CONVERGENCE ANALYSIS OF TSETLIN MACHINES FOR BASIC BOOLEAN OPERATORS UNDER NOISE-FREE AND NOISY TRAINING CONDITIONS

Anonymous authorsPaper under double-blind review

ABSTRACT

The Tsetlin Machine (TM) is an innovative machine learning algorithm grounded in propositional logic, achieving state-of-the-art performance in a variety of pattern recognition tasks. While previous studies have analyzed its convergence properties for the 1-bit operator under both noisy and noise-free training conditions, as well as for the XOR operator under noise-free conditions, this work extends the theoretical analysis to the AND and OR operators, thereby providing a more comprehensive analysis of fundamental Boolean operations. We show that the TM almost surely converges to correctly represent the AND and OR operators when trained on noise-free data over an infinite time horizon. Notably, our analysis of the OR operator reveals a distinct characteristic: the TM is capable of representing two sub-patterns jointly within a single clause, which contrasts with its behavior in the XOR case. Furthermore, we investigate the TM's behavior when learning AND, OR, and XOR operators under noisy training conditions, including the presence of mislabeled samples and irrelevant input variables. In the presence of incorrect labels, the TM does not converge to the exact target operators but remains capable of learning efficiently. In contrast, when irrelevant variables are present, the TM still converges almost surely to the correct operators. Together, these results offer a comprehensive theoretical foundation for understanding the convergence behavior of TMs across basic Boolean operators.

1 Introduction

The Tsetlin Machine (TM) is a classification algorithm. A TM (Granmo, 2018) organizes clauses, each associated with a team of Tsetlin Automata (TAs) (Tsetlin, 1961), to collaboratively capture distinct sub-patterns¹ for a certain class. A TA, which is the core learning entity of TM, is a kind of learning automata (Zhang et al., 2020; Yazidi et al., 2019; Omslandseter et al., 2022) that tackles the multi-armed bandit problem, learning the optimal action through the interaction with its environment which gives rewards and penalties. In a TM, all TAs play in a game orchestrated by the TM's feedback tables. Each TA takes care of one literal of input, which takes boolean values of either 0 or 1. Literals are basically features of the input data. A TA decides to "Include" or "Exclude" the literal, i.e., to consider or not to consider the feature in the final classification. A clause is a conjunction of all included literals, representing a sub-pattern of a certain class. Once distinct subpatterns are learned by a number of clauses, the overall pattern recognition task is completed by a voting scheme from the clauses.

The TM and its variations (Granmo et al., 2019; Abeyrathna et al., 2021; Darshana Abeyrathna et al., 2020; Sharma et al., 2023) have been employed in many applications, such as word sense disambiguation (Yadav et al., 2021c), aspect-based sentiment analysis (Yadav et al., 2021b), novelty detection (Bhattarai et al., 2021), text classification (Yadav et al., 2021a) with enhanced interpretability (Yadav et al., 2022), federated learning (Qi et al., 2025), signal classification (Jeeru et al., 2025a;b), and solving contextual bandit problems (Seraj et al., 2022), indicating better or competitive performance compared with most of the state-of-the-art techniques. Indeed, the TM is a symbolic AI algorithm that can be structured into deep connected architectures similar to neural

¹The concept of sub-pattern will be found in the example given in Section 2.

networks. The novel algorithm design gives it the potential to tackle complex machine learning tasks while maintaining transparent inference and learning (Bhattarai et al., 2024; Abeyrathna et al., 2023; Rafiev et al., 2022). Moreover, being entirely based on logical operations, TM is hardware-friendly and low in power consumption (Maheshwari et al., 2023; Rahman et al., 2022; Kishore et al., 2023; Tunheim et al., 2025a;b).

The TM is proven to almost surely convergence to the Identity/NOT operator with 1-bit input in (Zhang et al., 2022), where the role of the hyperparameter s is also revealed. In (Jiao et al., 2022), TM's convergence to the XOR operator with 2-bit input was proven, highlighting the functionality of the hyperparameter T. In this paper, we focus on analyzing the AND and OR operators with noise-free training samples, followed by an examination of the convergence properties of AND, OR, and XOR with noisy training samples, including the presence of wrong labels and irrelevant inputs.

This paper differs from previous studies in several key aspects. While (Zhang et al., 2022) used stationary distribution analysis of discrete-time Markov chains (DTMC), the current study focuses on absorbing states. For XOR (Jiao et al., 2022), where sub-patterns are bit-wise exclusive, TM learns and converges to sub-patterns individually. In contrast, the OR operator's sub-patterns share features (e.g., $[x_1 = 1, x_2 = 1]$ and $[x_1 = 1, x_2 = 0]$ share $x_1 = 1$), allowing joint representation. We show that TM can effectively learn and represent these shared features, making the convergence process distinct. Additionally, this paper examines the role of Type II feedback, which was omitted in the prior XOR convergence study. Most notably, we analyze the convergence properties of the AND, OR, and XOR operators under noisy training samples, substantially advancing the theoretical understanding and contribution of this work.

It is worth noting that learning 2-bit operators with or without noise, is a well-studied problem. For example, numerous studies in concept learning and probably approximately correct learning have extensively explored this topic (Valiant, 1984; Haussler et al., 1994; Mansour & Parnas, 1998; Belaid et al., 2025). While many elegant methods exist for learning conjunctions or disjunctions, their existence does not necessarily imply that the TM converges to such operators in the same manner. TM employs a unique approach, learning from samples to construct conjunctive expressions and coordinating these expressions across various sub-patterns, which merits its own dedicated analysis.

The remainder of the paper is organized as follows. In Section 2, we summarize the notation used for the TM. Sections 3 and 4 present the convergence analysis of the AND and OR operators, respectively, under noise-free conditions. In Section 5, we revisit the convergence proof for the XOR operator to underscore both its relevance and its differences. Section 6 provides the analytical results of the AND, OR, and XOR operators under noisy training conditions. Finally, the paper is concluded in Section 7.

2 NOTATIONS OF THE TM

To make the article self-contained, we present the TM notation. For more details on the inference and training concept, please refer to Appendix A.

The input of a TM is indicated as $\mathbf{X} = [x_1, x_2, \dots, x_o]$, where $x_k \in \{0, 1\}, k = 1, 2, \dots, o$, and o is the number of features. A literal is either x_k in the original form or its negation $\neg x_k$. A clause is a conjunction of literals. Each literal is associated with a TA. The TA is a 2-action learning automaton whose job is to decide whether to Include/Exclude its literal in/from the clause, based on the current state of the TA. A clause is associated with 2o TAs, forming a TA team. A TA team is denoted in general as $\mathcal{G}^i_j = \{\mathrm{TA}^{i,j}_{k'}|1 \leq k' \leq 2o\}$, where k' is the index of the TA, j is the index of the TA team/clause (multiple TA teams form a TM), and i is the index of the TM/class to be identified (A TM identifies a class, multiple TMs identify multiple classes).

Suppose we are investigating the i^{th} TM whose job is to identify class i, and that the TM is composed of m TA teams. Then $C^i_j(\mathbf{X})$ can be used to denote the output of the j^{th} TA team, which is a conjunctive clause:

$$\text{Training}: C_j^i(\mathbf{X}) = \left\{ \begin{pmatrix} \bigwedge_{k \in \xi_j^i} x_k \\ \\ 1, \end{pmatrix} \land \begin{pmatrix} \bigwedge_{k \in \bar{\xi}_j^i} \neg x_k \\ \\ 1, \end{pmatrix}, \quad \text{for } \xi_j^i, \; \bar{\xi}_j^i \neq \emptyset, \\ \text{for } \xi_j^i, \; \bar{\xi}_j^i = \emptyset.$$
 (1)

Testing:
$$C_j^i(\mathbf{X}) = \begin{cases} \begin{pmatrix} \bigwedge_{k \in \xi_j^i} x_k \\ 0, \end{pmatrix} \wedge \begin{pmatrix} \bigwedge_{k \in \bar{\xi}_j^i} \neg x_k \end{pmatrix}, & \text{for } \xi_j^i, \ \bar{\xi}_j^i \neq \emptyset, \\ 0, & \text{for } \xi_j^i, \ \bar{\xi}_j^i = \emptyset. \end{cases}$$
 (2)

In Eqs. (1) and (2), ξ_j^i and $\bar{\xi}_j^i$ are defined as the sets of indexes for the literals that have been included in the clause. ξ_j^i contains the indexes of included original inputs, x_k , whereas $\bar{\xi}_j^i$ contains the indexes of included negated inputs, $\neg x_k$.

Each clause represents a sub-pattern associated with class i by including a literal (a feature or its negation) if it contributes to the sub-pattern, or excluding it when deemed irrelevant. Multiple clauses, i.e., the TA teams, are assembled into a complete TM to sum up the outputs of the clauses $f_{\sum}(\mathcal{C}^i(\mathbf{X})) = \sum_{j=1}^m C^i_j(\mathbf{X})$, where $\mathcal{C}^i(\mathbf{X})$ is the set of clauses for class i. The output of the TM

is further determined by the unit step function: $\hat{y}^i = \begin{cases} 0, & \text{for } f_{\sum}(\mathcal{C}^i(\mathbf{X})) < Th \\ 1, & \text{for } f_{\sum}(\mathcal{C}^i(\mathbf{X})) \geq Th \end{cases}$, where Th is a predefined threshold for classification. This is indeed a voting scheme.

Note that the TM can assign polarity to each TA team (Granmo, 2018), and one can refer to Appendix A for more information. In this study, for ease of analysis, we consider only positive polarity clauses. Nevertheless, this does not change the nature of TM learning.

Example: We use TM learning the OR logic as an example. A sample is classified into the OR class if its two bits and label follow the OR logic: $0.1 \Rightarrow 1, 1.0 \Rightarrow 1, 1.1 \Rightarrow 1, \text{ or } 0.0 \Rightarrow 0$. Note that once the TM learns the pattern outputting 1, it inherently learns the complementary pattern outputting 0. Hence the TM's learning and reasoning can be understood primarily as identifying the pattern that results in an output of 1. In the OR logic, sub-patterns outputting 1 are 0.1, 1.0, 0, and 0.1, 0, and 0.1,

A clause is learned by a TA team, and a TM can be composed of multiple TA teams. A TA team is a set of TAs, each responsible for handling one literal. A literal is an input feature, in this example, x_1 or x_2 , or the negation of it: $\neg x_1$ or $\neg x_2$. In this example, each TA team consists of four TAs, managing four literals: x_1 , $\neg x_1$, x_2 , $\neg x_2$, respectively.

A TA decides, by its current state (which changes according to the feedback as shown in Table 1 and Table 2), whether to Include or Exclude its literal in/from the final clause. In a TA team of four TAs, if TA_1 includes x_1 , TA_2 excludes $\neg x_1$, TA_3 excludes x_2 , and TA_4 includes x_2 , the resulting clause from this TA team will be $x_1 \land \neg x_2$.

A TM learns the pattern of the OR relationship from the input samples that follow the OR logic (training). As the training result, some TA teams converge to clauses like $\neg x_1 \land x_2$, others to $x_1 \land \neg x_2$, or $x_1 \land x_2$, all outputting 1. The process of determining whether an input conforms to the OR logic involves summing the outputs of all the clauses. Let's assume we have three TA teams, each converging to one of the sub-patterns, then the sum is $sum = (\neg x_1 \land x_2) + (x_1 \land \neg x_2) + (x_1 \land x_2)$. If a test sample $\{[x_1, x_2], y\} = \{[0, 1], 1\}$ is put into the TM, the output will be $sum = (1 \land 1) + (0 \land 0) + (0 \land 1) = 1 + 0 + 0 = 1$, indicating one TA team votes for positive classification. If the threshold Th is defined as 1, as $sum \ge Th$, TM evaluates the sample following the OR logic (testing).

Training: In the training of a TM, the labeled data $(\mathbf{X} = [x_1, x_2, ..., x_o], \ y^i)$ is fed into the TM, where the TAs are guided by the feedback defined in Tables 1 and 2. Type I Feedback is triggered when the training sample has a positive label: $y^i = 1$, while Type II feedback is utilized when $y^i = 0$. s controls the granularity of the clauses. NA means not applicable. Examples demonstrating TA state transitions per feedback tables can be found in Section 3.1 in (Zhang et al., 2022). In brief, Type I feedback reinforces true positive and Type II feedback fights against false negative.

To avoid situations where a majority of the TA teams learn a *subset* of sub-patterns, forming an incomplete representation², the hyperparameter T is used to regulate the resource allocation. The strategy works as follows (Granmo, 2018):

²In the OR example, one should avoid to have a majority of TA teams converge to $\neg x_1 \land x_2$ to represent the sub-pattern of [0, 1], and ignore the other sub-patterns [1, 0] and [1, 1].

Value of the clause $C_i^i(\mathbf{X})$		1		0	
Value of the Literal $x_k/\neg x_k$		1	0	1	0
	P(Reward)	$\frac{s-1}{s}$	NA	0	0
Include Literal	P(Inaction)	$\frac{1}{s}$	NA	$\frac{s-1}{s}$	$\frac{s-1}{s}$
	P(Penalty)	ŏ	NA	$\frac{1}{s}$	$\frac{1}{s}$
	P(Reward)	0	$\frac{1}{s}$	$\frac{1}{s}$	$\frac{1}{s}$
Exclude Literal	P(Inaction)	$\frac{1}{s}$	$\frac{s-1}{s}$	$\frac{s-1}{s}$	$\frac{s-1}{s}$
	P(Penalty)	$\frac{s-1}{s}$	0	0	0

Table 1: Type I Feedback — Feedback upon receiving a sample with label $y^i = 1$, for a single TA to decide whether to Include or Exclude a given literal $x_k/\neg x_k$ into C^i_j . NA means not applicable (Granmo, 2018).

Value of the clause $C_i^i(\mathbf{X})$		1		0	
Value of the Literal $x_k / \neg x_k$		1	0	1	0
Include Literal	P(Reward)	0	NA	0	0
	P(Inaction)	1.0	NA	1.0	1.0
	P(Penalty)	0	NA	0	0
Exclude Literal	P(Reward)	0	0	0	0
	P(Inaction)	1.0	0	1.0	1.0
	P(Penalty)	0	1.0	0	0

Table 2: Type II Feedback — Feedback upon receiving a sample with label $y^i = 0$, for a single TA to decide whether to Include or Exclude a given literal $x_k/\neg x_k$ into C_i^i . (Granmo, 2018).

Generating Type I Feedback. If the label of the training sample X is $y^i = 1$, we generate, in probability, *Type I Feedback* for each clause $C_i^i \in \mathcal{C}^i$ according to:

$$u_1 = \frac{T - \max(-T, \min(T, f_{\sum}(\mathcal{C}^i(\mathbf{X}))))}{2T}.$$
 (3)

Generating Type II Feedback. If the label of the training sample **X** is $y^i = 0$, we generate, again, in probability, *Type II Feedback* to each clause $C_i^i \in \mathcal{C}^i$ according to:

$$u_2 = \frac{T + \max(-T, \min(T, f_{\sum}(\mathcal{C}^i(\mathbf{X}))))}{2T}.$$
 (4)

Briefly speaking, when the number of clauses representing one sub-pattern increases (reaches T), learning from samples that correspond to that sub-pattern will decrease (stops) as the probability of triggering update will decrease (becomes 0).

3 CONVERGENCE ANALYSIS OF THE AND OPERATOR

A TM has converged when the states of its TAs do not change any longer. We assume that the training samples are noise free, i.e., $P(y=1|x_1=1,x_2=1)=1, P(y=0|x_1=0,x_2=1)=1, P(y=0|x_1=1,x_2=0)=1, P(y=0|x_1=0,x_2=0)=1,$ where P denotes probability. We also assume the training samples are independently drawn at random, and the above four cases will appear with non-zero probability, which means that all of the four types of samples will appear for infinite times.

Because the considered AND operator has only one pattern of input, i.e., $x_1 = 1, x_2 = 1$, that will trigger a true output, we employ one clause in this TM, and we thus can ignore the indices of the classes and the clauses in our notation in the proof. After simplification, $TA_k^{i,j}$ becomes TA_k , and C_1^1 becomes C. Since there are two input parameters, namely x_1 and x_2 , we implement four TAs in the clause, i.e., TA_1 , TA_2 , TA_3 , and TA_4 . TA_1 has two actions, i.e., including or excluding x_1 . Similarly, TA_2 corresponds to including or excluding TA_1 . TA_2 and TA_3 and TA_4 determine the behavior of TA_3 and TA_4 respectively.

Once the TM converge correctly to the intended operation, the resulting clause will be $x_1 \wedge x_2$, with the actions of TA_1 , TA_2 , TA_3 , and TA_4 being I, E, I, and E, respectively. Here we use "I" and "E" as abbreviations for include and exclude respectively.

Theorem 1. Any clause will converge almost surely to $x_1 \wedge x_2$ given noise free AND training samples in infinite time when $u_1 > 0$ and $u_2 > 0$.

The complete proof of Theorem 1 is in Appendix B. We here outline the concept and the main steps of the proof.

The condition $u_1 > 0$ and $u_2 > 0$ guarantees that all types of samples are provided to the TM and no specific type is blocked by Eqs. (3) and (4) during training. The goal of the proof is to show that the system transitions will guarantee that there is a unique absorbing state of the TM and the absorbing state has the actions of TA_1 , TA_2 , TA_3 , and TA_4 to be I, E, I, E, respectively, corresponding to the expression $x_1 \wedge x_2$.

To simplify the analysis of joint TA transitions, we use quasi-stationary analysis by freezing the transitions of the TAs for the first input bit and focusing on the transitions of the TAs corresponding to the second input bit. Clearly, there are four possibilities when freezing the first bit x_1 . We name them as <u>cases</u>: Case 1: $TA_1 = E$, $TA_2 = I$, i.e., include $\neg x_1$. Case 2: $TA_1 = I$, $TA_2 = E$, i.e., include x_1 . Case 3: $TA_1 = E$, $TA_2 = E$, i.e., exclude both x_1 and $\neg x_1$. Case 4: $TA_1 = I$, $TA_2 = I$, i.e., include both x_1 and $TA_2 = I$, i.e., include both $x_2 = I$, i.e., include both $x_3 = I$, i.e., i.e.

In each of the above four cases, we analyze individually the transition of TA_3 (TA_4) with a given current action, under different actions of TA_4 (TA_3). We index the possibilities as <u>situations</u>: **Situation 1.** We study the transition of TA_3 when its current action is "Include", and when TA_4 is frozen to be "Include" or "Exclude". **Situation 2.** We study the transition of TA_3 when its current action is "Exclude", and when TA_4 is frozen to be "Include" or "Exclude". **Situation 3.** We study the transition of TA_4 when its current action is "Include", and when TA_3 is frozen to be "Include" or "Exclude". **Situation 4.** We study the transition of TA_4 when its current action is "Exclude", and when TA_3 is frozen to be "Include" or "Exclude".

Within each of the situation, there are 8 possible <u>instances</u>, determined by 4 possible combinations of the input samples of x_1 and x_2 , and the two possible frozen TA actions, i.e., Include and Exclude.

As an example, we randomly select an instance in Case 1, Situation 1. The selected instance is when the training sample is ($[x_1=1,x_2=1],y=1$), and TA_4 is E. For this instance, the training sample will trigger Type I feedback because y=1. Based on the current status of the TAs, the clause is in the form $C=\neg x_1 \wedge x_2$, which evaluates to 0 based on the input training sample. In Situation 1, the studied TA is TA_3 , whose corresponding literal is $x_2=1$. Given y=1, clause value 0, literal value 1, we go to Table 1, the third column of transition probabilities for "Include Literal", and find the transition of TA_3 to be: the penalty probability $\frac{1}{s}$ and the inaction probability $\frac{s-1}{s}$. To indicate the transitions of TA_3 , we have plotted the transition diagram in Fig. 1. Note that the overall transition probability is $u_1\frac{1}{s}$, where u_1 is defined in Eq. (3). Here, we have assumed $u_1>0$.

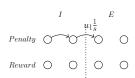


Figure 1: Transition of TA_3 when its current action is Include, TA_1 , TA_2 , and TA_4 's actions are Exclude, Include, and Exclude, respectively, upon a training sample $(x_1 = 1, x_2 = 1, y = 1)$.

Similar to the example instance, we derive a total of 128 transition instances, which can be further summarized into the overall transition behavior of TA_3 and TA_4 . These overall transitions reveal the directional dynamics of the two TAs, from which we observe that the unique absorbing state for TA_3 and TA_4 is (I, E), given that TA1 and TA2 are fixed in states I and E, respectively.

The transitions of TA_1 and TA_2 can be analyzed in the same manner as those of TA_3 and TA_4 . Based on this, we can conclude that the system has a unique absorbing state in the full dynamics of the system, where TA_1 , TA_2 , TA_3 , and TA_4 are in states of I, E, I, and E, respectively.

4 Convergence Analysis of the OR Operator

We assume the training samples for the OR operator are noise free (i.e., Eq. (5)), and are independently drawn at random. All these four cases will appear with non-zero probability.

$$P(y = 1|x_1 = 1, x_2 = 1) = 1, P(y = 1|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 1, P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(5)

Theorem 2. The clauses in a TM can almost surely learn the 2-bit OR logic given noise free training samples (shown in Eq. (5)) in infinite time, when $T \leq \lfloor \frac{m}{2} \rfloor$.

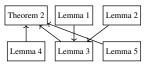


Figure 2: The dependence for the proof of the Theorem 2.

The proof of the theorem requires Lemma 1-Lemma 5 and their dependence is shown in Fig. 2. Clearly, there are three sub-patterns for the OR operator. In Lemma 1, we will show that any clause is able to converge to an intended sub-pattern when the training sample of only one sub-pattern is given, and when $u_1>0$ and $u_2>0$. In Lemma 2, we will show that the TM will become recurrent (not absorbing) when more sub-patterns jointly appear in the training samples and when $u_1>0$ and $u_2>0$. These two lemmas will be utilized in the proof of Lemma 3. Lemma 2 also reveals the recurrent nature of TM for the OR operator when the functionality of T is not enabled, i.e., when $u_1>0$ and $u_2>0$. This confirms the necessity of enabling the functionality of T in order to converge to an absorbing state that fulfills the OR operator, to be indicated by Lemma 3-Lemma 5. Specifically, Lemma 3-Lemma 5 analyze the system behavior when T is enabled and how T should be configured for the TM to converge to the OR operator. They guarantee that when the system reaches an absorbing state, the intended sub-patterns will have a number of clauses no less than T while the unintended sub-pattern will have 0 clause. Then the OR operator can be inferred by setting Th=T. In what follows, we will present and prove the lemmas.

Lemma 1. For any one of the three sub-patterns resulting in y = 1, shown in Eqs. (6)-(8), the TM can converge to the intended sub-pattern when noise free training samples following this sub-pattern are given, and when $u_1 > 0$, $u_2 > 0$.

$$P(y = 1|x_1 = 1, x_2 = 1) = P(y = 0|x_1 = 0, x_2 = 0) = 1,$$
 (6)

$$P(y = 1|x_1 = 0, x_2 = 1) = P(y = 0|x_1 = 0, x_2 = 0) = 1,$$
 (7)

$$P(y = 1|x_1 = 1, x_2 = 0) = P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
 (8)

The proof of Lemma 1 involves demonstrating convergence for three sub-patterns: those governed by Eqs. (6), (7), and (8). These analyses build upon the convergence proofs for the XOR and AND operators. For the sub-pattern in Eq. (6), transition diagrams in Appendix B confirm that the TAs converge to $TA_1 = I$, $TA_2 = E$, $TA_3 = I$, and $TA_4 = E$, when input samples $[x_1 = 0, x_2 = 1]$ and $[x_1 = 1, x_2 = 0]$ are excluded. The other two sub-patterns are proven using similar principles. Full details are provided in Appendix C.

From Lemma 1, we show that the clauses converge to the intended sub-pattern if the training samples following this particular sub-pattern are given. From Lemma 2, we will show that the system becomes recurrent if any two or more sub-patterns of training samples are given. Specifically, we show the TM is recurrent given samples following Eq. (5) and Eqs. (9)-(11), when $u_1 > 0$, $u_2 > 0$.

$$P(y = 1|x_1 = 1, x_2 = 1) = P(y = 1|x_1 = 1, x_2 = 0)$$

$$= P(y = 0|x_1 = 0, x_2 = 0) = 1,$$

$$P(y = 1|x_1 = 1, x_2 = 1) = P(y = 1|x_1 = 0, x_2 = 1)$$

$$= P(y = 0|x_1 = 0, x_2 = 0) = 1,$$
(10)

$$P(y=1|x_1=1,x_2=0) = P(y=1|x_1=0,x_2=1)$$

$$= P(y=0|x_1=0,x_2=0) = 1.$$
(11)

Lemma 2. The TM becomes recurrent if any two or more of the three sub-patterns jointly appear in the training samples, as shown in Eqs. (5), (9)-(11), when $u_1 > 0$, $u_2 > 0$.

The proof of Lemma 2 can be found in Appendix D. Lemma 2 tells us that if we always give TM the training samples from all sub-patterns without blocking the learnt patterns by using T via Eqs. (3) and (4), the system is recurrent. In other words, if we want to have the TM converge to

the OR operator in an absorbing state, it is critical to utilize the feature of T to block any incoming training samples from updating the learnt sub-patterns. Specifically, we need to configure T (1) so that the absorbing states exist and (2) confirm that the absorbing states follows the OR operator. In what follows, we will, through Lemmas 3-5, show how T via Eqs. (3) and (4) can guarantee the convergence and how the value of T should be configured.

Let's revisit the functionality of T. T can block the training samples from updating a learnt subpattern (clauses that have converged to one of the absorbing states) so that the clauses that have not converged can be guided to learn the other not-yet-learned sub-patterns. More specifically, if the number of the clauses reaches T for a certain sub-pattern, the new training samples of this subpattern will be blocked by the TM. There are three sub-patterns in OR operator. When the number of clauses for each of the three sub-patterns reaches T, all training samples associated with Type I feedback are blocked. Simultaneously, if none of the samples for Type II feedback trigger any change to the states of the TAs, the TM reaches an absorbing state. In Lemma 3, we detail the necessity and sufficiency of the absorbing state.

Lemma 3. The system is absorbed if and only if (1) the number of clauses for each intended subpattern reaches T, i.e., $f_{\Sigma}(C^{i}(\mathbf{X})) = T$, $\forall \mathbf{X} = [x_{1} = 0, x_{2} = 1]$ or $[x_{1} = 0, x_{2} = 1]$, and (2) no clause is formed only by a negated literal or negated literals.

The proof of Lemma 3 can be found in Appendix E. In Lemma 3, we find the conditions of the absorbing state. In the next Lemma, we will show how to set up the value of T so that the number of clauses for each intended sub-pattern can indeed reach T.

Lemma 4. $T \leq \lfloor m/2 \rfloor$ is required so that the number of clauses for each intended sub-pattern can reach T.

Proof of Lemma 4: There are three intended sub-patterns in the OR operator. Given m clauses in total, to make sure each one has at least T votes, we have $3T \le m$. This requires $T \le \lfloor m/3 \rfloor$ (T is an integer). However, the nature of the OR operator offers the possibility to represent 2 sub-patterns jointly. For example, T clauses in the form of x_1 will result in the number of clauses being T for each of the following sub-patterns, i.e., $[x_1 = 1, x_2 = 0]$ and $[x_1 = 1, x_2 = 1]$. If there are other T clauses representing the remaining sub-pattern, in total 2T clauses can garantee that each of the intended sub-patterns is represented by T clauses. We thus have $T \le \lfloor m/2 \rfloor$. Note that the fact that two sub-patterns can be jointly represented by one clause has been observed and confirmed in experiments shown in Section H.

When we have a smaller T, different sub-patterns may be represented by distinct clauses. However, when $T > \lfloor m/2 \rfloor$, there will always be one or two sub-patterns that cannot obtain a number of T clauses to represent them. For this reason, the maximum integer value is $T = \lfloor m/2 \rfloor$.

In Lemma 5, we show that the input sample $[x_1 = 0, x_2 = 0]$ will never cause the number of clauses associated with this unintended sub-pattern to reach or exceed T. This is to avoid any possible false positive upon input $[x_1 = 0, x_2 = 0]$ in testing.

Lemma 5. When absorbing, the sample from the unintended sub-pattern, i.e., $[x_1 = 0, x_2 = 0]$, will never lead to the number of clauses representing this unintended sub-pattern becoming greater than or equal to T.

Proof of Lemma 5: To have a positive output from $[x_1 = 0, x_2 = 0]$, the clause should be in the form of $C = \neg x_1$ or $C = \neg x_2$ or $C = \neg x_1 \land \neg x_2$. It has already shown in the proof of Lemma 3 that Type II feedback will eliminate such clauses. In fact, when the system is absorbed, no clause will be in the form of $C = \neg x_1$ or $C = \neg x_2$ or $C = \neg x_1 \land \neg x_2$. For this reason, $[x_1 = 0, x_2 = 0]$ will never lead to the number of clauses greater than or equal to T.

Proof of Theorem 2: Based on Lemmas 3–5, we understand that if $T \leq \lfloor m/2 \rfloor$ holds, Type I feedback will eventually be blocked and Type II feedback will eventually only give "inaction" feedback. In this situation, no actual transition will be triggered and thus the system reaches the absorbing state. Before absorbed, the system moves back and forth in the intermediate states. Once absorbed, any one of the intended sub-patterns will have the number of clauses for that sub-pattern no less than T and the unintended sub-pattern will have 0 clauses. We thus have the OR logic almost surely by setting a threshold Th = T and conclude the proof.

Now let's study a simple example with m=2, T=1. Here, $C_1=x_1$ and $C_2=x_2$ can be an instance for an absorbing case. $C_1=x_1$ and $C_2=\neg x_1 \wedge x_2$ also works. Clearly, the clauses can be in various forms, as long as the conditions in Lemma 3 fulfill. These converged clauses are not necessarily in the exact form of the three sub-patterns, which is distinct to that of the XOR operator.

Remark 1. Although both AND and OR operators converge, the approaches are different. For AND operator, the system is converged because the clauses become eventually absorbed to the intended pattern upon Type I and Type II feedback, even if the functionality of T is disabled $(u_1 > 0)$ and $u_2 > 0$. As the TM enables the functionality of T by default, the system will be absorbed when T clauses converge to $u_1 \wedge u_2$, before all clauses converge to this pattern. However, for the OR operator, the functionality of $u_1 > 0$ is critical because the TM is recurrent if $u_1 > 0$ and $u_2 > 0$. The absorbing state of the OR operator is achieved because the functionality of $u_1 > 0$ blocks all Type I feedback and Type II feedback gives only "Inaction" feedback. The concept of convergence for the OR operator is similar to that of XOR, but the form of clauses after absorbing varies due to the possible joint representation of sub-patterns in OR.

Remark 2. When T is greater than half the number of clauses, i.e., $T > \lfloor m/2 \rfloor$, the system will not have an absorbing state. We conjecture that the system can still learn the sub-patterns in an unbalanced manner, as long as T is not configured too close to the total number of clauses m.

Given $T > \lfloor m/2 \rfloor$, Type I feedback cannot be completely blocked and the TM is recurrent. Nevertheless, if T is not close to m, there will be clauses that possibly learn distinct sub-patterns. In addition, Type II feedback can avoid the form of $C = \neg x_1$ or $C = \neg x_2$ or $C = \neg x_1 \land \neg x_2$ from happening. Therefore, with Th > 0, the TM may still learn the OR operator with high probability.

5 REVISIT THE XOR OPERATOR

Let us revisit the proof of XOR operator. As stated in (Jiao et al., 2022), when the system is absorbed, the clauses follow the format $C=x_1\wedge \neg x_2$ or $C=\neg x_1\wedge x_2$ precisely. In other words, a clause with just one literal, such as $C=x_1$, cannot absorb the system. The reason is that the sub-patterns in XOR operator are mutual exclusive, i.e., the sub-patterns cannot be merged in any way. Although Type I feedback can be blocked when T clauses represent one sub-pattern using one literal, the Type II feedback can force the other missing literal to be included. For example, when T clauses happens to converge to $C=x_1$, the Type I feedback from any input samples of $([x_1=1,x_2=0],y=1)$ will be blocked. In this situation, the Type II feedback from $([x_1=1,x_2=1],y=0)$ will encourage the clause to include $\neg x_2$. This is because upon a sample $([x_1=1,x_2=1],y=0)$, we have Type II feedback, $C=x_1=1$, and the studied literal is $\neg x_2=0$. When the TA for excluding $\neg x_2$ is considered, a large penalty, i.e., a penalty in probability 1, is given to the TA, moving it towards action Include, and thus $C=x_1$ eventually becomes $C=x_1\wedge \neg x_2$. Following the same concept, we can analyze the development for $C=\neg x_1$, $C=x_2$, and $C=\neg x_2$, which will eventually converge to $C=\neg x_1\wedge x_2$ or $C=x_1\wedge \neg x_2$, upon Type II feedback.

6 Convergence Analysis under Random Noise

We studied the convergence properties of AND, OR, and XOR operators under training samples with noise. The noise type is *noisy completely at random* (Frénay & Verleysen, 2013), categorized as wrong labels and irrelevant input variables. A wrong label refers to an input that should be labeled as 1 but is instead labeled as 0, or vice versa. An irrelevant input variable, on the other hand, is one that does not contribute to the classification. We demonstrate that, with wrong labels, the TM does not converge to the intended operators but can still learn efficiently. With irrelevant variables, the TM converges to the intended operators almost surely. Experimental results confirmed these findings (Appendix I). We summarize the main findings in this section. The proof details can be found in Appendix F and Appendix G.

Theorem 3. The TM is recurrent given training samples with wrong labels for the AND, OR, and XOR operators.

Remark 3. The recurrent property of TM indicates that there is a non-zero probability that it cannot learn the intended operator. The primary reason for the recurrent behavior when wrong labels are present is the statistically conflicting labels for the same input samples. These inconsistency causes

the TAs within a clause to learn conflicting outcomes for the same input. When a clause learns to evaluate an input as 1 based on Type I feedback, samples with a label of 0 for the same input prompt it to learn the input as 0 through Type II feedback. This conflict in labels confuses the TM, leading to back-and-forth learning.

Remark 4. Although wrong labels will make the TM not converge (not absorbing with 100% accuracy for the intended logic), via experiments, we can still find that the TM are able to learn the operators efficiently, shown in Appendix I. This property aligns with the concept of PAC learnable (Mansour & Parnas, 1998) or ϵ -optimality (Zhang et al., 2020), although a formal proof remains open.

Theorem 4. The clauses in a TM can almost surely learn the 2-bit AND logic given training samples with k irrelevant input variables in infinite time, $0 < k < \infty$, when $T \le m$.

Theorem 5. The clauses in a TM can almost surely learn the 2-bit XOR and OR logic given training samples with k irrelevant input variables in infinite time, $0 < k < \infty$, when $T \le \lfloor m/2 \rfloor$.

When the number of irrelevant variables is large, the training set may not cover all possible examples due to the required exponential space. Although not yet theoretically proven, polynomial space for training samples seems feasible for TM, which has been confirm by experiments (Appendix I.3). This is because the TM can independently update the actions of a TA within a clause, as long as the clause value and the literal value are determined by the training sample. In other words, once the clause value and the literal value are known, the transitions triggered by Type I and Type II feedback are fully determined. As a result, the TM does not need to observe all possible combinations of irrelevant inputs to learn effectively. Instead, as long as the statistical irrelevance of certain inputs is demonstrated in the training samples, the corresponding TA transitions will be triggered accordingly. This enables the TM to learn without requiring exhaustive coverage of the input space.

7 CONCLUSIONS

In this article, we prove the convergence of the TM for the AND and OR operators with noise free training samples. Our proof for the OR operator highlights the TM's ability to learn joint subpatterns, showcasing the efficiency of its learning process. Additionally, we analyze the behavior of the TM for the AND, OR, XOR operators in the presence of random noise in the training data. Together with the convergence proofs in (Zhang et al., 2022) and (Jiao et al., 2022), this work provides a comprehensive analysis of TM convergence for the most common digital operators.

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A BRIEF OVERVIEW OF THE TM

We present the basics of TM here. Those who already are familiar with the concept and notations of TM can ignore this appendix.

A.1 BASIC CONCEPT OF THE TM

The input of a TM is denoted as $\mathbf{X} = [x_1, x_2, \dots, x_o]$, where $x_k \in \{0, 1\}, k = 1, 2, \dots, o$, and o is the number of features. A literal is either the x_k being 0 or 1 in the original form or its negation $\neg x_k$ being 1 or 0. A clause is a conjunction of literals, and each literal is associated with a TA. The TA is a 2-action learning automaton whose job is to decide whether to Include/Exclude its literal in/from the clause, and the decision is determined by the current state of the TA.

Figure 3 illustrates the structure of a TA with two actions and 2N states, where N is the number of states for each action. This study considers N as a finite number. When the TA is in any state between 0 to N-1, the action "Include" is selected. The action becomes "Exclude" when the TA is in any state between N to 2N-1. The transitions among the states are triggered by a reward or a penalty that the TA receives from the environment, which, in this case, is determined by different types of feedback defined in the TM (to be explained later).

A clause is associated with 2o TAs, forming a TA team. A TA team is denoted in general as $\mathcal{G}^i_j = \{ \mathrm{TA}^{i,j}_{k'} | 1 \leq k' \leq 2o \}$, where k' is the index of the TA, j is the index of the TA team/clause (multiple TA teams form a TM), and i is the index of the TM/class to be identified (A TM identifies a class, multiple TMs identify multiple classes).

Suppose we are investigating the i^{th} TM whose job is to identify class i, and that the TM is composed of m TA teams. Then $C_j^i(\mathbf{X})$ can be used to denote the output of the j^{th} TA team, which is a conjunctive clause:

For training:

$$C_j^i(\mathbf{X}) = \left\{ \begin{pmatrix} \bigwedge_{k \in \xi_j^i} x_k \end{pmatrix} \land \begin{pmatrix} \bigwedge_{k \in \bar{\xi}_j^i} \neg x_k \end{pmatrix}, & \text{for } \xi_j^i, \, \bar{\xi}_j^i \neq \emptyset, \\ 1, & \text{for } \xi_j^i, \, \bar{\xi}_j^i = \emptyset. \end{cases}$$
(12)

For testing:

$$C_j^i(\mathbf{X}) = \left\{ \begin{pmatrix} \bigwedge_{k \in \xi_j^i} x_k \\ 0, \end{pmatrix} \land \begin{pmatrix} \bigwedge_{k \in \bar{\xi}_j^i} \neg x_k \\ 0, \end{pmatrix}, \quad \text{for } \xi_j^i, \ \bar{\xi}_j^i \neq \emptyset, \\ \text{for } \xi_j^i, \ \bar{\xi}_j^i = \emptyset. \end{cases}$$
(13)

In Eqs. (12) and (13), ξ_j^i and $\bar{\xi}_j^i$ are defined as the sets of indexes for the literals that have been included in the clause. ξ_j^i contains the indexes of included original (non-negated) inputs, x_k , whereas $\bar{\xi}_j^i$ contains the indexes of included negated inputs, $\neg x_k$. ξ_j^i , $\bar{\xi}_j^i \neq \emptyset$ means not a single literal (feature) is included in the clause. Note that in propositional logic, an empty clause is typically defined as having a value of 1. However, empirical results indicate that TMs generally achieve higher test accuracy on new data when empty clauses are 0-valued. Therefore, during TM training, an "empty" clause outputs 1 to encourage the TAs to include literals, following the feedback mechanisms of the TM. In contrast, during TM testing, an "empty" clause outputs 0, indicating that it does not influence the final classification decision since it does not represent any specific sub-pattern.

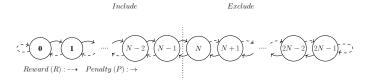


Figure 3: A two-action Tsetlin automaton with 2N states Jiao et al. (2022).

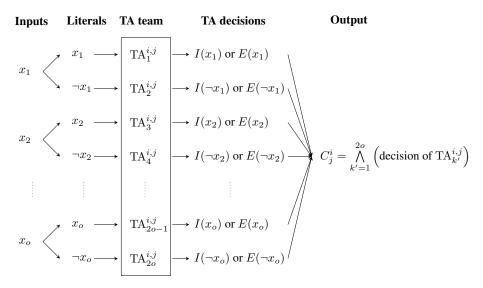


Figure 4: A TA team G_j^i consisting of 2o TAs Zhang et al. (2022). Here $I(x_1)$ means "include x_1 " and $E(x_1)$ means "exclude x_1 ".

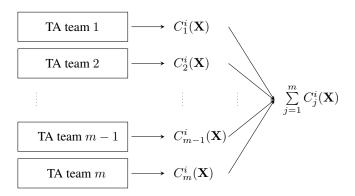


Figure 5: TM voting architecture Jiao et al. (2022).

Figure 4 illustrates the structure of a clause and its relationship to its literals. Here, for ease of notation, we define I(x) = x, $I(\neg x) = \neg x$, and $E(x) = E(\neg x) = 1$ in the analysis of the training procedure, with the latter meaning that an excluded literal does not contribute to the output.

Multiple clauses, i.e., the TA teams, each of which in conjunctive form, are assembled into a complete TM. There are two architectures for clause assembling: Disjunctive Normal Form Architecture and Voting Architecture. In this study, we focus on the latter one, as shown in Figure 5. The voting consists of summing the outputs of the clauses:

$$f_{\sum}(\mathcal{C}^{i}(\mathbf{X})) = \sum_{j=1}^{m} C_{j}^{i}(\mathbf{X}), \tag{14}$$

where $C^i(\mathbf{X})$ is the set of trained clauses for class i.

The output of the TM, in turn, is decided by the unit step function:

$$\hat{y}^{i} = \begin{cases} 0 & \text{for } f_{\sum}(\mathcal{C}^{i}(\mathbf{X})) < Th \\ 1 & \text{for } f_{\sum}(\mathcal{C}^{i}(\mathbf{X})) \ge Th \end{cases}$$
(15)

where Th is a predefined threshold for classification. For example, the classifier $(x_1 \land \neg x_2) + (\neg x_1 \land x_2)$ captures the XOR-relation given Th = 1, meaning if any sub-pattern is satisfied, the input will be identified as following the XOR logic.

Note that for the voting architecture, the TM can assign polarity to each TA team (Granmo, 2018). Specifically, TA teams with odd indices have positive polarity, learning from training samples with label 1, while those with even indices have negative polarity, learning from training samples with label 0. The only difference between these polarities is that the output of a clause associated with an even-indexed TA team will be flipped to its negative. The voting consists of summing the polarized clause outputs, and the threshold Th is set to zero. For example, for the XOR operator with four clauses, the learned clauses with positive polarity can be $C_1 = x_1 \land \neg x_2$ and $C_3 = \neg x_1 \land x_2$, while the ones with negative polarity can be $C_2 = x_1 \land x_2$ and $C_4 = \neg x_1 \land \neg x_2$. In this case, when the testing sample $[x_1 = 1, x_2 = 0]$ arrives, the sum of the clause values is 1. On the contrary, when the testing sample $[x_1 = 0, x_2 = 0]$ arrives, the sum of the clause values is -1. In this way, with Th = 0, the system's decision range and tolerance is expected to be larger.

In this study, we consider only positive polarity clauses. The reason is two-folds: firstly, in the AND/OR case, once the TM has learned out the pattern that outputs 1, it also has learned the pattern that outputs 0, as they are complementary. Therefore, the learning/reasoning process of TM can be explained from the perspective of learning the pattern that outputs 1. Secondly, for the sake of easy analysis and better understanding.

A.2 TRAINING PROCESS OF THE TM

The training process is built on letting all the TAs take part in a decentralized game. Training data $(\mathbf{X} = [x_1, x_2, ..., x_o], \ y^i)$ is obtained from a data set \mathcal{S} , distributed according to the probability distribution $P(\mathbf{X}, y^i)$. In the game, each TA is guided by Type I Feedback and Type II Feedback defined in Table 3 and Table 4, respectively. Type I Feedback is triggered when the training sample has a positive label, i.e., $y^i = 1$, meaning that the sample belongs to class i. When the training sample is labeled as not belonging to class i, i.e., $y^i = 0$, Type II Feedback is utilized for generating feedback. Examples demonstrating TA state transitions per feedback tables can be found in Section 3.1 in (Zhang et al., 2022). In brief, Type I feedback is to reinforce true positive and Type II feedback is to fight against false negative.

The parameter, s, controls the granularity of the clauses and a larger s encourages more literals to be included in each clause. A more detailed analysis on parameter s can be found in (Zhang et al., 2022).

Value of the clause $C_i^i(\mathbf{X})$		1		0	
Value of the Literal $x_k/\neg x_k$		1	0	1	0
Include Literal	P(Reward)	$\frac{s-1}{s}$	NA	0	0
	P(Inaction)	$\frac{1}{s}$	NA	$\frac{s-1}{s}$	$\frac{s-1}{s}$
	P(Penalty)	Ŏ	NA	$\frac{1}{s}$	$\frac{1}{s}$
	P(Reward)	0	$\frac{1}{s}$	$\frac{1}{s}$	$\frac{1}{s}$
Exclude Literal	P(Inaction)	$\frac{1}{s}$	$\frac{s-1}{s}$	$\frac{s-1}{s}$	$\frac{s-1}{s}$
	P(Penalty)	$\frac{s-1}{s}$	ő	Ŏ	Õ

Table 3: Type I Feedback — Feedback upon receiving a sample with label y=1, for a single TA to decide whether to Include or Exclude a given literal $x_k/\neg x_k$ into C_j^i . NA means not applicable (Granmo, 2018).

To avoid the situation that a majority of the TA teams learn only one sub-pattern (or a subset of sub-patterns) while ignore other sub-patterns, forming an incomplete representation³, the hyperparameter T is used to regulate the resource allocation. If the votes, i.e., the summation $f_{\sum}(\mathcal{C}^i(\mathbf{X}))$, for a certain sub-pattern \mathbf{X} already reach a total of T or more, neither rewards nor penalties are provided to the TAs when more training samples of this particular sub-pattern are given. In this way, we can ensure that each specific sub-pattern can be captured by a limited number, i.e., T, of available clauses, allowing sparse sub-pattern representations among competing sub-patterns. Formally, the strategy works as follows:

³For example, for the OR operator, one should avoid the situation that a majority of TA teams converge to $\neg x_1 \land x_2$ to represent the sub-pattern of [0, 1], and ignore the other sub-patterns [1, 0] and [1, 1], making the learning outcome biased/unbalanced. A proper configuration of T can avoid this situation.

Value of the clause $C^i_i(\mathbf{X})$		1		0	
Value of the Literal $x_k/\neg x_k$		1	0	1	0
	P(Reward)	0	NA	0	0
Include Literal	P(Inaction)	1.0	NA	1.0	1.0
	P(Penalty)	0	NA	0	0
	P(Reward)	0	0	0	0
Exclude Literal	P(Inaction)	1.0	0	1.0	1.0
	P(Penalty)	0	1.0	0	0

Table 4: Type II Feedback — Feedback upon receiving a sample with label y=0, for a single TA to decide whether to Include or Exclude a given literal $x_k/\neg x_k$ into C_j^i . NA means not applicable (Granmo, 2018).

Generating Type I Feedback. If the label of the training sample **X** is $y^i = 1$, we generate, in probability, *Type I Feedback* for each clause $C^i_j \in \mathcal{C}^i$. The probability of generating Type I Feedback is (Granmo, 2018):

$$u_1 = \frac{T - \max(-T, \min(T, f_{\sum}(\mathcal{C}^i(\mathbf{X}))))}{2T}.$$
(16)

Generating Type II Feedback. If the lable of the training sample **X** is $y^i = 0$, we generate, again, in probability, *Type II Feedback* to each clause $C_i^i \in \mathcal{C}^i$. The probability is (Granmo, 2018):

$$u_2 = \frac{T + \max(-T, \min(T, f_{\sum}(\mathcal{C}^i(\mathbf{X}))))}{2T}.$$
(17)

After Type I Feedback or Type II Feedback is generated for a clause, each individual TA within each clause is given a reward/penalty/inaction according to the probability defined in the Type I and Type 2 feedback tables, and then the state of the corresponding TA is updated.

B DETAILED PROOF OF THE CONVERGENCE OF THE AND OPERATOR

Proof: In this Appendix, we will prove Theorem 1. The condition $u_1 > 0$ and $u_2 > 0$ guarantees that all types of samples for AND operator, following Eq. (18), are always given and no specific type is blocked during training. The goal of the proof is to show that the system transitions will guarantee the actions of TA_1 , TA_2 , TA_3 , and TA_4 to be I, E, I, E, and these actions correspond to the unique absorbing state of the system.

$$P(y = 1|x_1 = 1, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 1, x_2 = 0) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(18)

In Subsections B.1, we will describe the transitions of the system in an exhaustive manner. Thereafter, in the Subsection B.2, we summarize the transitions in Subsection B.1 and reveal the absorbing state of the system, which is the intended AND operator.

B.1 THE TRANSITIONS OF THE TAS

In order to analyze the transitions of the system, we freeze the transition of the two TAs for the first bit of the input and study the transition of the second bit of input. Clearly, there are four <u>cases</u> for the first bit, x_1 , as:

- Case 1: $TA_1 = E$, $TA_2 = I$, i.e., include $\neg x_1$.
- Case 2: $TA_1 = I$, $TA_2 = E$, i.e., include x_1 .
- Case 3: $TA_1 = E$, $TA_2 = E$, i.e., exclude both x_1 and $\neg x_1$.
- Case 4: $TA_1 = I$, $TA_2 = I$, i.e., include both x_1 and $\neg x_1$.

In what follows, we will analyze the transition of the TAs for x_2 , given the TAs of x_1 frozen in the above four distinct cases, one by one.

B.1.1 CASE 1: INCLUDE $\neg x_1$

In this subsection, we assume that the TAs for first bit is frozen as $TA_1 = E$ and $TA_2 = I$, and thus the overall joint actions of TAs for the first bit give " $\neg x_1$ ". In this case, we have 4 <u>situations</u> to study, detailed below:

- Situation1: We study the transition of TA₃ when it has "Include" as its current action, given different actions of TA₄ (i.e., when the action of TA₄ is frozen as "Include" or "Exclude".).
- Situation 2: We study the transition of TA₃ when it has "Exclude" as its current action, given different actions of TA₄ (i.e., when the action of TA₄ is frozen as "Include" or "Exclude".).
- Situation 3: We study the transition of TA₄ when it has "Include" as its current action, given different actions of TA₃ (i.e., when the action of TA₃ is frozen as "Include" or "Exclude".).
- Situation 4: We study the transition of TA₄ when it has "Exclude" as its current action, given different actions of TA₃ (i.e., when the action of TA₃ is frozen as "Include" or "Exclude".).

In what follows, we will go through, exhaustively, the four situations.

B.1.1.1 Study TA₃ with Action Include

Here we study the transitions of TA_3 when its current action is *Include*, given different actions of TA_4 and input samples. For ease of expressions, the self-loops of the transitions are not depicted in the transition diagram. Clearly, this situation has 8 instances, depending on the variations of the training samples and the status of TA_4 , where the first four correspond to the instances with $TA_4 = E$ while the remaining four represent the instances with $TA_4 = I$.

Now we study the first instance, with $x_1=1$, $x_2=1$, y=1, and $TA_4=E$. Clearly, this training sample will trigger Type I feedback because y=1. Together with the current status of the other TAs, the clause is determined to be $C=\neg x_1 \wedge x_2=0$ and the literal is $x_2=1$. From Table 3, we know that the penalty probability is $\frac{1}{s}$ and the inaction probability is $\frac{s-1}{s}$. To indicate the transitions, we have plotted the diagram, with the transitions for penalty below. Note that the overall transition probability is $u_1 \frac{1}{s}$, where u_1 is defined in Eq. (3). Here, we have assumed $u_1>0$.

```
Condition: x_1 = 1, x_2 = 1, y = 1, TA_4 = E. Thus, Type I, x_2 = 1, C = \neg x_1 \land x_2 = 0.
```

```
\begin{array}{c|c} u_1\frac{1}{s} \\ \end{array} Penalty \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc
```

We here continue with analyzing another example shown below. In this instance, it covers the training samples: $x_1 = 1$, $x_2 = 0$, y = 0, and $TA_4 = E$. Clearly, the training sample will trigger Type II feedback because y = 0. The clause output becomes $C_3 = \neg x_1 \land x_2 = 0$. Because we now study TA_3 , the corresponding literal is $x_2 = 0$. Based on the information above, we can check from Table 4 and find the probability of "Inaction" is 1. For this reason, the transition diagram does not have any arrow, indicating that there is "No transition" for TA_3 .

```
Condition: x_1 = 1, x_2 = 0, y = 0, TA_4 = E.
Thus, Type II, x_2 = 0, C = \neg x_1 \land x_2 = 0.
```

```
864
865
         Penalty O O
866
                                             No transition
         Reward O O O
867
868
869
         The same analytical principle applies for all the other instances, and we therefore will not explain
870
         them in detail. Instead, we just list the transition diagrams.
871
         Condition: x_1 = 0, x_2 = 1, y = 0,
872
         TA_4 = E.
873
         Thus, Type II, x_2 = 1,
874
         C = \neg x_1 \land x_2 = 1.
875
                 I E
876
         Penalty O O
877
                                             No transition
878
         Reward O O O
879
880
         Condition: x_1 = 0, x_2 = 0, y = 0,
881
         TA_4 = E.
882
         Thus, Type II, x_2 = 0,
883
         C = \neg x_1 \land x_2 = 0.
884
                 I E
885
         Penalty O O
886
                                             No transition
887
         Reward O O O
888
889
         Condition: x_1 = 1, x_2 = 1, y = 1,
890
         TA_4 = I.
891
         Thus, Type I, x_2 = 1,
892
         C = \neg x_1 \wedge x_2 \wedge \neg x_2 = 0.
893
         894
895
         Reward O O O
897
898
         Condition: x_1 = 1, x_2 = 0, y = 0,
899
         TA_4 = I.
900
         Thus, Type II, x_2 = 0,
901
         C = \neg x_1 \wedge x_2 \wedge \neg x_2 = 0.
902
903
         Penalty O O O
904
                                             No transition
905
         Reward O O O
906
907
         Condition: x_1 = 0, x_2 = 1, y = 0,
908
         TA_4 = I.
909
         Thus, Type II, x_2 = 1,
910
         C = \neg x_1 \wedge x_2 \wedge \neg x_2 = 0.
911
912

        Penalty
        ○
        ○
        ○
        ○

        Reward
        ○
        ○
        ○
        ○

913
                                             No transition
914
915
916
```

```
Condition: x_1 = 0, x_2 = 0, y = 0,
918
919
        TA_4 = I.
920
        Thus, Type II, x_2 = 0,
        C = \neg x_1 \wedge x_2 \wedge \neg x_2 = 0.
921
922
923
         Penalty O O
                                           No transition
924
925
        Reward O O O
926
927
```

B.1.1.2 Study TA₃ with Action Exclude

Here we study the transitions of TA₃ when its current action is Exclude, given different actions of TA₄ and input samples. This situation has 8 instances, depending on the variations of the training samples and the status of TA₄. In this subsection and the following subsections, we will not plot the transition diagrams for "No transition".

```
932
        Condition: x_1 = 1, x_2 = 1, y = 1,
933
        TA_4 = E.
934
        Thus, Type I, x_2 = 1,
935
        C = \neg x_1 = 0.
936
937
938
        939
940
941
        Condition: x_1 = 0, x_2 = 0, y = 0,
942
        TA_4 = E.
943
        Thus, Type II, x_2 = 0,
944
        C = \neg x_1 = 1.
945
946
        Penalty \bigcirc u_2 \times 1
947
948
949
        Reward O O O
950
951
        Condition: x_1 = 1, x_2 = 1, y = 1,
952
        TA_4 = I.
953
        Thus, Type I, x_2 = 1,
        C = \neg x_1 \wedge \neg x_2 = 0.
954
955
956
        957
958
959
960
        Condition: x_1 = 0, x_2 = 0, y = 0,
961
        TA_4 = I.
962
        Thus, Type II, x_2 = 0,
963
        C = \neg x_1 \wedge \neg x_2 = 1.
964
965
966
        Penalty O O
967
```

B.1.1.3 Study TA₄ with Action Include

Reward O O

Here we list the transitions for TA₄ when its current action is *Include*.

```
972
         Condition: x_1 = 1, x_2 = 1, y = 1,
973
         TA_3 = E.
974
         Thus, Type I, \neg x_2 = 0,
975
         C = \neg x_1 \wedge \neg x_2 = 0.
         I \qquad \qquad E Penalty \qquad \qquad \square \qquad \square \qquad \square
976
977
978
979
         Reward O O
980
981
         Condition: x_1 = 1, x_2 = 1, y = 1,
982
         TA_3 = I.
983
         Thus, Type I, \neg x_2 = 0,
         C = \neg x_1 \wedge x_2 \wedge \neg x_2 = 0.
984
         I \qquad \qquad E \\ u_1 \frac{1}{s} \qquad \qquad \\ Penalty \qquad \qquad \bigcirc \qquad \bigcirc \qquad \bigcirc
985
986
987
988
         Reward O O O
989
990
         B.1.1.4 Study TA<sub>4</sub> with Action Exclude
991
         Here we list the transitions for TA_4 when its current action is Exclude.
992
993
         Condition: x_1 = 1, x_2 = 1, y = 1,
994
         TA_3 = E.
995
         Thus, Type I, \neg x_2 = 0,
         C = \neg x_1 = 0.
996
997
                 I E
998
         999
1000
1001
1002
         Condition: x_1 = 0, x_2 = 1, y = 0,
1003
         TA_3 = E.
1004
         Thus, Type II, \neg x_2 = 0,
1005
         C = \neg x_1 = 1.
1006
         1007
1008
1009
         Reward O O
1010
1011
         Condition: x_1 = 1, x_2 = 1, y = 1,
1012
         TA_3 = I.
1013
         Thus, Type I, \neg x_2 = 0,
1014
         C = \neg x_1 \land x_2 = 0.
1015
                 I E
1016
         1017
1018
1019
1020
         Condition: x_1 = 0, x_2 = 1, y = 0,
1021
         TA_3 = I.
1022
         Thus, Type II, \neg x_2 = 0,
```

 $C = \neg x_1 \wedge x_2 = 1.$

```
1026
1027
1028
1029
        Reward O
1030
1031
1032
        B.1.2 CASE 2: INCLUDE x_1
1033
1034
        For Case 2, we assume that the actions of the TAs for the first bit are frozen as TA_1 = I and
1035
        TA_2 = E, and thus the overall joint action for the first bit is "x_1". Similar to Case 1, we also have 4
1036
        situations.
1037
        B.1.2.1 Study TA<sub>3</sub> with Action Include
1038
1039
        Condition: x_1 = 1, x_2 = 1, y = 1,
1040
        TA_4 = E.
        Thus, Type I, x_2 = 1,
1041
        C = x_1 \wedge x_2 = 1.
1042
1043
1044
        Penalty O O
1045
        1046
1047
1048
        Condition: x_1 = 1, x_2 = 1, y = 1,
1049
        TA_4 = I.
1050
        Thus, Type I, x_2 = 1,
1051
        C = x_1 \wedge x_2 \wedge \neg x_2 = 0.
1052
1053
1054
1055
        Reward O O
1056
1057
        B.1.2.2 Study TA<sub>3</sub> with Action Exclude
1058
1059
        Condition: x_1 = 1, x_2 = 1, y = 1,
1060
        TA_4 = E.
        Thus, Type I, x_2 = 1,
1061
        C = x_1 = 1.
1062
1063
1064
1065
1066
        Reward O O
1067
1068
        Condition: x_1 = 1, x_2 = 0, y = 0,
1069
        TA_4 = E.
        Thus, Type II, x_2 = 0,
1070
        C = x_1 = 1.
1071
1072
1073
        Penalty O
1074
```

Reward O O O

Thus, Type I, $x_2 = 1$, $C = x_1 \land \neg x_2 = 0$.

 $TA_4 = I$.

Condition: $x_1 = 1, x_2 = 1, y = 1,$

```
1080
1081
         1082
1083
1084
1085
         Condition: x_1 = 1, x_2 = 0, y = 0,
1086
1087
         Thus, Type II, x_2 = 0,
1088
         C = x_1 \land \neg x_2 = 1.
1089
                  I E
1090
         Penalty \bigcirc u_2 \times 1
1091
1092
         Reward O O O
1093
1094
         B.1.2.3 Study TA<sub>4</sub> with Action Include
1095
         Condition: x_1 = 1, x_2 = 1, y = 1,
1096
         TA_3 = E.
1097
         Thus, Type I, \neg x_2 = 0,
1098
         C = x_1 \land \neg x_2 = 0.
1099
         I \qquad \qquad E Penalty  \qquad \qquad \square 1 \frac{1}{s} \qquad \qquad \square
1100
1101
1102
1103
         Reward O O O
1104
1105
         Condition: x_1 = 1, x_2 = 1, y = 1,
1106
         TA_3 = I.
         Thus, Type I, \neg x_2 = 0,
1107
         C = x_1 \wedge x_2 \wedge \neg x_2 = 0.
1108
         I \qquad \qquad E \\ u_1 \frac{1}{s} \qquad \qquad \\ Penalty \qquad \qquad \\ \bigcirc \qquad \qquad \\ \bigcirc
1109
1110
1111
1112
         Reward O O O
1113
1114
         B.1.2.4 Study TA<sub>4</sub> with Action Exclude
1115
         Condition: x_1 = 1, x_2 = 1, y = 1,
1116
         TA_3 = E.
1117
         Thus, Type I, \neg x_2 = 0,
1118
         C = x_1 = 1.
1119
1120
         Penalty O O
1121
         1122
1123
1124
1125
         Condition: x_1 = 1, x_2 = 1, y = 1,
1126
         TA_3 = I.
         Thus, Type I, \neg x_2 = 0,
1127
         C = x_1 \wedge x_2 = 1.
1128
1129
1130
        1131
1132
1133
```

```
1134
       B.1.3 Case 3: Exclude Both \neg x_1 and x_1
1135
1136
1137
1138
       For Case 3, we assume that the actions of TAs for the first bit are frozen as TA_1 = E and TA_2 = E,
1139
       with 4 situations. Note that in the training process, when all literals are excluded, C is assigned to 1.
1140
       B.1.3.1 Study TA<sub>3</sub> with Action Include
1141
1142
       Condition: x_1 = 1, x_2 = 1, y = 1,
1143
       TA_4 = E.
1144
       Thus, Type I, x_2 = 1,
1145
       C = x_2 = 1.
1146
1147
        Penalty O O
1148
       1149
1150
1151
       Condition: x_1 = 1, x_2 = 1, y = 1,
1152
       TA_4 = I.
1153
       Thus, Type I, x_2 = 1,
1154
       C=0.
1155
1156
1157
1158
        Reward O O
1159
1160
       B.1.3.2 Study TA<sub>3</sub> with Action Exclude
1161
1162
       Condition: x_1 = 1, x_2 = 1, y = 1,
1163
       TA_4 = E.
1164
       Thus, Type I, x_2 = 1,
1165
       C=1.
1166
1167
1168
1169
        Reward O O O
1170
1171
       Condition: x_1 = 1, x_2 = 0, y = 0,
1172
       TA_4 = E.
       Thus, Type II, x_2 = 0,
1173
1174
       C=1.
1175
        Penalty \bigcirc u_2 \times 1
1176
1177
1178
        Reward O O O
1179
1180
       Condition: x_1 = 0, x_2 = 0, y = 0,
1181
       TA_4 = E.
1182
       Thus, Type II, x_2 = 0,
1183
1184
       1185
1186
1187
```

```
1188
        Condition: x_1 = 1, x_2 = 1, y = 1,
1189
        TA_4 = I.
1190
        Thus, Type I, x_2 = 1,
        C=0.
1191
                I E
1192
        1193
1194
1195
1196
1197
        Condition: x_1 = 1, x_2 = 0, y = 0,
1198
        TA_4 = I.
1199
        Thus, Type II, x_2 = 0,
1200
        C=1.
1201
1202
1203
1204
        Reward O O O
1205
1206
        Condition: x_1 = 0, x_2 = 0, y = 0,
1207
        TA_4 = I.
1208
        Thus, Type II, x_2 = 0,
1209
        C=1.
        1210
1211
1212
1213
        Reward O O O
1214
1215
        B.1.3.3 Study TA<sub>4</sub> with Action Include
1216
        Condition: x_1 = 1, x_2 = 1, y = 1,
1217
        TA_3 = E.
1218
        Thus, Type I, \neg x_2 = 0,
1219
        C = \neg x_2 = 0.
1220
        I \qquad u_1 \frac{1}{s} \qquad E
Penalty \( \bigcirc \)
1221
1222
1223
        Reward O O O
1224
1225
        Condition: x_1 = 1, x_2 = 1, y = 1,
1226
        TA_3 = I.
1227
        Thus, Type I, \neg x_2 = 0,
1228
        C = \neg x_2 \land x_2 = 0.
1229
        I \qquad \qquad E Penalty  \bigcirc \qquad \bigcirc \qquad \bigcirc
1230
1231
1232
1233
        Reward \( \cap \)
1234
        B.1.3.4 Study TA<sub>4</sub> with Action Exclude
1235
1236
        Condition: x_1 = 1, x_2 = 1, y = 1,
1237
        TA_3 = E.
1238
        Thus, Type I, \neg x_2 = 0,
1239
        C=1.
```

```
1242
      1243
1245
1246
1247
      Condition: x_1 = 0, x_2 = 1, y = 0,
1248
1249
      Thus, Type II, \neg x_2 = 0,
1250
      C=1.
1251
      1252
1253
1254
      Reward O O O
1255
1256
      Condition: x_1 = 1, x_2 = 1, y = 1,
1257
      TA_3 = I.
1258
      Thus, Type I, \neg x_2 = 0,
1259
      C=1.
1260
            I E
1261
      Penalty O O
1262
      1263
1264
1265
      Condition: x_1 = 0, x_2 = 1, y = 0,
1266
      TA_3 = I.
1267
      Thus, Type II, \neg x_2 = 0,
1268
      C = 1.
1269
      1270
1271
1272
      Reward O O O
1273
1274
```

B.1.4 Case 4: Include Both $\neg x_1$ and x_1

For Case 4, we assume that the actions of TAs for the first bit are frozen as $TA_1 = I$ and $TA_2 = I$, and thus $C = \mathbf{0}$ always. Similarly, we also have 4 situations, detailed below.

B.1.4.1 Study TA₃ with Action Include

```
1283
1284
       Condition: x_1 = 1, x_2 = 1, y = 1,
1285
       TA_4 = E.
1286
       Thus, Type I, x_2 = 1,
1287
       C=0.
1288
1289
1290
1291
        Reward O O O
1292
1293
       Condition: x_1 = 1, x_2 = 1, y = 1,
1294
       TA_4 = I.
1295
       Thus, Type I, x_2 = 1,
       C=0.
```

```
1350
      1351
1352
1353
1354
1355
      Condition: x_1 = 1, x_2 = 1, y = 1,
1356
1357
      Thus, Type I, \neg x_2 = 0,
1358
      C=0.
1359
1360
      1361
1362
1363
1364
```

So far, we have gone through, exhaustively, the transitions of TA_3 and TA_4 for all the cases (all possible training samples and system states). Hereafter, we can summarize the direction of transitions and study the convergence properties of the system for the given training samples, to be detailed in the next subsection.

B.2 SUMMARIZE OF THE DIRECTIONS OF TRANSITIONS IN DIFFERENT CASES

Based on the analysis above, we summarize here what happens to TA_3 and TA_4 , given different status (Cases) of TA_1 and TA_2 . More specifically, we will summarize here the directions of the transitions for the TAs. For example, " $TA_3 \Rightarrow E$ " means that TA_3 will move towards the action "Exclude", while " $TA_4 \Rightarrow E$ or I" means TA_4 transits towards either "Exclude" or "Include".

Scenario 1: Study $TA_3 = I$ and $TA_4 = I$.

```
Case 1, we have:Case 3, we have:TA_3 \Rightarrow E.TA_3 \Rightarrow E.TA_4 \Rightarrow E.TA_4 \Rightarrow E.Case 2, we have:Case 4, we have:TA_3 \Rightarrow E.TA_3 \Rightarrow E.TA_4 \Rightarrow E.TA_4 \Rightarrow E.
```

From the facts presented above, we can confirm that regardless the state of TA_1 and TA_2 , if $TA_3 = I$ and $TA_4 = I$, they (TA_3 and TA_4) will eventually move out of their states.

Scenario 2: Study $TA_3 = I$ and $TA_4 = E$.

```
Case 1, we have:Case 3, we have:TA_3 \Rightarrow E.TA_3 \Rightarrow I.TA_4 \Rightarrow E or I.TA_4 \Rightarrow E or I.Case 2, we have:Case 4, we have:TA_3 \Rightarrow I.TA_3 \Rightarrow E.TA_4 \Rightarrow E.TA_4 \Rightarrow E.
```

For Scenario 2 Case 2, we can observe that if $TA_3 = I$, $TA_4 = E$, $TA_1 = I$, and $TA_2 = E$, TA_3 will move deeper to "include" and TA_4 will go deeper to "exclude". It is not difficult to derive also that TA_1 will move deeper to "include" and TA_2 will transfer deeper to "exclude" in this circumstance. This tells us that the TAs in states $TA_3 = I$, $TA_4 = E$, $TA_1 = I$, and $TA_2 = E$, reinforce each other to move deeper to their corresponding directions and they therefore construct an absorbing state of the system. If it is the only absorbing state, we can conclude that the TM converge to the intended "AND" operation.

In Scenario 2, we can observe for Cases 1, 3, and 4, the actions for TA_3 and TA_4 are not absorbing because the TAs will not be reinforced to move monotonically deeper to the states of the corresponding actions for difference cases.

For Scenario 2, Case 3, TA_4 has two possible directions to transit, I or E, depending on the input of the training sample. For action exclude, it will be reinforced when training sample $x_1=1$ and $x_2=1$ is given, based on Type I feedback. However, TA_4 will transit towards "include" side when training sample $x_1=0$ and $x_2=1$ is given, due to Type II feedback. Therefore, the direction of the transition for TA_4 is I or E, depending on the training samples. In the following paragraphs, when "or" appears in the transition direction, the same concept applies.

```
Scenario 3: Study TA_3 = E and TA_4 = I.
```

```
1412
                                               Case 3, we have:
             Case 1, we have:
1413
            TA_3 \Rightarrow E \text{ or } I.
                                               TA_3 \Rightarrow E \text{ or } I.
1414
            TA_4 \Rightarrow E.
                                               TA_4 \Rightarrow E.
1415
             Case 2, we have:
                                               Case 4, we have:
1416
            TA_3 \Rightarrow E \text{ or } I.
                                               TA_3 \Rightarrow E.
                                               TA_4 \Rightarrow E.
            TA_4 \Rightarrow E.
1417
```

In Scenario 3, we can see that the actions for $TA_3 = E$ and $TA_4 = I$ are not absorbing because the TAs will not be reinforced to move deeper to the states of the corresponding actions.

Scenario 4: Study $TA_3 = E$ and $TA_4 = E$.

```
Case 1, we have:Case 3, we have:TA_3 \Rightarrow I or E.TA_3 \Rightarrow I.TA_4 \Rightarrow I or E.TA_4 \Rightarrow I or E.Case 2, we have:Case 4, we have:TA_3 \Rightarrow I.TA_3 \Rightarrow E.TA_4 \Rightarrow E.TA_4 \Rightarrow E.
```

In Scenario 4, we see that, the actions for $TA_3 = E$ and $TA_4 = E$ seem to be an absorbing state, because the states of TAs will move deeper in Case 4. After a revisit of the condition for Case 4, i.e., include both $\neg x_1$ and x_1 , we understand that this condition is not absorbing. In fact, when TA_1 and TA_2 both have "Include" as their actions, they monotonically move towards "Exclude". Therefore, from the overall system's perspective, the system state $TA_1 = I$, $TA_2 = I$, $TA_3 = E$, and $TA_4 = E$ is not absorbing. For the other cases in this scenario, there is no absorbing state.

Based on the above analysis, we understand that there is only one absorbing condition in the system, namely, $TA_1 = I$, $TA_2 = E$, $TA_3 = I$, and $TA_4 = E$, for the given training samples with AND logic. The same conclusion applies when we freeze the transition of the two TAs for the second bit of the input and study behavior of the first bit of input. Therefore, we can conclude that the TM with a clause can learn to be the intended AND operator, almost surely, in infinite time horizon. We thus complete the proof of Theorem 1.

C PROOF OF LEMMA 1

The probability of the training samples for the noise-free OR operator can be presented by the following equations.

$$P(y = 1|x_1 = 1, x_2 = 1) = 1,$$

$$P(y = 1|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(19)

Clearly, there are three sub-patterns of x_1 and x_2 that will give y=1, i.e., $[x_1=1,\ x_2=1]$, $[x_1=1,\ x_2=0]$, and $[x_1=0,\ x_2=1]$. More specifically, Eq. (19) can be split into three cases, corresponding to the three sub-patterns:

$$P(y=1|x_1=1,x_2=1)=1, (20)$$

1455
$$P(y=0|x_1=0,x_2=0)=1,$$
1456
$$P(y=1|x_1=0,x_2=1)=1,$$
1457
$$P(y=0|x_1=0,x_2=0)=1,$$
(21)

```
1458
         and
1459
1460
                                                P(y = 1|x_1 = 1, x_2 = 0) = 1,
                                                                                                                      (22)
1461
                                                P(y = 0|x_1 = 0, x_2 = 0) = 1.
1462
1463
         In what follows, we will show the convergence of the three sub-patterns, i.e., Lemma 1.
1464
         The convergence analyses of the above three sub-patterns can be derived by reusing the analyses
1465
         of the sub-patterns of the XOR operator plus the AND operator. For the sub-pattern described by
1466
          Eq. (20), we can confirm that the TAs will indeed converge to TA_1 = I, TA_2 = E, TA_3 = I,
1467
          and TA_4 = E, by studying the transition diagrams in Subsection B when input samples of [x_1 = 0,
1468
         x_2 = 1] and [x_1 = 1, x_2 = 0] are removed. In this case, the directions of the transitions for different
1469
         scenarios are summarized below.
1470
         Scenario 1: Study TA_3 = I and TA_4 = I.
1471
1472
          Case 1, we have:
1473
         TA_3 \Rightarrow E.
         TA_4 \Rightarrow E.
1474
         Case 2, we have:
1475
         TA_3 \Rightarrow E.
1476
         TA_4 \Rightarrow E.
1477
         Case 3, we have:
1478
         TA_3 \Rightarrow E.
1479
         TA_4 \Rightarrow E.
1480
          Case 4, we have:
1481
         TA_3 \Rightarrow E.
1482
         TA_4 \Rightarrow E.
1483
1484
1485
         Scenario 2: Study TA_3 = I and TA_4 = E.
1486
         Case 1, we have:
1487
         TA_3 \Rightarrow E.
1488
         TA_4 \Rightarrow E.
1489
         Case 2, we have:
1490
         TA_3 \Rightarrow I.
         TA_4 \Rightarrow E.
1491
1492
         Case 3, we have:
1493
         TA_3 \Rightarrow I.
1494
         TA_4 \Rightarrow E.
1495
         Case 4, we have:
1496
         TA_3 \Rightarrow E.
1497
         TA_4 \Rightarrow E.
1498
1499
          Scenario 3: Study TA_3 = E and TA_4 = I.
1500
          Case 1, we have:
1501
         TA_3 \Rightarrow E \text{ or } I.
1502
         TA_4 \Rightarrow E.
1503
         Case 2, we have:
1504
         TA_3 \Rightarrow E.
1505
         TA_4 \Rightarrow E.
1506
          Case 3, we have:
1507
         TA_3 \Rightarrow E \text{ or } I.
1508
         TA_4 \Rightarrow E.
1509
         Case 4, we have:
1510
         TA_3 \Rightarrow E.
```

 $TA_4 \Rightarrow E$.

```
1512
         Scenario 4: Study TA_3 = E and TA_4 = E.
1513
         Case 1, we have:
1514
         TA_3 \Rightarrow I \text{ or } E.
1515
         TA_4 \Rightarrow E.
1516
         Case 2, we have:
1517
         TA_3 \Rightarrow I.
1518
         TA_4 \Rightarrow E.
1519
         Case 3, we have:
1520
         TA_3 \Rightarrow I.
1521
         TA_4 \Rightarrow E.
1522
         Case 4, we have:
1523
         TA_3 \Rightarrow E.
1524
         TA_4 \Rightarrow E.
1525
1526
         Comparing the analysis with the one in Subsection B.2, there is apparently another possible ab-
         sorbing case, which can be observed in Scenario 2, Case 3, where TA_3 = I and TA_4 = E, given
1527
         TA_1 = E and TA_2 = E. However, given TA_3 = I and TA_4 = E, the TAs for the first bit, i.e.,
1528
         TA_1 = E and TA_2 = E, will not move only towards Exclude. Therefore, they do not reinforce
1529
         each other to move to deeper states for their current actions. For this reason, the system in TA_3 = I,
1530
         TA_4 = E, TA_1 = E, and TA_2 = E, is not in an absorbing state. In addition, given TA_3 = I and
1531
         TA_4 = E, TA_1 and TA_2 with actions E and E will transit towards I and E, encouraging the overall
1532
         system to move towards I, E, I, and E. Consequently, the system state with TA_1 = I, TA_2 = E,
1533
         TA_3 = I, and TA_4 = E is still the only absorbing case for the given training samples following
1534
1535
         For Eq. (21), similar to the proof of in Lemma 1 in (Jiao et al., 2022), we can derive that the TAs will
1536
         converge in TA_1 = E, TA_2 = I, TA_3 = I, and TA_4 = E. The transition diagrams for the samples
1537
         of Eq. (21) are in fact a subset of the ones presented in Subsection 3.2.1 and Appendix 2 of (Jiao
1538
         et al., 2022), when the input samples of [x_1 = 1 \text{ and } x_2 = 1] are removed. We summarize below
1539
         only the directions of transitions.
1540
         The directions of the transitions of the TAs for the second input bit, i.e., x_2/\neg x_2, when the TAs
1541
         for the first input bit are frozen, are summarized as follows (based on the subset of the transition
1542
         diagrams in Subsection 3.2.1 of (Jiao et al., 2022)).
1543
         Scenario 1: Study TA_3 = I and TA_4 = I.
1544
1545
         Case 1: we have
1546
         TA_3 \rightarrow E
1547
         TA_4 \to E
1548
         Case 2: we have
1549
         TA_3 \rightarrow E
         TA_4 \rightarrow E
1550
1551
         Case 3: we have
1552
         TA_3 \rightarrow E
1553
         TA_4 \rightarrow E
         Case 4: we have
1554
         TA_3 \rightarrow E
1555
         TA_4 \rightarrow E
1556
1557
1558
         Scenario 2: Study TA_3 = I and TA_4 = E.
1559
         Case 1: we have
1560
         TA_3 \rightarrow I
1561
         TA_4 \rightarrow E
1562
         Case 2: we have
1563
         TA_3 \rightarrow E
1564
         TA_4 \rightarrow E
```

```
1566
         Case 3: we have
1567
         TA_3 \rightarrow I
1568
         TA_4 \rightarrow E
         Case 4: we have
1569
         TA_3 \rightarrow E
1570
         TA_4 \to E
1571
1572
1573
         Scenario 3: Study TA_3 = E and TA_4 = I.
1574
         Case 1: we have
1575
         TA_3 \rightarrow I, or E
1576
         TA_4 \rightarrow E
1577
         Case 2: we have
1578
         TA_3 \to E
         TA_4 \to E
1579
1580
         Case 3: we have
1581
         TA_3 \rightarrow I, or E
1582
         TA_4 \to E
         Case 4: we have
1583
1584
         TA_3 \rightarrow E
         TA_4 \rightarrow E
1585
1586
1587
         Scenario 4: Study TA_3 = E and TA_4 = E.
1588
         Case 1: we have
1589
         TA_3 \to I
1590
         TA_4 \to E
1591
         Case 2: we have
1592
         TA_3 \rightarrow E
1593
         TA_4 \rightarrow E
1594
         Case 3: we have
1595
         TA_3 \rightarrow I
1596
         TA_4 \rightarrow E
1597
         Case 4: we have
1598
         TA_3 \to E
1599
         TA_4 \to E
1600
1601
         The directions of the transitions of the TAs for the first input bit, i.e., x_1/\neg x_1, when the TAs for
1602
         the second input bit are frozen, are summarized as follows (based on the subset of the transition
1603
         diagrams in Appendix 2 of (Jiao et al., 2022)).
1604
         Scenario 1: Study TA_1 = I and TA_2 = I.
1605
1606
         Case 1: we have
1607
         TA_1 \to E
1608
         TA_2 \rightarrow E
         Case 2: we have
1609
         TA_1 \rightarrow E
1610
         TA_2 \to E
1611
1612
         Case 3: we have
1613
         TA_1 \rightarrow E
         TA_2 \rightarrow E
1614
         Case 4: we have
1615
         TA_1 \rightarrow E
1616
         TA_2 \to E
1617
1618
```

Scenario 2: Study $TA_1 = I$ and $TA_2 = E$.

```
1620
           Case 1: we have
1621
          TA_1 \rightarrow E
1622
          TA_2 \rightarrow E
           Case 2: we have
1623
           TA_1 \rightarrow E
1624
          TA_2 \rightarrow E
1625
           Case 3: we have
1626
          TA_1 \rightarrow E
1627
          TA_2 \rightarrow E
1628
           Case 4: we have
1629
          TA_1 \rightarrow E
1630
          TA_2 \rightarrow E
1631
1632
1633
          Scenario 3: Study TA_1 = E and TA_2 = I.
1634
          Case 1: we have
1635
          TA_1 \rightarrow I, or E
1636
          TA_2 \to E
1637
           Case 2: we have
1638
          TA_1 \rightarrow E
1639
          TA_2 \rightarrow I
1640
           Case 3: we have
1641
          TA_1 \rightarrow I
1642
          TA_2 \rightarrow I
1643
          Case 4: we have
          TA_1 \rightarrow E
1644
          TA_2 \rightarrow E
1645
1646
1647
          Scenario 4: Study TA_1 = E and TA_2 = E.
1648
           Case 1: we have
1649
          TA_1 \rightarrow I, or E
1650
          TA_2 \rightarrow E
1651
          Case 2: we have
1652
          TA_1 \rightarrow E
1653
          TA_2 \rightarrow I
1654
           Case 3: we have
1655
          TA_1 \rightarrow E
1656
          TA_2 \rightarrow E
1657
          Case 4: we have
1658
          TA_1 \rightarrow E
1659
          TA_2 \rightarrow E
```

1662

1663

1664

1665

1668

1669

1670

1671

1672

1673

By analyzing the transitions of TAs for the two input bits with samples following Eq. (21), we can conclude that $TA_1 = E$, $TA_2 = I$, $TA_3 = I$, and $TA_4 = E$ is an absorbing state, as the actions of TA_1 – TA_4 reinforce each other to transit to deeper states for the current actions upon various input samples. There are a few other cases in different scenarios that seem to be absorbing, but in fact not. For example, the status $TA_3 = I$ and $TA_4 = E$ seems also absorbing in Scenario 2, Case 3, i.e., when $TA_1 = E$ and $TA_2 = E$ hold. However, to make $TA_1 = E$ and $TA_2 = E$ absorbing, the condition is $TA_3 = I$ and $TA_4 = I$, or $TA_3 = E$ and $TA_4 = E$. Clearly, the status $TA_3 = I$ and $TA_4 = I$ is not absorbing. For $TA_3 = E$ and $TA_4 = E$ to be absorbing, it is required to have $TA_1 = I$ and $TA_2 = I$ to be absorbing, or $TA_1 = I$ and $TA_2 = E$ to be absorbing, which are not true. Therefore, all those absorbing-like states are not absorbing. In fact, when $TA_3 = I$, $TA_4 = E$, $TA_1 = E$, and $TA_2 = E$ hold, the condition $TA_3 = I$, $TA_4 = E$ will reinforce TA_1 and TA_2 to move towards $TA_1 = I$ and $TA_2 = I$ to the absorbing state of the system. Based on the above analysis on the transition directions, we can thus confirm the convergence of $TA_1 = I$ when training samples from Eq. (21) are given.

Following the same principle, we can also confirm that the TAs will converge to $TA_1 = I$, $TA_2 = E$, $TA_3 = E$, and $TA_4 = I$ when training samples from Eq. (22) are given, according to the proof of Lemma 2 in (Jiao et al., 2022).

D PROOF OF LEMMA 2

Proof of Lemma 2: To show the recurrent property when samples following Eq. (9) are given, we need to show that the absorbing states for Eq. (6) disappear when $([x_1 = 1, x_2 = 0], y = 1)$ is given in addition, and the same applies for Eq. (8) when $([x_1 = 1, x_2 = 1], y = 1)$ is given.

We first show that the absorbing state of $TA_1 = I$, $TA_2 = E$, $TA_3 = I$, $TA_4 = E$, for sub-pattern ($[x_1 = 1, x_2 = 1]$, y = 1) as shown in Eq. (6), disappears when sub-pattern ($[x_1 = 1, x_2 = 0]$, y = 1) is given in addition. Indeed, TA_3 will move toward E when ($[x_1 = 1, x_2 = 0]$, y = 1) is given, because a penalty is given to TA_3 as shown in Fig. 6.

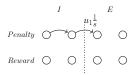


Figure 6: Transition of TA_3 when its current action is Include, TA_1 , TA_2 , and TA_4 's actions are Include, Exclude, and Exclude, respectively, upon a training sample $(x_1 = 1, x_2 = 0, y = 1)$.

Clearly, when $([x_1=1,x_2=0],y=1)$ is given in addition, TA_3 has a non-zero probability to move towards "Exclude". Therefore, "Include" is not the only direction that TA_3 moves to upon the new input. In other words, $([x_1=1,x_2=0],y=1)$ will make the state $TA_1=I$, $TA_2=E$, $TA_3=I$, $TA_4=E$, not absorbing any longer. For other states, the newly added training sample will not remove any transition from the previous case. For this reason, the system will not have any new absorbing state. Therefore, when $([x_1=1,x_2=0],y=1)$ is given in addition, the absorbing state disappears and the system will not have any new absorbing state.

Following the same concept, we show that the absorbing state for $([x_1 = 1, x_2 = 0], y = 1)$ shown in Eq. (8), i.e., $TA_1 = I$, $TA_2 = E$, $TA_3 = E$, $TA_4 = I$, disappears when sub-pattern $([x_1 = 1, x_2 = 1], y = 1)$ is given in addition. Indeed, TA_4 will also move towards E when $([x_1 = 1, x_2 = 1], y = 1)$ is given, as shown in Fig. 7.

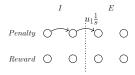


Figure 7: Transition of TA_4 when its current action is Include, TA_1 , TA_2 , and TA_3 's actions are Include, Exclude, and Exclude, respectively, upon a training sample $(x_1 = 1, x_2 = 1, y = 1)$.

Understandably, because of the newly added sub-patterns, the absorbing states in Eqs. (6) and (8) disappear and no new absorbing states are generated. In other words, the TM trained based on samples from Eq. (9) becomes recurrent.

Following the same concept, we can show that the system becomes recurrent for Eqs. (5), (10), and (11) as well. For the sake of conciseness, we will not provide the details here. In general, any newly added sub-pattern will involve a probability for the learnt sub-pattern to move outside the learnt state, making the system recurrent.

E PROOF OF LEMMA 3

Proof of Lemma 3: In Lemma 2, the TM is recurrent if the functionality of T is disabled (i.e., $u_1 > 0$, $u_2 > 0$). Therefore, for the OR operator to converge, the functionality of T is critical to block any feedback in order to form an absorbing state.

By design, TM will either be updated via Type I feedback or Type II feedback. We show via (1) the condition when Type I feedback is blocked and then show via (2) when any update from Type II feedback is not triggered. When both happen, the system will not be updated anymore and thus absorbed.

To prove (1) in Lemma 3, we show that the system is not absorbed when 0 or 1 intended sub-pattern is blocked by T. When 2 intended sub-patterns are blocked, the system will guide the clauses to learn the remaining intended sub-pattern. Only when all 3 intended sub-patterns are blocked by T, the system will stop updating based on Type I feedback.

Clearly, when no intended sub-pattern is blocked by T, the training samples provided to the system follow Eq. (5). In other words, no samples corresponding to a specific sub-pattern are blocked. Under such training conditions, as shown in Lemma 2, the TM is recurrent. When only 1 intended sub-pattern is blocked by T, the system is updated based on samples following Eqs. (9), (10), or (11), which is also recurrent.

We look at the cases when two intended sub-patterns are blocked by T but the third one is not blocked. In other words, the number of clauses for each of the two intended sub-patterns reaches at least T, and the number of clauses for the remaining sub-pattern is less than T. In this case, only one type of samples from Eqs. (6) or (7) or (8) will be provided to the TM^4 . Based on Lemma 1, we understand that all clauses, including the ones that have learnt the two blocked sub-patterns, will be forced to learn the not-yet-blocked sub-pattern. This is due to the fact that only the samples following the not-yet-blocked sub-pattern are triggering the update for the TM. In this circumstance, as soon as the not-yet-blocked sub-pattern also has T clauses, i.e., when all three sub-patterns are blocked by T at the same time, Type I feedback are blocked completely.

Note that the samples corresponding to the not-yet-blocked sub-pattern will encourage the learnt clauses (i.e., the clauses for the blocked sub-patterns) to move out from the learnt sub-patterns, and this may cause the number of clauses for the blocked sub-pattern being lower than T (thus unblocked), again. If this happens before the number of clauses for the not-yet-blocked sub-pattern reaches T, at least two sub-patterns will be in the non-blocked state, and the system becomes one of the three cases described by Eqs. (9), (10) or (11). In other words, even if an absorbing state exists after two intended sub-patterns are blocked by T, the system may not monotonically move towards the absorbing state. Nevertheless, as soon as all three intended sub-patterns are blocked by reaching T clauses, the Type I feedback will be blocked.

Here we prove (2) in Lemma 3. Type II feedback is only triggered by training sample ($[x_1=0,x_2=0],y=0$) in the OR operator. For Type II feedback, based on Table 2, a transition is triggered only when a penalty occurs, i.e., when the excluded literal has a value of 0 and the clause evaluates to 1. Specifically for the OR operation, this only happens when $C=\neg x_1 \wedge \neg x_2$ or $C=\neg x_1$ or $C=\neg x_2$. For $C=\neg x_1 \wedge \neg x_2$, based on the Type II feedback, the TA with the action "excluding x_1 " and the TA with the action "excluding x_2 " will be penalized. In other words, the actions of the two TAs for x_1 and x_2 will be encouraged to move from exclude to include side. As soon as one of the TAs (or occasionally both of them) becomes include, the clause will become $C=x_1 \wedge \neg x_1 \wedge \neg x_2$ or $C=\neg x_1 \wedge x_2 \wedge \neg x_2$ (or occasionally $C=x_1 \wedge \neg x_1 \wedge x_2 \wedge \neg x_2$). In this case, input $[x_1=0,x_2=0]$ will always result in 0 as the clause value and then the Type II feedback will not update the system any longer. Following the same concept, for $C=\neg x_2$, the Type II feedback will encourage the excluded x_1 to be included so that the clause becomes $C=x_1 \wedge \neg x_2$. The same applies to $C=\neg x_1$, which will eventually become $C=\neg x_1 \wedge x_2$ upon Type II feedback. When all clauses in $C=\neg x_2$ or $C=\neg x_1$ are also updated to $C=x_1 \wedge \neg x_2$ or $C=\neg x_1 \wedge x_2$, no Type II feedback is triggered up on any input sample.

We summarize the requirements for an absorbing state:

⁴More precisely speaking, all samples will be fed into the TM, but only samples corresponding to the notyet-blocked sub-pattern will be used by the TM for training purpose.

- For any sample X following sub-pattern $[x_1 = 1, x_2 = 1]$, or $[x_1 = 1, x_2 = 0]$, or $[x_1 = 0, x_2 = 1]$, the number of clauses for that sub-pattern, i.e., $f_{\Sigma}(\mathcal{C}^i(\mathbf{X}))$, must be at least T, no matter in which form the clauses are constructed. This will block Type I feedback.
- There are no clauses with literal(s) in only negated form, such as $C = \neg x_1$ or $C = \neg x_2$ or or $C = \neg x_1 \land \neg x_2$. This guarantees that no transition will happen upon any Type II feedback.

F ANALYSIS OF THE TM WITH WRONG TRAINING LABELS

In this appendix, we analyze the transition properties of the TM when training samples contain wrong labels.

There are two types of wrong labels:

- Inputs labeled as 0, which should be 1.
- Inputs labeled as 1, which should be 0.

We begin by examining the first type of wrong label, followed by the second type, and then address the general case.

F.1 THE AND OPERATOR WITH THE FIRST TYPE OF WRONG LABELS

To formally define training samples with the first type of wrong label, we use the following formulas:

$$P(y = 1|x_1 = 1, x_2 = 1) = a, a \in (0, 1)$$

$$P(y = 0|x_1 = 1, x_2 = 1) = 1 - a,$$

$$P(y = 0|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 1, x_2 = 0) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(23)

In this case, the label for training samples representing the intended logic $[x_1 = 1, x_2 = 1]$ is y = 1 with probability a and y = 0 with probability 1 - a. In other words, in addition to the training samples detailed in Subsection B, a new training sample will appear to the system, namely $([x_1 = 1, x_2 = 1], y = 0)$.

Lemma 6. The TM exhibits recurrence for the training samples defined in Eq. (23).

Proof: To prove this lemma, we analyze the TM's transitions as follows. First, we examine the transitions assuming $u_1 > 0$ and $u_2 > 0$, similar to the analysis in Subsection B, as detailed in Subsection F.1.1. Next, we study the impact of T to determine whether it leads to convergence (absorption), as discussed in Subsection F.1.2.

F.1.1 Transition of TM with AND Operator Given $u_1>0$ and $u_2>0$

Following the approach in Subsection B, we examine the transitions of TA_3 and TA_4 when the additional training sample ($[x_1=1,x_2=1],y=0$) is introduced, considering Cases 1 to 4 as defined in Subsection B. Since y=0 for this sample, only Type II feedback can be triggered to cause transitions. As TA_3 is responsible for the literal x_2 , which is always 1 for this sample, Type II feedback does not trigger any transitions for TA_3 . Therefore, we focus on studying the potential transitions of TA_4 in the four cases defined in Subsection B.1.

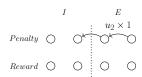
In Case 1, where $TA_1 = E$ and $TA_2 = I$, the clause value will always be 0 for the training sample because $\neg x_1$ is included in the clause, regardless of the action TA_4 takes. According to the Type II feedback transition table, no transition occurs when C = 0, so no transitions are triggered for TA_4 . Similarly, in Case 4, where $TA_1 = I$ and $TA_2 = I$, the clause value will always be 0 due to the presence of $x_1 \land \neg x_1$ in the clause. As a result, there are no transitions for TA_4 .

In Case 2, where $TA_1 = I$ and $TA_2 = E$, the literal x_1 will always appear in the clause. When $TA_4 = I$, the clause includes the literal $\neg x_2$, which results in a clause value of 0. Therefore, no

transition is triggered. However, when $TA_4 = E$, the literal x_1 will always appear in the clause, and the value of x_2 is 1, making the clause value 1 regardless of TA_3 's action (whether it includes or excludes x_2). According to the Type II feedback table, with the literal value of $\neg x_2$ being 0 and the clause value being 1, the transition for $TA_4 = E$ is:

```
Condition: x_1 = 1, x_2 = 1, y = 0.
Thus, Type II, \neg x_2 = 0,
```

C = 1.

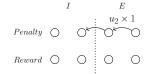


In Case 3, where $TA_1 = E$ and $TA_2 = E$, the clause value is fully determined by TA_3 and TA_4 . When TA_4 's action is to include, the clause value is 0 for this sample because it includes the literal $\neg x_2$, resulting in no transition for TA_4 . However, when TA_4 's action is to exclude, the clause value is always 1, regardless of TA_3 's action. Specifically, when TA_4 includes x_2 , the clause value is 1, as the literal value of x_2 is 1. When it is exclude, all literals are excluded and then the clause value becomes 1 by definition. By examining the transitions of TA_4 , we can summarize the following graph:

Condition:
$$x_1 = 1, x_2 = 1, y = 0.$$

Thus, Type II,
$$\neg x_2 = 0$$
,

C = 1.



We summarize the directions of the transitions when the new wrongly labeled sample is added, with the newly added actions highlighted in red.

Scenario 1: Study $TA_3 = I$ and $TA_4 = I$.

 $\begin{array}{lll} \textbf{Case 1}, \text{ we have:} & \textbf{Case 3}, \text{ we have:} \\ TA_3 \Rightarrow E. & TA_4 \Rightarrow E. \\ \textbf{Case 2}, \text{ we have:} & \textbf{Case 4}, \text{ we have:} \\ TA_3 \Rightarrow E. & TA_3 \Rightarrow E. \\ TA_4 \Rightarrow E. & TA_4 \Rightarrow E. \\ TA_4 \Rightarrow E. & TA_4 \Rightarrow E. \end{array}$

Scenario 2: Study $TA_3 = I$ and $TA_4 = E$.

 $\begin{array}{lll} \textbf{Case 1}, \text{ we have:} & \textbf{Case 3}, \text{ we have:} \\ TA_3 \Rightarrow E. & TA_3 \Rightarrow I. \\ TA_4 \Rightarrow E \text{ or I.} & TA_4 \Rightarrow E \text{ or I.} \\ \textbf{Case 2}, \text{ we have:} & \textbf{Case 4}, \text{ we have:} \\ TA_3 \Rightarrow I. & TA_3 \Rightarrow E. \\ TA_4 \Rightarrow E \text{ or I.} & TA_4 \Rightarrow E. \end{array}$

Scenario 3: Study $TA_3 = E$ and $TA_4 = I$.

Case 1, we have: Case 3, we have: $TA_3 \Rightarrow E \text{ or } I.$ $TA_3 \Rightarrow E \text{ or } I.$ $TA_4 \Rightarrow E$. $TA_4 \Rightarrow E$. Case 2, we have: Case 4, we have: $TA_3 \Rightarrow E \text{ or } I.$ $TA_3 \Rightarrow E$. $TA_4 \Rightarrow E$. $TA_4 \Rightarrow E$.

Scenario 4: Study $TA_3 = E$ and $TA_4 = E$.

Case 1, we have:Case 3, we have: $TA_3 \Rightarrow I$ or E. $TA_3 \Rightarrow I$. $TA_4 \Rightarrow I$ or E. $TA_4 \Rightarrow I$ or E.Case 2, we have: $TA_3 \Rightarrow I$. $TA_3 \Rightarrow I$. $TA_3 \Rightarrow E$. $TA_4 \Rightarrow E$ or I. $TA_4 \Rightarrow E$.

Clearly, the only absorbing state (TA₃ = I and TA₄ = E) becomes recurrent due to the newly added transition (the red I for TA₄). As a result, the system is recurrent when $u_1 > 0$ and $u_2 > 0$.

F.1.2 Transition of TM with AND Operator when T can block Type I feedback

Based on the above analysis, we understand that the system is recurrent when $u_1 > 0$ and $u_2 > 0$. Next, we examine whether it is possible for the system to become absorbing when T can block Type I feedback.

When T clauses have learned the intended pattern $\mathbf{X} = [x_1 = 1, x_2 = 1]$, i.e., when $f_{\sum}(\mathcal{C}^i(\mathbf{X})) = T$, then $u_1 = 0$ holds, and Type I feedback is blocked. In this situation, only Type II feedback can occur. Due to the presence of the wrong label, i.e., $([x_1 = 1, x_2 = 1], y = 0)$, Type II feedback triggers transitions in the TAs that have already learned the intended logic $(([x_1 = 1, x_2 = 1], y = 1))$. For example, Type II feedback will cause a transition in TAs of a learned clause $C = x_1 \wedge x_2$, making the clause deviate from its learned state (e.g., changing from $x_1 \wedge x_2$ to $x_1 \wedge x_2 \wedge \neg x_2$). Once this happens, $u_1 > 0$ holds, and Type I feedback is triggered by samples of $([x_1 = 1, x_2 = 1], y = 1)$, encouraging TAs in this clause to move back toward the action Exclude. Thus, even when T blocks all Type I feedback samples (setting $u_1 = 0$), the system remains recurrent due to the wrong label and Type II feedback. Notably, no value of $f_{\sum}(\mathcal{C}^i(\mathbf{X}))$ can make both $u_1 = 0$ and $u_2 = 0$ simultaneously⁵. Therefore, Type I and Type II feedback cannot be blocked simultaneously, ensuring the system is recurrent.

F.2 THE AND OPERATOR WITH THE SECOND TYPE OF WRONG LABELS

To properly define the training samples with the second type of wrong label, we employ the following formulas:

$$P(y = 1|x_1 = 1, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 1, x_2 = 0) = a, a \in (0, 1)$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 1 - a,$$

$$P(y = 0|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(24)

In this case, clearly, label of the training samples $[x_1 = 1, x_2 = 0]$ are wrongly labeled as 1 with probability 1-a. In other words, in addition to the training samples detailed in Subsection B, a new type (wrongly labeled) of training sample will appear to the system, namely $([x_1 = 1, x_2 = 0], y = 1)$.

Lemma 7. The TM is recurrent for the training samples given by Eq. (24).

Proof: Similar to the proof of Lemma 6, we first consider the transitions of TM with $u_1 > 0$ and $u_2 > 0$, and then examine the impact of T for the system transition.

When $u_1 > 0$ and $u_2 > 0$, there is a non-zero probability in which the training sample ($[x_1 = 1, x_2 = 0], y = 1$) will appear to the system. The appearance of this sample will involve transition of TA₃ moving from action Include toward Exclude, as shown in Fig. 6, making the system recurrent.

When T clauses have learned the intended pattern $\mathbf{X} = [x_1 = 1, x_2 = 1]$, i.e., $f_{\sum}(\mathcal{C}^i(\mathbf{X})) = T$, then $u_1 = 0$, and thus Type I feedback is blocked for this training sample. In this situation, the TM

⁵In this study, we focus only on positive polarity thus $u_2 > 0$ always holds. When negative polarity is enabled (i.e., when a set of clauses learns sub-patterns with label y = 0), u_2 becomes 0 when T clauses learn a sample with y = 0. However, it remains true that no value of $f_{\sum}(\mathcal{C}^i(\mathbf{X}))$ can make both u_1 and u_2 equal to 0 simultaneously.

can only see the training samples of the following:

$$P(y = 0|x_1 = 1, x_2 = 0) = a, a \in (0, 1)$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 1 - a,$$

$$P(y = 0|x_1 = 0, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(25)

Following the same concept as the proof of Lemma 6, we can conclude that the TM is recurrent for the samples in Eq. (25). Clearly, the system is recurrent, regardless of the value of u_1 . Therefore, we can conclude that the TM is recurrent for the training samples described in Eq. (24).

Following the same principle, we can also prove that the TM is recurrent when other training samples, i.e., $[x_1 = 0, x_2 = 1]$, and $[x_1 = 0, x_2 = 0]$, or their combinations, have wrong labels. We thus can conclude that the TM is recurrent for the second type of wrong labels.

So far, we have proven that the TM is recurrent when only one type of wrong label exists for the AND operator. It is straightforward to conclude that the TM remains recurrent when both types of wrong labels are present. The key reason is that adding both types of wrong labels does not eliminate any transitions between system states in recurrent systems. Therefore, the TM is recurrent for training samples with general wrong labels for the AND operator. Using the same reasoning, we can extend this conclusion to the XOR and OR operators. Thus, the following theorem holds.

Theorem 6. The TM is recurrent given training samples with wrong labels for the AND, OR, and XOR operators.

Remark 5. The primary reason for the recurrent behavior of the TM when wrong labels are present is the introduction of statistically conflicting labels for the same input samples. These inconsistency causes the TAs within a clause to learn conflicting outcomes for the same input due to the corresponding Type I and Type II feedback for label 1 and 0 respectively. When a clause learns to evaluate an input as 1 based on Type I feedback, samples with a label of 0 for the same input prompt it to learn the input as 0 through Type II feedback. This conflict in labels confuses the TM, leading to back-and-forth learning.

Remark 6. Note that although wrong labels will make the TM not converge (not absorbing with 100% accuracy for the intended logic), via simulations, we find that the TM can still learn the operators efficiently, which has been demonstrated in Section I, especially when the probability of wrong label is small. Interestingly, when the probability of the second type of wrong label is large, TM will consider it as a sub-pattern, and learn it, which aligns with the nature of learning.

G Analysis of the TM with an Irrelevant Input Variable

In this appendix, we examine the impact of irrelevant input noise on the TM. Irrelevant noise refers to an input bit with a random value that does not affect the classification result. For instance, in the AND operator, a third input bit, x_3 , may appear in the training sample with random 1 and 0 values, but its value does not influence the output of the AND operator. In other words, the output is entirely determined by the values of x_1 and x_2 . Formally, we have:

$$P(y = 1 | x_1 = 1, x_2 = 1, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 0 | x_1 = 1, x_2 = 0, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 0 | x_1 = 0, x_2 = 1, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 0 | x_1 = 0, x_2 = 0, x_3 = 0 \text{ or } 1) = 1.$$
(26)

Here $x_3 = 0$ or 1 means $P(x_3 = 0) = a$, $P(x_3 = 1) = 1 - a$, $a \in (0, 1)$.

G.1 Convergence Analysis of the AND Operator with Irrelevant Variable

Theorem 7. The clauses in a TM can almost surely learn the AND logic given training samples in Eq. (26) in infinite time, when $T \leq m$.

Proof: The proof of Theorem 7 consists of two steps: (1) Identifying a set of absorbing conditions and confirming that the TM, when in these conditions, satisfies the requirements of the AND operator. (2) Demonstrating that any state of the TM that deviates from the conditions defined in step (1) is not absorbing.

The TM will be absorbed when the following conditions fulfill:

- 1. Condition to block Type I feedback: For any input sample $\mathbf{X} = [x_1 = 1, x_2 = 1, x_3]$, regardless of whether $x_3 = 1$ or 0, the TM has at least T clauses that output 1.
- 2. Conditions to guarantee no action upon Type II feedback:
 - (a) When x_3 or $\neg x_3$ appears in a clause in the TM: The literals that are included in the clause for the first two input variables must result in a clause value of 0 for the input samples $\mathbf{X} = [x_1 = 0, x_2 = 1, x_3]$, $\mathbf{X} = [x_1 = 1, x_2 = 0, x_3]$ and $\mathbf{X} = [x_1 = 0, x_2 = 0, x_3]$. This ensures that C = 0 for these input samples, regardless of the value of x_3 , thereby preventing transitions caused by any Type II feedback. The portion of the clause involving the first two input variables can be, e.g., $x_1 \land x_2$ or $x_1 \land \neg x_1 \land x_2$, while the overall clauses can be, e.g., $C = x_1 \land x_2 \land x_3$, or $C = x_1 \land \neg x_1 \land x_2 \land \neg x_3$, as long as the resulted clause value is 0 for those input samples.
 - (b) When x_3 or $\neg x_3$ does NOT appear in a clause in the TM: There is no clause that is in the form of $C=x_1$, $C=x_2$, $C=x_1 \land \neg x_2$, $C=\neg x_1 \land x_2$, $C=\neg x_1$, $C=\neg x_2$, or $C=\neg x_1 \land \neg x_2$.

Clearly, when the above conditions fulfill, the system has absorbed because no feedback appears to the system. Additionally, this absorbing state follows AND operator. Based on the statement of the condition to block Type I feedback, there are at least T clauses that output 1 for input sample $\mathbf{X} = [x_1 = 1, x_2 = 1, x_3]$, regardless $x_3 = 1$ or 0. Studying the conditions for Type II feedback, we can conclude that the clause outputs 0 for all input samples $\mathbf{X} = [x_1 = 1, x_2 = 0, x_3]$, $\mathbf{X} = [x_1 = 0, x_2 = 1, x_3]$, or $\mathbf{X} = [x_1 = 0, x_2 = 0, x_3]$. We can then setup the Th = T to confirm the AND logic.

The next step is to show that any state of the TM deviating from the above conditions is not absorbing. To demonstrate this, we can simply confirm that transitions, which might change the current actions of the TAs, will occur due to updates from Type I or Type II feedback.

When literal x_3 or literal $\neg x_3$ is included as a part of the clause, the probability for C=0 is non-zero due to the randomness of input variable x_3 . As a result, Type I Feedback will encourage the TA for the included literal x_3 or $\neg x_3$ to move away from its current action, thus preventing the system from becoming absorbing.

For the case where literal x_3 or literal $\neg x_3$ is not included in the clause, the system operates purely based on the first two input variables, namely x_1 and x_2 . According our previous analysis for the noise free AND case (Theorem 1), there is only one absorbing status, which is $C=x_1 \wedge x_2$. However, this absorbing state disappears because Type I feedback will encourage the excluded literal x_3 to be included when $x_3=1$, and similarly encourage the excluded literal $\neg x_3$ to be included when $x_3=0$. Once either x_3 or $\neg x_3$ is included, the analysis in the previous paragraph applies, and thus the system is not absorbing.

From the above discussion, it is clear that Type I feedback is the key driver of action changes in non-absorbing cases. If Type I feedback is not blocked, the system cannot reach an absorbing state. Therefore, blocking Type I feedback is critical for achieving convergence. The condition T < m is to guarantee that T should not be greater than the total number of clauses, making it feasible to block Type I feedback.

Remark 7. Due to the existence of the noisy input x_3 , the system requires the functionality of T to block Type I feedback in order to converge. This contrasts with the noise-free case, where the TM will almost surely converge to the AND operator even when Type I feedback is consistently present $(u_1 > 0)$.

G.2 CONVERGENCE ANALYSIS OF THE OR OPERATOR WITH IRRELEVANT VARIABLE

For the OR case, we have

$$P(y = 1|x_1 = 1, x_2 = 1, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 1|x_1 = 1, x_2 = 0, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 1|x_1 = 0, x_2 = 1, x_3 = 0 \text{ or } 1) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 0, x_3 = 0 \text{ or } 1) = 1.$$
(27)

Theorem 8. The clauses in a TM can almost surely learn the OR logic given training samples in Eq. (27) in infinite time, when $T \leq \lfloor m/2 \rfloor$.

Proof: The proof of Theorem 8 follows a similar structure to that of the AND case and involves two steps: (1) Identifying a set of absorbing conditions and verifying that, under these conditions, the TM satisfies the requirements of the OR operator. (2) demonstrating that any state of the TM deviating from these conditions is not absorbing.

- 1. Condition to block Type I feedback: For any input sample $\mathbf{X} = [x_1 = 1, x_2 = 1, x_3]$, $\mathbf{X} = [x_1 = 1, x_2 = 0, x_3]$, and $\mathbf{X} = [x_1 = 0, x_2 = 1, x_3]$ regardless of whether $x_3 = 1$ or 0, the TM has at least T clauses that output 1.
- 2. Conditions to guarantee no action upon Type II feedback:
 - (a) When x_3 or $\neg x_3$ appears in a clause in the TM: The literals included in the clause for the first two input variables must ensure a clause value of 0 for the input samples $\mathbf{X} = [x_1 = 0, x_2 = 0, x_3]$. This is to guarantee that C = 0 for those input samples, irrespective of the value of x_3 , thereby preventing any transitions caused by Type II feedback. The portion of the clause involving the first two input variables can take the form such as $x_1, x_1 \land \neg x_2, x_1 \land x_2, x_1 \land \neg x_1 \land x_2$. Correspondingly, the overall clauses can take the form such as $C = x_1 \land \neg x_3, C = x_1 \land \neg x_2 \land x_3, C = x_1 \land x_2 \land x_3$, or $C = x_1 \land \neg x_1 \land x_2 \land \neg x_3$, as long as the resulted clause value is 0 for those input samples.
 - (b) When x_3 or $\neg x_3$ does not appear in a clause in the TM: There are no clauses with literal(s) in only negated form, such as $C = \neg x_1$, $C = \neg x_2$, or $C = \neg x_1 \land \neg x_2$.

Clearly, when the above conditions fulfill, the system is absorbing because no feedback appears to the system. Additionally, this absorbing state adheres to the OR operator. Based on the condition required to block Type I feedback, there are at least T clauses that output 1 for input sample $\mathbf{X} = [x_1 = 1, x_2 = 1, x_3]$, $\mathbf{X} = [x_1 = 1, x_2 = 0, x_3]$, or $\mathbf{X} = [x_1 = 0, x_2 = 1, x_3]$ regardless of whether $x_3 = 1$ or 0. Analyzing the conditions for Type II feedback, we find that the clause outputs 0 for all input samples $\mathbf{X} = [x_1 = 0, x_2 = 0, x_3]$. We can then setup the Th = T to confirm the OR logic.

The next step is to demonstrate that any state of the TM that deviates from the above conditions outlined above is not absorbing. To do this, we can confirm that transitions which may alter the current actions of the TAs will occur due to updates from Type I and Type II feedback.

When literal x_3 or literal $\neg x_3$ is included in the clause, there is a non-zero probability for C=0 due to the randomness of the input variable x_3 . In this case, Type I Feedback will move the included literal x_3 or $\neg x_3$ towards action Exclude, preventing the system from being absorbing.

For the case where literal x_3 or literal $\neg x_3$ is not included as a part of the clause, the system operates purely based on the first two input variables, namely x_1 and x_2 . Based on our previous analysis for the noise free OR case shown in Lemma 2, the system is recurrent. This recurrent behavior will eventually lead the system to a state where the excluded literal, either x_3 or $\neg x_3$, is encouraged to be included. For example, if the TM has a clause $C = x_1 \land x_2$, upon a training sample $\mathbf{X} = [x_1 = 1, x_2 = 1, x_3 = 0]$, the Type I feedback will encourage the excluded literal $\neg x_3$ to be included. Once one of the excluded literal, x_3 or $\neg x_3$, is included, the analysis in the previous paragraph applies, meaning the system is not absorbing.

Clearly, if Type I feedback is not blocked, the system will not be absorbing. As blocking Type I feedback is critical, condition $T \leq |m/2|$ is necessary, refer to Lemma 4.

When T clauses have learned the intended sub-patterns of OR operation, the Type I feedback will be blocked. At the same time, Type II feedback will eliminate all clauses that output 1 for input sample following $\mathbf{X} = [x_1 = 0, x_2 = 0, x_3]$, removing false positives. At this point, the system has converged. The presence of x_3 does not change the convergence feature, but it adds more dynamics to the TM.

G.3 CONVERGENCE ANALYSIS OF THE XOR OPERATOR WITH IRRELEVANT VARIABLE

Theorem 9. The clauses in a TM can almost surely learn the XOR logic given training samples in Eq. (28) in infinite time, when $T \leq \lfloor m/2 \rfloor$.

 $P\left(y=0|x_1=1,x_2=1,x_3=0\ or\ 1\right)=1, \\ P\left(y=1|x_1=1,x_2=0,x_3=0\ or\ 1\right)=1, \\ P\left(y=1|x_1=0,x_2=1,x_3=0\ or\ 1\right)=1, \\ P\left(y=0|x_1=0,x_2=0,x_3=0\ or\ 1\right)=1.$

The proof for XOR follows the same principles as the AND and OR cases, and therefore, we do not present it explicitly here.

G.4 CONVERGENCE ANALYSIS OF THE OPERATORS WITH MULTIPLE IRRELEVANT VARIABLES

In the previous subsections, we demonstrated that if a single irrelevant bit is present in the training samples, the system will almost surely converge to the intended operators. This conclusion can be readily extended to scenarios involving multiple irrelevant variables. Here, "multiple irrelevant variables" refers to the presence of additional variables, beyond x_3 , in the training samples that do not contribute to the classification.

Theorem 10. The clauses in a TM can almost surely learn the 2-bit AND logic given training samples with k irrelevant input variables in infinite time, $0 < k < \infty$, when $T \le m$.

Theorem 11. The clauses in a TM can almost surely learn the 2-bit XOR and OR logic given training samples with k irrelevant input variables in infinite time, $0 < k < \infty$, when $T \le \lfloor m/2 \rfloor$.

Proof: The proofs of Theorems 10 and 11 are straightforward. It suffices to verify whether the conditions for blocking Type I and Type II feedback remain valid when multiple irrelevant variables are present.

The condition for blocking Type I feedback remains valid because Type I feedback is only determined by the first two input bits and is not a function of the irrelevant variables. For Type II feedback, its effect depends on whether the literals for the irrelevant inputs are present in the clause. In cases where the literals of the irrelevant bits are not included in the clause, the analysis holds, as those literals are absent. When the literals of the irrelevant bits are included, their number does not impact the analysis. This is because the clause value is entirely determined by the first two bits, and the clause value remains C=0, regardless of the number of irrelevant variables.

H EXPERIMENT RESULTS WITH NOISE-FREE TRAINING SAMPLES

To validate the theoretical analyses, we here present the experiment results⁶ for both the AND and the OR operators.

Figure 8 shows the convergence of TM for the AND operator when $m=7,\,T=5,\,s=4,$ and $N=50\,(N)$ is the number of states for each action in each Tsetlin Automaton). More specifically, we plot the number of clauses that learn the AND operator, namely, $x_1=x_2=1,$ and the number of system updates as a function of epochs. From these figures, we can clearly see that after a few epochs, the TM has 5 clauses that learn the AND operator and then the system stops updating because no update is triggered anymore. Note that if we control T so that $u_1>0$ always holds, all clauses will converge to the AND operator, which has been validated via experiments. These observations confirm Theorem 1. Although the theorem says it may require infinite time in principle, the actual convergence can be much faster.

In Fig. 9, we illustrate the number of clauses in distinct sub-patterns when we employ m=7, T=3, s=4, and N=50 for the OR operator. Based on the analytical result, i.e., Theorem 2, the system will be absorbed, where each sub-pattern will have at least 3 clauses and no update will happen afterwards. From the figure, we can clearly observe that after a few epochs, the system becomes indeed absorbed as no updates are observed. When absorbed, the three intended sub-patterns have 3, 4, 5 clauses to represent them respectively, while the unintended sub-pattern has 0

 $^{^6}$ The code for validating the convergence of AND and OR operators can be found at https://github.com/JaneGlim/Convergence-of-Tsetline-Machine-for-the-AND-OR-operators.

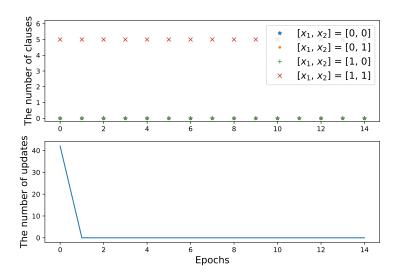


Figure 8: The convergence of a TM with 7 clauses when T=5 for the AND operator.

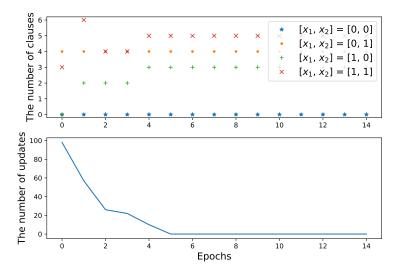


Figure 9: The convergence of a TM with 7 clauses when T=3 for the OR operator.

clause, which is consistent with the theorem. Indeed, the list of the converged clauses are: $C_1=x_1$, $C_2=x_1$, $C_3=x_2$, $C_4=x_1 \land \neg x_2$, $C_5=x_1 \land x_2$, $C_6=x_2$, and $C_7=\neg x_1 \land x_2$, explaining the number of converged clauses in different sub-patterns shown in the figure. Clearly, in this example, some clauses, i.e., C_1 , C_2 , C_3 and C_6 , can each cover multiple sub-patterns. This indicates that in real world applications, if distinct sub-patterns have certain bits in common, which can be used to differentiate it from other classes, it is possible for TM to learn those bits as joint features, confirming the efficiency of the TM.

Note that there are many other possible absorbing states that are different from the shown example, which have been observed when we run multiple instances of the experiments. As long as each intended sub-pattern is represented by at least T clauses in the OR operator, the system converges.

In Fig. 10, the configuration is identical to that in Fig. 9 except that T=4. In this case, as stated in Remark 2, the system will not become absorbing, but will still cover the intended sub-patterns with high probability. From this figure, we can observe that each intended sub-pattern is represented by at least two clauses, and that the unintended sub-pattern has zero clause. At the same time, the TAs do not stop updating their states, which can be seen in the bottom figure. It is worth mentioning that we have occasionally observed in other rounds of experiments, that one intended sub-pattern is covered

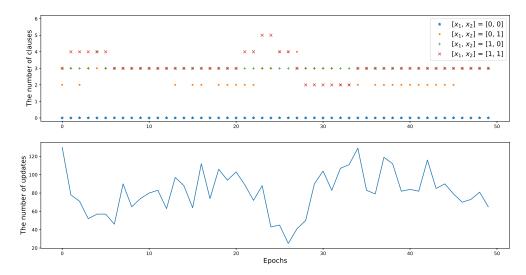


Figure 10: The behavior of a TM with 7 clauses when T=4 for the OR operator.

by only 1 clause. In this case, it is still possible to set up $Th \ge 1$ to have successful classification. Nevertheless, there is no guarantee that each intended sub-pattern will be represented by at least one (or Th) clause(s) in this configuration, thus no guaranteed successful classification.

I EXPERIMENT RESULTS WITH NOISY TRAINING SAMPLES

We present the experimental results for the operators under noisy conditions. First, we show the results when incorrect labels are present, followed by the results involving irrelevant variables. The final subsection addresses a case where both incorrect labels and irrelevant variables are present.

I.1 EXPERIMENT RESULTS FOR WRONG LABELS

To evaluate the performance of the TM when exposed to mislabeled samples, we introduced incorrectly labeled data into the system. The key observation is that the TM does not converge to the intended logic, meaning it does not absorb into a state where the correct logic is consistently represented. However, with carefully chosen hyperparameters, the TM can still learn the intended logic with high probability.

To demonstrate the TM's behavior, we first conduct experiments on the OR operator, which satisfies the following equation:

$$P(y = 1|x_1 = 1, x_2 = 1) = 90\%,$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 90\%,$$

$$P(y = 1|x_1 = 1, x_2 = 0) = 90\%,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(29)

In this scenario, 10% of the input samples that should be labeled as 1 were incorrectly labeled as 0. To train the TM and evaluate its performance, we used the following hyperparameters: m=7, T=4, Th=2, s=3, and N=100. Fig. 11 shows the number of updates and the number of clauses that learn distinct sub-patterns, as a function of epochs. As shown in Fig. 11, the number of updates is big, and thus the system did not converge. Nevertheless, when examining the number of clauses associated with each sub-pattern, we observed that each sub-pattern was covered by at least two clauses, ensuring that the OR operator remained valid. Similar results were observed in experiments conducted on the AND and XOR operators.

Interestingly and understandably, when the proportion of mislabeled samples increases to an extreme level, where inputs that should be labeled as 0 are instead labeled as 1, the TM begins to treat the

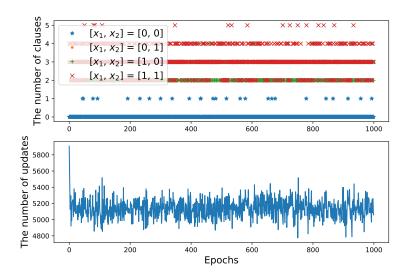


Figure 11: The behavior of TM when m = 7, T = 4 for the OR operator with wrong training labels.

noise as a sub-pattern. For instance, consider the AND operator with input $\mathbf{X} = [x_1 = 0, x_2 = 1]$, which is mislabeled as 1 in 90% of the cases, as shown in Eq. (30). Using the hyperparameters m=7, T=3, s=3.0, and N=100, we observed from experiments that the TM generates three clauses with an output of 1 for $\mathbf{X} = [x_1 = 0, x_2 = 1]$ and another three clauses with an output of 1 for $\mathbf{X} = [x_1 = 1, x_2 = 1]$. This behavior indicates that the TM has incorporated the noise as a learned sub-pattern. Such outcomes align with the TM's underlying principle of learning, where it identifies and models sub-patterns associated with the label 1.

$$P(y = 1|x_1 = 1, x_2 = 1) = 1,$$

$$P(y = 0|x_1 = 1, x_2 = 0) = 1,$$

$$P(y = 0|x_1 = 0, x_2 = 1) = 10\%,$$

$$P(y = 0|x_1 = 0, x_2 = 0) = 1.$$
(30)

I.2 EXPERIMENT RESULTS FOR IRRELEVANT VARIABLE

To confirm the convergence property of TM with irrelevant variable, we setup the experiments for the AND, OR, and XOR operators when one irrelevant variable, namely, x_3 , exists. The probability of x_3 being 1 in the training and testing samples is 50%.

For the AND operator, we use the hyperparameters $m=5,\,T=2,\,s=3,\,Th=2,$ and N=100. Fig. 12 illustrates the convergence of TM for the AND operator in the presence of an irrelevant bit. The results confirm that the TM can correctly learn the AND operator without uncertainty, validating the correctness of Theorem 7.

Interestingly, upon convergence, the form of the included literals varies. For instance, with the aforementioned hyperparameters, we observe that the converged TM includes two clauses of the form $x_1 \wedge x_2 \wedge x_3$ and another two clauses of the form $x_1 \wedge x_2 \wedge \neg x_3$. This suggests that, instead of excluding the irrelevant bit x_3 , the TM includes at least T clauses containing x_3 and at least T clauses containing x_3 , which ensures correct classification regardless of the value of x_3 . However, when the hyperparameters are set to m=1, T=1, s=3, Th=1, and N=100, where only a single clause exists in the TM, the converged clause takes the form $x_1 \wedge x_2$, excluding the literals x_3 and x_3 .

As T increases (T > m/2), we observe that convergence becomes challenging. This difficulty arises because the TM cannot simultaneously learn T clauses containing x_3 and another T clauses

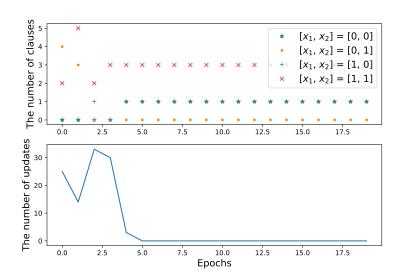


Figure 12: Convergence of TM when m = 5, T = 2 for the AND operator with an irrelevant label.

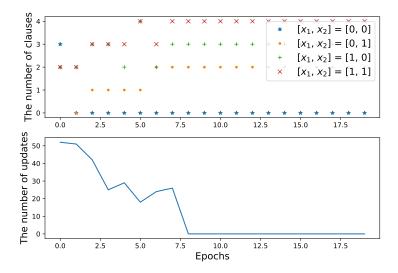


Figure 13: Convergence of TM when m = 5, T = 2 for the OR operator with an irrelevant label.

containing $\neg x_3$. In such cases, the TM must rely on T clauses in the form $x_1 \land x_2$ to achieve convergence, which can be particularly demanding.

For the OR operator, we use the hyperparameters m=5, T=2, s=3, Th=2, and N=100. Figure 13 illustrates the convergence of the TM for the OR operator in the presence of an irrelevant bit. The results confirm that TM successfully learns the OR operator without ambiguity, validating the correctness of Theorem 8. The results also confirm that the TM is capable of presenting two sub-patterns jointly.

Indeed, the OR operator has multiple absorbing states, corresponding to multiple clause forms. Some clause forms may include x_3 or $\neg x_3$, depending on the hyperparameter configuration. Regardless of the value of x_3 , as long as the vote sum of the clauses is greater than or equal to T, the correct classification can be guaranteed.

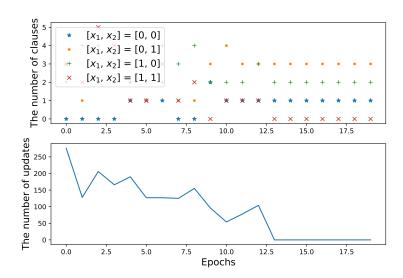


Figure 14: Convergence of TM when m = 7, T = 2 for the XOR operator with an irrelevant label.

We have also studied the XOR operator. The convergence instance is shown in Fig. 14, confirming Theorem 9. Here we use m = 7, T = 2, s = 3, Th = 2.

1.3 EXPERIMENT RESULTS FOR BOTH WRONG LABELS AND IRRELEVANT VARIABLES

In this experiment, we assess the performance of the TM in the presence of both mislabeled data and irrelevant variables. Specifically, we evaluate the TM's ability to learn the XOR operator when 40% of the samples are incorrectly labeled, and 10 irrelevant variables are added. The input comprises 12 bits, with only the first two bits determining the output based on the XOR logic.

The hyperparameters are configured as follows: $m=20,\,T=15,\,s=3.9,$ and N=100 with polarity enabled. Experimental results reveal that the TM successfully learns the XOR operator in 99% of 200 independent runs. These findings demonstrate the robustness of the TM training in noisy environments.

In another experiment, we configured the TM to learn a noisy XOR function with 2 useful input bits and 18 irrelevant input bits (hyper parameters: $N=128,\,m=20,\,T=10,\,s=3$, label noise 0.1). Remarkably, the TM was still able to learn the XOR operator with 100% accuracy using just 5000 training samples. If all possible input combinations were required in the training samples, it would require $2^{20}=1048576$ samples. Clearly, the TM does not rely on the entire combinatorial input space to learn effectively.