^{\mathrm{T}} + S_0V_0^{\mathrm{T}}S_0S_0^{\mathrm{T}} \quad \text{and} \quad \Omega_0 = \frac{1}{2}S_0^{\mathrm{T}}V_0. \tag{11}$$

So, $C_0 = e^{X_0} S_0 e^{\Omega_0}$ defines the control point.

We now proceed to the construction of the second quasi-geodesic β_1 that joins C_0 to S_1, using Theorem 7 with the obvious adaptations. That is,

$$\beta_1(t) = e^{tX_1} C_0 e^{t\Omega_1},$$

where

$$X_1 = \frac{1}{2}\log\big((\mathbf{I} - 2S_1S_1^{\mathrm{T}}) \cdot (\mathbf{I} - 2C_0C_0^{\mathrm{T}})\big) \quad \text{and} \quad \Omega_1 = \log(C_0^{\mathrm{T}} \cdot e^{-X_1} \cdot S_1).$$

The first curve segment, joining S_0 to S_1 with prescribed initial velocity equal to V_0 can now be obtained from quasi-linear interpolation of β_0 and β_1. More precisely, we can state the following result for the first spline segment.

Theorem 14. *The curve γ in the Stiefel manifold, defined by*

$$\gamma(t) = e^{tX(t)} \beta_0(t) e^{t\Omega(t)},$$

where

$$X(t) = \tfrac{1}{2}\log\big((\mathbf{I} - 2\beta_1(t)\beta_1^{\mathrm{T}}(t)) \cdot (\mathbf{I} - 2\beta_0(t)\beta_0^{\mathrm{T}}(t))\big),$$

$$\Omega(t) = \log(\beta_0^{\mathrm{T}}(t) \cdot e^{-X(t)} \cdot \beta_1(t)),$$

satisfies the boundary conditions

$$\gamma(0) = S_0, \quad \gamma(1) = S_1, \quad \dot\gamma(0) = V_0,$$

so it solves Problem 2 when $m = 1$. Moreover, $\dot\gamma(1) = 2(X_1S_1 + S_1\Omega_1)$.

Proof. Clearly, $\gamma(0) = \beta_0(0) = S_0$. Since $X(1) = \frac{1}{2}\log\big((\mathbf{I} - 2S_1S_1^{\mathrm{T}}) \cdot (\mathbf{I} - 2C_0C_0^{\mathrm{T}})\big) = X_1$ and $\Omega(1) = \log(C_0^{\mathrm{T}} \cdot e^{-X_1} \cdot S_1) = \Omega_1$, then $\gamma(1) = e^{X_1}C_0e^{\Omega_1} = e^{X_1}C_0C_0^{\mathrm{T}}e^{-X_1}S_1 = S_1S_1^{\mathrm{T}}S_1 = S_1$. Finally, for the initial velocity, notice that γ is a particular case of the curve described in Section 5 for which Proposition 13 holds. As a consequence, $\dot\gamma(0) = 2\dot\beta_0(0) = 2(X_0S_0 + S_0\Omega_0)$. Now it is enough to use (11) to write X_0 and Ω_0 in terms of V_0 and S_0, and the condition $V_0^{\mathrm{T}}S_0 = -S_0^{\mathrm{T}}V_0$ for a vector $V_0 \in \mathbf{T}_{S_0}\mathscr{S}_{n,k}$, to obtain, after simplifications, $\dot\gamma(0) = V_0$. Finally, the expression for $\dot\gamma(1)$ also follows from Proposition 13. $\square$

6.3. Generating consecutive spline segments:

After having generated the first spline segment, one continues in a similar way for the second spline segment. Since the interpolating curve is required to be $\mathcal{C}^1$-smooth, the initial velocity for this second curve segment must equal the end velocity of the previous curve segment, which is known. So, we repeat the steps in the proof of last theorem for the boundary data S_1, S_2 and $V_1 = 2(X_1S_1 + S_1\Omega_1)$ instead of S_0, S_1 and $V_0 = 2(X_0S_0 + S_0\Omega_0)$. The control point is now $C_1 = e^{X_1}S_1e^{\Omega_1}$.

The other $m-2$ consecutive segments are generated similarly. The solution of Problem 2 is the spline curve resulting from the concatenation of the m consecutive segments.

Remark 15. *The mapping $\psi\colon \mathscr{S}_{n,k} \to \mathscr{G}_{n,k}$ defined by $\psi(S) := SS^{\mathrm{T}}$ transforms quasi-geodesics on the Stiefel manifold into geodesics on the Graßmann manifold. As a consequence, the algorithm presented in this section projects on the Graßmann manifold $\mathscr{G}_{n,k}$ as the true Casteljau algorithm presented in Subsection 4.1.*

7. Illustrative examples

In this section we include the results of two simple experiments in order to illustrate the importance of the interpolating method proposed in the last section. In order to understand the results of these experiments we need to explain how data can be represented as points in the Stiefel or in the Graßmann manifolds.

Figure 7 below represents the set of $3D$ points that draw the number 8 and the number 3 in $3D$-space, in five different configurations. In each one of the configurations, the representation of the number varies not just in orientation but also on scale and shape. Each configuration is represented by a $n \times m$ matrix X_i, where $i = 1, \cdots, 5$, $n = 3$, and $m = 200$. Non-centered principal

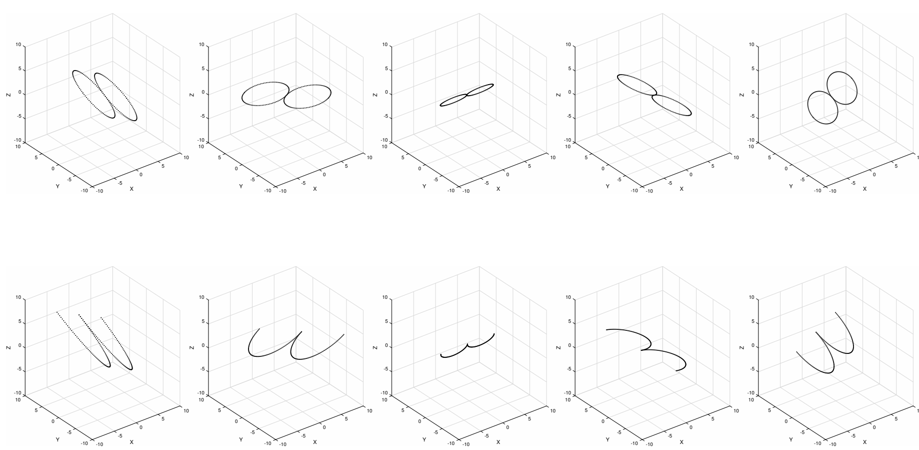

Figure 7: Five representations of the number 8 (top row) and of the number 3 (bottom row) in $3D$-space.

component analysis (PCA) is applied to all matrices X_i, producing i orthogonal matrices Θ_i in $\mathbb{SO}(3)$. By projecting each Θ^i on the Stiefel manifold $\mathscr{S}_{3,2}$, we produce i matrices $S_i = \pi(\Theta^i) = \Theta^i\Delta$, representing i orthonormal $2D$-frames in $3D$-space. Each one of these frames generates a subspace of dimension 2, that is, a point in $\mathscr{G}_{3,2}$. After applying the algorithm presented in the previous section we select K frames or subspaces by sampling the interpolating curves. To illustrate the smoothness of the interpolating curve on the Stiefel manifold, the $3D$ samples of one of the initial configurations X_i are projected in the K sampled frames through $X_j = S_j^{\mathrm{T}}\ X_i, j = 1, \dots K$.

Instead of interpolating the points S_i on the Stiefel manifold we could first project them on the Graßmann manifold, through $\psi(S_i) = S_i S_i^{\mathrm{T}} = P_i$, and then generate the smooth curve on $\mathscr{G}_{3,2}$ interpolating the points (subspaces) P_i. In order to lift this curve to $\mathscr{S}_{3,2}$, one can perform an SVD of the P_K interpolating subspaces to obtain one point S_K in the Stiefel manifold such that $S_K S_K^{\mathrm{T}} = P_K$. However, since $\psi^{-1}(P_K) = \{S_K e^{\Omega} : \Omega \in \mathfrak{so}(k)\}$, that is, there are infinitely many frames generating the same subspace, smoothness and even continuity of the lifted curve is lost. This is the main drawback of performing the interpolation on the Graßmann manifold when the objective is to have smooth changes of the initial configurations in the reduced space.

Figures 8 and 9 show several sequences resulting from the interpolation on the Stiefel manifold (using the modified Casteljau algorithm), compared with the same sequences resulting from the interpolation on the Graßmann manifold. Note that smooth changes are observed in the Stiefel case and jumps are observed in the Graßmann case. These experiments illustrate the advantages of solutions obtained with the modified Casteljau algorithm when compared with solutions that result from interpolating the projected data on the Graßmann manifold and then projecting back on the Stiefel manifold. While the former guarantees a smooth interpolating curve, the later doesn't guarantee smoothness, not even continuity, of the curve interpolating the subspaces representing the data sets.

8. Conclusion

The generalisation of the classical Casteljau algorithm can be used to generate interpolating polynomial splines on manifolds. This algorithm is based on successive linear interpolation and can be successfully used whenever explicit formulas for the geodesic joining two points are available. We have implemented the Casteljau algorithm on the Graßmann manifold. However, since explicit formulas for geodesics on the Stiefel manifold that join two points are not known, this algorithm can not be applied. To overcome this problem, we presented a convenient modification of the Casteljau algorithm, where instead of geodesics we use quasi-geodesics. These curves possess very interesting properties. In particular, they have constant speed, constant covariant acceleration and constant geodesic curvature not greater than one. The result of the implementation of the modified Casteljau algorithm is a smooth curve interpolating data on the Stiefel manifold. The theoretical results have been tested through several experiments. The advantage of using interpolation on the Stiefel manifold as opposed to interpolation on the Graßmann manifold is illustrated in the previous section with two simple experiments.

The work presented here opens up new directions for research, from numerical issues to more abstract questions, such as: Are quasi-geodesics optimal in some sense? How can quasi-geodesics be described in terms of the action of the full isometry group of the Stiefel manifold? It is our intention to address these issues in the near future.

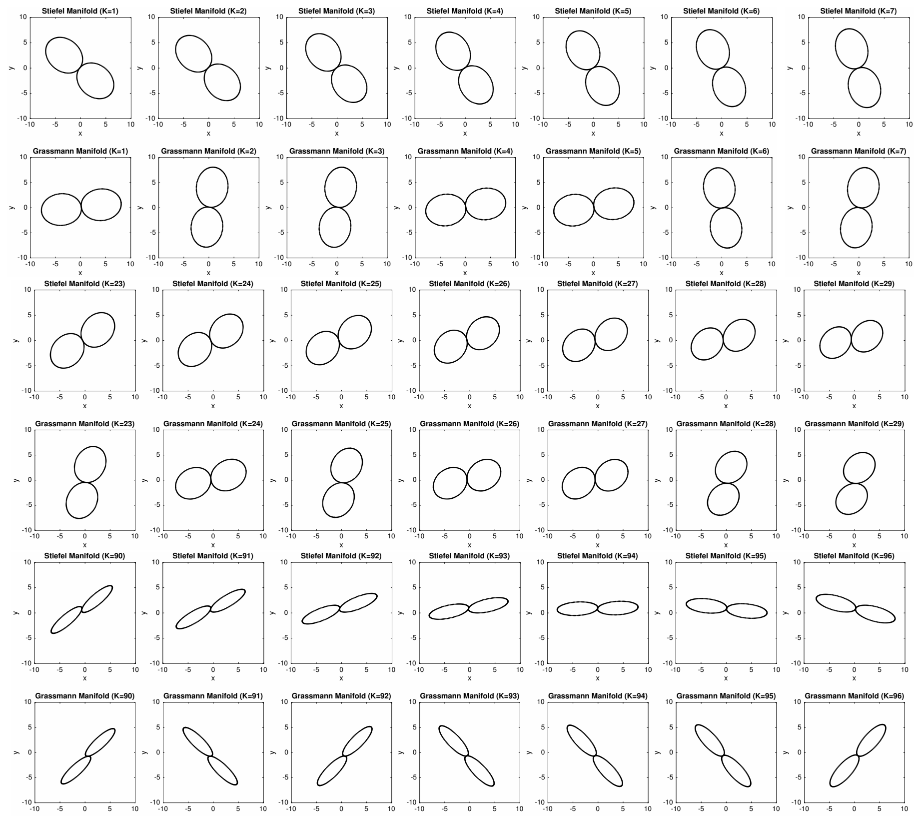

Figure 8: What is observed in the reduced 2-space when the modified Casteljau algorithm is applied to data on Stiefel manifold $\mathscr{S}_{3,2}$ representing figures 8, compared with what is observed in the reduced 2-space when the Casteljau altorithm is applied to the data projected on the Graßmann manifold $\mathscr{G}_{3,2}$ and then lifted to the Stiefel manifold. Odd rows show results from the Stiefel and even rows show results from the Graßmann.

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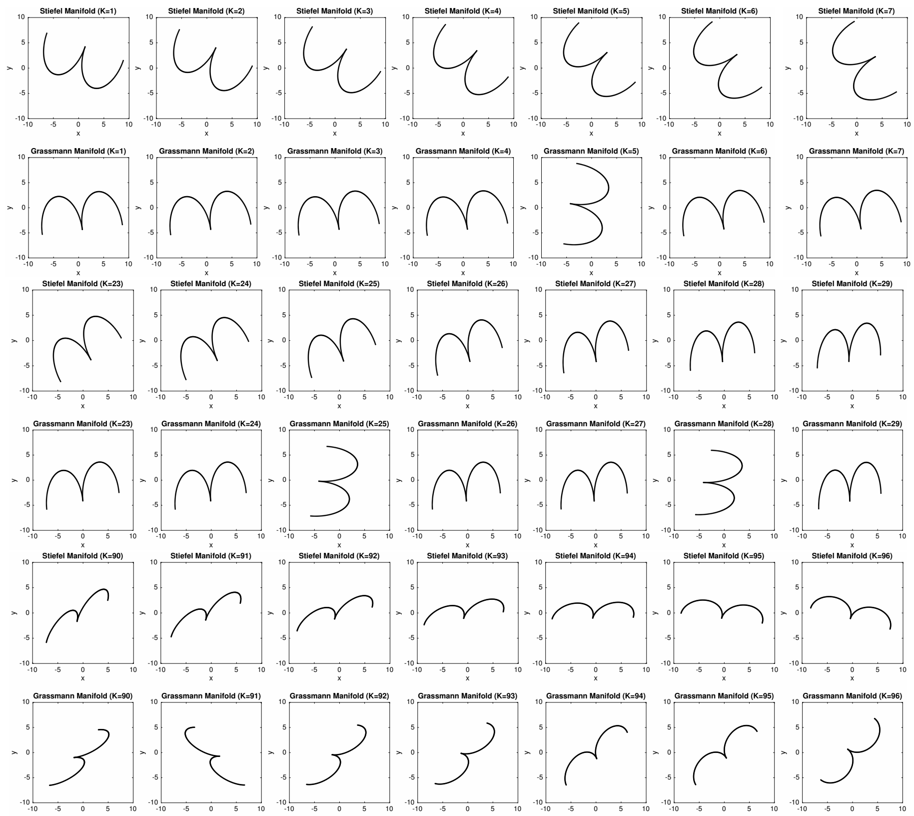

Figure 9: What is observed in the reduced 2-space when the modified Casteljau algorithm is applied to data on Stiefel manifold $\mathscr{S}_{3,2}$ representing figures 3, compared with what is observed in the reduced 2-space when the Casteljau altorithm is applied to the data projected on the Graßmann manifold $\mathscr{G}_{3,2}$ and then lifted to the Stiefel manifold. Odd rows show results from the Stiefel and even rows show results from the Graßmann.

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