Design of Pulse Shapes Based on Sampling with Gaussian Prefilter

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Abstract—**Two new pulse shapes for communications are presented. The first pulse shape generates a set of pulses without intersymbol interference (ISI) or ISI-free for short. In the neighborhood of the origin it is similar in shape to the classical cardinal sine function but is of exponential decay at infinity. This pulse shape is identical to the interpolating function of a generalized sampling theorem with Gaussian prefilter. The second pulse shape is obtained from the first pulse shape by spectral factorization. Besides being also of exponential decay at infinity, it has a causal appearance since it is of superexponential decay for negative times. It is closely related to the orthonormal generating function considered earlier by Unser in the context of shift-invariant spaces. This pulse shape is not ISI-free but it generates a set of orthonormal pulses. The second pulse shape may also be used to define a receive matched filter so that at the filter output the ISI-free pulses of the first kind are recovered.**

I. Introduction

Unser [1] extended the standard sampling paradigm to the representation or even approximation of functions by elements of shift-invariant function spaces. These function spaces are defined by a generating function φ, which has to satisfy certain conditions. By one of them, the partition of unity condition, Gaussian functions are actually precluded from being used as generators. In [2] it was substantiated that Gaussian functions still could be useful generators. Consequently, following [1, Tab. 1], the interpolating generating function and the dual generating function have been computed for a Gaussian generator in [2]. The computation of the corresponding orthonormal generating function, φ_{ortho}, is accomplished in Section III of the present paper. Rather than extracting the square root as suggested in [1], our approach is based on spectral factorization and leads, interestingly enough, to expressions in terms of q-analogs [3]. As an application, two new pulse shapes are proposed and discussed in Section IV.

The following notations and conventions are adopted: $L^2(\mathbb{R})$ is the space of square-integrable functions (or finite-energy signals) $f : \mathbb{R} \to \mathbb{C} \cup \{\infty\}$ with inner product $\langle f_1, f_2\rangle = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,\mathrm{d}x$ and norm $\|f\| = \langle f, f\rangle^{1/2}$. For the Fourier transform we use the definition $\hat{f}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} f(x)\,\mathrm{d}x$, where x denotes time and ω angular frequency. $\ell^2(\mathbb{Z})$ is the space of square-summable, complex-valued sequences indexed by the integers. Finally, $\delta_n = 0$, $n \in \mathbb{Z}\setminus\{0\}$, and $\delta_0 = 1$.

II. Sampling in Shift-Invariant Spaces and Localization Spaces Revisited

The purpose of the present section is to motivate the pulse shapes presented in Section IV. To this end, we give a brief overview of sampling in shift-invariant spaces and localization spaces with an emphasis on those spaces defined by a Gaussian generator or a Gaussian prefilter respectively.

Suppose that $\varphi \in L^2(\mathbb{R})$ is a continuous function satisfying $\varphi(x) = O(|x|^{-1-\epsilon})$ as $x \to \pm\infty$ for some $\epsilon > 0$ and for any $\lambda > 0$ the system of functions $\{\varphi(\cdot - n\lambda); n \in \mathbb{Z}\}$ forms a Riesz basis in $L^2(\mathbb{R})$. Furthermore, suppose that for any $\lambda > 0$

$$\sum_{n=-\infty}^{\infty} \varphi(n\lambda)\mathrm{e}^{-\mathrm{i}n\lambda\omega} \neq 0,\ \omega \in \mathbb{R}. \tag{1}$$

The shift-invariant space $V_\lambda(\varphi)$ is the subspace of $L^2(\mathbb{R})$ defined as

$$V_\lambda(\varphi) = \left\{ f; f(x) = \sum_{n\in\mathbb{Z}} c_n \varphi(x - n\lambda),\ c \in \ell^2(\mathbb{Z}) \right\}. \tag{2}$$

Then, the following sampling theorem applies.

Theorem 1: For any $\lambda > 0$ and $f \in V_\lambda(\varphi)$ it holds that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n\lambda)\varphi_{\text{int}}(x - n\lambda),\ x \in \mathbb{R}, \tag{3}$$

where the interpolating function $\varphi_{\text{int}} \in V_\lambda(\varphi)$ is given by

$$\begin{aligned}\hat{\varphi}_{\text{int}}(\omega) &= \frac{\hat{\varphi}(\omega)}{\sum_{n\in\mathbb{Z}} \varphi(n\lambda)\mathrm{e}^{-\mathrm{i}n\lambda\omega}} \\ &= \frac{\hat{\varphi}(\omega)}{\frac{\Lambda}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \hat{\varphi}(\omega + k\Lambda)}, \quad \Lambda = \frac{2\pi}{\lambda}.\end{aligned} \tag{4}$$

This theorem is due to Walter [4], who originally proved it for orthonormal bases $\{\varphi(\cdot - n); n \in \mathbb{Z}\}$. The theorem was later extended to Riesz bases by Unser [1], who also considered alternative generating functions for $V_\lambda(\varphi)$ like φ_{int} as above and φ_{ortho} (see below). In an attempt to retain the flavor of the original Whittaker–Kotelnikov–Shannon (WKS) sampling theorem [5], in [6] prior to sampling a prefilter

$$(\boldsymbol{P}_\varphi f)(x) = \int_{-\infty}^{\infty} f(y)\overline{\varphi(y - x)}\,\mathrm{d}y \tag{5}$$

with prefilter function $\varphi \in L^2(\mathbb{R})$ has been applied to an arbitrary finite-energy signal $f \in L^2(\mathbb{R})$. The so-called localization space

$$\mathcal{P}_\varphi = \{g = \boldsymbol{P}_\varphi f; f \in L^2(\mathbb{R})\}$$

then corresponds to the space of bandlimited, finite-energy signals in the classical WKS sampling theorem. The goal is to recover the filter output signal $g = \boldsymbol{P}_\varphi f$ from sample values $g(n\lambda)$, $n \in \mathbb{Z}$, either perfectly or at least with an acceptable error. To this end, the autocorrelation function Φ of φ,

$$\Phi = \boldsymbol{P}_\varphi \varphi \in \mathcal{P}_\varphi \overset{\text{Fourier}}{\longleftrightarrow} \hat{\Phi}(\omega) = \sqrt{2\pi}|\hat{\varphi}(\omega)|^2, \tag{6}$$

is needed. The (second) interpolating function $\Phi_{\text{int}} \in \mathcal{P}_\varphi$ is defined by its Fourier transform

$$\hat{\Phi}_{\text{int}}(\omega) = \frac{\hat{\Phi}(\omega)}{\frac{\Lambda}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{\Phi}(\omega + k\Lambda)}, \tag{7}$$

where Λ is as in (4). Note that because of the Riesz basis condition still imposed on φ (see [6] for the full set of assumptions), which is equivalent to the existence of positive constants A and B (possibly depending on λ) so that [7]

$$0 < A \leq \Lambda \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + k\Lambda)|^2 \leq B < \infty,\ \omega \in \mathbb{R}, \tag{8}$$

the denominator in (7) never will vanish. We remark that in general $\Phi_{\text{int}} \neq \boldsymbol{P}_\varphi \varphi_{\text{int}}$. In the special case of a Gaussian prefilter function,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}(1/\beta)}\, \mathrm{e}^{-\frac{x^2}{2(1/\beta)^2}} \overset{\text{Fourier}}{\longleftrightarrow} \hat{\varphi}(\omega) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{\omega^2}{2\beta^2}}, \tag{9}$$

where the parameter $\beta > 0$ controls effective bandwidth, the following generalized sampling theorem has been obtained [2], [6].

Theorem 2: For any $\lambda > 0$ let the interpolating function Φ_{int} be defined by (7). Then for any function $g \in \mathcal{P}_\varphi$, where $g = \boldsymbol{P}_\varphi f$, $f \in L^2(\mathbb{R})$, the function $\tilde{g}$ given by

$$\tilde{g}(x) = \sum_{n\in\mathbb{Z}} g(n\lambda)\Phi_{\text{int}}(x - n\lambda),\ x \in \mathbb{R}, \tag{10}$$

is again in $\mathcal{P}_\varphi$, it perfectly reconstructs g at the sampling instants $x_n = n\lambda$, $n \in \mathbb{Z}$, and for all other $x \in \mathbb{R}$ the squared relative error $\left(|g(x) - \tilde{g}(x)|/\|f\|\right)^2$ becomes small as soon as $\lambda \leq 1/\beta$, then decaying superexponentially to zero as $\lambda \to 0$.

For the Gaussian prefilter function φ as given in (9) (which will be assumed for the rest of the paper) one has

$$\hat{\Phi}_{\text{int}}(\omega) = \frac{\lambda}{\sqrt{2\pi}} \frac{\mathrm{e}^{-\frac{i\pi}{\tau}(\omega/\Lambda)^2}}{\sqrt{-i\tau}\vartheta_3(\omega/\Lambda, \tau)}, \tag{11}$$

where

$$\vartheta_3(z, \tau) = \frac{1}{\sqrt{-\mathrm{i}\tau}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}\pi}{\tau}(z+n)^2} \tag{12}$$

is a Jacobi theta function [8] with parameter τ given by

$$\tau = \mathrm{i}(\lambda\beta)^2/(4\pi). \tag{13}$$

By inversion of the Fourier transform we obtain after use of Jacobi's $\tau \to -1/\tau$ transformation that [2], [9]

$$\Phi_{\text{int}}(x) = \frac{\mathrm{i}\pi\tau}{\vartheta_1'(0, -1/\tau)} \frac{\vartheta_1(x/\lambda, -1/\tau)}{\sinh(\mathrm{i}\pi\tau x/\lambda)}, \tag{14}$$

where $\vartheta_1(z, \tau) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}(-1)^n \sin[(2n+1)\pi z]$, $q = \mathrm{e}^{\mathrm{i}\pi\tau}$, $\Im(\tau) > 0$, is another theta function [8]. When $\lambda \leq 1/\beta$, it holds with high accuracy that $\Phi_{\text{int}}(x) \approx S_0(x)$, where [9]

$$S_0(x) \triangleq \mathrm{i}\tau \frac{\sin(\pi x/\lambda)}{\sinh(\mathrm{i}\pi\tau x/\lambda)},\ x \in \mathbb{R}. \tag{15}$$

Note that in our context $\mathrm{i}\tau$ is always a negative real number and that $\Phi_{\text{int}}(x)$ decays exponentially to zero as $x \to \pm\infty$.

Fig. 1 shows the interpolating function Φ_{int} for $\beta = 100$ and $\lambda = 1/\beta$; actually, the approximation (15) for $\Phi_{\text{int}}(x)$ has been used.

III. Spectral Factorization of the Interpolating Function

We start by compiling a few prerequisites [3].

Definition 1: For any $a \in \mathbb{R}$ and $q \in \mathbb{C}$ with $|q| < 1$ the q-Pochhammer symbol $(a; q)_n$ is defined by

$$\begin{aligned}
(a; q)_n &= (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),\ n = 1, 2, \ldots, \\
(a; q)_\infty &= \lim_{n\to\infty} (a; q)_n\,, \\
(a; q)_0 &= 1.
\end{aligned}$$

The following identity of Euler holds true for any $q \in \mathbb{C}, |q| < 1$, and $z \in \mathbb{C}$:

$$1 + \sum_{n=1}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q; q)_n} z^n = \prod_{n=0}^{\infty} (1 + zq^n). \tag{16}$$

Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}z^{-1})(1 + q^{2n-1}z) \tag{17}$$

will play a central role in the proof of the next theorem. In our case always

$$q = \mathrm{e}^{\mathrm{i}\pi\tau}, \tag{18}$$

where the parameter τ is as in (13). The special q-Pochhammer symbols

$$(q^2; q^2)_n = \prod_{k=1}^{n} (1 - q^{2k}), \quad Q_0 \triangleq (q^2; q^2)_\infty$$

will occur frequently.

Theorem 3: For the function

$$\varphi_{\text{ortho}}(x) = \frac{1}{\sqrt{(q^2; q^2)_\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(q^2; q^2)_n}\, \phi(x - n\lambda), \tag{19}$$

where

$$\phi(x) = \frac{\beta^{1/2}}{\pi^{1/4}}\, \mathrm{e}^{-\frac{x^2}{2(1/\beta)^2}} \tag{20}$$

is the Gaussian function (9) normalized to unit energy, i.e. $\int_{\mathbb{R}} |\phi(x)|^2\, \mathrm{d}x = 1$, it holds that

$$\hat{\Phi}_{\text{int}}(\omega) = \sqrt{2\pi}|\hat{\varphi}_{\text{ortho}}(\omega)|^2,\ \omega \in \mathbb{R}. \tag{21}$$

Proof: The definition (11) of the interpolating function Φ_{int} may be written in the Fourier domain as

$$\hat{\Phi}_{\text{int}}(\omega) = \sqrt{2\pi}\lambda \frac{|\hat{\varphi}(\omega)|^2}{\sqrt{-\mathrm{i}\tau}\vartheta_3(\omega/\Lambda, \tau)}.$$

Then, Eq. (21) becomes

$$|\hat{\varphi}_{\mathrm{ortho}}(\omega)|^2 = \lambda \frac{|\hat{\varphi}(\omega)|^2}{\sqrt{-\mathrm{i}\tau}\vartheta_3(\omega/\Lambda,\tau)}.$$

Since the theta function (12) has the second representation [8]

$$\vartheta_3(x,\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos(2\pi n x) = \sum_{n=-\infty}^{\infty} q^{n^2}\mathrm{e}^{2\pi \mathrm{i} n x},$$

we obtain by means of Eq. (17), putting

$$P(z) = \prod_{n=1}^{\infty}(1 + q^{2n-1}z^{-1})$$

and subsequently $z = \mathrm{e}^{2\pi\mathrm{i}x}$, for $\vartheta_3(x,\tau)$ the factorization

$$\vartheta_3(x,\tau) = Q_0^{1/2}P(\mathrm{e}^{2\pi\mathrm{i}x}) \cdot \overline{Q_0^{1/2}P(\mathrm{e}^{2\pi\mathrm{i}x})},\ x \in \mathbb{R}.$$

Since the function $x \mapsto \vartheta_3(x,\tau)$ is real-valued and positively lower bounded on $\mathbb{R}$ so is the function $x \mapsto |P(\mathrm{e}^{2\pi\mathrm{i}x})|$. As a consequence, the definition of φ_{ortho} in the Fourier domain by

$$\hat{\varphi}_{\mathrm{ortho}}(\omega) = \lambda^{\frac{1}{2}} \frac{\hat{\varphi}(\omega)}{(-\mathrm{i}\tau)^{1/4}Q_0^{1/2}P(\mathrm{e}^{2\pi\mathrm{i}\omega/\Lambda})}$$

will result in a function $\varphi_{\mathrm{ortho}} \in L^2(\mathbb{R})$ satisfying Eq. (21).

Now, we need to invert the Fourier transform. Since $x \mapsto 1/P(\mathrm{e}^{2\pi\mathrm{i}x})$ is a bounded 1-periodic function, it is in $L^2([0,1))$ and thus has a Fourier series expansion $1/P(\mathrm{e}^{2\pi\mathrm{i}x}) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{-2\pi\mathrm{i}nx}$, $a \in \ell^2(\mathbb{Z})$, with the coefficients

$$a_n = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}nx}\frac{1}{P(\mathrm{e}^{2\pi\mathrm{i}x})}\,\mathrm{d}x,\ n \in \mathbb{Z}. \tag{22}$$

By inverse Fourier transform we now obtain (putting $c = \lambda^{1/2}(-\mathrm{i}\tau)^{-1/4}Q_0^{-1/2}$) that

$$\begin{aligned}
\varphi_{\mathrm{ortho}}(x) &= \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}x\omega}\frac{\hat{\varphi}(\omega)}{P(\mathrm{e}^{2\pi\mathrm{i}\omega/\Lambda})}\,\mathrm{d}\omega \\
&= c\sum_{n=-\infty}^{\infty}\frac{a_n}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(x-n\lambda)\omega}\hat{\varphi}(\omega)\,\mathrm{d}\omega \\
&= c\sum_{n=-\infty}^{\infty}a_n\varphi(x-n\lambda) \\
&= \frac{1}{Q_0^{1/2}}\sum_{n=-\infty}^{\infty}a_n\phi(x-n\lambda).
\end{aligned}$$

The computation of the coefficients (22) is carried out in the complex domain.

Case $n = -1, -2, \dots$: After substitution in (22) of x by $1-x$ we obtain

$$\begin{aligned}
a_n &= \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}nx}\frac{1}{P(\mathrm{e}^{-2\pi\mathrm{i}x})}\,\mathrm{d}x \\
&= \frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{z^{-n-1}}{P(z^{-1})}\,\mathrm{d}z,
\end{aligned}$$

where integration in the contour integral is performed counterclockwise around the unit circle. Since the integrand function is analytic within a neighbourhood of the closed unit disc, we obtain by means of Cauchy's integral theorem that $a_n = 0$, $n = -1, -2, \dots$

Case $n = 0, 1, \dots$: Eq. (22) now directly yields by transition to a contour integral that

$$\begin{aligned}
a_n &= \frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{z^{n-1}}{P(z)}\,\mathrm{d}z \\
&= \lim_{M\to\infty}\underbrace{\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{z^{n-1}}{P_M(z)}\,\mathrm{d}z}_{I_M(n)},
\end{aligned}$$

where

$$P_M(z) = \prod_{k=0}^{M-1}(1 + q^{2k+1}z^{-1}),\ M = 1, 2, \dots,$$

and the path of integration is same as before. The integrand function in the integral defining $I_M(n)$ has simple poles at

$$z_m = -q^{2m+1},\ m = 0, 1, \dots, M-1,$$

lying inside of the unit circle. By means of the theorem of residues we obtain

$$I_M(n) = \sum_{m=0}^{M-1}\mathrm{Res}_{z_m}\frac{z^{n-1}}{P_M(z)},$$

whence

$$a_n = \lim_{M\to\infty}I_M(n) = \sum_{m=0}^{\infty}\mathrm{Res}_{z_m}\frac{z^{n-1}}{P(z)}.$$

We compute that

$$\begin{aligned}
\mathrm{Res}_{z_m}\frac{z^{n-1}}{P(z)} &= \mathrm{Res}_{z_m}\frac{z^n}{zP(z)} \\
&= \mathrm{Res}_{z_m}\frac{z^n}{(z-z_m)\prod_{k=0,k\neq m}^{\infty}(1+q^{2k+1}z^{-1})} \\
&= \frac{z_m^n}{\prod_{k=0,k\neq m}^{\infty}(1+q^{2k+1}z_m^{-1})} \\
&= \frac{(-q)^n q^{2mn}}{(q^2;q^2)_\infty\prod_{k=1}^{m}(1-q^{-2k})} \\
&= \frac{(-q)^n}{(q^2;q^2)_\infty}\frac{(-1)^m q^{m(m+1)}(q^{2n})^m}{\prod_{k=1}^{m}(1-q^{2k})},
\end{aligned}$$

treating in the case of $m = 0$ the empty product as one. After replacement of q in Eq. (16) with q^2 we get

$$1 + \sum_{m=1}^{\infty}\frac{q^{m(m-1)}}{(q^2;q^2)_m}z^m = \prod_{m=0}^{\infty}(1 + zq^{2m}).$$

Hence

$$\begin{aligned}
a_n &= \frac{(-q)^n}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{m(m-1)}}{(q^2;q^2)_m}(-q^{2(n+1)})^m \\
&= \frac{(-q)^n}{(q^2;q^2)_\infty}\prod_{m=0}^{\infty}(1 - q^{2(n+1)}q^{2m}) \\
&= \frac{(-q)^n}{(q^2;q^2)_n},
\end{aligned}$$

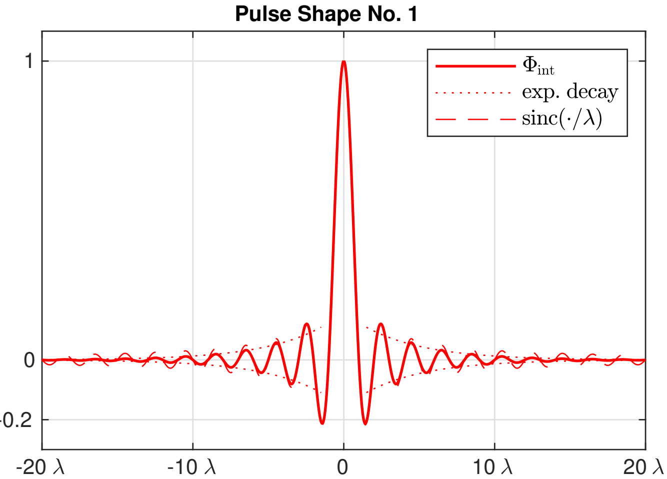

Fig. 1. Φ_{int} for $\beta = 100$ and $\lambda = 1/\beta$; sinc is the cardinal sine function defined by $\text{sinc}(x) = \sin(\pi x)/(\pi x)$.

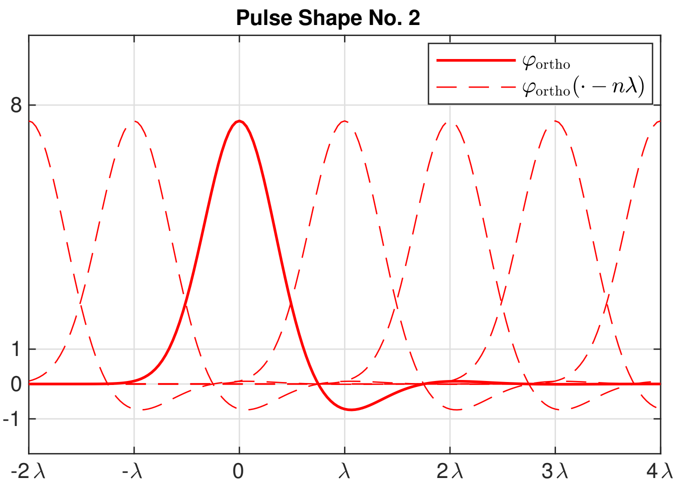

Fig. 2. φ_{ortho} together with some of its orthogonal translates for $\beta = 100$ and $\lambda = 3/\beta$.

which concludes the proof of Theorem 3. ∎

It can be shown that $V_\lambda(\varphi_{\text{ortho}}) = V_\lambda(\varphi)$, where $V_\lambda(\varphi)$ is again the shift-invariant space (2) with Gaussian generator φ as given in (9). In Fig. 2, the function φ_{ortho} together with some of its translates is depicted for bandwidth parameter $\beta = 100$ and $\lambda = 3/\beta$.

IV. Applications

A. ISI-Free Pulses

The function Φ_{int} in (14) may be used as a pulse shape to generate ISI-free pulses. Indeed, from representation (7) it is readily seen that for all $\omega \in \mathbb{R}$ it holds

$$\sum_{n\in\mathbb{Z}} \Phi_{\text{int}}(n\lambda)\mathrm{e}^{-in\lambda\omega} = \frac{\Lambda}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{\Phi}_{\text{int}}(\omega + k\Lambda) = 1 \qquad (23)$$

(the first equation being Poisson's summation formula; e.g., [1]), which is equivalent to $\Phi_{\text{int}}(n\lambda) = \delta_n$, $n \in \mathbb{Z}$. Therefore, the set of shifted pulses $\{\Phi_{\text{int}}(x - n\lambda); n \in \mathbb{Z}\}$ is ISI-free at points in time $x_n = n\lambda$. The second equation in (23) is, of course, the well-known Nyquist criterion for ISI-free pulses [10]. Concerning the use of the also ISI-free pulses generated by (15) (for *arbitrary* positive parameters β and λ) see [11].

B. Orthonormal Pulses with ISI-Free Matched Filter Output

1) Orthonormal Pulses: For any function $\varphi \in L^2(\mathbb{R})$ the system of functions $\{\varphi(x - n\lambda); n \in \mathbb{Z}\}$ forms an orthonormal system in $L^2(\mathbb{R})$ if and only if in (8) $A = B = 1$ may be chosen [1]. For the function φ_{ortho} of Theorem 3 this is true because of Eq. (21) and the second equation in (23). Therefore, φ_{ortho} may be used as a pulse shape to generate a set $\{\varphi_{\text{ortho}}(x - n\lambda);\ n \in \mathbb{Z}\}$ of orthonormal pulses.

2) Matched Filter: At the receiver, the filter $\boldsymbol{P}_{\varphi_{\text{ortho}}}$ obtained by replacing the prefilter function φ in (5) with φ_{ortho} forms a matched filter allowing optimal detection of the pulses $\varphi_{\text{ortho}}(x - n\lambda)$ at points in time $x_n = n\lambda$, $n \in \mathbb{Z}$, in the presence of noise. Due to orthogonality, overlap of adjacent pulses is of no relevance. Moreover, since [cf. (6) and Eq. (21)]

$$\boldsymbol{P}_{\varphi_{\text{ortho}}}\varphi_{\text{ortho}} = \Phi_{\text{int}},$$

the ISI-free pulses $\Phi_{\text{int}}(x - n\lambda)$, $n \in \mathbb{Z}$, of Section IV-A are recovered at the matched filter output.

In the light of the last application, the two proposed pulse shapes Φ_{int} and φ_{ortho} are seen to correspond to the classical pulse shapes produced by a raised-cosine filter or a root raised-cosine filter respectively [10]. Recall that the proposed pulse shapes decay (super-)exponentially to zero as $x \to \pm\infty$ whereas the classical pulse shapes merely decay order of x^{-3}.

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