Convergence Analysis of ADMM for Multi-Affine Trajectory Optimization

Yutong Chao
School of Computation, Information and Technology
Technical University of Munich
Munich, Germany
yutong.chao@tum.de

Michal Ciebielski
Munich Institute of Robotics and Machine Intelligence
Technical University of Munich
Munich, Germany
michal.ciebielski@tum.de

Jalal Etesami
School of Computation, Information and Technology
Technical University of Munich
Munich, Germany
j.etesami@tum.de

Majid Khadiv
Munich Institute of Robotics and Machine Intelligence
Technical University of Munich
Munich, Germany
majid.khadiv@tum.de

***Abstract*—In this paper, we study a class of non-convex optimization problems known as multi-affine quadratic equality constrained problems, which appear in various applications– from generating feasible force trajectories in robotic locomotion and manipulation to training neural networks. Although these problems are generally non-convex, they exhibit convexity or related properties when all variables except one are fixed. Under mild assumptions, we prove that the alternating direction method of multipliers (ADMM) converges when applied to this class of problems. Furthermore, when the "degree" of non-convexity in the constraints remains within certain bounds, we show that ADMM achieves a linear convergence rate. We validate our theoretical results through practical examples in robotic locomotion.**

***Index Terms*—Nonconvex Optimization, Robot Locomotion**

I. Introduction

Non-convex optimization serves as a fundamental concept in modern machine learning, such as reinforcement learning [1], [2] and robotics [3], [4]. The non-convexity in these applications may arise from the objective function, the constraint set, or both. Finding a solution to a non-convex problem is, in general, NP-hard [5]. As a step to manage this complexity, a common practice is to study problems with additional structural assumptions under which particular solvers, such as gradient-based methods, are guaranteed to converge to a stationary point. Subsequently, various relaxations of the objective and/or constraints have been proposed to transform the original problem into a more tractable problem. For instance, the objective function has been studied under assumptions such as weak strong convexity [6], restricted secant inequality [7], error bound [8], and quadratic growth [9]. On the other hand, optimization problems with various types of non-linear constraints have been investigated, such as quadratically constrained quadratic programs (QCQP) [10], [11], geometric programming (GP) [12], [13], mixed-integer nonlinear programming (MINLP) [14], [15], and equilibrium constraints problem [16], [17].

[0]This is the short version of the paper Last-iterate Convergence of ADMM on Multi-affine Quadratic Equality Constrained Problem.

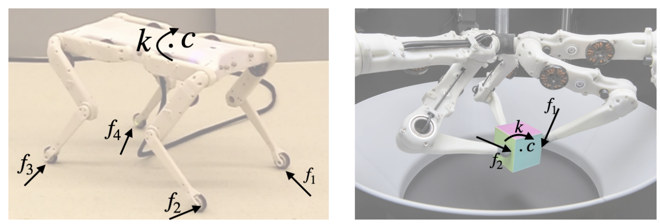


Fig. 1. Examples of locomotion and manipulation settings, (left) Solo [24] and (right) Trifinger [25]

Recently, there has been growing interest in analyzing non-convex optimization problems with specific block structures, driven by their broad range of applications. Although such problems are generally non-convex, they often exhibit convexity or related properties when all but one block of variables is fixed. Various structural properties of these problems have been studied, including multi-convexity in minimization settings [18]–[20], PL-strongly concave [21], and PL-PL [22], [23] in min-max formulations. Motivated by two well-known applications in robotics, in this work, we study multi-affine equality-constrained optimization problems (see Problem in (1)).

In particular, locomotion and manipulation problems in robotics (Figure 1) involve intermittent contact interactions with the world. Due to the hybrid nature of these interactions, generating dynamically-consistent trajectories for such systems leads to a set of non-convex problems, which remains an open challenge. In general, the problem of planning through contact is handled in two ways; contact-implicit and contact-explicit. The first approach directly incorporates the complementarity constraints arising from the contacts, either by relaxing them within the problem formulation [26] or at the solver level [27]. While this approach has recently shown considerable promise in practice [28]–[30], providing convergence guarantees remains an open problem due to the presence of multiple sources of non-convexity. The second approach handles contact in the trajectory optimization problem by casting it as a mixed-integer optimization [31], [32]. In this approach, the hybrid nature of interaction is explicitly taken into account with integer variables, and thus, a combinatorial search is required to decide over the integer decision variables, while the continuous trajectory optimization problem ensures the kinematic and dynamic feasibility of the problem. This approach has also shown great success in recent years in both locomotion [33]–[35] and contact-rich manipulation tasks [36]–[38].

The optimization problem in the contact-explicit setting exhibits additional structure. In particular, the dynamics can be decomposed into underactuated and actuated components [39]. This implies that, to generate a feasible force trajectory, the kinematics can be abstracted away. Assuming the robot can produce any desired contact force, it is sufficient to consider only the Newton-Euler dynamics to generate dynamically consistent trajectories for the robot's center of mass (CoM) in locomotion and for the object's CoM in manipulation. In this setting, the non-convexity in the dynamics has a special form, namely it is multi-affine [40]. This renders the trajectory optimization problem a multi-affine equality-constrained optimization problem. These types of problems also appear in other applications such as matrix factorization, graph theory, and neural network training process [41]–[44].

II. Problem Setting

Definitions and Assumptions: In this work, we consider the following *multi-affine quadratic equality constrained* problem:

$$\min_{x,z} F(x) + \phi(z), \quad \textit{s.t.} \quad A(x) + Qz = 0, \tag{1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^{n_x}$ is partitioned into n blocks, with each block $x_i \in \mathbb{R}^{n_i}$. Q is a matrix in $\mathbb{R}^{n_c \times n_z}$, and z is a vector in $\mathbb{R}^{n_z}$. $A(x)$ is a multi-affine quadratic operator.

Definition 1. *Function $A(\cdot) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_c}$ is called a multi-affine quadratic operator when for each $i \in \{1, ..., n_c\}$, there exist $C_i \in \mathbb{R}^{n_x \times n_x}$, $d_i \in \mathbb{R}^{n_x}$, and $e_i \in \mathbb{R}$ such that*

$$(A(x))_i := \frac{x^T C_i x}{2} + d_i^T x + e_i, \tag{2}$$

and moreover, $A(x_j; x_{-j})$ is an affine function for x_j when x_{-j} are fixed, $\forall x_{-j},\ j \in [n]$.

Note that the set of constraints in (1) comprises the linear ones and encompasses a much broader class in nonlinear settings. From definition, it is obvious that the diagonal blocks of the matrix C_i is zero matrix. We provide a simple example of the multi-affine quadratic equality constrained problem.

Example 2. *Consider the following problem*

$$\min_{x,z} x_1^2 + x_2^2 + z_1^2 + z_2^2,$$

$$\textit{s.t.} \quad x_1x_2 + x_1 + 1 + z_1 = 0, \ -x_1x_2 + x_2 + 1 + z_2 = 0.$$

This problem can be reformulated in the form of (1) by considering Q to be the identity matrix, $F(x) := x_1^2 + x_2^2$, $\phi(z) := z_1^2 + z_2^2$, $e_1 = 1, e_2 = 1$ and

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Algorithm: To solve (1), we consider the augmented Lagrangian ADMM, introducing a dual variable $w \in \mathbb{R}^{n_c}$ and a quadratic penalty term for the constraints with the coefficient ρ. This results in the following Lagrangian function.

Definition 3. *The corresponding augmented Lagrangian of (1), $L(x, z, w)$ is given by*

$$F(x) + \phi(z) + \langle w, A(x) + Qz \rangle + \frac{\rho}{2} \|A(x) + Qz\|^2, \tag{3}$$

where $\rho > 0$ is the penalty parameter.

Algorithm 1 ADMM

Require: $(x_1^0, \ldots, x_n^0), z^0, w^0, \rho$
  for $k = 0, 1, 2, \ldots$ **do**
    for $i = 1, \ldots, n$ **do**
      $x_i^{k+1} \in \arg\min_{x_i} L(x_{1:i-1}^{k+1}, x_i, x_{i+1:n}^k, z^k, w^k)$
    end for
    $z^{k+1} \in \arg\min_z L(x^{k+1}, z, w^k)$
    $w^{k+1} = w^k + \rho(A(x^{k+1}) + Qz^{k+1})$
  end for

ADMM is a powerful algorithm that can iteratively find a stationary point of the above Lagrangian function. Algorithm 1 summarizes the steps of this algorithm. At each iteration, it sequentially updates the current estimate by minimizing the augmented Lagrangian with respect to the variables in block x_i, $i \in \{1, ..., n\}$, and z when all other blocks are fixed at their current estimates. Afterwards, the dual variable w is updated depending on how much the constraints are violated. It is important to note that the minimization sub-problems for updating each block are derived by fixing all other blocks. This results in an augmented Lagrangian corresponding to a linearly constrained problem, which is tractable and can be solved efficiently [45], [46].

III. Application in Robotics

In the locomotion problem, the robot's centroidal momentum dynamics are considered [47], [48]. The location, velocity, and angular momentum generated around the CoM are denoted by $\mathbf{c}$, $\dot{\mathbf{c}}$, and $\mathbf{k}$. We aim to optimize the objective function subject to the physics constraints, i.e., Newton-Euler equations. By discretizing the Newton-Euler equations and fixing the contact sequence and its timing, the optimal control problem for locomotion can be written in the following unified way.

$$\min_{\mathbf{c}, \dot{\mathbf{c}}, \mathbf{k}, \mathbf{f}} \quad \sum_{i=0}^{T-1} \phi_t(\mathbf{c}_i, \dot{\mathbf{c}}_i, \mathbf{k}_i, \mathbf{f}_i) + \phi_T(\mathbf{c}_T, \dot{\mathbf{c}}_T, \mathbf{k}_T), \tag{4}$$

$$\text{s.t. } \mathbf{c}_{i+1} = \mathbf{c}_i + \dot{\mathbf{c}}_i \Delta t, \ \dot{\mathbf{c}}_{i+1} = \dot{\mathbf{c}}_i + \sum_{j=1}^{N} \frac{\mathbf{f}_i^j}{m} \Delta t + \mathbf{g} \Delta t,$$

$$\dot{\mathbf{c}}_0 = \dot{\mathbf{c}}_{init}, \ \mathbf{c}_0 = \mathbf{c}_{init}, \ \mathbf{f}_i^j \in \Omega_i^j, \ \forall i, j,$$

$$\mathbf{k}_{i+1} = \mathbf{k}_i + \sum_{j=1}^{N} (\mathbf{r}_i^j - \mathbf{c}_i) \times \mathbf{f}_i^j \Delta t, \quad \mathbf{k}_0 = \mathbf{k}_{init},$$

where Δt is the time discretization, subscript i stands for time index, T being the last one. Superscript j specifies the index of the end-effector in contact with the environment, and N is the number of the end-effector. Variables $\mathbf{c}_i, \dot{\mathbf{c}}_i, \mathbf{k}_i, \mathbf{f}_i^j$ denote the location, speed, angular momentum of the center of mass and the friction force at j-th contact at i-th discretization. The location of the end-effector in contact $\mathbf{r}$ is known. The initial conditions for the CoM are given by $\mathbf{c}_{init}, \dot{\mathbf{c}}_{init}, \mathbf{k}_{init}$. Function ϕ_t represents the running cost, ϕ_T is the terminal cost, and the friction $\mathbf{f}_i^j$ is constrained to lie within a safe region Ω_i^j, which we assume it is cone and use polyhedral approximation to represent it. Note that this is a multi-affine equality constraint due to the term $\mathbf{c} \times \mathbf{f}$ in the angular momentum dynamics.

Next, we reformulate the problem into the form of (1) and provide sufficient conditions under which ADMM applied to this formulation achieves a provable convergence rate. First, notice that the variables $\mathbf{c}_i$ and $\dot{\mathbf{c}}_i$ can be rewritten as functions of $\mathbf{f} = \{\mathbf{f}_i\}$. Second, by defining a new set of variables $\mathbf{k}' = \{\mathbf{k}'_i\}$ as $\mathbf{k}'_{i+1} := \mathbf{k}_{i+1} - \mathbf{k}_i$ for $i \geq 0$ and $\mathbf{k}'_0 := \mathbf{k}_{init}$ and assuming that the running and terminal costs can be decomposed into $f(\mathbf{f}) + \phi(\mathbf{k}')$, we obtain the following equivalent problem.

$$\min_{\mathbf{k}',\mathbf{f}} \quad f(\mathbf{f}) + \phi(\mathbf{k}') + \sum_{i=0}^{T} I_i(\mathbf{f}_i), \tag{5}$$

$$\text{s.t. } \mathbf{k}'_0 = \mathbf{k}_{init}, \quad \mathbf{k}'_1 = \sum_{j=1}^{N} \Big(\big(\mathbf{r}_0^j - \mathbf{c}_{\text{init}}\big) \times \mathbf{f}_0^j \Big) \Delta t,$$

$$\mathbf{k}'_2 = \sum_{j=1}^{N} \Big(\big(\mathbf{r}_1^j - \mathbf{c}_{\text{init}} - \dot{\mathbf{c}}_{\text{init}}\, \Delta t\big) \times \mathbf{f}_1^j \Big) \Delta t,$$

$$\mathbf{k}'_{i+1} = \sum_{j=1}^{N} \Big(\big(\mathbf{r}_i^j - \mathbf{c}_{\text{init}} - \dot{\mathbf{c}}_{\text{init}}\, i(\Delta t) -$$

$$\sum_{i'=0}^{i-2} (i-1-i')(\sum_{l=1}^{N} \frac{\mathbf{f}_{i'}^l}{m} + \mathbf{g})(\Delta t)^2\big) \times \mathbf{f}_i^j \Big) \Delta t, i \geq 2,$$

Problem in (5) has the same form as in (1). This can be seen by defining z and x in (1) to be $z := [\mathbf{k}'_0, \mathbf{k}'_1, \cdots, \mathbf{k}'_T]^T$ and $x := [x_0, \cdots, x_T]^T$, where $x_i := [\mathbf{f}_i^1, \cdots, \mathbf{f}_i^N]$. By denoting the corresponding Lagrangian function of the above problem as $L(\mathbf{f}, \mathbf{k}', w) = L(x, z, w)$.

The following set of assumptions is introduced to restrict the objective function in (1), making it more closely resemble the form presented in (5). Namely, we assume that the function $F(x)$ can be decomposed into a C^2 strongly convex function and a group of indicator functions that are block-separable.

Assumption 4. *$F(x) = f(x) + \sum_{i=0}^{T} I_i(x_i)$ is subanalytic, where f is C^2, μ_f-strongly convex with minimizer $x_f^\star$ and L_f with and I_i is the indicator function of a convex and closed set $X_i \subseteq \mathbb{R}^{n_i}$. Function $\phi(z)$ is also C^2, μ_ϕ-strongly convex and L_ϕ-smooth with minimizer $z_\phi^\star$.*

Theorem 5. *Under Assumption 4 with sufficiently large ρ, the iterates of the ADMM applied to the problem in* (5) *satisfy*

$$L(x^k, z^k, w^k) - L(x^\star, z^\star, w^\star) \in o(1/k).$$

We also extend the result of Theorem 5 by requiring that the blocks of the initial point x^0, the global minimizer of $f(x)$ and $\phi(z)$, $x_f^\star$ and $\phi_z^\star$ are all bounded, i.e.,

$$\|x^0\|^2, \|x_f^\star\|^2 \in \mathcal{O}(n_x), \quad \|z_\phi^\star\|^2 \in \mathcal{O}(n_z). \tag{6}$$

This requirement holds in almost all physical problems. Note that (5) is a multi-affine quadratic constrained problem with the nonlinear term proportional to $(\Delta t)^3$. As $\Delta t \to 0$, the nonlinear term decays and consequently, ADMM is expected to converge linearly, i.e., at a faster rate. This behavior is shown in the following result.

Theorem 6. *Under the assumptions of Theorem 5, if* (6) *holds and $L(x, z, w)$ is second-order differentiable at the limit point $(x^\star, z^\star, w^\star)$, then there exists $c > 1$ and $t_0 > 0$, such that the iterates of the ADMM applied to the problem in* (5) *with $\Delta t \leq t_0$ satisfy*

$$L(x^k, z^k, w^k) - L(x^\star, z^\star, w^\star) \in \mathcal{O}(c^{-k}),$$

Furthermore, $(x^\star, z^\star)$ is a local minimum of problem 5.

IV. Experiments

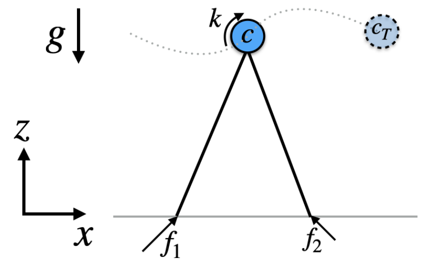
Herein, we apply the ADMM algorithm to simplified 2D example of locomotion and dynamic locomotion.

2D Locomotion problem: Figure 2a depicts a 2D locomotion problem in which the goal is to achieve smooth walking behaviors, potentially involving varying step lengths at different time steps. To demonstrate the performance of the ADMM for finding the optimal trajectories, i.e., $\{\mathbf{f}_i, \mathbf{k}_i\}$, we considered this 2D version of the problem in (4).

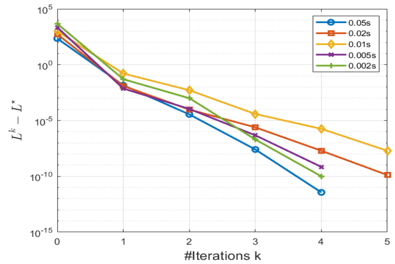
In this experiment, we selected a set of parameters that are close to a realistic application, namely, we set $m = 2$ kg (small-size robot in [24]). We used the cost terms $f_i(\mathbf{f}_i) = \frac{1}{2} \sum_{i=0}^{T} \|\mathbf{f}_i\|^2 + I_i(\mathbf{f}_i)$, and $\phi(z) = 5 \sum_{i=0}^{T} \|k'_i\|^2$. The constraints on $\mathbf{f}$ are designed to ensure that the center of mass remains within a specified target area.

Figure 2b illustrates the convergence rate for different discretization time values Δt. Note that the y-axis is on the log scale. As suggested by the result of Corollary 6, for small enough Δt, linear convergence is guaranteed by the ADMM. While $\Delta t = 0.005$ sec as suggested by Corollary 6, our empirical results in Fig. 2b indicate that the bound provided in the corollary is conservative. In practice, ADMM exhibits a linear convergence rate even for significantly larger values of Δt. As shown in Fig. 2c, ADMM consistently converges linearly regardless of the initial configuration.

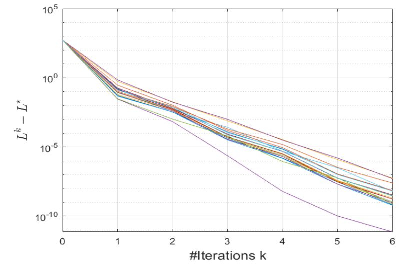
Dynamic locomotion problem: Figure 2d depicts dynamic motions executed on a humanoid and quadrupedal robot. These motions can be described by a fixed contact sequence and transition times, which can be used to formulate (4). The resulting CoM trajectory $(\mathbf{c}, \dot{\mathbf{c}}, \mathbf{k})$ can then be tracked via a kinematics optimization in order be applied on a robotic system as depicted in the figure.



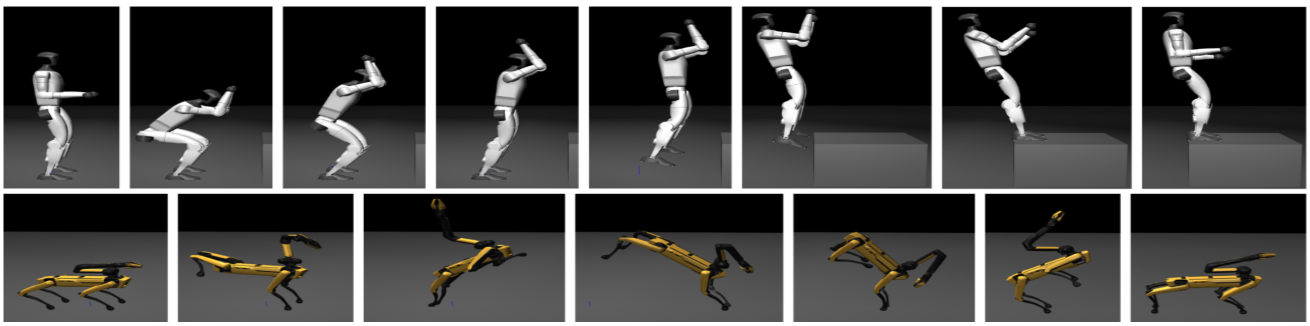
(a) 2D locomotion schematic



(b) ADMM performance convergence rate



(c) ADMM convergence rate



(d) Robot experiment snapshots

Fig. 2. **Top row:** Left: Schematic of a 2D locomotion problem. The robot has two contacts with friction f_1 and f_2. The location and angular momentum are $\mathbf{c}$ and $\mathbf{k}$. Center: Performance of ADMM for different Δt. Right: Convergence rate of ADMM for the 2D problem with random initialization. **Bottom row:** Snapshots of the robot experiments. The top row shows a humanoid robot performing a vertical jump. The bottom row illustrates a quadruped robot executing a bounding gait. In both cases, centroidal trajectories and forces are found using (4), and then a kinematic optimization tracks the planned centroidal trajectory.

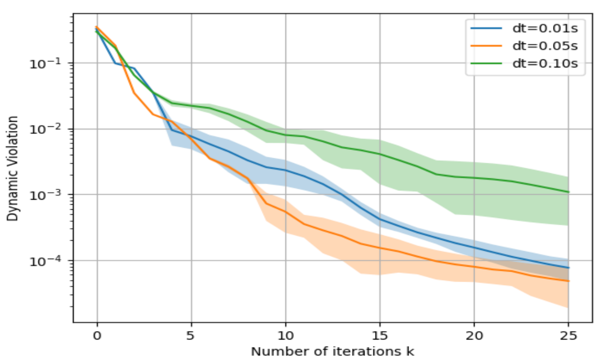


Fig. 3. Mean and standard deviation of dynamic violation values over optimization iterations. Results are shown for three different time discretizations. The x-axis shows the iteration number k.

In this experiment, we show successful transfer of the centroidal trajectories found using (4) or its equivalent in (5) via Algorithm 1 to high-dimensional robotics systems. The kinematics optimization is executed using an open source implementation of Differential Dynamic Programming (DDP) [49]. We report the centroidal dynamics constraint violation per iteration of Algorithm 1 for the jumping motion of the humanoid for three different discretization values Δt. The results are depicted in Figure 3, displaying the mean and standard deviation for each Δt over 10 trials with randomized initial conditions.

V. Conclusion

In this paper, we provided theoretical guarantees for the convergence rate of ADMM when applied to a class of multi-affine quadratic equality-constrained problems. We proved that the sublinear convergence of the Lagrangian always holds, and every block of the limit point is the optimal solution when other blocks are fixed. We further proved that when the degree of non-linearity is small enough, the convergence will be linear. In addition, the limit point is a local minimum of the problem. Moreover, we applied our result to the locomotion problem in robotics. Our experimental results validated the correctness and robustness of our theorem. Extending our results to problems with higher-order nonlinearities in the constraints, as well as relaxing the convexity assumption on the objective, constitute directions for future work.

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