Pseudo-inversion of integration-based time encoding using POCS

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***Abstract*—While event-based sampling allows the use of sampling circuits of higher precision and lower power consumption, it faces the difficult problem of signal reconstruction from generalized nonuniform samples. An ideal solution to this problem is to perform the pseudo-inversion of the linear operator that maps the input signals into the sequences of samples. We show in this article that this is possible with all time-encoding schemes based on input integration, using the method of projection onto convex sets (POCS). This includes multi-channel time encoding.**

I. Introduction

A method of event-based sampling that is currently attracting particular attention is the time encoding of integrals of a bandlimited input $x(t)$. This consists in detecting the instants $(t_k)_{k\in\mathsf{Z}}$ at which a certain integral function of $x(t)$ reaches a given threshold (Z is assumed to be an index set of consecutive integers, finite or infinite). Based on this, one is able to extract from $x(t)$ integral values of the type

$$\mathrm{s}_k := \int_{t_{k-1}}^{t_k} x(t) f_k(t) \mathrm{d}t, \qquad k \in \mathsf{Z} \tag{1}$$

where $f_k(t)$ is some known function that may depend on k. This applies to asynchronous Sigma-Delta modulation (ASDM) [1], [2] where $f_k(t)=1$, and leaky integrate-and-fire encoding (LIF) [3], [4] where $f_k(t)=e^{\alpha(t_{k-1}-t)}$ with $\alpha \geq 0$. The problem is to recover $x(t)$ from $(\mathrm{s}_k)_{k\in\mathsf{Z}}$ that we view as generalized samples of $x(t)$. For the *given* instants $(t_k)_{k\in\mathsf{Z}}$, $(\mathrm{s}_k)_{k\in\mathsf{Z}}$ can be presented as the transformation of $x(t)$ by the linear operator

$$\begin{array}{rccc} S: & \mathcal{B} & \rightarrow & \mathbb{R}^{\mathsf{Z}} \\ & x & \mapsto & (\mathrm{s}_k)_{k\in\mathsf{Z}} \end{array} \tag{2}$$

where $\mathcal{B}$ designates the considered space of bandlimited signals. To encompass all situations of data acquisition, including uniqueness of reconstruction, incomplete sampling and noisy sampling, the ultimate solution of reconstruction is to find the transformation of $(\mathrm{s}_k)_{k\in\mathsf{Z}}$ by the pseudo-inverse of S.

As the functions of $\mathcal{B}$ can be equivalently described as discrete-time signals, a basic engineering approach is to view S as a matrix, for which pseudo-inversion is performed mostly by algebraic manipulations [5]. However, even though Z is always finite in practice, it is typically of prohibitive size for algebraic inversions. The scope of this article is to consider numerical methods that perform the pseudo-inversion of S in the generalized sense of linear operators. The numerical methods considered until now for the present problem have been mainly proposed in [1], [3] but have been limited to sampling conditions guaranteeing S to be exactly invertible (while not being necessary conditions for invertibility). The method of projection onto convex sets (POCS) was more recently considered in [6], [7] as an alternative technique of reconstruction for the same encoding schemes. While consisting of an iteration of similar structure, its limit was shown to systematically reach the pseudo-inverse of S applied to $(\mathrm{s}_k)_{k\in\mathsf{Z}}$. The connection between the POCS method and the pseudo-inversion of a linear operator was first made in [8] in the finite-dimensional and simpler version of Kaczmarz's algorithm. The goal of this article is to extend the potential pseudo-inversion action of the POCS method to the most general sampling scheme possible.

The basic property that was utilized in [6], [7] is that the samples $(\mathrm{s}_k)_{k\in\mathsf{Z}}$ of (1) are of the form

$$\mathrm{s}_k := \langle x, g_k \rangle, \qquad k \in \mathsf{Z} \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product of $L^2(\mathbb{R})$ and $(g_k(t))_{k\in\mathsf{Z}}$ are *orthogonal* in $L^2(\mathbb{R})$. Indeed, these functions are respectively supported by the intervals $([t_{k-1}, t_k))_{k\in\mathsf{Z}}$, which are non-overlapping. For the sake of finding the ultimate generalizations, we revisit the techniques of [6], [7] under the most abstract conditions of sampling. This includes the assumption that x belongs to some general Hilbert space $\mathcal{A}$ that is not necessarily separable, while $(g_k)_{k\in\mathsf{Z}}$ is orthogonal in a larger Hilbert space $\mathcal{H} \supset \mathcal{A}$. We show the fundamental mechanisms in the POCS method that lead to the pseudo-inversion of the resulting sampling operator. As applications, we show how the POCS iteration can be rigorously discretized even when $\mathcal{A}$ does not have a countable basis (non-separable case), and show how the elaborate multi-channel time encoding scheme introduced in [9], [10] fits perfectly in this framework.

II. Pseudo-inverse of sampling operator

A. Abstract problem setting

We state the exact conditions under which we study the sampling problem:

- All continuous-time signals belong to a general Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ (that is not necessarily separable).

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- The input x to be sampled belongs to a closed linear subspace $\mathcal{A} \subset \mathcal{H}$.
- The kernel sampling functions form an *orthogonal* family $(g_k)_{k\in\mathsf{Z}}$ of $\mathcal{H}$.

The goal is to estimate x from the samples $(\mathrm{s}_k)_{k\in\mathsf{Z}}$ of (3).

B. Normalized sampling operator

For proper analysis, it is always possible to work with an orthonormal version of $(g_k)_{k\in\mathsf{Z}}$. Indeed, (3) is equivalent to

$$\mathrm{x}_k = \langle x, h_k\rangle, \qquad k \in \mathsf{Z} \tag{4}$$

with $\qquad h_k := g_k/\|g_k\| \quad$ and $\quad \mathrm{x}_k := \mathrm{s}_k/\|g_k\|$

where $\|\cdot\|$ is the norm induced by $\langle\cdot,\cdot\rangle$. In this setting, the sampling operator to invert takes the form

$$\begin{array}{rccl} S: & \mathcal{A} & \rightarrow & \ell^2(\mathsf{Z}) \\ & x & \mapsto & \big(\langle x, h_k\rangle\big)_{k\in\mathsf{Z}}. \end{array} \tag{5}$$

As opposed to (2), note here that the destination space of S is more specifically the space $\ell^2(\mathsf{Z})$ of square-summable sequences $(\mathrm{x}_k)_{k\in\mathsf{Z}}$. By orthonormality of $(h_k)_{k\in\mathsf{Z}}$ and Bessel's inequality, not only do we have the guarantee that $Sx \in \ell^2(\mathsf{Z})$, but we also have $\|Sx\|_2 \le \|x\|$ where $\|\cdot\|_2$ is the canonical norm of $\ell^2(\mathsf{Z})$.

C. Pseudo-inverse $S^\dagger$

As mentioned in the introduction, the approach of this paper is to estimate x from (4) by attempting an inversion of the operator S. Specifically, we consider the Moore-Penrose pseudo-inverse $S^\dagger$ of S [11, §6.11]. Assuming that S has closed range, $S^\dagger$ is defined by

$$\forall \mathbf{x} \in \ell^2(\mathsf{Z}), \qquad S^\dagger\mathbf{x} := \underset{u\in\mathcal{M}_\mathbf{x}}{\operatorname{argmin}} \|u\| \tag{6}$$

$$\text{where} \quad \mathcal{M}_\mathbf{x} := \Big\{u \in \mathcal{A} : \|Su - \mathbf{x}\|_2 \text{ is minimized}\Big\}. \tag{7}$$

The closed range assumption for S is necessary for $\mathcal{M}_\mathbf{x}$ to be defined. In this case, $\mathcal{M}_\mathbf{x}$ is then a closed affine subspace and hence yields a unique minimum-norm element $S^\dagger\mathbf{x}$. This assumption is satisfied by default when Z is finite, as is always the case in practice. When Z is infinite, S has closed range if and only if there exists $\alpha > 0$ such that $\alpha\|u\| \le \|Su\|$ for all $u \in \mathrm{null}(S)^\perp$, where $\mathrm{null}(S)$ is the null space of S [12, §2].

III. Pseudo-inversion by means of orthogonal projections

A. Orthogonal projection onto closed affine subspace

As a closed affine subspace, $\mathcal{M}_\mathbf{x}$ yields an orthogonal projection. For any closed affine subspace $\mathcal{S}$, the orthogonal projection of $u \in \mathcal{H}$ onto $\mathcal{S}$ is the unique element $P_\mathcal{S}u$ of $\mathcal{S}$ such that $P_\mathcal{S}u-u$ is orthogonal to $P_\mathcal{S}u-v$ for all $v \in \mathcal{S}$. By the Pythagorean theorem, one obtains that $\|P_\mathcal{S}u - v\| < \|u - v\|$ for any $v \in \mathcal{S}$ and $u \notin \mathcal{S}$. One simultaneously concludes that $P_\mathcal{S}u$ is the unique element of $\mathcal{S}$ that minimizes the distance $\|P_\mathcal{S}u - u\|$. As a result,

$$\forall \mathbf{x} \in \ell^2(\mathsf{Z}), \qquad S^\dagger\mathbf{x} = P_{\mathcal{M}_\mathbf{x}}0. \tag{8}$$

B. Case $\mathcal{A} = \mathcal{H}$

This is a trivial case which may not be realized in practice but is pedagogical to start with. Due to the orthonormality of $(h_k)_{k\in\mathsf{Z}}$, we have the implications

$$u = \sum_{k\in\mathsf{Z}} \mathrm{x}_k\, h_k \;\;\Rightarrow\;\; \forall k \in \mathsf{Z},\ \langle u, h_k\rangle = \mathrm{x}_k \;\;\Rightarrow\;\; Su = \mathbf{x}. \tag{9}$$

As a result, the minimum value of $\|Su-\mathbf{x}\|_2$ in the description of $\mathcal{M}_\mathbf{x}$ is 0 since there exists $u \in \mathcal{A} = \mathcal{H}$ such that $Su = \mathbf{x}$. Therefore,

$$\mathcal{M}_\mathbf{x} = \mathcal{C}_\mathbf{x}$$

where

$$\begin{aligned} \mathcal{C}_\mathbf{x} &:= \big\{u \in \mathcal{H} : Su = \mathbf{x}\big\} \\ &= \big\{u \in \mathcal{H} : \forall k \in \mathsf{Z},\ \langle u, h_k\rangle = \mathrm{x}_k\big\}. \end{aligned} \tag{10}$$

Based on this description and the orthonormality of $(h_k)_{k\in\mathsf{Z}}$, one easily finds that

$$\forall u \in \mathcal{H}, \qquad P_{\mathcal{C}_\mathbf{x}}u = u + \sum_{k\in\mathsf{Z}} \big(\mathrm{x}_k - \langle u, h_k\rangle\big)\, h_k. \tag{11}$$

Hence,

$$\forall \mathbf{x} \in \ell^2(\mathsf{Z}), \qquad S^\dagger\mathbf{x} = P_{\mathcal{C}_\mathbf{x}}0 = \sum_{k\in\mathsf{Z}} \mathrm{x}_k h_k = S^*\mathbf{x} \tag{12}$$

where S^* is the adjoint operator of S (see [13, §3.1] with $U = S^*$ and hence $U^* = S$).

C. Case $\mathcal{A} \subsetneq \mathcal{H}$ and $\mathbf{x} \in \mathrm{ran}(S)$

This is for example the case assumed in the introduction where $\mathcal{A}$ is a space of bandlimited signals and $\mathcal{H} = L^2(\mathbb{R})$. The condition that $\mathbf{x}$ belongs to $\mathrm{ran}(S)$, the range of S, is satisfied by default from (4). The importance of this condition is in the following equivalence

$$\mathbf{x} \in \mathrm{ran}(S) \qquad \Leftrightarrow \qquad \mathcal{A} \cap \mathcal{C}_\mathbf{x} \neq \emptyset. \tag{13}$$

When this is realized, the minimal value of $\|Su - \mathbf{x}\|_2$ with $u \in \mathcal{A}$ in the description of (7) is then 0, which is achieved if and only if $u \in \mathcal{A} \cap \mathcal{C}_\mathbf{x}$. This implies that

$$\mathcal{M}_\mathbf{x} = \mathcal{A} \cap \mathcal{C}_\mathbf{x}.$$

It then follows from (8) that

$$S^\dagger\mathbf{x} = P_{\mathcal{A}\cap\mathcal{C}_\mathbf{x}}0.$$

The projection $P_{\mathcal{A}\cap\mathcal{C}_\mathbf{x}}$ is unfortunately not explicitly accessible. However, it is known from the POCS method that the iteration of

$$u^{(n+1)} = P_\mathcal{A} P_{\mathcal{C}_\mathbf{x}} u^{(n)}, \qquad n \ge 0 \tag{14}$$

converges to $P_{\mathcal{A}\cap\mathcal{C}_\mathbf{x}}u^{(0)}$. Then, by choosing $u^{(0)} = 0$, $u^{(n)}$ tends to

$$u^{(\infty)} = S^\dagger\mathbf{x}. \tag{15}$$

From the explanations of Section III-A with $v = x$, a supplementary attractive property is that $\|u^{(n)} - x\|$ strictly decreases with n as long as $u^{(n)} \notin \mathcal{A}\cap\mathcal{C}_\mathbf{x}$. In other words, each iteration of (14) contributes to estimate improvements, unless convergence has already occurred. This time, the composed projection $P_{\mathcal{A}} P_{\mathcal{C}_{\mathbf{x}}}$ has an easy derivation. After noticing from (14) that $u^{(n)}$ remains in $\mathcal{A}$ (including $n=0$ since we choose $u^{(0)}=0$), it is clear from (11) and (14) that

$$u^{(n+1)} = u^{(n)} + \sum_{k\in\mathsf{Z}} \big(\mathrm{x}_k - \langle u^{(n)}, h_k\rangle\big)\, \widetilde{h}_k, \qquad n\geq 0 \qquad (16)$$

where

$$\widetilde{h}_k := P_{\mathcal{A}} h_k,$$

D. Linear operator approach

Given that (16) is known to converge, there is another way to see that $u^{(n)}$ should tend to $S^\dagger \mathbf{x}$. This consists in expressing (16) in terms of S and its adjoint S^*. A slight difficulty to express S^* is that the vectors $(h_k)_{k\in\mathsf{Z}}$ in (5) are not in the domain $\mathcal{A}$ of S. This can be fixed by noting that

$$\forall x\in\mathcal{A}, \qquad \langle x, h_k\rangle = \langle x, \widetilde{h}_k\rangle \qquad (17)$$

since $\widetilde{h}_k - h_k$ is orthogonal to $\mathcal{A}$. Then, S and S^* can be presented as

$$S: \begin{array}{ccc} \mathcal{A} & \rightarrow & \ell^2(\mathsf{Z}) \\ x & \mapsto & \big(\langle x, \widetilde{h}_k\rangle\big)_{k\in\mathsf{Z}} \end{array} \quad \text{and} \quad S^*: \begin{array}{ccc} \ell^2(\mathsf{Z}) & \rightarrow & \mathcal{A} \\ (\mathrm{x}_k)_{k\in\mathsf{Z}} & \mapsto & \sum\limits_{k\in\mathsf{Z}} \mathrm{x}_k\, \widetilde{h}_k \end{array}. \qquad (18)$$

One then easily rewrites (16) as

$$u^{(n+1)} = u^{(n)} + S^*(\mathbf{x} - S u^{(n)}), \qquad n\geq 0 \qquad (19)$$

where $\mathbf{x} := (\mathrm{x}_k)_{k\in\mathsf{Z}}$. At the limit of n towards ∞, one then finds that

$$S^* S u^{(\infty)} = S^*\mathbf{x}. \qquad (20)$$

If S^*S is invertible, then

$$u^{(\infty)} = (S^*S)^{-1} S^*\mathbf{x} \qquad (21)$$

It is known that

$$S^\dagger = (S^*S)^{-1} S^* \qquad (22)$$

whenever S^*S is invertible [11, §6.11]. Thus, (21) leads to (15) in this case. In fact, both (21) and (22) remain true in all cases, provided that the inverse $(S^*S)^{-1}$ is performed on the restriction of S^*S to $\mathrm{ran}(S^*)$. This is justified in Appendix A.

E. General case $\mathcal{A} \subsetneq \mathcal{H}$

The result of (21) was in fact only based on the assumption that the iteration of (19) is convergent. Until now, we have this convergence from Section III-C when $\mathbf{x}\in\mathrm{ran}(S)$. When $\mathbf{x}\notin\mathrm{ran}(S)$, $\mathcal{A}\cap\mathcal{C}_{\mathbf{x}}$ becomes empty according to (13), and the convergence of (19) can no longer be justified based on the arguments of Section III-C. We show in the present section that the convergence is still guaranteed when $\mathbf{x}\notin\mathrm{ran}(S)$. This is of particular interest in practice as the samples of (4) are more generally of the form

$$\mathrm{x}_k := \langle x, h_k\rangle + \epsilon_k, \qquad k\in\mathsf{Z}$$

where $(\epsilon_k)_{k\in\mathsf{Z}}$ is some sequence of errors due to noise. This may prevent $\mathbf{x} = (\mathrm{x}_k)_{k\in\mathsf{Z}}$ from remaining in $\mathrm{ran}(S)$. However, let $\bar{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto $\mathrm{ran}(S)$ in the sense of the inner product of $\ell^2(\mathsf{Z})$. By construction

$$\bar{\mathbf{x}} - \mathbf{x} \in \mathrm{ran}(S)^\perp = \mathrm{null}(S^*)$$

where $\mathrm{null}(S^*)$ designates the null space of S^* and the identity is a standard result of bounded operators in Hilbert spaces [11, §6.6]. As a result,

$$S^*\mathbf{x} = S^*\bar{\mathbf{x}}.$$

Thus, (19) can be equivalently written with $\bar{\mathbf{x}}$ in place of $\mathbf{x}$. This alone proves that (19) is convergent. Hence, (15) remains valid. But we propose here to prove (15) directly from the definition of $S^\dagger$ in (6) without the knowledge of (22).

Proposition 3.1: The iteration of (19) starting from $u^{(0)}=0$ converges to $u^{(\infty)} = S^\dagger\mathbf{x}$ for any given $\mathbf{x}\in\ell^2(\mathsf{Z})$.

Proof: Replacing $\mathbf{x}$ by $\bar{\mathbf{x}}\in\mathrm{ran}(S)$ in (19), we know from Section III-C and (8) that $u^{(n)}$ tends to $u^{(\infty)} = S^\dagger\bar{\mathbf{x}} = P_{\mathcal{M}_{\bar{\mathbf{x}}}}0$. For any $u\in\mathcal{A}$, $Su-\bar{\mathbf{x}}$ is in $\mathrm{ran}(S)$ and is therefore orthogonal to $\bar{\mathbf{x}}-\mathbf{x}$. Then, by the Pythagorean theorem, $\|Su-\bar{\mathbf{x}}\|_2^2 + \|\bar{\mathbf{x}}-\mathbf{x}\|_2^2 = \|Su-\mathbf{x}\|_2^2$, which implies that $\|Su-\bar{\mathbf{x}}\|_2$ is then minimized in terms of $u\in\mathcal{A}$ if and only if $\|Su-\mathbf{x}\|_2$ is minimized. This proves that $\mathcal{M}_{\bar{\mathbf{x}}} = \mathcal{M}_{\mathbf{x}}$. Hence, $u^{(\infty)} = P_{\mathcal{M}_{\mathbf{x}}}0 = S^\dagger\mathbf{x}$. ■

IV. Discrete-time implementation of iteration

A. General derivation

We show that (19) can be implemented by means of discrete-time iteration even when $\mathcal{A}$ does not have a countable basis (case of non-separable space). As noted in Section III-D and seen in (19), $u^{(n)}$ remains in the range of S^* with the initial iterate $u^{(0)}=0$. So, for any $n\geq 0$, we have

$$u^{(n)} = S^*\mathbf{c}^{(n)} \qquad (23)$$

for some $\mathbf{c}^{(n)}\in\ell^2(\mathsf{Z})$. Then (19) is equivalent to

$$u^{(n+1)} = S^*\mathbf{c}^{(n)} + S^*\big(\mathbf{x} - SS^*\mathbf{c}^{(n)}\big)$$

and hence to

$$S^*\mathbf{c}^{(n+1)} = S^*\big(\mathbf{c}^{(n)} - SS^*\mathbf{c}^{(n)} + \mathbf{x}\big). \qquad (24)$$

This relation is guaranteed by recursively constructing $\mathbf{c}^{(n)}$ as

$$\mathbf{c}^{(n+1)} = \mathbf{c}^{(n)} - SS^*\mathbf{c}^{(n)} + \mathbf{x}. \qquad (25)$$

Starting from $u^{(0)}=0$, suppose that $u^{(m)}$ is the targeted estimate. Instead of iterating (19) m times, one can then alternatively iterate (25) m times starting from $\mathbf{c}^{(0)}=0$, and apply (23) only once at $n=m$. The outstanding advantage of this procedure is that (25) is a pure discrete-time operation as $\mathbf{c}^{(n)}\in\ell^2(\mathsf{Z})$. In it, SS^* is a linear operator of $\ell^2(\mathsf{Z})$ and can be presented as the square matrix of coefficients

$$SS^* = \Big[\langle\widetilde{h}_{k'}, h_k\rangle\Big]_{(k,k')\in\mathsf{Z}^2} \text{ where } \langle\widetilde{h}_{k'}, h_k\rangle = \frac{\langle\widetilde{g}_{k'}, g_k\rangle}{\|g_{k'}\|\|g_k\|}.$$

In practice, the values of $\|g_k\|$ and $\langle\widetilde{g}_{k'}, g_k\rangle$ are to be predetermined before the iteration.

B. Application

For illustration, we give the values of $\|g_k\|$ and $\langle\widetilde{g}_{k'}, g_k\rangle$ that were derived in [7] in the case of LIF. We recall from the introduction that, in this case, $g_k(t)$ is equal to $e^{\alpha(t_{k-1}-t)}$ in $[t_{k-1}, t_k)$ and 0 outside. Assuming bandlimited functions of Nyquist period 1 and defining $T_{k',k} := t_{k'} - t_k$, it was found that

$$\|g_k\|^2 = \tfrac{1}{2\alpha}\big(1 - e^{-2\alpha T_{k,k-1}}\big)$$
$$\langle\widetilde{g}_{k'}, g_k\rangle = e^{-\alpha T_{k,k-1}}\big(f(T_{k',k-1}) - e^{-\alpha T_{k',k'-1}} f(T_{k'-1,k-1})\big) - \big(f(T_{k',k}) - e^{-\alpha T_{k',k'-1}} f(T_{k'-1,k})\big) \qquad (26)$$

where $\quad f(t) := \frac{1}{\alpha}\int_0^t \sinh(\alpha(t{-}s))\,\sin(\pi s)/(\pi s)\,\mathrm{d}s.$

The values of $f(t)$ can be precalculated at a high enough discrete resolution of t, and stored in a lookup table. Then $\|g_k\|$, $\|g_{k'}\|$ and $\langle\widetilde{g}_{k'}, g_k\rangle$ are just functions of $t_k, t_{k-1}, t_{k'}, t_{k'-1}$. The case of ASDM was derived in [6] and is simply obtained here by taking the limit of α towards 0.

V. Application to Multi-Channel Integration-Based Time Encoding

To illustrate the power of our general constructions, we apply it in this section on the elaborate multi-channel time encoding system introduced in [9] and represented in Fig. 1.

A. Problem description and resolution

We present the problem of [9] with some adaptation to be aligned with our formalism. The input that is time-encoded is an M-channel signal

$$\boldsymbol{x}(t) = \big(x^i(t)\big)_{1\le i\le M} = (x^1(t),\cdots,x^M(t)), \qquad t\in\mathbb{R}.$$

For each $i = 1,\cdots,M$, $x^i(t)$ is fed into an ASDM which outputs a sequence of impulses at instants $(t^i_j)_{j\in\mathsf{Z}_i}$. The integral values

$$\mathrm{s}_{i,j} := \int_{t^i_{j-1}}^{t^i_j} x^i(t)\,\mathrm{d}t, \qquad j\in\mathsf{Z}_i \qquad (27)$$

can then extracted from these instants using the ASDM encoding equations presented in [1]. It is assumed that $\boldsymbol{x}(t)$ is of the form

$$\boldsymbol{x}(t) = \mathbf{A}\boldsymbol{y}(t)$$

where $\mathbf{A}$ is a full rank $M{\times}N$ matrix with $N\le M$, $\boldsymbol{y}(t) = (y^1(t),\cdots,y^N(t)) \in \mathcal{B}^N$ and $\mathcal{B}$ is a space of bandlimited signals. The problem of [9] is to reconstruct $\boldsymbol{x}(t)$ from the sample values $(\mathrm{s}_{i,j})_{(i,j)\in\mathsf{Z}}$ with $\mathsf{Z} := \{(i,j) : 1\le i\le M \text{ and } j\in\mathsf{Z}_i\}$, provided a sufficiently high density of samples. Starting from $\boldsymbol{u}^{(0)}(t) = 0$, this is achieved numerically in [9] by iterating

$$\boldsymbol{u}^{(n+1)}(t) = P_3P_2P_1\boldsymbol{u}^{(n)}(t), \qquad n\ge 0 \qquad (28)$$

where for any $\boldsymbol{u}(t) = \big(u^1(t),\cdots,u^M(t)\big)\in(L^2(\mathbb{R}))^M$,

$$P_1\boldsymbol{u}(t) := \Big(u^i(t) + \sum_{j\in\mathsf{Z}_i}\frac{\mathrm{s}_{i,j} - \langle u^i, g^i_j\rangle}{t^i_j - t^i_{j-1}}\, g^i_j(t)\Big)_{1\le i\le M},$$
$$P_2\boldsymbol{u}(t) := \mathbf{A}\mathbf{A}^+\boldsymbol{u}(t), \qquad P_3\boldsymbol{u}(t) := \big(P_{\mathcal{B}}u^i(t)\big)_{1\le i\le M},$$

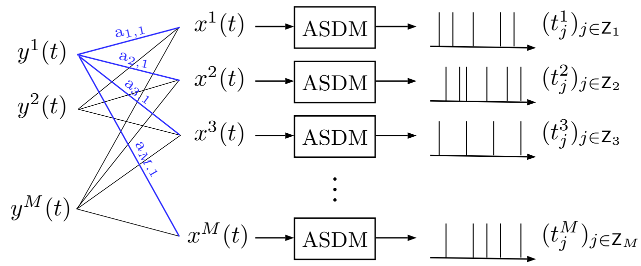

Fig. 1. Multi-channel time-encoding system from [9], [10].

$g^i_j(t)$ is the indicator function of the time interval $[t^i_{j-1}, t^i_j)$, $\langle\cdot,\cdot\rangle$ is the inner product of $L^2(\mathbb{R})$, and $\mathbf{A}^+ = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$ is the matrix pseudo-inverse of $\mathbf{A}$.

B. Formalization

To bring this problem back to our framework, the above descriptions can be formalized as follows. In the Hilbert space $\mathcal{H} := (L^2(\mathbb{R}))^M$, the input $\boldsymbol{x}(t)$ belongs to the subspace

$$\mathcal{A} := \big\{\boldsymbol{u}(t)\in\mathcal{B}^M : \forall t\in\mathbb{R},\ \boldsymbol{u}(t)\in\operatorname{ran}(\mathbf{A})\big\}.$$

The inner product of $\mathcal{H}$ is naturally defined by

$$\big\langle\boldsymbol{u}(t),\boldsymbol{v}(t)\big\rangle := \textstyle\sum_{i=1}^M \big\langle u^i(t), v^i(t)\big\rangle, \qquad \boldsymbol{u},\boldsymbol{v}\in\mathcal{H}$$

where $\langle\cdot,\cdot\rangle$ in the right hand side is the inner product of $L^2(\mathbb{R})$. As $\mathrm{s}_{i,j}$ from (27) is equal to $\langle x^i(t), g^i_j(t)\rangle$, it can be presented as

$$\mathrm{s}_{i,j} = \big\langle\boldsymbol{x}(t), g^i_j(t)\mathbf{e}^i\big\rangle$$

where $\mathbf{e}^i$ is the ith coordinate vector of $\mathbb{R}^M$. By normalization, this is equivalent to

$$\mathrm{x}_{i,j} = \big\langle\boldsymbol{x}(t),\boldsymbol{h}_{i,j}(t)\big\rangle$$

with $\quad \boldsymbol{h}_{i,j}(t) := g^i_j(t)\mathbf{e}^i/\|g^i_j\| \quad$ and $\quad \mathrm{x}_{i,j} := \mathrm{s}_{i,j}/\|g^i_j\|.$

The family $(\boldsymbol{h}_{i,j})_{(i,j)\in\mathsf{Z}}$ is easily seen to be orthonormal in $\mathcal{H}$. The POCS method resulting from Section III-C then consists in iterating

$$\boldsymbol{u}^{(n+1)}(t) = P_{\mathcal{A}}P_{\mathcal{C}_{\mathbf{x}}}\boldsymbol{u}^{(n)}(t), \qquad n\ge 0 \qquad (29)$$

where $\quad \mathcal{C}_{\mathbf{x}} := \big\{\boldsymbol{u}\in\mathcal{H} : \forall(i,j)\in\mathsf{Z},\ \langle\boldsymbol{u},\boldsymbol{h}_{i,j}\rangle = \mathrm{x}_{i,j}\big\}.$

It can be verified that the two iterations of (28) and (29) are the same. More specifically, $P_3P_2 = P_{\mathcal{A}}$ and $P_1 = P_{\mathcal{C}_{\mathbf{x}}}$.

C. Convergence of POCS method

We thus know that, *regardless* of the sampling condition, $\boldsymbol{u}^{(n)}(t)$ from (28) tends to $S^\dagger(\mathrm{x}_{i,j})_{(i,j)\in\mathsf{Z}}$ where S is the operator

$$\begin{array}{rccl} S: & \mathcal{A} & \to & \ell^2(\mathsf{Z}) \\ & \boldsymbol{x} & \mapsto & \big(\langle\boldsymbol{x},\boldsymbol{h}_{i,j}\rangle\big)_{(i,j)\in\mathsf{Z}}. \end{array}$$

Beyond the condition of perfect reconstruction in [9], [10], this convergence includes the situations of insufficient sampling and/or sampling with noise.

D. Discrete-time implementation of iteration

Another consequence is the reduction of (28) to the discrete-time iteration of (25). In it, the operator SS^* is the matrix of coefficients $\langle \widetilde{\boldsymbol{h}}_{i',j'}, \boldsymbol{h}_{i,j}\rangle$, where $\widetilde{\boldsymbol{h}}_{i',j'} := P_{\mathcal{A}}\boldsymbol{h}_{i',j'}$. It can be shown that

$$\langle \widetilde{\boldsymbol{h}}_{i',j'}, \boldsymbol{h}_{i,j}\rangle = \frac{\langle \widetilde{g}^{i'}_{j'}, g^i_j\rangle}{\|g^{i'}_{j'}\|\|g^i_j\|}\,\mathrm{p}_{ii'}$$

where $\widetilde{g}^{i'}_{j'} = P_{\mathcal{B}} g^{i'}_{j'}$ and $(\mathrm{p}_{ii'})_{i,i'}$ are the entries of the matrix $\mathbf{P} = \mathbf{A}\mathbf{A}^+$. While $\|g^i_j\|^2 = t^i_j - t^i_{j-1}$, $\langle \widetilde{g}^{i'}_{j'}, g^i_j\rangle$ can be derived in a way similar to (26) with $\alpha = 0$, which gives

$$\langle \widetilde{g}^{i'}_{j'}, g^i_j\rangle = f(T^{i',i}_{j',j-1}) - f(T^{i',i}_{j'-1,j-1}) - f(T^{i',i}_{j',j}) + f(T^{i',i}_{j'-1,j})$$

where $T^{i',i}_{j',j} := t^{i'}_{j'} - t^i_j$ and $f(t) := \int_0^t (t-s)\,\sin(\pi s)/(\pi s)\,\mathrm{d}s$.

VI. Conclusion

In the most general form of sampling, uniform or not, the samples of an input can be viewed as its inner products with known kernel functions. So, globally, they are the transformation of the input by a known linear operator S. An ultimate solution to signal reconstruction is then to find the pseudo-inverse of S applied to the samples, as this covers all encoding situations, including perfect reconstruction, insufficient sampling and/or noisy sampling. Given the signal processing context of virtually infinite inputs, pseudo-inversion cannot be performed algebraically and is accessible only by numerical methods. The present article shows that this pseudo-inversion is systematically achievable by the method of POCS whenever the sampling kernel functions are orthogonal in a Hilbert space that may be larger than that of the input. This covers for example the currently trendy time encoding of bandlimited signals based on integration, including multi-channel encoding. But the abstract assumptions made in this paper gives the most general framework where this action of the POCS method remains operational, for the exploration of future sampling schemes. Besides covering all sampling situations, the POCS iteration has an intrinsic and rigorous discrete-time implementation, solely based on the sampling nature of the encoding. This does not involve any pre-existing discrete expansion of the continuous-time input signals, such as the Shannon sampling expansion of bandlimited signals for example, and thus allows input spaces with no countable bases (case of non-separable Hilbert spaces).

Appendix

A. Explicit expression of $S^\dagger$

The goal of this appendix is to justify the general validity of (21) and (22) when S^*S in these expressions is restricted to $\mathrm{ran}(S^*)$ before inversion. The invertibility of this restriction simply follows from the bijectivity of S between $\mathrm{ran}(S^*)$ and $\mathrm{ran}(S)$, which implies by similarity the bijectivity of S^* between $\mathrm{ran}(S)$ and $\mathrm{ran}(S^*)$. The former result is based on the following properties: (i) $\mathrm{ran}(S^*)$ is closed since $\mathrm{ran}(S)$ is assumed to be closed (see Lemma 2.5.2 of [13]); (ii) the closure of $\mathrm{ran}(S^*)$ is equal to $\mathrm{null}(S)^\perp$ (see Theorem 3 of [11, §6.6]); (iii) S is a bijection between $\mathrm{null}(S)^\perp$ and $\mathrm{ran}(S)$ (see the proof of Lemma 2.5.1 in [13]). In this circumstance, (22) follows from the known result that $\mathrm{ran}(S^\dagger) = \mathrm{null}(S)^\perp$ (see (2.10) in [13]), which is equal to $\mathrm{ran}(S^*)$. Finally, $u^{(\infty)} \in \mathrm{ran}(S^*)$ as $u^{(n)}$ can be seen from (16) to remain in $\mathrm{ran}(S^*)$ given that $u^{(0)} = 0$. This justifies (21).

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