Gauge symmetry and dissipative dynamics in probability spaces

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Abstract

We show that the wave-function parametrization allows us to implement gauge symmetries as exact constraints in a standard probability space without attributing to the constrained space null-measure. The wave-function parametrization of classical dissipative dynamics is also studied.

Introduction

Egison is a programming language based on patterns. The documentation[1] is a PhD thesis and it says:

> "Non-free data types are data types whose data have no canonical forms. For example, multisets are non-free data types because the multiset {a, b, b} has two other equivalent but literally different forms {b, a, b} and {b, b, a}. We developed the pattern-match facility for non-free data types."

Using the concept of non-free data type we can define manifolds or virtually any other mathematical object in the language Egison (thus also only conceptually or mathematically). This data type is called "non-free" because it is built from canonical forms (which are "free") by adding "patterns" which are actually gauge symmetries (which are a particular case of constraints, as we will see). For example, the constraint {a, b, b} = {b, a, b}. Since numbers always have canonical form, this is valid for defining manifolds not only symbolically but also numerically. The conclusion is that manifolds can be very useful in some cases, but if we are looking for a practical way to represent data on a computer then "patterns"/gauge symmetries are simpler and more general. Even if we only want manifolds, a good way to represent them on a computer (symbolically or numerically) is using "patterns"/gauge symmetries.

Now consider a continuous sample space (say, the real line). In this article we will show that quantum constraints allow us to implement exact gauge symmetries in a separable probability space without attributing to the constrained space null probability measure. Note that we can always define a wave-function by taking the square-root of the probability measure, the Koopman-von Neumann version of classical statistical mechanics[2][3][4][5] defines classical statistical mechanics as a particular case of quantum mechanics where the algebra of observable operators is necessarily commutative (because the time-evolution is deterministic).

In the next section we will study a simple example which shows another application of the wave-function parametrization of a probability measure, in the case of classical dissipative dynamics.

Wave-function parametrization of dissipative dynamics

The existence of a probability measure allows us to define a quantum Hamiltonian and quantum constraints, which are more general than the classical ones because they do not need to commute with the variables defining the sample space.

As a simple example of a motion which cannot be derived from a Lagrangian (and thus also not from a classical Hamiltonian without enlarging the system) we can cite[6] the oscillations of two classical coupled oscillators with different frequencies and different damping constants:

$$\begin{cases} \ddot{x}_1 + \lambda_1 \dot{x}_1 + \omega_1^2 x_1 - c_2 x_2 = 0 \\ \ddot{x}_2 + \lambda_2 \dot{x}_2 + \omega_2^2 x_2 - c_1 x_1 = 0 \end{cases}, \tag{1}$$

where c_1 and c_2 are constants. Nevertheless, if we consider a probability measure for this classical system then we can define a quantum Hamiltonian for it. Since this system is dissipative, it cannot have a conserved Energy but the pendulums do not disappear and thus the probability is conserved. Another example is when a classical Hamiltonian can be defined but it is much less useful than a quantum Hamiltonian, due to non-linear dynamics[7][8].

The quantum formalism is the most general formalism whenever there is a conserved probability, thus it is expected that it can be applied even when classical Hamiltonians (due to energy dissipation[6], or non-linear dynamics[7][8]) or classical exact constraints (due to the fact that the constrained space has null probability measure) cannot.

Gauge transformations, constrained systems and conditioned probability

Since in standard measure spaces it is always possible to define regular conditional probabilities[9], then in principle it is always possible to implement exact constraints in a separable probability space without attributing to the constrained space null probability measure.

But often the explicit solutions of the constraint equations cannot be found[10], as it is required to define conditional probabilities. In those cases, there is still the possibility of defining the probability measure of the constrained space as a pushforward measure from the unconstrained to the constrained space. Then, the constraint equations would not need to be solved explicitly, at the cost that the constraints are somehow special.

This includes the case of parametrizations, since these are surjective but often there are two or more point in the space of parameters which corresponds to the same point in the parametrized space[11], which is harmless when we evaluate functions at a point but it leads to problems when defining a measure for the parametrized space since the same point cannot be counted twice. As a matter of principle[12], for all parametrizations we can define a gauge group transforming points in the parameter space without modifying the corresponding point in the parametrized space. Thus all parametrizations are solutions to constraint equations requiring gauge invariance.

First-class constraints are the generators of a unitary gauge group (all operators of the group are constrained to be the identity operator). In case the gauge group is infinite-dimensional, there is some ambiguity in its definition [13][14]. In this article we will only address the finite-dimensional case, the infinite-dimensional case was studied in another article by the same author.

Note that it is a subalgebra of the commutative von Neumann algebra that is gauge-invariant and not the Hilbert space. In fact, in many cases it would be impossible for a cyclic state of the Hilbert space to be gauge-invariant, as it was noted long ago:

> "*So we have the situation that we cannot define accurately the vacuum state. We therefore have to work with a standard ket $|S\rangle$ which is ill-defined. One can, however, do many calculations without using the accurate conditions* [vacuum is gauge invariant] *and the successes of quantum electrodynamics are obtained in this way.*"
>
> Paul Dirac (1955)[15]

Indeed, there are some symmetries of the commutative von Neumann algebra of operators which necessarily the probability measure cannot have (see also [16]), since the probability measure is normalized to 1. For instance, consider an infinite-dimensional discrete basis $\{e_k\}$ of an Hilbert space (indexed by the integer numbers k) and the gauge symmetry group generated by the transformation $e_k \to e_{k+1}$ (translation). There is no normalized wave-function (and thus no probability measure) which is translation-invariant. On the other hand, there is a translation-invariant subalgebra of the commutative von Neumann algebra (starting with the identity operator) where the expectation value of each operator is the same for an equivalence class of wave-functions defined up to a gauge symmetry.

The process of representing each equivalence class of operators (in a commutative von Neumann algebra) defined up to a gauge symmetry with at least one operator (in another commutative von Neumann algebra) is called gauge-fixing. We define gauge-fixing as complete whenever it crosses at most once each equivalence class, i.e. whenever there is no remnant gauge symmetry in the equivalence classes. We define gauge-fixing as unconstrained whenever the gauge generators are necessarily excluded from the commutative von Neumann algebra and thus do not impose constraints on the spectrum of the commutative algebra.

The Dirac brackets require the gauge-fixing to be both unconstrained and complete (as if the gauge symmetry could be eliminated), which is not possible in general due to the Gribov ambiguity [17]. Note that the Gribov ambiguity is not solved by the BRST cohomology [18]:

> "*Being gauge invariant, the BFV-PI necessarily reduces to an integral over modular space, irrespective of the gauge fixing choice. Nevertheless, which domain and integration measure over modular space are thereby induced are function of the choice of gauge fixing conditions. The BFV-PI is not totally independent of the choice of gauge fixing fermion Ψ.*"

Thus, the generators of the gauge group cannot be interpreted literally, that is, as mere constraints in a too large phase-space whose "non-physical" degrees of freedom need to be eliminated. Moreover, this picture makes little sense in standard measure spaces, where the dimension of the phase-space is not fixed *a priori*. Moreover, this picture makes little sense in infinite-dimensions: in gauge field theory, the gauge potentials can be fully reconstructed from the algebra of gauge-invariant functions, apart from the gauge potential and its derivatives at one specific arbitrary point in space-time [19]; thus the number of "non-physical" degrees of freedom would be finite at most, which clearly does not match with the uncountable infinite number of constraints.

If we consider instead a commutative von Neumann algebra and its spectrum[1], such that any non-trivial gauge transformation necessarily modifies any point of the spectrum while conserving the commutative von Neumann algebra (e.g. the commutative von Neumann algebra with spectrum given by the gauge field A_μ which is a function of space-time), then such commutative von Neumann algebra is one example of an incomplete unconstrained gauge-fixing. The gauge-fixing is incomplete because the remnant gauge symmetry is a faithful (thus non-trivial) representation of the original gauge symmetry. Such commutative algebra has the crucial advantage that the gauge generators are necessarily excluded from the algebra, so that it can be used to define a separable Hilbert space compatible with the gauge group because the expectation value of any operator of the commutative algebra which commutes with the gauge generators is the same at each equivalence class, saving us the need to eliminate the "non-physical" of degrees of freedom. Note that it is crucial that the von Neumann algebra used in the gauge-fixing is commutative.

Even when a solution exists, setting non-abelian gauge generators to zero in the wave-function would require to solve a non-linear partial differential equation with no obvious solution [20][21][22][23]. On the other hand, since bounded commuting normal operators can always be simultaneously diagonalized[24] there is always one basis where the gauge unitary transformations are a function of the spectrum and (if the gauge-fixing is unconstrained) there is another basis where the gauge unitary transformations are not a function of the spectrum and thus there are no constraints. The expectation values of the gauge-invariant operators in both basis are related by the unitary transformation relating both basis. Thus, for any wave-function for the unconstrained basis there is one corresponding wave-function for the constrained basis which gives the same expectation values for the gauge-invariant operators. This establishes a correspondence between gauge generators as a particular case of constraints, that is, whenever there is a basis where the gauge generators do not depend only on the spectrum then such basis effectively solves the constraint equations when at least one solution exists. We now just need to check that the constrained spectrum is not empty: for a locally compact gauge group (a Lie group, for instance), a constant measure (Haar measure) always exists which allows to create a functional which is gauge invariant.

Finally, since the gauge-invariant operators commute with all diagonal operators in the unconstrained basis, then the diagonal operators in the unconstrained basis may modify the value of the constraint operator in the constrained basis but they conserve the values of the gauge-invariant operators at every point of the spectrum. Thus, we can define a measurable function which projects the spectrum in the constrained basis to another spectrum where the constraints are null (built using the Haar measure, which is a limit of the average of the action of the constraint operators on the diagonal operators in the unconstrained basis). Then, the pushforward measure using such measurable function implements the exact constraints in a separable probability space without attributing to the constrained space null probability measure (which is our goal).

Note that it suffices to constrain to zero the Casimir operators of the (eventually non-commutative) Lie algebra of constraints, this imposes the constraints without the need for the constraints to be part of the

[1] The word *spectrum* is used in the sense of the Gelfand representation: there is an isomorphism between a commutative C*-algebra A and the algebra of continuous functions of the spectrum of A.

commutative von Neumann algebra, only the Casimir operators are included in the commutative algebra.

The gauge symmetry is different from anomalies. An anomaly is a failure of a symmetry of the wave-function to be restored in the limit in which a symmetry-breaking parameter (usually introduced due to the mathematical consistency of the theory) goes to zero. On the other hand, in the case of a gauge symmetry, there is no way to introduce a symmetry-breaking parameter because we only consider expectation values of gauge-invariant operators, thus we can never observe an anomaly.

Quantization of a classical Gauge Mechanics system

An example of the quantization of a classical system is the following. The Hilbert space is $L^2(\mathbb{R}^2\times\mathbb{Z}_2))$ [25][26]. The $\mathbb{Z}_2$ corresponds to the ghost degree of freedom (k) which can be $k=0$ or $k=1$. The $\mathbb{R}^2$ degrees of freedom correspond to one complex operator (ϕ).

The Hamiltonian for the gauge mechanics quantum theory has the same form as the classical Hamiltonian Action:

$$\begin{aligned}
&H=-\frac{1}{2}\left[\frac{1}{2}|\pi|^2+V(|\phi|^2)+\text{ h.c.}\right]\\
&\Omega=(\pi\phi+\pi^*\phi^*)\psi^\dagger\\
&\psi^\dagger\{\Psi\}(\phi,\phi^*,k)=\Psi(\phi,\phi^*,1)\delta_{k0}\\
&\psi\{\Psi\}(\phi,\phi^*,k)=\Psi(\phi,\phi^*,0)\delta_{k1}\\
&[\phi,\pi]=\phi\pi-\pi\phi=i\\
&[\phi,\pi^*]=\phi\pi^*-\pi^*\phi=0\\
&\{\psi,\psi^\dagger\}=\psi\psi^\dagger+\psi^\dagger\psi=1
\end{aligned}\tag{2}$$

where (ϕ,π) is a complex field/conjugate momentum field (correspondingly). $V(|\phi|^2)$ is a functional (a polynomial or other) on $|\phi|^2$. The BRST charge is Ω, where ψ is the ghost field.

The gauge generator in the gauge mechanics system is the charge operator:

$$Q=\pi\phi+\pi^*\phi^*\tag{3}$$

The wave-function needs not be gauge-invariant, just the operators corresponding to observables need to commute with the gauge generator. We need to separate the gauge generator from the gauge-invariant algebra, that is, not only the gauge-invariant algebra must commute with the gauge generator, but also the gauge generator cannot be included in the gauge-invariant algebra. This is guaranteed by unconstrained gauge-fixing: the gauge-invariant algebra is a sub-algebra of the commutative von Neumann algebra with spectrum given by the operators ϕ,ϕ^*,k, such that the generator necessarily modifies any point of the spectrum up to a set with null measure. The conjugate fields π,π^* or the gauge generator are not part of the gauge-invariant algebra, since they do not commute with the corresponding operators ϕ,ϕ^* which are included in the commutative algebra.

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