Causal Identification with Relaxed Positivity

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Abstract

Identifying and estimating a causal effect is a fundamental task when researchers want to infer a causal effect using an observational study without experiments. A conventional assumption is the strict positivity of the given distribution, or so called positivity (or overlap) under the unconfounded assumption that the probabilities of treatments are positive. However, there exist many environments where neither observational data exhibits strict positivity nor unconfounded assumption holds. In this work, we examine the graphical counterpart of the conventional positivity condition so as to license the use of an identification formula without strict positivity. In particular, we explore various approaches, including analysis in a post-hoc manner, do-calculus, Q-decomposition, and algorithmic, to yielding a positivity condition for an identification formula. We relate these approaches, providing a comprehensive view.

Introduction The causal effect of a set of treatment variables $\mathbf{X}$ on a disjoint set of outcome variables $\mathbf{Y}$ is said to be identifiable from a causal graph $\mathcal{G}$ if the quantity $P_{\mathbf{x}}(\mathbf{y}) = P(\mathbf{y} \mid do(\mathbf{x}))$ can be uniquely computed from any positive distribution over the observed variables [10, 9]. One simple, widely adopted identification condition is the *adjustment criterion* (backdoor criterion [5]; g-computation [8]). It yields the following form of the formula for $P_x(y)$

$$P_x(y) = \sum_z P(y \mid x, \mathbf{z}) P(\mathbf{z}), \tag{1}$$

where $\mathbf{Z}$ is an admissible set. Here, *positivity* $P(x \mid \mathbf{z}) > 0$ is assumed to license the use of the above formula for the causal effect. In the context of estimating average treatment effect, Hernán and Robins [2] state that, for each value of the covariate in the population, there are some subjects that

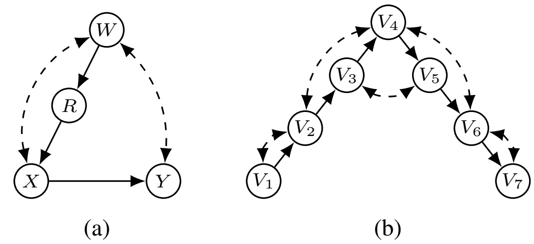

Figure 1: (a) Napkin graph; (b) Anchor graph

received the treatment—i.e., $P(X \mid \mathbf{z}) > 0$ for all $\mathbf{z}$ with $P(\mathbf{z}) \neq 0$. With all individuals receiving the same treatment, it would be impossible to estimate the causal effect from observed data [2, 1]. This leads to the original positivity condition for the adjustment:

$$\forall \mathbf{z} (P(\mathbf{z}) = 0 \vee P(x \mid \mathbf{z}) > 0), \tag{2}$$

which we will denote by $\mathrm{adj}(x; \mathbf{Z})$. In this study, we investigate the graphical equivalent of the conventional positivity condition to allow the use of identification formulas without requiring strict positivity $P(\mathbf{V}) > 0$.

Contributions (i) We provide a comprehensive view of eliciting a positivity condition over an observational distribution for an identifiable causal query given an arbitrary causal graph. (ii) In particular, we offer positivity conditions for do-calculus and generalized Q-decomposition, which are the main drivers of sound and complete identification algorithms, providing a foundation for obtaining positivity for causal effect identification. (iii) We devise an algorithmic approach to eliciting positivity through incorporating a relaxed version of generalized Q-decomposition into an existing identification algorithm. We establish a connection to post-hoc analysis of positivity[1].

There are two ways to verify if a given causal graph $\mathcal{G}$ is identifiable over the strict positivity $P(\mathbf{V})$: (i) do-calculus

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[1]This work outlines our recently accepted manuscript [3].

[6] and (ii) Q-decomposition [11]. We utilize these two methods to devise more generalized and principled ways of validating identification with relaxed positivity conditions. To begin with, we can informally examine a positivity condition under which the identification formula itself is well-defined.

Post-hoc Approach to Positivity We introduce the concept of well-definedness of a formula through a causal diagram called Napkin (Fig. 1a) [7] where its formula is $P_x(y) = \frac{\sum_w P(y,x|r,w)P(w)}{\sum_w P(x|r,w)P(w)}$. Based on this formula, we consider the (1) numerator and (2) denominator separately. If the denominator is zero, the formula is undefined. We derive the positivity condition as follows:

$$\exists r \frac{\sum_w P(y,x|r,w)P(w)}{\sum_w P(x|r,w)P(w)} \geq 0 \Leftarrow \exists r(① \geq 0 \wedge ② > 0),$$

where ① and ② is a numerator and denominator, respectively. Each condition can be expressed as

$$\begin{aligned} ① \geq 0 &\Leftarrow \mathrm{adj}(r;W), \\ ② > 0 &\Leftarrow \mathrm{adj}(r;W) \wedge \exists w(P(x \mid r,w)P(w) > 0) \\ &\Leftarrow \mathrm{adj}(r;W) \wedge \exists w(P(x \mid r,w) > 0 \wedge P(w) > 0) \\ &\Leftarrow \mathrm{adj}(r;W) \wedge \exists w(P(x,r,w) > 0) \\ &\Leftarrow \mathrm{adj}(r;W) \wedge P(x,r) > 0. \end{aligned}$$

Hence, the sufficient condition for the well-definedness of the formula is $\exists r\big(\mathrm{adj}(r;W) \wedge P(x,r) > 0\big)$. While it is true that the positivity condition derived directly from a formula ensures that the formula is well-defined, yet its validity is unclear for now since the formula is derived under strict positivity, and there *might* be some conditions that cannot be read off from the formula. We formally provide the validity of post-hoc analysis in the extended version [3].

Positivity for Do-calculus We consider developing a general approach for deriving a positivity condition by examining the conditions for the application of do-calculus [6]. Pearl [6] implicitly stated that a causal effect is identifiable from any $P(\mathbf{V}) > 0$ in a model characterized by a graph $\mathcal{G}$ if there exists a finite sequence of transformations, each conforming to one of the three rules of do-calculus. However, we can relax the positivity conditions for do-calculus taking advantage of the results from [4, 12].

Proposition 1. *Let $\mathcal{G}$ be the directed acyclic graph (DAG) associated with a causal model, and let $P(\cdot)$ be the probability distribution induced by the model. Then,*

Rule 1: $P_{\mathbf{x}}(\mathbf{y} \mid \mathbf{z},\mathbf{w}) = P_{\mathbf{x}}(\mathbf{y} \mid \mathbf{w})$ *if* $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{W})_{(\mathcal{G} \setminus \mathbf{X})}$ *and* $P_{\mathbf{x}}(\mathbf{z},\mathbf{w}) > 0$

Rule 2: $P_{\mathbf{x},\mathbf{z}}(\mathbf{y} \mid \mathbf{w}) = P_{\mathbf{x}}(\mathbf{y} \mid \mathbf{z},\mathbf{w})$ *if* $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{W})_{(\mathcal{G} \setminus \mathbf{X})_{\underline{\mathbf{Z}}}}$ *and* $P_{\mathbf{x}}(\mathbf{z},\mathbf{w}) > 0$

Rule 3: $P_{\mathbf{x},\mathbf{z}}(\mathbf{y} \mid \mathbf{w}) = P_{\mathbf{x}}(\mathbf{y} \mid \mathbf{w})$ *if* $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z} \mid \mathbf{W})_{(\mathcal{G} \setminus \mathbf{X})_{\overline{\mathbf{Z}(\mathbf{W})}}}$ *and* $P_{\mathbf{x}}(\mathbf{w}) > 0$.

For instance, we derive the condition for Napkin:

$$\begin{aligned} P_x(y) &= P_{w,r,x}(y) \\ &= P_{w,r}(y \mid x) && \text{if } P_{w,r}(x) > 0 \\ &= P_{w,r}(y,x)/P_{w,r}(x) && \text{if } P_{w,r}(x) > 0 \\ &= P_r(y,x)/P_r(x) \\ &= \frac{\sum_{w'} P(y,x|r,w')P(w')}{\sum_{w'} P(x|r,w')P(w')}. && \text{if } \mathrm{adj}(r;W) \end{aligned}$$

This derivation eventually yields the same result as the one from the post-hoc analysis.

Relaxed Q-decomposition We modify Q-decomposition [11] so that it does not rely on the strict positivity. The intuition behind the generalization is that the product of fractions often can be shortened by canceling out terms depending on the topological order. We illustrate an example in Fig. 1b. Here, $Q[\mathbf{H}]$ is factorized as $Q[\mathbf{H}] = Q[\mathbf{H}_1] \cdot Q[\mathbf{H}_2]$ where $\mathbf{H}_1 = \{V_1, V_2, V_4, V_6, V_7\}$ and $\mathbf{H}_2 = \{V_3, V_5\}$. Denoting $Q[\mathbf{H}^{\preceq i}]$ as Q_i for brevity, if $Q[\mathbf{H}] = Q_7 > 0$, then $Q[\mathbf{H}_1] = \frac{Q_7}{Q_6} \cdot \frac{Q_6}{Q_5} \cdot \frac{Q_4}{Q_3} \cdot \frac{Q_2}{Q_1} \cdot \frac{Q_1}{Q_0}$ and $Q[\mathbf{H}_2] = \frac{Q_5}{Q_4} \cdot \frac{Q_3}{Q_2}$ by [11, Lemma. 4]. Since Q_6 and Q_1 can be canceled out, we can write $Q[\mathbf{H}_1] = \frac{Q_7}{Q_5} \cdot \frac{Q_4}{Q_3} \cdot \frac{Q_2}{Q_0}$. We show that this expression is valid if $Q_5 > 0$, and further show that it is still possible to identify $Q[\mathbf{H}_1]$ when some of the denominators are 0, i.e., $Q_5 = 0$ or $Q_3 = 0$, relaxing the strict positivity condition of $Q[\mathbf{H}] > 0$ in Tian and Pearl [11].

Theorem 1. *Given $\mathbf{H} \subseteq \mathbf{V}$, let $\mathbf{H}' \in \mathrm{cc}(\mathcal{G}[\mathbf{H}])$ where $I_{\mathcal{G}[\mathbf{H}],\prec}(\mathbf{H}') = \{(l_d, r_d)\}_{d=1}^{T}$ and $\mathrm{cc}(\cdot)$ is c-components. Then, the following holds: (i) If $Q[\mathbf{H}^{\preceq l_T - 1}] > 0$, then $Q[\mathbf{H}'] = \prod_{d=1}^{T} \frac{Q[\mathbf{H}^{\preceq r_d}]}{Q[\mathbf{H}^{\preceq l_d - 1}]}$. (ii) If $Q[\mathbf{H}^{\preceq r_m}] = 0$ and $Q[\mathbf{H}^{\preceq l_m - 1}] > 0$ for some m, then $Q[\mathbf{H}'] = 0$.*

This generalizes Tian and Pearl [11] where irrelevant c-factors can be canceled out taking the positivity of $Q[\mathbf{H}^{\preceq l_T - 1}]$. Further, even when such positivity assumption is violated, still it provides a condition where the c-factor is identified as zero. In Fig. 1b, our theorem states that $Q[\mathbf{H}_1] = \frac{Q_7}{Q_5} \cdot \frac{Q_4}{Q_3} \cdot \frac{Q_2}{Q_0}$ if $Q_5 > 0$, and $Q[\mathbf{H}_2] = \frac{Q_5}{Q_4} \cdot \frac{Q_3}{Q_2}$ if $Q_4 > 0$. If $Q_4 = 0$ and $Q_3 > 0$, $Q[\mathbf{H}_1] = 0$. Similarly, if $Q_2 = 0$ then $Q[\mathbf{H}_1] = 0$. On the other hand, if $Q_4 > 0$ and $Q_5 = 0$, we cannot make a conclusion on $Q[\mathbf{H}_1]$.

This characterization of Q-decomposition with respect to positivity allows us to construct an identification algorithm named IDENTIFY+, which simultaneously returns a positivity condition implied by the resulting identification formula. In the extended version of this paper [3], we formally present IDENTIFY+ and provide its soundness.

Conclusion We offer a thorough approach to establishing a positivity condition essential for identifying causal effects in graphical models.

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