Small-loss Adaptive Regret for Online Convex Optimization

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Abstract

To deal with changing environments, adaptive regret has been proposed to minimize the regret over every interval. Previous studies have established a small-loss adaptive regret bound for general convex functions under the smoothness condition, offering the advantage of being much tighter than minimax rates for benign problems. However, it remains unclear whether similar bounds are attainable for other types of convex functions, such as exp-concave and strongly convex functions. In this paper, we first propose a novel algorithm that achieves a small-loss adaptive regret bound for exp-concave and smooth function. Subsequently, to address the limitation that existing algorithms can only handle one type of convex functions, we further design a universal algorithm capable of delivering small-loss adaptive regret bounds for general convex, exp-concave, and strongly convex functions simultaneously. That is challenging because the universal algorithm follows the meta-expert framework, and we need to ensure that upper bounds for both meta-regret and expert-regret are of small-loss types. Moreover, we provide a novel analysis demonstrating that our algorithms are also equipped with minimax adaptive regret bounds when functions are non-smooth.

1. Introduction

Online convex optimization (OCO) is a powerful framework for online learning, which enjoys both theoretical guarantees and practical applications (Hazan, 2016). According to the protocol of OCO, it can be seen as a structured repeated game. At each round t, the online learner chooses a decision $\mathbf{w}_t$ from a given convex set $\mathcal{W}$. After submitting this decision, a convex loss function $f_t : \mathcal{W} \mapsto \mathbb{R}$ is revealed and the online learner suffers a loss $f_t(\mathbf{w}_t)$. The learner aims to minimize the cumulative loss over T rounds. To measure the performance, static regret is typically used:

$$\text{Regret}(T) = \sum_{t=1}^{T} f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})$$

which is defined as the difference between the cumulative loss of the online learner and that of the best decision chosen in hindsight.

Since the seminal work of Zinkevich (2003), various algorithms have been developed to minimize the static regret (Shalev-Shwartz, 2011; Hazan, 2016). However, the static regret is not suitable to changing environments because it chooses a fixed comparator. To address this limitation, recent advances in OCO introduce adaptive regret (Hazan & Seshadhri, 2007), which measures the performance with respect to a changing comparator. The goal is to minimize the static regret of every interval $[r, s] \subseteq [T]$, i.e.,

$$\text{Regret}([r, s]) = \sum_{t=r}^{s} f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w}).$$

In the literature, several online algorithms have been proposed to minimize the adaptive regret of convex functions, which attain $O(\sqrt{(s-r)\log s})$, $O(\frac{d}{\alpha}\log s \log(s-r))$ and $O(\frac{1}{\lambda}\log s\log(s-r))$ adaptive regret for general convex, α-exponentially concave (abbr. α-exp-concave) and λ-strongly convex functions (Hazan & Seshadhri, 2007; Jun et al., 2017a; Zhang et al., 2018) respectively, where d is the dimensionality. Furthermore, when the loss functions are also smooth, Zhang et al. (2019) demonstrate that an $O(\sqrt{L^*_{r,s}\log L^*_{1,s}\log L^*_{r,s}})$ *small-loss* adaptive regret bound (also known as first-order bound) is attainable for general convex functions, where

$$L^*_{r,s} = \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w}) \tag{1}$$

is the total loss of the best decision over interval $[r, s]$. Such bound is comparable to the $O(\sqrt{(s-r)\log s})$ bound in the worst case, but could be much tighter when the comparator has a small loss. Note that in the studies of static regret, we also have small-loss bounds for exp-concave and strongly

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Table 1. Comparative overview of related work on adaptive regret and our paper.

Method	Loss Type	Adaptive Regret
Hazan & Seshadhri (2009)	α-exp-concave	$\frac{d}{\alpha}\log s \log(s-r)$
Zhang et al. (2019)	convex & smooth	$\sqrt{L^*_{r,s}\log L^*_{1,s}\log L^*_{r,s}}$
Zhang et al. (2021)	convex	$\sqrt{(s-r)\log s \log(s-r)}$
	α-exp-concave	$\frac{d}{\alpha}\log s \log(s-r)$
	λ-strongly convex	$\frac{1}{\lambda}\log s \log(s-r)$
FLHS (this work)	α-exp-concave & smooth	$\frac{d}{\alpha}\log L^*_{1,s}\log L^*_{r,s}$
	α-exp-concave	$\frac{d}{\alpha}\log s \log(s-r)$
USIA (this work)	convex & smooth	$\sqrt{L^*_{r,s}\log L^*_{1,s}\log L^*_{r,s}}$
	α-exp-concave & smooth	$\frac{d}{\alpha}\log L^*_{1,s}\log L^*_{r,s}$
	λ-strongly convex & smooth	$\frac{1}{\lambda}\log L^*_{1,s}\log L^*_{r,s}$
	convex	$\sqrt{(s-r)\log s \log(s-r)}$
	α-exp-concave	$\frac{d}{\alpha}\log s \log(s-r)$
	λ-strongly convex	$\frac{1}{\lambda}\log s \log(s-r)$

convex functions (Orabona et al., 2012; Wang et al., 2020). As a result, it is natural to ask whether small-loss adaptive regret bounds can also be established for them which motivates the study of this paper.

To minimize the adaptive regret, we adopt the two-level framework where multiple experts are created dynamically and aggregated by a meta-algorithm (Hazan & Seshadhri, 2007; Zhang et al., 2019; Wang et al., 2024). For exp-concave and smooth functions, we propose Follow-the-Leading-History for Smooth functions (FLHS) by combining Follow-the-Leading-History (Hazan & Seshadhri, 2007) and problem-dependent intervals (Zhang et al., 2019). Our method achieves an $O(\frac{d}{\alpha}\log L^*_{1,s}\log L^*_{r,s})$ small-loss adaptive regret for α-exp-concave and smooth functions. Furthermore, we adapt FLHS to λ-strongly convex and smooth functions by replacing its expert-algorithm with S^2OGD (Wang et al., 2020), resulting in an $O(\frac{1}{\lambda}\log L^*_{1,s}\log L^*_{r,s})$ small-loss adaptive regret for λ-strongly convex and smooth functions.

Combined with the result of Zhang et al. (2019), small-loss adaptive regret bounds have been successfully established for general convex, exp-concave, and strongly convex functions. However, the associated algorithms lack universality and can only handle one type of convex functions, which drives us to develop a Universal algorithm for exploiting Smoothness to Improve the Adaptive regret (USIA). First, we address the variability of function characteristics by designing three types of expert-algorithms, specifically designed for convex, strongly convex, and exp-concave functions under the smoothness condition. Second, we construct a set of problem-dependent intervals based on the cumulative loss of the meta-algorithm, which is different from Zhang et al. (2019) that rely on the performance of the expert. These intervals are created dynamically, and each of them is associated with multiple experts. Third, inspired by universal algorithms for static regret (Zhang et al., 2022b), we extend Adapt-ML-Prod (Gaillard et al., 2014) to combine the predictions of dynamically created experts, thereby equipping our algorithm with second-order bounds that can exploit exp-concavity and strong convexity. In this way, USIA attains $O(\sqrt{L^*_{r,s}\log L^*_{1,s}\log L^*_{r,s}})$, $O(\frac{d}{\alpha}\log L^*_{1,s}\log L^*_{r,s})$ and $O(\frac{1}{\lambda}\log L^*_{1,s}\log L^*_{r,s})$ small-loss adaptive regret bounds over any interval $[r,s]\subseteq[T]$ for general convex, α-exp-concave and λ-strongly convex functions respectively.

Furthermore, we also introduce an additional enhancement to our algorithms, i.e., FLHS and USIA. Notably, previous studies on adaptive regret for convex and smooth functions (Zhang et al., 2019; Zhao et al., 2022) fall short in providing theoretical guarantees for non-smooth functions. To avoid this limitation, we provide a novel analysis demonstrating that our algorithms maintain the minimax adaptive regret bounds when functions are *non-smooth*.

Technical Challenge. The technical challenges in this paper can be summarized into two aspects: (i) the first challenge lies in algorithm design, specifically in enhancing Adapt-ML-Prod as the meta-algorithm to aggregate dynamically constructed experts and also enjoy universality. Additionally, the dynamic construction of experts requires careful design to yield problem-dependent regret bounds; (ii) the second challenge is to adapt the proposed algorithms (designed for smooth functions) to non-smooth functions, which is unexplored in the literature. Table 1 summarizes our results in comparison with existing studies.

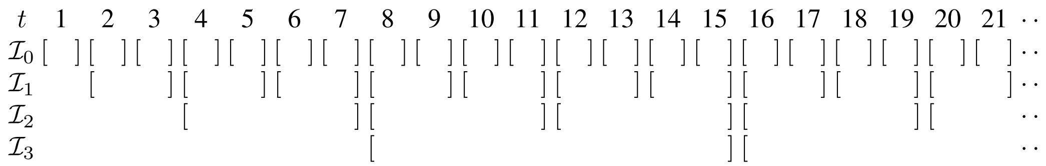

Figure 1. Geometric covering (GC) intervals. In the figure, each interval is denoted by [ ].

2. Related Work

We briefly review the development of adaptive regret, universal algorithms and parameter-free algorithms. The omitted related work on static regret is included in Appendix A.

2.1. Adaptive Regret

Adaptive regret has been well studied in various settings (Littlestone & Warmuth, 1994; Freund et al., 1997; Adamskiy et al., 2012; György et al., 2012; Wan et al., 2022; Zhang et al., 2022a), such as prediction with expert advice (PEA) and online convex optimization (OCO). Hazan & Seshadhri (2007) first introduce the adaptive regret for OCO,

$$\text{A-Regret}(T) = \max_{[r,s]\subseteq[T]} \text{Regret}([r,s])$$

which is the maximum regret over any contiguous interval, and propose Follow-the-Leading-History (FLH) to attain an $O(\frac{d}{\alpha}\log^2 T)$ adaptive regret bound for α-exp-concave functions. FLH is a two-level algorithm and contains 3 components:

- An expert-algorithm, which minimizes the static regret over a given interval;
- Dynamically created intervals, each of which is associated with an expert-algorithm;
- A meta-algorithm, which aggregates the decisions of active experts in each round.

FLH creates a set of intervals based on a data streaming algorithm, and runs an instance of ONS as an expert over every interval. Then, it uses a meta-algorithm based on Fixed-Share algorithm (Herbster & Warmuth, 1998) to track the best expert. Adamskiy et al. (2012) point out that FLH creates and removes experts dynamically, which can be modeled by sleeping experts (Freund et al., 1997). For λ-strongly convex functions, we can also use online gradient descent (OGD) as the expert-algorithm in FLH, and obtain an $O(\frac{1}{\lambda}\log^2 T)$ adaptive regret bound (Zhang et al., 2018; Wang et al., 2018). For general convex functions, Hazan & Seshadhri (2007) modify FLH by replacing ONS with OGD as the expert-algorithm, thereby achieving an $O(\sqrt{T\log^3 T})$ adaptive regret bound. In this way, efficient FLH achieves an $O(\sqrt{T\log^3 T})$ adaptive regret bound. The limitation of the above bound is that it is not guaranteed to perform well on short intervals, because the upper bound is meaningless for intervals of length $O(\sqrt{T})$.

To ensure a good performance on every interval, Daniely et al. (2015) propose the strongly adaptive regret,

$$\text{SA-Regret}(T,\tau) = \max_{[s,s+\tau-1]\subseteq[T]} \text{Regret}([s,s+\tau-1])$$

which is defined as the maximum static regret over intervals of length τ. Furthermore, they propose a novel way to construct the set of intervals, named as geometric covering (GC) intervals. Mathematically, GC intervals are defined as

$$\mathcal{I} = \bigcup_{k\in\mathbb{N}\cup\{0\}} \mathcal{I}_k, \tag{2}$$

where $\mathcal{I}_k = \left\{[i\cdot 2^k, (i+1)\cdot 2^k - 1] : i \in \mathbb{N}\right\}$. A graphical illustration of GC intervals is given in Fig. 1. We observe that each $\mathcal{I}_k$ is a partition of $\mathbb{N}\setminus\{1,\cdots,2^k-1\}$ to consecutive intervals of length 2^k. For each of GC intervals, Daniely et al. (2015) run an instance of online gradient descent (OGD) as the expert-algorithm. Then, they introduce a novel meta-algorithm, named as Strongly Adaptive Online Learner (SAOL), which is similar to the multiplicative weights method (Arora et al., 2012). SAOL attains an $O(\sqrt{\tau}\log T)$ strongly adaptive regret bound, which is further enhanced to $O(\sqrt{\tau\log T})$ (Jun et al., 2017a).

When the loss functions are also smooth, the adaptive regret of general convex and smooth functions can be improved to $O(\log s\sqrt{L_{r,s}^*})$ (Jun et al., 2017b). To deliver tighter results, Zhang et al. (2019) propose problem-dependent intervals, which make the number of experts dependent on the total loss of the expert-algorithm instead of the time length. Then, they develop SACS by running multiple instances of SOGD over problem-dependent intervals and using AdaNormalHedge (Luo & Schapire, 2015) as the meta-algorithm. For general convex and smooth functions, SACS attains an $O(\sqrt{L_{r,s}^*\log L_{1,s}^*\log L_{r,s}^*})$ small-loss adaptive regret bound. Following this research, Zhao et al. (2022) propose an efficient method for adaptive regret that reduces projection complexity. To achieve this goal, they refine the construction of problem-dependent intervals and attain $\min\{O(\sqrt{L_{r,s}^*\log L_{1,s}^*\log L_{r,s}^*}), O(\sqrt{(s-r)\log T})\}$ for general convex and smooth functions. However, both of them did not consider non-smooth functions. Moreover, under the smoothness condition, small-loss adaptive regret bounds for exp-concave functions and strongly convex functions are still unknown in the literature.

2.2. Universal Algorithms

Although there exist plenty of algorithms under the setting of OCO, most of them can only handle one type of convex functions and need to know the moduli of strong convexity and exp-concavity beforehand. The requirement of domain knowledge for choosing the optimal algorithm hinders their applications to real-world problems, motivating the development of universal algorithms for OCO.

The first universal algorithm for OCO is adaptive online gradient descent (AOGD) (Bartlett et al., 2008). AOGD is able to automatically interpolate between the $O(\sqrt{T})$ regret bound of general convex functions and the $O(\log T)$ regret bound of strongly convex functions. However, AOGD suffers two restrictions, including its need for the modulus of strongly convexity in each round and its suboptimal $O(\sqrt{T})$ regret bound for exp-concave functions. Another milestone of universal algorithms is the multiple eta gradient algorithm (MetaGrad) (van Erven & Koolen, 2016; Mhammedi et al., 2019; van Erven et al., 2021), which adapts to a much broader class of functions. Specifically, MetaGrad attains $O(\frac{d}{\alpha}\log T)$ and $O(\sqrt{T \log\log T})$ regret bound for α-exp-concave and general convex functions, respectively. However, it treats strongly convex functions as exp-concave, thus obtaining a suboptimal $O(\frac{d}{\lambda}\log T)$ regret bound. Wang et al. (2019) overcome this problem by developing Maler, which achieves minimax optimal regret bounds for three types of convex functions simultaneously. In a subsequent work, Wang et al. (2020) extend Maler to support smoothness. Their universal algorithm, named as UFO, obtains $O(\sqrt{L_T^*})$, $O(\frac{d}{\alpha}\log L_T^*)$ and $O(\frac{1}{\lambda}\log L_T^*)$ regret bounds for general convex, α-exp-concave and λ-strongly convex functions respectively.

However, most of existing universal methods (van Erven & Koolen, 2016; Wang et al., 2019; 2020) require the experts to process the surrogate losses, making it difficult to exploit the structure of the original problem and utilize previous algorithms. To address this limitation, Zhang et al. (2022b) propose a simple yet universal strategy for OCO (USC), which allows the experts to process the original online functions directly. The key idea is to run multiple expert-algorithms to process the original online functions, and deploy Adapt-ML-Prod (Gaillard et al., 2014) over linearized loss to aggregate the decisions. In this way, USC can utilize the property of exp-concavity and strongly convexity to yield a negligible regret bound for the meta-algorithm. Advancing this line of research, Yan et al. (2023) propose a multi-layer universal algorithm equipped with gradient-variation bounds. Furthermore, they construct novel surrogate losses to reduce gradient query complexity. For adaptive regret, we note that Zhang et al. (2021) have proposed a universal algorithm for minimizing the adaptive regret of convex functions, but they did not consider the smoothness condition.

2.3. Parameter-free Algorithms

Most of the existing online learning algorithms require the knowledge about functions to set their parameters, such as step size and the norm of gradients, thereby driving the advancement of parameter-free online learning algorithms (Orabona, 2014; Orabona & Pál, 2016; Cutkosky & Boahen, 2016; 2017; Cutkosky & Orabona, 2018; Mhammedi & Koolen, 2020). The study of parameter-free algorithms mainly focuses on general convex functions, which is complementary to the development of universal algorithms.

3. Main Results

We first present necessary preliminaries, and then provide our proposed algorithms.

3.1. Preliminaries

We introduce the following standard assumptions used in the studies of OCO (Hazan, 2016).

Assumption 3.1. The gradients of all functions are bounded by G, i.e.,

$$\max_{\mathbf{w}\in\mathcal{W}} \|\nabla f_t(\mathbf{w})\| \leq G, \forall t \in [T]. \tag{3}$$

Assumption 3.2. The diameter of the domain $\mathcal{W}$ is bounded by D, i.e.,

$$\max_{\mathbf{w},\mathbf{w}'\in\mathcal{W}} \|\mathbf{w}-\mathbf{w}'\| \leq D. \tag{4}$$

Assumption 3.3. All the online functions are nonnegative and H-smooth over $\mathcal{W}$, i.e.,

$$\|\nabla f_t(\mathbf{w}) - \nabla f_t(\mathbf{w}')\| \leq H\|\mathbf{w}-\mathbf{w}'\|. \tag{5}$$

for all $\mathbf{w},\mathbf{w}' \in \mathcal{W}$, $t \in [T]$.

Assumption 3.4. The value of each function is bounded by F, i.e.,

$$0 \leq f_t(\mathbf{w}) \leq F, \forall \mathbf{w} \in \mathcal{W}, \forall t \in [T]. \tag{6}$$

Clarifications on Assumption 3.4 It is worth mentioning that our algorithms does not need to know the value of F; instead, it is only used in the theoretical analysis, which is consistent with previous work for small-loss bounds (Orabona et al., 2012; Zhang et al., 2019; Wang et al., 2020).

Next, we state the definitions of strongly convexity and exp-concavity (Boyd & Vandenberghe, 2004; Cesa-Bianchi & Lugosi, 2006), and introduce an important property of exp-concave functions (Hazan et al., 2007, Lemma 3).

Definition 3.5. A function $f : \mathcal{W} \mapsto \mathbb{R}$ is λ-strongly convex if

$$f(\mathbf{w}') \geq f(\mathbf{w}) + \langle \nabla f(\mathbf{w}), \mathbf{w}'-\mathbf{w}\rangle + \frac{\lambda}{2}\|\mathbf{w}'-\mathbf{w}\|^2,$$

for all $\mathbf{w},\mathbf{w}' \in \mathcal{W}$.

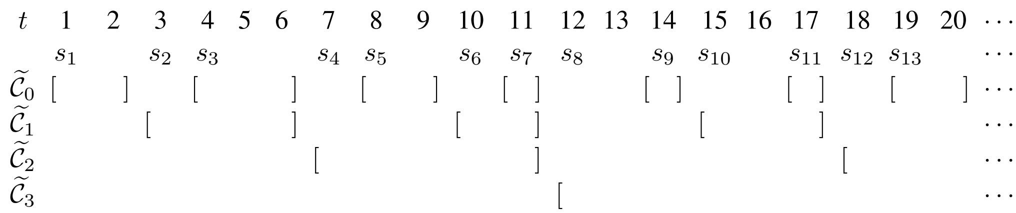

Figure 2. Compact problem-dependent geometric covering (CPGC) intervals. In the figure, each interval is denoted by [ ].

Definition 3.6. A function $f : \mathcal{W} \mapsto \mathbb{R}$ is α-exp-concave if $\exp(-\alpha f(\cdot))$ is concave over $\mathcal{W}$.

Lemma 3.7. *For a function $f : \mathcal{W} \mapsto \mathbb{R}$, where $\mathcal{W}$ has diameter D, such that $\forall \mathbf{w} \in \mathcal{W}$, $\|\nabla f(\mathbf{w})\| \leq G$ and $\exp(-\alpha f(\cdot))$ is concave, we have*

$$f(\mathbf{w}') \geq f(\mathbf{w}) + \langle \nabla f(\mathbf{w}), \mathbf{w}' - \mathbf{w} \rangle + \frac{\beta}{2} \langle \nabla f(\mathbf{w}), \mathbf{w}' - \mathbf{w} \rangle^2,$$

for all $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, where $\beta = \frac{1}{2} \min\{\frac{1}{4GD}, \alpha\}$.

3.2. Follow-the-leading-history for Smooth Functions

To utilize exp-concavity, we follow the structure of FLH (Hazan & Seshadhri, 2009), which runs multiple experts over a set of intervals and aggregates them by a meta-algorithm. The regret over interval $[r, s]$ can be decomposed as the sum of the meta-regret and the expert-regret, i.e.,

$$\begin{aligned} &\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \min_{\mathbf{w} \in \mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w}) \\ =& \underbrace{\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}_t')}_{\texttt{meta-regret}} + \underbrace{\sum_{t=r}^{s} f_t(\mathbf{w}_t') - \min_{\mathbf{w} \in \mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w})}_{\texttt{expert-regret}} \end{aligned}$$

where $\mathbf{w}_t$ and $\mathbf{w}_t'$ denote the output of the meta-algorithm and an expert-algorithm in the t-th round. For expert-regret, FLH runs ONS (Hazan et al., 2007) as the expert-algorithm to attain an $O(\frac{d}{\alpha} \log(s - r))$ regret bound. For meta-regret, it uses a variant of Fixed-Share (Herbster & Warmuth, 1998) as the meta-algorithm to achieve an $O(\frac{1}{\alpha} \log s)$ regret bound. To obtain small-loss regret over interval $[r, s]$, we need to turn both bounds to depend on the cumulative loss.

Under the smoothness condition, ONS naturally attains a small-loss regret bound for exp-concave functions (Orabona et al., 2012). Thus, we can directly utilize theoretical guarantee of ONS to obtain small-loss expert-regret. However, the meta-regret over interval $[r, s]$ of FLH depends on the number of intervals created till round s, which is problem-independent and cannot exploit the smoothness property. Inspired by Zhang et al. (2019), we construct intervals in a *problem-dependent* way. In the following, we first review the key procedures of problem-dependent intervals.

Review of Problem-dependent Intervals The basic idea of problem-dependent intervals is to generate intervals based on the real-time performance of an expert-algorithm. Specifically, we run an instance of the expert-algorithm in round s_i, where $s_1 = 1$. When its cumulative loss becomes larger than a threshold C in round $s_i + \alpha$, we set $s_{i+1} = s_i + \alpha + 1$ and restart the expert-algorithm. By repeating this process, a sequence of points $s_1, s_2, s_3 \cdots$ are generated, and we refer to them as *markers*. Instead of using original time points, we construct problem-dependent geometric covering (PGC) intervals based on markers. The advantage of PGC intervals is that the number of intervals depends on the *total loss* of expert-algorithm instead of the time length.

Mathematically, PGC intervals are given by

$$\widetilde{\mathcal{I}} = \bigcup_{k \in \mathbb{N} \cup \{0\}} \widetilde{\mathcal{I}}_k, \quad \widetilde{\mathcal{I}}_k = \left\{ [s_{i \cdot 2^k}, s_{(i+1) \cdot 2^k} - 1] : i \in \mathbb{N} \right\}$$

where for all $k \in \mathbb{N} \cup \{0\}$

$$\widetilde{\mathcal{I}}_k = \left\{ [s_{i \cdot 2^k}, s_{(i+1) \cdot 2^k} - 1] : i \in \mathbb{N} \right\}.$$

Moreover, we can further simplify PGC intervals by removing overlapping intervals with the same point (Zhang et al., 2019). To facilitate understanding, we first explain how to compress the GC intervals in (2) to obtain compact GC (CGC) intervals. Mathematically, CGC intervals are defined as $\mathcal{C} = \bigcup_{k \in \mathbb{N} \cup \{0\}} \mathcal{C}_k$, where $\mathcal{C}_k = \{[i \cdot 2^k, (i+1) \cdot 2^k - 1] : i \text{ is odd}\}$. Following this idea, we can compress PGC intervals similarly, and the compact PGC (CPGC) intervals are given by

$$\widetilde{\mathcal{C}} = \bigcup_{k \in \mathbb{N} \cup \{0\}} \widetilde{\mathcal{C}}_k, \quad \widetilde{\mathcal{C}}_k = \left\{ [s_{i \cdot 2^k}, s_{(i+1) \cdot 2^k} - 1] : i \text{ is odd} \right\}.$$

A graphical illustration of CPGC intervals is given in Fig. 2.

Our Approach According to the framework of FLH, our algorithm also contains 3 components, including an expert-algorithm, a set of intervals and a meta-algorithm. First, we use ONS as our expert-algorithm. Then, we construct CPGC intervals based on the real-time performance of ONS, and associate each interval $I = [s_p, s_q - 1] \in \widetilde{\mathcal{C}}$ with an instance of ONS that minimizes the regret during I. Finally,

Algorithm 1 Follow-the-Leading-History for Smooth functions (FLHS)

1: Initialize indicator *NewInterval=true*, the total number of intervals $m = 0$, the index of the latest expert $n = 0$
2: **for** $t = 1$ **to** T **do**
3: **if** *NewInterval* is *true* **then**
4: Create an expert E_t by running the algorithm of ONS, and set $L_{t-1} = 0$, $p_{t,t} = \frac{1}{m+1}$
5: Add E_t to the set of active experts: $\mathcal{A}_t = \mathcal{A}_{t-1} \cup \{E_t\}$
6: Reset the indicator *NewInterval=false*
7: Update the total number of intervals $m = m + 1$
8: Set $g_t = j$ such that $[m, j-1] \in \mathcal{C}$ and record the index of the latest expert $n = t$
9: **end if**
10: Receive output $\mathbf{w}_{t,i}$ from each expert $E_i \in \mathcal{A}_t$
11: Submit $\mathbf{w}_t$ in $\mathbf{w}_t = \sum_{E_j \in \mathcal{A}_t} p_{t,j}\mathbf{w}_{t,j}$
12: Update the cumulative loss of the latest expert: $L_t = L_{t-1} + f_t(\mathbf{w}_{t,n})$
13: **if** $L_t > C$ **then**
14: Set the indicator *NewInterval=true*
15: Remove experts whose ending times are $s_{m+1} - 1$: $\mathcal{A}_t = \mathcal{A}_t \setminus \{E_i | g_i = m + 1\}$
16: **end if**
17: **for all** $E_i \in \mathcal{A}_t$ **do**
18: Update weight in (7)
19: **if** *NewInterval* is *true* **then** $p_{t+1,i} = (1 - (m+1)^{-1})\hat{p}_{t+1,i}$ **else** $p_{t+1,i} = \hat{p}_{t+1,i}$
20: **end for**
21: **end for**

we modify the meta-algorithm of FLH to accommodate the new intervals. Let E_t be the expert-algorithm created in round t. FLH initializes the weight of E_t to be $1/t$ and multiplies the weight of other active experts by $(1 - 1/t)$ which lead to problem-independent terms. To yield a problem-dependent meta-regret, both the initial weight and the updating rule need to be modified to benefit from CPGC intervals. Specifically, we denote the number of dynamically constructed experts by m, and then modify the initial and updating weights to $1/m$ and $(1 - 1/m)$ accordingly.

Our **F**ollow-the-**L**eading-**H**istory for **S**mooth functions (FLHS) is summarized in Algorithm 1. To generate CPGC intervals, we introduce a Boolean variable *NewInterval* to indicate whether a new interval should be created. Furthermore, the total number of intervals and the index of the latest expert are denoted by m and n, respectively. In each round t, if *NewInterval* is true, we will create an expert E_t by running an instance of ONS, and introduce L_{t-1} to record the cumulative loss of the latest expert. In Step 4, we initialize the weight of E_t to be $p_{t,t} = 1/(m+1)$ which is problem-dependent. We add E_t to the active set A_t in Step 5. Then, we reset the indicator to be false and update the total number of intervals (Steps 6 to 7). According to CPGC intervals, we know that the m-th marker $s_m = t$ and the ending time s_j satisfies $[m, j-1] \in \mathcal{C}$. So we set $g_t = j$ and record the index of the latest expert $n = t$ in Step 8 and remove expert E_t when m is going to reach g_t in Step 15. In Step 10, we collect the predictions of all the active experts, and aggregate them in Step 11. We keep track of the latest expert and record its cumulative loss L_t. When L_t is larger than the threshold C, we set *NewInterval* to be true and remove experts whose ending times are $s_{m+1} - 1$ (Steps 13 to 16). In Step 18, FLHS updates the weight of each active expert, i.e.,

$$\hat{p}_{t+1,i} = \frac{p_{t,i} e^{-\eta f_t(\mathbf{w}_{t,i})}}{\sum_{j \in \mathcal{A}_t} p_{t,j} e^{-\eta f_t(\mathbf{w}_{t,j})}} \tag{7}$$

where η is the learning rate. To ensure that the sum of all weights is 1, FLHS multiplies each weight by $(1 - (m+1)^{-1})$ if *NewInterval* is *true* because a new expert will be created at next round.

In the following, we present theoretical guarantees of FLHS for α-exp-concave and smooth functions.

Theorem 3.8. *Under Assumptions 3.1, 3.2, 3.3 and 3.4, if the online functions are α-exp-concave and appropriate parameters are set, for any interval $[r, s] \subseteq [T]$ and any $\mathbf{w} \in \mathcal{W}$, FLHS satisfies*

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\frac{d}{\alpha} \log L_{1,s}^* \log L_{r,s}^*\right)$$

where $L_{1,s}^*$ and $L_{r,s}^*$ are defined in (1).

Remark In the literature, the state-of-art adaptive regret bound for exp-concave functions is $O(\frac{d}{\alpha} \log s \log(s-r))$ of Hazan & Seshadhri (2009). We observe that the problem-independent term $\log s \log(s-r)$ is improved to $\log L_{1,s}^* \log L_{r,s}^*$ in Theorem 3.8.

Then, we also prove that our proposed algorithm is equipped with the following theoretical guarantee when dealing with non-smooth functions.

Theorem 3.9. *Under Assumptions 3.1, 3.2 and 3.4, if the online functions are α-exp-concave and appropriate parameters are set, for any interval $[r, s] \subseteq [T]$ and any $\mathbf{w} \in \mathcal{W}$, FLHS achieves an $O(\frac{d}{\alpha} \log s \log(s-r))$ minimax adaptive regret bound for non-smooth functions.*

Remark When functions are non-smooth (without Assumption 3.3), we provide a novel analysis for problem-dependent intervals to demonstrate that our algorithm can also deliver a minimax adaptive regret bound, conferring an advantage over previous studies on adaptive regret for smooth functions (Zhang et al., 2019; Zhao et al., 2022).

Parameter Setting According to the procedure of FLHS, we need to set threshold C to construct CPGC intervals, where C requires the problem-dependent parameters, such as G, D, H and α. For non-smooth functions, we can set H be any constant, e.g., 1. The detailed requirements of threshold and the exact regret bounds of FLHS are included in Appendix B.1.

Strongly Adaptive Regret We would like to further clarify that our work focuses on strongly adaptive regret, which was introduced to measure the performance of each interval, with its bound reflecting the length of the interval. Notice that our analysis is conducted for every interval $[r, s] \subseteq [T]$, and our small-loss bound includes the cumulative loss $L^*_{r,s}$, which in turn depends on the length of the interval. In particular, when functions are non-smooth, we can derive the existing strongly adaptive regret bounds from our theoretical guarantee. By substituting the regret bound from Theorem 3.9 into the definition of strong adaptive regret, we prove that FLHS achieves an $O(\frac{d}{\alpha} \log \tau \log T)$ strongly adaptive regret for α-exp-concave functions.

3.3. From Exp-concave to Strongly Convex

We proceed to discuss how to modify FLHS to obtain a small-loss adaptive regret bound for λ-strongly convex and smooth functions. It is established that strongly convex functions with bounded gradients are also exp-concave (Zhang et al., 2018, Lemma 2). Therefore, we can reuse the meta-algorithm of FLHS. For the expert-algorithm, we replace ONS with S^2OGD (Wang et al., 2020), which is specially designed for strongly convex and smooth functions. Next, we establish the following theoretical guarantee λ-strongly convex functions under the smoothness condition.

Theorem 3.10. *Under the same assumptions as Theorem 3.8, if the online functions are λ-strongly convex and the learning rate $\eta = \frac{\lambda}{G^2}$, for any interval $[r, s] \subseteq [T]$ and any $\mathbf{w} \in \mathcal{W}$, FLHS with S^2OGD expert satisfies*

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\frac{1}{\lambda} \log L^*_{1,s} \log L^*_{r,s}\right)$$

where $L^*_{r,s}$ is defined in (1). Furthermore, when functions are non-smooth (without Assumption 3.3), FLHS achieves an $O(\frac{1}{\lambda} \log s \log(s - r))$ adaptive regret bound.

Remark Compared with the $O(\frac{1}{\lambda} \log s \log(s - r))$ bound of Zhang et al. (2018), our algorithm attains a small-loss adaptive regret bound for λ-strongly convex and smooth functions. Similar to Theorem 3.9, our algorithm can also achieve a minimax adaptive regret bound when functions are non-smooth.

3.4. A Universal Algorithm for Exploiting Smoothness

In this section, we develop a universal algorithm for exploiting smoothness to improve the adaptive regret.

Expert-algorithm We run multiple experts simultaneously to deal with the uncertainty of the functions types, as well as the modulus of exponentially concavity and strong convexity. Specifically, We create three types of experts by running the algorithm of SOGD, S^2OGD and ONS to address the uncertainty of the functions types, where SOGD is designed for convex and smooth functions. To approximate the modulus of exp-concavity and strong convexity, we construct two finite sets $\mathcal{P}_{\text{str}}$ and $\mathcal{P}_{\text{exp}}$ comprising possible values of the modulus, which are served as the input parameters for S^2OGD and ONS. Taking λ-strongly convex functions as an example, assume that T is fixed in advance and the unknown modulus λ is both upper bounded and lower bounded, i.e., $\lambda \in [1/T, 1]$, we construct $\mathcal{P}_{\text{str}} = \{1/T, 2/T, 2^2/T, \cdots, 2^N/T\}$ to be set of possible values, where $N = \lceil \log_2 T \rceil$. The detailed procedure of creating multiple experts is summarized in Appendix D.2. Additionally, we also provide justifications in Appendix G regarding the assumption of the bounded modulus, which is commonly accepted in the majority of scenarios.

Meta-algorithm We choose Adapt-ML-Prod (Gaillard et al., 2014) as Zhang et al. (2022b) have proved that its meta-regret can automatically exploit the exp-concavity and strong convexity by utilizing the linearized loss. Following this idea, we extend Adapt-ML-Prod to support sleeping experts so that our meta-algorithm can aggregate all the decisions from dynamically created experts, and yield a problem-dependent meta-regret.

Construction of Problem-dependent Intervals compared with FLHS, we need to run multiple experts rather than a single expert over an interval. In this scenario, we explain that generating CPGC intervals based on the cumulative loss of the expert-algorithm as Zhang et al. (2019) will result in a suboptimal regret bound for convex and non-smooth functions. Specifically, if we construct intervals based on the total loss of the expert, we can obtain an optimal regret bound over a CPGC interval $[s_{a-1}, s_a - 1] \in \widetilde{\mathcal{C}}$. However, the problem arises when bounding the regret of a sub-interval $[r, s_a - 1]$,

$$\begin{aligned}
&\sum_{t=r}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=r}^{s_a-1} f_t(\mathbf{w}) \\
\leq &\underbrace{\sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_{t,s_{a-1}})}_{\texttt{meta-regret}} + \underbrace{\sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_{t,s_{a-1}})}_{\leq C+F}
\end{aligned}$$

Algorithm 2 A Universal Algorithm for Exploiting Smoothness to Improve the Adaptive Regret (USIA)

1: Initialize indicator *NewInterval=true*, the total number of intervals $m = 0$
2: **for** $t = 1$ **to** T **do**
3: **if** *NewInterval* is *true* **then**
4: Create multiple experts for three types of convex functions and add all the experts to the set active experts: $\mathcal{A}_t = \mathcal{A}_{t-1} \cup \{E_t^k\}$
5: Reset the indicator *NewInterval=false* and update the total number of intervals $m = m + 1$
6: Initialize $\gamma_t^k = \ln(2m+1)$, $x_{t-1,t}^k = 1$ and $L_{t-1,t}^k = 0$ for all the experts
7: Set $g_t = j$ such that $[m, j-1] \in \mathcal{C}$ and $\hat{L}_{t-1} = 0$
8: **end if**
9: Set $\eta_{t,i}^k$ and calculate the weight $p_{t,i}^k$ by (8) for each expert $E_i^k \in \mathcal{A}_t$
10: Receive output $\mathbf{w}_{t,i}^k$ from each expert $E_i^k \in \mathcal{A}_t$
11: Calculate $\mathbf{w}_t = \sum_{E_j^k \in \mathcal{A}_t} p_{t,j}^k \mathbf{w}_{t,j}^k$ and evaluate the gradient $\nabla f_t(\mathbf{w}_t)$
12: Update the cumulative loss of the latest expert: $\hat{L}_t = \hat{L}_{t-1} + f_t(\mathbf{w}_t)$
13: **if** $\hat{L}_t > \mathcal{G}(m)$ **then**
14: Set the indicator *NewInterval=true*
15: Remove experts whose ending times are $t+1$: $\mathcal{A}_t = \mathcal{A}_t \setminus \{E_i^k | g_i = m+1\}$
16: **end if**
17: Observe the normalized linearized loss $\ell_{t,i}^k$ of each expert $E_i^k \in \mathcal{A}_t$ by (9)
18: Observe the meta loss $\ell_t = \sum_{E_j^k \in \mathcal{A}_t} p_{t,j}^k \ell_{t,j}^k$
19: Update $L_{t,i}^k$ and $x_{t,i}^k$ of each expert $E_i^k \in \mathcal{A}_t$ by (10)
20: **end for**

where $s_{a-1} \le r < s_a$. According to the theoretical guarantee of Adapt-ML-Prod for convex and non-smooth functions, the meta-regret will be

$$\sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_{t,s_{a-1}}) \le O\left(\sqrt{s_a - s_{a-1}}\right).$$

Since there is no relationship between s_{a-1} and r, this bound could be loose for convex functions over interval $[r, s_a - 1]$. To address this problem, we provide another approach to bound the regret of a sub-interval $[r, s_a - 1]$ by

$$\sum_{t=r}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=r}^{s_a-1} f_t(\mathbf{w}) \le \sum_{t=r}^{s_a-1} f_t(\mathbf{w}_t) \le \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_t),$$

which motivates us to directly control the cumulative loss of the output $\mathbf{w}_t$ from *meta-algorithm* during the construction of CPGC intervals. Since the meta-algorithm aggregates the experts over all the intervals, the cumulative loss of its output depends on the number of intervals created till current round, thereby rendering a fixed threshold of Zhang et al. (2019) inadequate for intervals construction. Therefore, we instead construct a threshold function $\mathcal{G}(a)$ based on the cumulative loss varying with the number of intervals, where a is the interval index. We also mention that such technique also appears in Zhao et al. (2022), but their goal is to reduce projection complexity, which is different from ours. According to the theoretical guarantee, the meta-regret exhibits only a logarithmic dependency on the number of intervals, thus ensuring a tight regret bound within each interval segment.

Our **U**niversal algorithm for exploiting **S**moothness to **I**mprove the **A**daptive regret (USIA) is summarized in Algorithm 2. To generate CPGC intervals, we introduce a Boolean variable *NewInterval* to indicate whether a new interval should be created and denote the total number of intervals by m. In each round t, if *NewInterval* is *true*, we create multiple experts E_t^k in Step 4, where $k = 1, \cdots, 1 + 2\lceil \log_2 T \rceil$. Then, we reset the indicator to be false and update the total number of intervals in Step 5. Next, we initialize the γ_t^k, $x_{t-1,t}^k$ and $L_{t-1,t}^k$ for each expert E_t^k which are parameters for Adapt-ML-Prod in Step 6. In Step 7, we introduce $g_t = j$ and $\hat{L}_{t-1}$ to record the ending time of the m-th marker and the cumulative loss, repsectively. In Step 9, we set the learning rate for each expert in the active set and calculate the weight $p_{t,i}^k$:

$$\eta_{t-1,i}^k = \min\left\{\frac{1}{2}, \sqrt{\frac{\gamma_i^k}{1 + L_{t-1,i}^k}}\right\}, \quad p_{t,i}^k = \frac{\eta_{t-1,i}^k x_{t-1,i}^k}{\sum_{E_j^k \in \mathcal{A}_t} \eta_{t-1,j}^k x_{t-1,j}^k}. \tag{8}$$

In Step 10, USIA collects the predictions of all the active experts, and aggregates them in Step 11. We keep track of the aggregated prediction $\mathbf{w}_t$ and record its cumulative loss $\hat{L}_t$. When $\hat{L}_t$ is larger than the threshold function $\mathcal{G}(m)$, we set *NewInterval* to be true and remove experts whose ending times are $s_{m+1} - 1$ (Steps 13 to 16). After evaluating the gradient $\nabla f_t(\mathbf{w}_t)$, we observe the normalized linearized loss for all the active experts in Step 17, which is formulated as,

$$\ell_{t,i} = \frac{\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_{t,i} - \bar{\mathbf{w}} \rangle + GD}{2GD} \in [0, 1]. \tag{9}$$

In Step 18, we calculate the weighted average of $\ell_{t,i}^k$ as the loss of the meta-algorithm suffered in the t-th round. Finally, we update the parameter $L_{t,i}^k$ and $x_{t,i}^k$ in the set of active experts according to the rule of Adapt-ML-Prod:

$$L_{t,i}^k = L_{t-1,i}^k + (\ell_t - \ell_{t,i}^k)^2, \quad x_{t,i}^k = \left(x_{t-1,i}^k \left(1 + \eta_{t-1,i}^k (\ell_t - \ell_{t,i}^k)\right)\right)^{\frac{\eta_{t,i}^k}{\eta_{t-1,i}^k}}. \tag{10}$$

Subsequently, we present the following theoretical guarantee of USIA, and defer the details of expert-algorithms and analysis in Appendix D.

Theorem 3.11. *Under Assumptions 3.1, 3.2, 3.3 and 3.4, for any interval $[r, s] \subseteq [T]$ and any $\mathbf{w} \in \mathcal{W}$, with appropriate parameters set, if the online functions are α-exp-concave with $\alpha \in [1/T, 1]$, USIA satisfies*

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\frac{d}{\alpha} \log L_{1,s}^* \log L_{r,s}^*\right),$$

if the online functions are λ-strongly convex with $\lambda \in [1/T, 1]$, USIA satisfies

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\frac{1}{\lambda} \log L_{1,s}^* \log L_{r,s}^*\right),$$

if the online functions are general convex, USIA satisfies

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\sqrt{L_{r,s}^* \log L_{1,s}^* \log L_{r,s}^*}\right),$$

where $L_{r,s}^*$ is defined in (1). Moreover, when functions are non-smooth, USIA achieves $O(\frac{d}{\alpha} \log s \log(s-r))$, $O(\frac{1}{\lambda} \log s \log(s-r))$ and $O(\sqrt{(s-r) \log s \log(s-r)})$ adaptive regret for α-exp-concave, λ-strongly convex, and general convex functions, respectively.

Remark Theorem 3.11 demonstrates that USIA is able to attain small-loss adaptive regret bounds for three types of loss functions simultaneously. Moreover, USIA is implemented without knowing the modulus of exp-concavity and strong convexity. When dealing with changing environments, USIA can also handle the case that the type of functions changes between rounds. For example, under the smoothness condition, suppose the online functions are general convex during interval $[r_1, s_1]$, then become α-exp-concave in $[r_2, s_2]$, and finally switch to λ-strongly convex in $[r_3, s_3]$. When facing this function sequence, USIA achieves $O(\sqrt{L_{r_1,s_1}^* \log L_{1,s_1}^* \log L_{r_1,s_1}^*})$, $O(\frac{d}{\alpha} \log L_{1,s_2}^* \log L_{r_2,s_2}^*)$ and $O(\frac{1}{\lambda} \log L_{1,s_3}^* \log L_{r_3,s_3}^*)$ regrets over intervals $[r_1, s_1]$, $[r_2, s_2]$ and $[r_3, s_3]$, respectively. Furthermore, when the online functions are non-smooth, our algorithm can also attain adaptive regret bounds for three types of loss functions simultaneously.

Parameter Setting According to the procedure of USIA, we need to set threshold $\mathcal{G}(m)$ to construct CPGC intervals, where $\mathcal{G}(m)$ requires G, D, H and T in USIA. For non-smooth functions, we can set H be any constant, e.g., 1. The detailed requirements of threshold and the exact regret bounds of USIA are included in Appendix D.1.

Reduce Gradient Query Complexity According to the description of Algorithm 2, we need to construct $O(\log T)$ CPGC intervals to adapt to the changing environment, and each interval is associated with $O(\log T)$ expert-algorithms to address the uncertainty of the function. Therefore, USIA maintains $O(\log^2 T)$ expert-algorithms which is the same as that of existing universal algorithm for adaptive regret (Zhang et al., 2021). Within the meta-expert framework, each expert-algorithm needs to query the function gradient once and evaluate the function value once per round, thus leading to significant concerns about computational efficiency. To address this limitation, we introduce an improved implementation of USIA with *one* gradient query and *one* value estimation per round in Appendix E.

4. Conclusion

In this paper, we develop an adaptive algorithm, named as FLHS, which achieves a small-loss adaptive regret bound for exp-concave functions. Additionally, we point out that a small-loss adaptive regret bound for strongly-convex functions is attainable by changing the expert-algorithm. Furthermore, we propose a universal algorithm for exploiting smoothness to improve the adaptive regret, namely USIA. Under the smoothness condition, it delivers small-loss adaptive regret bounds for general convex, exp-concave and strongly convex functions simultaneously. Finally, we prove that our algorithms can also attain minimax optimal regret bounds when functions are non-smooth, which offers a significant advantage over previous studies.

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Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none of which we feel must be specifically highlighted here.

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A. Omitted Related Work

In this section, we discuss some related work on static regret. For general convex functions, online gradient descent (OGD) with step size $\eta_t = O(1/\sqrt{t})$ attains an $O(\sqrt{T})$ regret bound (Zinkevich, 2003). For α-exp-concave functions, ONS is equipped with $O(\frac{d}{\alpha}\log T)$ regret bound, where d is the dimensionality (Hazan et al., 2007). For λ-strongly convex functions, an $O(\frac{1}{\lambda}\log T)$ regret bound is achievable by applying OGD with step size $\eta_t = O(1/[\lambda t])$ (Shalev-Shwartz et al., 2007). While the above regret bounds are minimax optimal for the corresponding type of functions (Abernethy et al., 2008), tighter bounds are attainable if the loss functions are smooth. For general convex and smooth functions, OGD with a constant step size attains an $O(\sqrt{L})$ regret bound (Srebro et al., 2010), where L is the upper bound of $\sum_{t=1}^{T} f_t(\mathbf{w})$ for any $\mathbf{w} \in \mathcal{W}$. For general convex and smooth functions, OGD with a constant step size attains an $O(\sqrt{L})$ regret bound (Srebro et al., 2010), where L is the upper bound of $\sum_{t=1}^{T} f_t(\mathbf{w})$ for any $\mathbf{w} \in \mathcal{W}$. However, the modulus of smoothness and upper bound L are required to set the step size, which are typically unavailable in practice. Scale-free online gradient descent (SOGD) (Zhang et al., 2019) is proposed to address this limitation, which is a special case of the Scale-free online mirror descent (SOMD) (Orabona & Pál, 2018). SOGD achieves an $O(\sqrt{L_T^*})$ small-loss regret bound for general convex functions, where $L_T^* = \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})$. For α-exp-concave and smooth functions, ONS is able to attain an $O(\frac{d}{\alpha}\log L_T^*)$ small-loss regret bound (Orabona et al., 2012). For λ-strongly convex and smooth functions, Wang et al. (2020) introduce smooth and strongly convex online gradient descent (S^2OGD) which yields an $O(\frac{1}{\lambda}\log L_T^*)$ small-loss regret bound.

B. Follow-the-Leading-History for Exp-concave and Smooth functions

B.1. Exact Bounds for α-exp-concave and (Non-)Smooth Functions

Due to page limit in the main body, we present bounds using the big-O notation in the theorems. Here, we provide the exact bounds.

For α-exp-concave and smooth functions, suppose

$$C \geq \frac{d}{\beta}\sqrt{\frac{4H\beta D^2}{d} + 4H\beta D^2 \log\frac{4H\beta D^2}{e} + 2} + \frac{1}{\beta} + dHD^2 \tag{11}$$

where $\beta = \frac{1}{2}\min\{\frac{1}{4GD}, \alpha\}$. Under this condition, FLHS achieves the following regret bound

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq \left(1 + \left\lceil \log_2\left(2 + \frac{4}{C}\sum_{t=r}^{s} f_t(\mathbf{w})\right)\right\rceil\right)(\text{MR}(s) + \text{ER}(r,s)) + C + F$$

where

$$\text{MR}(s) = \frac{2}{\alpha}\ln\left(1 + \frac{4}{C}\sum_{t=1}^{s} f_t(\mathbf{w})\right)$$

$$\text{ER}(r,s) = \frac{d}{2\beta}\log\left(\frac{8H\beta^2 D^2}{d}\sum_{t=r}^{s} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2\log\frac{4H\beta D^2}{e} + 2\right) + \frac{1}{2\beta}.$$

For α-exp-concave and non-smooth functions, FLHS achieves the following regret bound

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w})$$
$$\leq \left(1 + \left\lceil \log_2\left(\frac{4F}{C}(s-r+1) + 2\right)\right\rceil\right)\left(\frac{2}{\alpha}\ln\left(1 + \frac{F}{C}s\right) + \frac{d}{2\beta}\log\left(\frac{\beta^2 G^2 D^2}{d}(s-r) + 1\right) + \frac{1}{2\beta}\right) + C + F.$$

B.2. Algorithm Description: Follow-the-Leading-History for Smooth functions

Highlight First of all, we make clarifications that the expert-algorithm must attain an *anytime* regret bound over CPGC intervals, because we need to construct CPGC intervals based on the *real-time performance* of the expert-algorithm.

For exp-concave functions, we use Online Newton Step (ONS) as our expert-algorithm to minimize the regret during interval $[s_p, s_q - 1]$. The generalized projection $\Pi_{\mathcal{W}}^{\Sigma}(\cdot)$ is defined as

$$\Pi_{\mathcal{W}}^{\Sigma}(\mathbf{x}) = \underset{\mathbf{w}\in\mathcal{W}}{\text{argmin}}(\mathbf{w}-\mathbf{x})^\top \Sigma(\mathbf{w}-\mathbf{x})$$

which is used in Step 4 of Algorithm 3.

Algorithm 3 Expert E_{s_p}: Online Newton Step (ONS)

1: Initialize $\mathbf{w}_{s_p}$ be any point in $\mathcal{W}$ and $\Sigma_{s_p-1} = \frac{1}{\beta^2 D^2}\mathbf{I}_d$,
2: **for** $t = s_p$ **to** $s_q - 1$ **do**
3: Update $\Sigma_t = \Sigma_{t-1} + \mathbf{g}_t\mathbf{g}_t^\top$ where $\mathbf{g}_t = \nabla f_t(\mathbf{w}_{t,s_p})$
4: Calculate

$$\mathbf{w}_{t+1,s_p} = \Pi_{\mathcal{W}}^{\Sigma_t}\left(\mathbf{w}_{t,s_p} - \frac{1}{\beta}\Sigma_t^{-1}\mathbf{g}_t\right)$$

5: **end for**

B.3. Proof of Theorem 3.8

First, we start with the meta-regret.

Lemma B.1. *Under Assumptions 3.1, 3.2 and 3.3, if the online functions are α-exp-concave and the learning rate $\eta = \alpha$, for any interval $[s_p, s_q - 1] \in \widetilde{\mathcal{C}}$, the meta-regret of FLHS with respect to E_{s_p} satisfies*

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) \leq \frac{2}{\alpha}\ln\left(1 + \frac{4}{C}\sum_{t=1}^{s_q-1} f_t(\mathbf{w})\right)$$

where C is defined in (11).

Remark Compared with the problem-independent bound $O\left(\frac{1}{\alpha}\log(s_q - 1)\right)$ of efficient FLH, the meta-regret of our FLHS is problem-dependent, since it depends on the cumulative loss of the expert instead of the time length.

For each CPGC interval $I = [s_p, s_q - 1] \in \widetilde{\mathcal{C}}$, we create an instance of Online Newton Step (ONS) (Hazan et al., 2007) as expert-algorithm because it can attain a small-loss regret bound for exp-concave and smooth functions during I (Orabona et al., 2012, Theorem 1).

Lemma B.2. *Let $\beta = \frac{1}{2}\min\{\frac{1}{4GD}, \alpha\}$. Under Assumptions 3.1, 3.2 and 3.3, if the online functions are α-exp-concave, for any interval $[s_p, s_q - 1] \in \widetilde{\mathcal{C}}$ and any $\mathbf{w} \in \mathcal{W}$, expert E_{s_p} satisfies*

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \frac{d}{2\beta}\log\left(\frac{8H\beta^2D^2}{d}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2\log\frac{4H\beta D^2}{e} + 2\right) + \frac{1}{2\beta}.$$

For simplicity, we denote

$$\mathrm{MR}(s_q) = \frac{2}{\alpha}\ln\left(1 + \frac{4}{C}\sum_{t=1}^{s_q-1} f_t(\mathbf{w})\right),$$

$$\mathrm{ER}(s_p, s_q) = \frac{d}{2\beta}\log\left(\frac{8H\beta^2D^2}{d}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2\log\frac{4H\beta D^2}{e} + 2\right) + \frac{1}{2\beta}.$$

Combining the meta-regret and expert-regret, we can obtain a small-loss bound for α-exp-concave and smooth functions over any CPGC interval. Next, we extend this bound to any interval $[r, s] \subseteq [T]$.

The analysis is similar to the proof of Theorem 4 of Zhang et al. (2019). Let s_a be the smallest marker that is larger than r, and s_b be the largest marker that is not larger than s. Then, we have

$$s_{a-1} \leq r < s_a, \text{ and } s_b \leq s < s_{b+1}.$$

First, we bound the regret over interval $[r, s_a - 1]$. We have

$$\begin{aligned}&\sum_{t=r}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=r}^{s_a-1} f_t(\mathbf{w}) \leq \sum_{t=r}^{s_a-1} f_t(\mathbf{w}_t) \leq \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_t)\\ \leq &\sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_t) - \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_{t,s_{a-1}}) + \sum_{t=s_{a-1}}^{s_a-1} f_t(\mathbf{w}_{t,s_{a-1}})\\ \leq &\mathrm{MR}(s_a - 1) + C + F\end{aligned}$$

where the last step is due to the construction rule of markers and Assumption 3.4. Next, we bound the regret over interval $[s_a, s]$. To proceed, we introduce the following property of CPGC intervals (Zhang et al., 2019, Lemma 11).

Lemma B.3. *Let $[s_a, s_b] \subseteq [T]$ be an interval that starts from an marker s_a and ends at another marker s_b. Then, we can find a sequence of consecutive intervals*

$$I_1 = [s_{i_1}, s_{i_2} - 1], I_2 = [s_{i_2}, s_{i_3} - 1], \cdots, I_v = [s_{i_v}, s_{i_{v+1}} - 1] \in \widetilde{\mathcal{C}}$$

such that

$$i_1 = a, i_v \leq b < i_{v+1}, \text{ and } v \leq \lceil \log_2(b - a + 2) \rceil.$$

Note that

$$b < i_{v+1} \Rightarrow b + 1 \leq i_{v+1} \Rightarrow s_{b+1} - 1 \leq s_{i_v+1} - 1 \Rightarrow s \leq s_{i_v+1} - 1.$$

Thus, the interval $[s_a, s]$ is also covered by the sequence of intervals in the above lemma. Then, for the first $v - 1$ intervals,

$$\sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w}_t) - \sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w}) \leq \mathrm{MR}(s_{i_{k+1}} - 1) + \mathrm{ER}(s_{i_k}, s_{i_{k+1}} - 1), \forall k \in [v - 1].$$

And for the last interval, we have

$$\sum_{t=s_{i_v}}^{s} f_t(\mathbf{w}_t) - \sum_{t=s_{i_v}}^{s} f_t(\mathbf{w}) \leq \mathrm{MR}(s) + \mathrm{ER}(s_{i_v}, s).$$

By adding them together, we have

$$\begin{aligned}\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) &\leq (v + 1)\mathrm{MR}(s) + v\mathrm{ER}(1, s) + C + F\\ &\leq (1 + \lceil \log_2(b - a + 2) \rceil) (\mathrm{MR}(s) + \mathrm{ER}(1, s)) + C + F.\end{aligned} \tag{12}$$

Finally, we provide an upper bound of $b - a$ by summing (19) over $i = a, \cdots, b - 1$ and arrive at

$$\sum_{t=s_a}^{s_b-1} f_t(\mathbf{w}) \geq \frac{C}{4}(b - a) \Rightarrow b - a \leq \frac{4}{C} \sum_{t=s_a}^{s_b-1} f_t(\mathbf{w}) \leq \frac{4}{C} \sum_{t=r}^{s} f_t(\mathbf{w}).$$

B.4. Proof of Theorem 3.9

When functions are non-smooth, i.e., removing Assumption 3.3, the following analysis shows that FLHS with ONS expert can also attain an adaptive regret bound for α-exp-concave functions. For expert-regret, since ONS is equipped with a *squared gradient-norm* regret bound, we have

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \frac{d}{2\beta} \log\left(\frac{\beta^2 G^2 D^2}{d}(s_q - s_p) + 1\right) + \frac{1}{2\beta}. \tag{13}$$

For meta-regret, from the construction of markers, we have

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) \geq C.$$

Summing the above bound over $i = 1, \ldots, m-1$, we attain

$$m \leq 1 + \frac{1}{C} \sum_{i=1}^{m-1} \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) \leq 1 + \frac{1}{C} \sum_{t=s_1}^{s_m-1} F \leq 1 + \frac{F}{C}(s_q - 1)$$

which implies

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) \leq \frac{2}{\alpha} \ln\left(1 + \frac{F}{C}(s_q - 1)\right).$$

Combining the meta-regret and expert-regret, we can easily attain an adaptive regret bound for α-exp-concave and non-smooth functions. Then, we extend to $[r, s] \subseteq [T]$ by repeating the above analysis.

We start with (12), and obtain

$$\begin{aligned}
&\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \\
\leq &(1 + \lceil \log_2(b - a + 2) \rceil) \left(\frac{2}{\alpha} \ln\left(1 + \frac{F}{C} s\right) + \frac{d}{2\beta} \log\left(\frac{\beta^2 G^2 D^2}{d}(s - r) + 1\right) + \frac{1}{2\beta}\right) \\
&+ C + F.
\end{aligned} \tag{14}$$

For the upper bound of $b - a$, we get

$$b - a \leq \frac{4}{C} \sum_{t=s_a}^{s_b-1} f_t(\mathbf{w}) \leq \frac{4}{C} \sum_{t=r}^{s} f_t(\mathbf{w}) \leq \frac{4F}{C}(s - r + 1).$$

Substituting the above bound into (14), we finish the proof.

B.5. Proof of Lemma B.1

According to Definition 3.6, we have

$$e^{-\alpha f_t(\mathbf{w}_t)} = e^{-\alpha f_t\left(\sum_{E_j \in \mathcal{A}_t} p_{t,j} \mathbf{w}_{t,j}\right)} \geq \sum_{E_j \in \mathcal{A}_t} p_{t,j} e^{-\alpha f_t(\mathbf{w}_{t,j})}$$

which implies

$$f_t(\mathbf{w}_t) \leq -\frac{1}{\alpha} \ln \sum_{E_j \in \mathcal{A}_t} p_{t,j} e^{-\alpha f_t(\mathbf{w}_{t,j})}.$$

Then, we have

$$\begin{aligned}
f_t(\mathbf{w}_t) - f_t(\mathbf{w}_{t,s_p}) &\leq \frac{1}{\alpha}\left(\ln e^{-\alpha f_t(\mathbf{w}_{t,s_p})} - \ln \sum_{E_j \in \mathcal{A}_t} p_{t,j} e^{-\alpha f_t(\mathbf{w}_{t,j})}\right) \\
&= \frac{1}{\alpha} \ln \frac{e^{-\alpha f_t(\mathbf{w}_{t,s_p})}}{\sum_{E_j \in \mathcal{A}_t} p_{t,j} e^{-\alpha f_t(\mathbf{w}_{t,j})}} \\
&= \frac{1}{\alpha} \ln\left(\frac{1}{p_{t,i}} \cdot \frac{p_{t,i} e^{-\alpha f_t(\mathbf{w}_{t,s_p})}}{\sum_{E_j \in \mathcal{A}_t} p_{t,j} e^{-\alpha f_t(\mathbf{w}_{t,j})}}\right) \\
&= \frac{1}{\alpha} \ln \frac{\hat{p}_{t+1,s_p}}{p_{t,s_p}}.
\end{aligned} \tag{15}$$

When $t = s_p$, we have

$$f_{s_p}(\mathbf{w}_{s_p}) - f_{s_p}(\mathbf{w}_{s_p,s_p}) \leq \frac{1}{\alpha}\left(\ln \hat{p}_{s_p+1,s_p} + \ln m_{s_p}\right)$$

where m_{s_p} denotes the number of experts created till round s_p. When $t \neq s_p$, we prove the following inequality

$$f_t(\mathbf{w}_t) - f_t(\mathbf{w}_{t,s_p}) \leq \frac{1}{\alpha}\left(\ln \hat{p}_{t+1,s_p} - \ln p_{t,s_p}\right) \leq \frac{1}{\alpha}\left(\ln \hat{p}_{t+1,s_p} - \ln \hat{p}_{t,s_p} + \mathbb{1}_{\{NewInterval\}}\frac{2}{m_t}\right)$$

where the last step is due to $-\ln(1-x) \leq 2x$ when $0 < x \leq \frac{1}{2}$.

Summing (15) over $t = s_p, \cdots, s_q - 1$, we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p})\\
\leq& f_{s_p}(\mathbf{w}_{s_p}) - f_{s_p}(\mathbf{w}_{s_p,s_p}) + \frac{1}{\alpha}\left(\sum_{t=s_p+1}^{s_q-1}(\ln \hat{p}_{t+1,s_p} - \ln \hat{p}_{t,s_p}) + \sum_{t=s_p+1}^{s_q-1}\mathbb{1}_{\{NewInterval\}}\cdot\frac{2}{m_t}\right)\\
\leq& \frac{1}{\alpha}(\ln m_{s_p} + \underbrace{\ln \hat{p}_{s_q+1,s_p}}_{\leq 0}) + \frac{1}{\alpha}\sum_{t=m_{s_p+1}}^{m_{s_q-1}}\frac{2}{t} \leq \frac{1}{\alpha}\ln m_{s_p} + \frac{2}{\alpha}\ln m_{s_q-1} - \frac{2}{\alpha}\ln m_{s_p+1}\\
\leq& \frac{2}{\alpha}\ln m_{s_q-1}.
\end{aligned} \tag{16}$$

Next, we need to provide a small-loss upper bound for m_{s_q-1}. Note that in each interval $[s_i, s_{i+1}-1]$, an expert E_{s_i} is created by running ONS. According to (29), we have

$$\begin{aligned}
\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) - \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) &\leq \frac{d}{2\beta}\log\left(\frac{8H\beta^2 D^2}{d}\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) + \Xi\right) + \frac{1}{2\beta}\\
&\leq \frac{d}{2\beta}\sqrt{\frac{8H\beta^2 D^2}{d}\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) + \Xi} + \frac{1}{2\beta}\\
&\leq \frac{1 + d\sqrt{\Xi}}{2\beta} + \frac{dHD^2}{2} + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w})
\end{aligned} \tag{17}$$

where $\Xi = \frac{4H\beta D^2}{d} + 4H\beta D^2 \log \frac{4H\beta D^2}{e} + 2$. On the other hand, from the construction of markers, we have

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) \geq C.$$

Suppose

$$C \geq \frac{d}{\beta}\sqrt{\frac{4H\beta D^2}{d} + 4H\beta D^2 \log\frac{4H\beta D^2}{e} + 2} + \frac{1}{\beta} + dHD^2. \tag{18}$$

Thus, we have

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) \geq \frac{1}{2}\left(C - \left(\frac{1+d\sqrt{\Xi}}{2\beta} + \frac{dHD^2}{2}\right)\right) \overset{(18)}{\geq} \frac{C}{4}. \tag{19}$$

Let m is the number of experts created till round t. Summing (19) over $i = 1, \cdots, m-1$, we have

$$\sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}) \geq \frac{C}{4}(m-1)$$

implying

$$m \leq 1 + \frac{4}{C} \sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}) \leq 1 + \frac{4}{C} \sum_{t=1}^{t} f_t(\mathbf{w}). \tag{20}$$

Combining (16) and (20), we can achieve the meta-regret of FLHS,

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) \leq \frac{2}{\alpha} \ln \left(1 + \frac{4}{C} \sum_{t=1}^{s_q-1} f_t(\mathbf{w})\right) = \text{MR}(s_q - 1).$$

B.6. Proof of Lemma B.2

The analysis is similar the proofs of Theorem 1 of Orabona et al. (2012). For the convenience of notations, we suppose $\mathbf{w}_t = \mathbf{w}_{t,s_p}$, let $\mathbf{y}_{t+1} = \mathbf{w}_t - \frac{1}{\beta}\Sigma_t^{-1}\mathbf{g}_t$, and $\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}^{\Sigma_t}(\mathbf{y}_{t+1})$, we have

$$\begin{aligned} \mathbf{y}_{t+1} - \mathbf{w} = \mathbf{w}_t - \mathbf{w} - \frac{1}{\beta}\Sigma_t^{-1}\mathbf{g}_t, \\ \Sigma_t(\mathbf{y}_{t+1} - \mathbf{w}) = \Sigma_t(\mathbf{w}_t - \mathbf{w}) - \frac{1}{\beta}\mathbf{g}_t. \end{aligned} \tag{21}$$

And we combine above equalities and arrive at

$$\begin{aligned} (\mathbf{y}_{t+1} - \mathbf{w})^\top \Sigma_t (\mathbf{y}_{t+1} - \mathbf{w}) &= (\mathbf{w}_t - \mathbf{w})^\top \Sigma_t (\mathbf{w}_t - \mathbf{w}) - \frac{2}{\beta}\mathbf{g}_t^\top (\mathbf{w}_t - \mathbf{w}) + \frac{1}{\beta^2}\mathbf{g}_t^\top \Sigma_t^{-1}\mathbf{g}_t \\ &\geq (\mathbf{w}_{t+1} - \mathbf{w})^\top \Sigma_t (\mathbf{w}_{t+1} - \mathbf{w}). \end{aligned} \tag{22}$$

Thus,

$$\mathbf{g}_t^\top (\mathbf{w}_t - \mathbf{w}) \leq \frac{1}{2\beta}\mathbf{g}_t^\top \Sigma_t^{-1}\mathbf{g}_t + \frac{\beta}{2}(\mathbf{w}_t - \mathbf{w})^\top \Sigma_t (\mathbf{w}_t - \mathbf{w}) - \frac{\beta}{2}(\mathbf{w}_{t+1} - \mathbf{w})^\top \Sigma_t (\mathbf{w}_{t+1} - \mathbf{w}). \tag{23}$$

Then summing over $t = s_p, \cdots, s_q - 1$ and we attain

$$\begin{aligned} \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top (\mathbf{w}_t - \mathbf{w}) \leq& \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top \Sigma_t^{-1}\mathbf{g}_t + \frac{\beta}{2}(\mathbf{w}_{s_p} - \mathbf{w})^\top \Sigma_{s_p} (\mathbf{w}_{s_p} - \mathbf{w}) \\ &+ \frac{\beta}{2} \sum_{t=s_p+1}^{s_q-1} (\mathbf{w}_t - \mathbf{w})^\top (\Sigma_t - \Sigma_{t-1})(\mathbf{w}_t - \mathbf{w}) \\ &- \frac{\beta}{2}(\mathbf{w}_{s_q} - \mathbf{w})^\top \Sigma_{s_q-1} (\mathbf{w}_{s_q} - \mathbf{w}) \\ \leq& \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top \Sigma_t^{-1}\mathbf{g}_t + \frac{\beta}{2} \sum_{t=s_p}^{s_q-1} \langle \mathbf{g}_t, \mathbf{w}_t - \mathbf{w} \rangle^2 \\ &+ \frac{\beta}{2}(\mathbf{w}_{s_p} - \mathbf{w})^\top (\Sigma_{s_p} - \mathbf{g}_{s_p}\mathbf{g}_{s_p}^\top)(\mathbf{w}_{s_p} - \mathbf{w}). \end{aligned} \tag{24}$$

After that,

$$
\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top (\mathbf{w}_t - \mathbf{w}) - \frac{\beta}{2} \sum_{t=s_p}^{s_q-1} \langle \mathbf{g}_t, \mathbf{w}_t - \mathbf{w} \rangle^2 \\
\leq & \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top \Sigma_t^{-1} \mathbf{g}_t + \frac{\beta}{2} (\mathbf{w}_{s_p} - \mathbf{w})^\top (\Sigma_{s_p} - \mathbf{g}_{s_p} \mathbf{g}_{s_p}^\top)(\mathbf{w}_{s_p} - \mathbf{w}) \\
\leq & \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top \Sigma_t^{-1} \mathbf{g}_t + \frac{1}{2\beta} = \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \Sigma_t^{-1} \mathbf{g}_t \mathbf{g}_t^\top + \frac{1}{2\beta} \\
= & \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \Sigma_t^{-1} (\Sigma_t - \Sigma_{t-1}) + \frac{1}{2\beta} \\
\leq & \frac{1}{2\beta} \sum_{t=s_p}^{s_q-1} \log \frac{|\Sigma_t|}{|\Sigma_{t-1}|} + \frac{1}{2\beta} = \frac{1}{2\beta} \log \frac{|\Sigma_{s_q-1}|}{|\Sigma_{s_p-1}|} + \frac{1}{2\beta}
\end{aligned}
\tag{25}
$$

where the last inequality is obtained by Hazan et al. (2007, Lemma 12). It is easy to bound $|\Sigma_{s_q-1}|$ with sum of gradient norm.

$$
\begin{aligned}
\operatorname{tr}(\Sigma_{s_q-1}) = \operatorname{tr}(\Sigma_{s_p-1}) + \sum_{t=s_p}^{s_q-1} \|\mathbf{g}_t\|^2 = \frac{d}{\beta^2 D^2} + \sum_{t=s_p}^{s_q-1} \|\mathbf{g}_t\|^2 \\
|\Sigma_{s_q-1}| = \left(\frac{1}{\beta^2 D^2} + \sum_{t=s_p}^{s_q-1} \|\mathbf{g}_t\|^2 / d \right)^d
\end{aligned}
\tag{26}
$$

where the last inequality is obtained by Jensen's inequality. According to the definition of Σ_{s_p-1} and Lemma 3.7, we have

$$
\begin{aligned}
\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq & \sum_{t=s_p}^{s_q-1} \mathbf{g}_t^\top (\mathbf{w}_t - \mathbf{w}) - \frac{\beta}{2} \sum_{t=s_p}^{s_q-1} \langle \mathbf{g}_t, \mathbf{w}_t - \mathbf{w} \rangle^2 \\
\leq & \frac{d}{2\beta} \log \left(\frac{\beta^2 D^2}{d} \sum_{t=s_p}^{s_q-1} \|\mathbf{g}_t\|^2 + 1 \right) + \frac{1}{2\beta}.
\end{aligned}
\tag{27}
$$

Next, we introduce the self-bounding property of smooth functions (Srebro et al., 2010, Lemma 3.1).

Lemma B.4. *For an H-smooth and nonnegative function,*

$$
\|\nabla f(\mathbf{w})\| \leq \sqrt{4Hf(\mathbf{w})}, \forall \mathbf{w} \in \mathcal{W}.
$$

In this way, we attain

$$
\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \frac{d}{2\beta} \log \left(\frac{4H\beta^2 D^2}{d} \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) + 1 \right) + \frac{1}{2\beta}. \tag{28}
$$

To obtain the small-loss regret bound, we need the follow lemma (Orabona et al., 2012, Corollary 5).

Lemma B.5. *Let $a, b, c, d, x > 0$ satisfy*

$$
x - d \leq a \ln(bx + c).
$$

Then

$$
x - d \leq a \ln \left(2 \left(ab \ln \frac{2ab}{\mathrm{e}} + db + c \right) \right).
$$

Finally, we finished the proof by combining (28) and Lemma B.5,

$$
\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\
\leq&\frac{d}{2\beta}\log\left(\frac{8H\beta^2D^2}{d}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2\log\frac{4H\beta D^2}{e} + 2\right) + \frac{1}{2\beta} \\
=&\mathrm{ER}(s_p, s_q-1).
\end{aligned} \tag{29}
$$

C. Follow-the-Leading-History for Strongly-convex and Smooth functions

C.1. Algorithm Description: Smooth and Strongly Convex OGD (S^{2}OGD)

For λ-strongly convex functions, we use Smooth and Strongly Convex OGD (S^{2}OGD) as our expert-algorithm, since it is equipped with small-loss regret bound. The projection operator $\Pi_{\mathcal{W}}(\cdot)$ is defined as

$$\Pi_{\mathcal{W}}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{w}\in\mathcal{W}} \|\mathbf{w}-\mathbf{x}\|.$$

Algorithm 4 Expert E_{s_p}: Smooth and Strongly Convex OGD (S^{2}OGD)

1: **for** $t = s_p$ **to** $s_q - 1$ **do**

2: Update

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}\left(\mathbf{w}_t - \alpha_t \mathbf{g}_t\right)$$

where

$$\alpha_t = \frac{\gamma}{\delta + \sum_{i=s_p}^{t}\|\mathbf{g}_i\|^2}, \quad \mathbf{g}_t = \nabla f_t(\mathbf{w}_t).$$

3: **end for**

For each CPGC interval $I = [s_p, s_q-1] \in \widetilde{\mathcal{C}}$, we create an instance of Smooth and Strongly Convex OGD (S^{2}OGD) as expert-algorithm because it can attain a small-loss regret bound for strongly convex and smooth functions during I (Wang et al., 2020, Theorem 1).

Lemma C.1. *Let $\gamma = \frac{G^2}{\lambda}$ and $\delta = G^2$. Under Assumptions 3.1 and 3.2, if the online functions are λ-strongly convex, for any interval $[s_p, s_q-1] \in \widetilde{\mathcal{C}}$ and any $\mathbf{w}\in\mathcal{W}$, expert E_{s_p} satisfies*

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \lambda D^2 + \frac{G^2}{2\lambda}\log\left(\frac{1}{G^2}\sum_{t=s_p}^{s_q-1}\|\nabla f_t(\mathbf{w}_{t,s_p})\|^2\right).$$

Furthermore, under Assumption 3.3, we have

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \frac{G^2}{\lambda}\ln\left(\frac{8H}{G^2}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \mu\right) + \lambda D^2$$

where

$$\mu = \frac{8\lambda H D^2}{G^2} + \frac{4H}{\lambda}\ln\frac{4H+\lambda e}{e\lambda} + 2 \tag{30}$$

is a constant.

C.2. Proof of Theorem 3.10

We introduce the following property of strongly convex functions (Zhang et al., 2018, Lemma 2).

Lemma C.2. *Suppose a function $f : \mathcal{W} \mapsto \mathbb{R}$ is λ-strongly convex and $\|\nabla f(\mathbf{w})\| \leq G$ for all $\mathbf{w} \in \mathcal{W}$. Then, $f(\cdot)$ is $\frac{\lambda}{G^2}$-exp-concave.*

The above lemma implies that strongly convex functions with bounded gradients are also exp-concave. Thus, for λ-strongly convex and smooth functions, when we set learning rate $\eta = \frac{\lambda}{G^2}$ in FLHS, (16) implies

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) \leq \frac{2G^2}{\lambda} \ln m_{s_q-1}. \tag{31}$$

Then, we bound m_{s_q-1} by repeating the analysis of Theorem 3.8. Note that in each interval $[s_i, s_{i+1}-1]$, an expert E_{s_i} is created by running S^{2}OGD. According to Lemma C.1, we have

$$\begin{aligned}\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) - \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) &\leq \frac{G^2}{\lambda} \ln \left(\frac{8h}{G^2} \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) + \mu \right) + \lambda D^2 \\ &\leq \frac{G^2}{\lambda} \sqrt{\frac{8h}{G^2} \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) + \mu} + \lambda D^2 \\ &\leq \frac{G^2}{\lambda}\sqrt{\mu} + \frac{2hG^2}{\lambda^2} + \lambda D^2 + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w})\end{aligned} \tag{32}$$

where $\mu = \frac{8\lambda h D^2}{G^2} + \frac{4h}{\lambda} \ln \frac{4h+\lambda e}{e\lambda} + 2$. Suppose

$$C \geq \frac{2G^2}{\lambda}\sqrt{\mu} + \frac{4hG^2}{\lambda^2} + 2\lambda D^2. \tag{33}$$

Thus, we have

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) \geq \frac{1}{2}\left(C - \left(\frac{G^2}{\lambda}\sqrt{\mu} + \frac{2hG^2}{\lambda^2} + \lambda D^2 \right) \right) \overset{(33)}{\geq} \frac{C}{4}. \tag{34}$$

Let m is the number of experts created till round t. Summing (19) over $i = 1, \cdots, m-1$, we have

$$\sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}) \geq \frac{C}{4}(m-1)$$

implying

$$m \leq 1 + \frac{4}{C} \sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}) \leq 1 + \frac{4}{C} \sum_{t=1}^{t} f_t(\mathbf{w}). \tag{35}$$

Combining the meta-regret in (31) and expert-regret of Lemma C.1, we have

$$\begin{aligned}&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\ \leq& \underbrace{\frac{2G^2}{\lambda} \ln \left(1 + \frac{4}{C} \sum_{t=1}^{s_q-1} f_t(\mathbf{w}) \right)}_{\text{MR}(s_q-1)} + \underbrace{\frac{G^2}{\lambda} \ln \left(\frac{8H}{G^2} \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \mu \right) + \lambda D^2}_{\text{ER}(s_p, s_q-1)}.\end{aligned}$$

Finally, we finish the proof by repeating the analysis of Theorem 3.8. Furthermore, we can also attain an adaptive regret bound for λ-strongly convex and non-smooth functions. Due to the theoretical guarantee of S^{2}OGD, we have

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \lambda D^2 + \frac{G^2}{2\lambda} \log(s_q - s_p) = \text{ERs}(s_p, s_q).$$

when functions are non-smooth, (31) implies that

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}) \leq \frac{2G^2}{\lambda} \ln\left(1 + \frac{F}{C}(s_q - 1)\right) = \mathrm{MRs}(s_q).$$

because

$$m \leq 1 + \frac{1}{C} \sum_{i=1}^{m-1} \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_{t,s_i}) \leq 1 + \frac{1}{C} \sum_{t=s_1}^{s_m-1} F \leq 1 + \frac{F}{C}(s_q - 1).$$

Combining the meta-regret and expert-regret, we have

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \frac{2G^2}{\lambda} \ln\left(1 + \frac{F}{C}(s_q - 1)\right) + \lambda D^2 + \frac{G^2}{2\lambda} \log\left(s_q - s_p\right).$$

Then, we extend the above bound to any interval $[r, s] \subseteq [T]$ by repeating the analysis of Theorem 3.8.

D. Universal Algorithm for Exploiting the Smoothness to Improve the Adaptive Regret

D.1. Exact Bounds of USIA for (Non)-Smooth Functions

Due to page limit in the main body, we present bounds using the big-O notation in the theorems. Here, we provide the exact bounds.

Suppose for the m-th interval,

$$\begin{aligned}\mathcal{G}(m) = 4GD\Gamma(m) + 10HD^2 \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2}\right)^2 \\ + 2\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2}\right) D\sqrt{4G^2 + 8HGD\Gamma(m)}\end{aligned}$$

where $\Gamma(m) = \ln(2m+1) + 2\ln(3 + 2\log_2 T)$. For α-exp-concave and smooth functions, USIA satisfies

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq (\mathrm{MR}(s) + \mathrm{ER}(r,s) + \Xi)\log_2\left(2 + \frac{4}{\mathcal{G}(1)} \sum_{t=r}^{s} f_t(\mathbf{w})\right) + \mathcal{G}(a) + F$$

where

$$\begin{aligned}\mathrm{MR}(s) &= \left(4GD + \frac{1}{2\beta}\right) \ln\left(3 + \frac{8}{\mathcal{G}(1)} \sum_{t=1}^{s} f_t(\mathbf{w})\right) \\ \mathrm{ER}(r,s) &= \frac{d}{\beta} \log\left(\frac{8H\beta^2D^2}{d} \sum_{t=r}^{s} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2 \log\frac{4H\beta D^2}{e} + 2\right) \\ \Xi &= \left(4GD + \frac{4}{\beta}\right)\epsilon(T) + \frac{2}{\beta}\epsilon^2(T) + \frac{1}{\beta} \\ \mathcal{G}(a) + F &\leq O\left(\log \sum_{t=1}^{s_a-1} f_t(\mathbf{w})\right) \leq O\left(\log \sum_{t=1}^{s} f_t(\mathbf{w})\right).\end{aligned}$$

For general convex and smooth functions, USIA has

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq A(s)v(r,s) + B(s)\sqrt{v(r,s)\sum_{t=r}^{s} f_t(\mathbf{w})} + \mathcal{G}(a) + F \tag{36}$$

where

$$\begin{aligned}
\mathrm{A}(s) =& 4HD^2 \ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s} f_t(\mathbf{w})\right) + (8\sqrt{2}HD^2 + 2GD + 1)\sqrt{\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s} f_t(\mathbf{w})\right)} \\
&+ \left(16HD^2(\sqrt{2}+1) + 4GD + 2\right)\epsilon(T) + 16HD^2\epsilon^2(T) + 24HD^2 + 2\sqrt{2}GD \\
\mathrm{B}(s) =& \sqrt{4HD^2}\sqrt{\left(\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s} f_t(\mathbf{w})\right) + 4(\sqrt{2}+1)\epsilon(T) + 4\epsilon^2(T)\right)} \\
\epsilon(T) =& \ln\left(3 + 2\log_2 T\right) \\
v(r,s) =& \log_2\left(2+\frac{4}{\mathcal{G}(1)}\sum_{t=r}^{s} f_t(\mathbf{w})\right) \\
\mathcal{G}(a) + F \leq& O\left(\log \sum_{t=1}^{s_a-1} f_t(\mathbf{w})\right) \leq O\left(\log \sum_{t=1}^{s} f_t(\mathbf{w})\right).
\end{aligned}$$

When deal with non-smooth functions, for α-exp-concave functions, USIA satisfies

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq (\mathrm{MR}(s) + \mathrm{ER}(r,s) + \Xi)\log_2\left(2 + \frac{4F}{\mathcal{G}(1)}(s-r+1)\right) + \mathcal{G}(a) + F$$

where

$$\begin{aligned}
\mathrm{MR}(s) =& \left(4GD + \frac{1}{2\beta}\right)\ln\left(3 + \frac{8F}{\mathcal{G}(1)}s\right) \\
\mathrm{ER}(r,s) =& \frac{d}{2\beta}\log\left(\frac{\beta^2G^2D^2}{d}(r-s)+1\right) + \frac{1}{2\beta} \\
\Xi =& \left(4GD + \frac{4}{\beta}\right)\epsilon(T) + \frac{2}{\beta}\epsilon^2(T) + \frac{1}{\beta} \\
\mathcal{G}(a) + F \leq& O\left(\log s\right).
\end{aligned}$$

For general convex functions, USIA satisfies

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\
\leq& 2GD\left(\mathrm{MR}^c(s) + \epsilon(T)\right)\log_2\left(2 + \frac{4F}{\mathcal{G}(1)}(s-r+1)\right) + G(a) + F \\
&+ GD\left(\sqrt{2} + \epsilon(T) + \sqrt{\mathrm{MR}^c(s)}\right)\sqrt{(s-r+4)\log_2\left(2 + \frac{4F}{\mathcal{G}(1)}(s-r+1)\right)}
\end{aligned}$$

where $\mathrm{MR}^c(s) = \ln\left(3 + \frac{2F}{\mathcal{G}(1)}(s)\right)$ and $\epsilon(T) = 2\ln(3 + \log_2 T)$

D.2. Expert-algorithms for USIA

The projection operator $\Pi_{\mathcal{W}}(\cdot)$ is defined as

$$\Pi_{\mathcal{W}}(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{w}\in\mathcal{W}} \|\mathbf{w} - \mathbf{x}\|.$$

Algorithm 5 Expert E_{s_p}: Scale-free online gradient descent (SOGD)

for $t = s_p$ **to** $s_q - 1$ **do**

Update

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}} \left(\mathbf{w}_t - \eta_t \mathbf{g}_t \right)$$

where

$$\eta_t = \frac{\gamma}{\delta + \sum_{i=s_p}^{t} \|\mathbf{g}_i\|^2}, \quad \mathbf{g}_t = \nabla f_t(\mathbf{w}_t).$$

end for

The procedure of creating multiple experts is summarized as below. In Step 1, we initialize $k = 1$ to record the number index for the expert-algorithm.

$$\mathcal{P}_{str} = \left\{ 1/T, 2/T, 2^2/T, \cdots, 2^N/T \right\}, \quad \mathcal{P}_{exp} = \left\{ 1/T, 2/T, 2^2/T, \cdots, 2^N/T \right\}$$

where $N = \lceil \log_2 T \rceil$. For general convex functions, we create an expert by running the algorithm of SOGD to minimize the linearized loss in Step 2. For λ-strongly convex functions, we construct a finite set $\mathcal{P}_{str}$ containing possible values of the modulus, which are served as the input parameters for S^{2}OGD in Step 4. Similarly, for α-exp-concave, we construct a finite set $\mathcal{P}_{exp}$ to be the input parameters for ONS in Step 7.

Algorithm 6 Expert-algorithms for USIA

1: **Input:** Initialize the number index of the expert-algorithm $k = 1$
2: Create an expert E_t^1 by running the algorithm of SOGD to minimize $\hat{\ell}_t(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} \rangle$
3: **for all** $\lambda \in \mathcal{P}_{str}$ and $k = k + 1$ **do**
4: Create an expert E_t^k by running the algorithm of S^{2}OGD with λ
5: **end for**
6: **for all** $\alpha \in \mathcal{P}_{exp}$ and $k = k + 1$ **do**
7: Create an expert E_t^k by running the algorithm of ONS with α
8: **end for**

D.3. Proof of Theorem 3.11

We start with the meta-regret of USIA over any CPGC interval.

Lemma D.1. *Under Assumptions 3.1 and 3.2, for the m-th interval $[s_p, s_q - 1] \in \widetilde{\mathcal{C}}$ we created and any $\mathbf{w} \in \mathcal{W}$, the meta-regret of USIA with respect to $E_{s_p}^k$ satisfies*

$$\sum_{t=s_p}^{s_q-1} \ell_t - \sum_{t=s_p}^{s_q-1} \ell_{t,s_p}^k \leq \frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} \sqrt{1 + \sum_{t=s_p}^{s_q-1} (\ell_t - \ell_{t,s_p}^k)^2} + \Gamma(m)$$

where $\Gamma(m) = \ln(2m + 1) + 2\ln(3 + 2\log_2 T)$.

The main advantage of the above meta-regret is that it automatically utilizes the property of exp-concavity and strongly convexity. We take α-exp-concave functions as an example to explain the reason why this meta-regret can deal with exp-concavity. The definition of ℓ_t in Step 26 implies

$$\ell_t = \sum_{E_j^k \in \mathcal{A}_t} p_{t,j}^k \ell_{t,j}^k = \frac{\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \bar{\mathbf{w}} \rangle + GD}{2GD}. \tag{37}$$

Combing Lemma D.1 with the definition of ℓ_t and ℓ^k_{t,s_p} in (37) and (9), we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle \\
\leq& \frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} \sqrt{4G^2D^2 + \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle^2} + 2GD\Gamma(m) \\
\leq& 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \sqrt{\frac{\Gamma^2(m)}{\ln(2m+1)} \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle^2} \\
\leq& 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{2\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{\beta}{2} \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle^2.
\end{aligned} \tag{38}$$

In the above derivation, the second inequality follows from the basic inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and the last inequality utilizes $2ab \leq a^2 + b^2$. Thus, when the online functions are α-exp-concave, Lemma 3.7 implies

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}^k_{t,s_p}) \\
\leq& \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle - \frac{\beta}{2} \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle^2 \\
\overset{(38)}{\leq}& 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{2\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} \\
\leq& 4GD \left(\ln(2m+1) + \ln(3 + 2\log_2 T) \right) + \frac{1}{2\beta} \left(\ln(2m+1) + 8\ln(3 + \log_2 T) + \frac{4\ln^2(3 + 2\log_2 T)}{\ln(2m+1)} \right) \\
\leq& O\left(\frac{1}{\beta} \log m \right)
\end{aligned} \tag{39}$$

where the last step is because we can treat the double logarithmic factor as constant (Chernov & Vovk, 2010; Luo & Schapire, 2015). From above discussions, we observe that USIA attains a logarithmic term regret bound for α-exp-concave functions without knowing value of α. Based on Definition 3.5, the above derivation also holds for strongly convex functions.

Small-loss bound for m-th interval. Next, we provide a small-loss upper bound for m-th interval. First, from the first-order condition of convex functions, we bound the meta-regret in Lemma D.1 by

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle \\
\leq& \frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} \sqrt{4G^2D^2 + \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p} \rangle^2} + 2GD\Gamma(m) \\
\leq& 2GD\Gamma(m) + \frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} \sqrt{4G^2D^2 + \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 \|\mathbf{w}_t - \mathbf{w}^k_{t,s_p}\|^2} \\
\leq& 2GD\Gamma(m) + \frac{\Gamma(m)D}{\sqrt{\ln(2m+1)}} \sqrt{4G^2 + \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2} \\
\leq& 2GD\Gamma(m) + \frac{\Gamma(m)D}{\sqrt{\ln(1+m)}} \sqrt{4G^2 + 4H \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t)}.
\end{aligned} \tag{40}$$

Note that we run SOGD in each CPGC interval, thus we can directly use the theoretical guarantee of SOGD in each interval $[s_i, s_{i+1}-1]$ (Zhang et al., 2019, Theorem 2).

Lemma D.2. *Set $\delta > 0$ and $\alpha = D/\sqrt{2}$. Under Assumptions 3.1, 3.2 and 3.3, for any interval $[s_i, s_{i+1}-1]$ and any $\mathbf{w} \in \mathcal{W}$, we have*

$$\sum_{t=s_i}^{s_{i+1}-1} \langle \nabla \hat{\ell}_t(\mathbf{w}_{t,s_i}^k), \mathbf{w}_{t,s_i}^k - \mathbf{w} \rangle \le \sqrt{2D^2} \sqrt{\delta + \sum_{t=s_i}^{s_{i+1}-1} \|\nabla \hat{\ell}_t(\mathbf{w}_{t,s_i}^k)\|^2}$$

The theoretical guarantee of SOGD implies that

$$\sum_{t=s_i}^{s_{i+1}-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_{t,s_i}^k - \mathbf{w} \rangle \le \sqrt{2D^2} \sqrt{\delta + 4H \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t)}. \tag{41}$$

Combining (40) and (41) (set $\delta = 4G^2$) on interval $[s_i, s_{i+1}-1]$ ($s_i = s_p$), we have

$$\begin{aligned}
&\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t) - \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) \le \sum_{t=s_i}^{s_{i+1}-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle \\
\le& 2GD\Gamma(m) + \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right) D \sqrt{4G^2 + 4H \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t)} \\
\le& 2GD\Gamma(m) + \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right) \sqrt{4HD^2} \sqrt{\frac{G^2}{H} + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t)}.
\end{aligned}$$

Then we need the following lemma (Shalev-Shwartz, 2007, Lemma 19).

Lemma D.3. *Let $x, b, c \in \mathbb{R}_+$. Then,*

$$x - c \le b\sqrt{x} \implies x - c \le b^2 + b\sqrt{c}.$$

Let

$$\begin{aligned}
x &= \frac{G^2}{H} + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t), \\
c &= \frac{G^2}{H} + 2GD\Gamma(m) + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}), \\
b &= \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right) \sqrt{4HD^2},
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t) - \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) \\
\le& 2GD\Gamma(m) + 4HD^2 \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right)^2 \\
&+ \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right) \sqrt{4HD^2} \sqrt{\frac{G^2}{H} + 2GD\Gamma(m) + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w})} \qquad (42) \\
\le& 2GD\Gamma(m) + 5HD^2 \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right)^2 \\
&+ \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2} \right) D\sqrt{4G^2 + 8HGD\Gamma(m)} + \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}).
\end{aligned}$$

Then, we have

$$\mathcal{G}(m) \leq \sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t) \leq P(m) + 2\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w})$$

where

$$\begin{aligned} P(m) = & 2GD\Gamma(m) + 5HD^2\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2}\right)^2 \\ & + \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \sqrt{2}\right) D\sqrt{4G^2 + 8HGD\Gamma(m)} \\ \Gamma(m) = & \ln(2m+1) + 2\ln(3 + 2\log_2 T). \end{aligned}$$

Suppose $\mathcal{G}(m) = 2P(m)$,

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}) \geq \frac{1}{2}\left(\mathcal{G}(m) - P(m)\right) \geq \frac{1}{4}\mathcal{G}(m). \tag{43}$$

Summing the above inequality over $i = 1, \cdots, m-1$, we have

$$\sum_{t=1}^{s_q-1} f_t(\mathbf{w}) \geq \sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}) \geq \frac{1}{4}\sum_{i=1}^{m-1}\mathcal{G}(i) \geq \frac{\mathcal{G}(1)}{4}(m-1) \Rightarrow m \leq 1 + \frac{4}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1} f_t(\mathbf{w}) \tag{44}$$

where the last inequality utilizes the property of nondecreasing in $\mathcal{G}(\cdot)$.

Small-loss bound for α-exp-concave functions. Combining (39) and (44), we can attain a small-loss adaptive regret bound over any CPGC interval for α-exp-concave functions,

$$\begin{aligned} & \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}^k) \\ \leq & \left(4GD + \frac{1}{2\beta}\right)\ln\left(3 + \frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1} f_t(\mathbf{w})\right) + \left(4GD + \frac{4}{\beta}\right)\epsilon(T) + \frac{2}{\beta}\epsilon^2(T) \end{aligned}$$

where $\epsilon(T) = \ln(3 + 2\log_2 T)$. Then we combine the above bound with the small-loss bound of ONS expert. Let E_{t,s_p}^k be the ONS expert with $\hat{\alpha} \in \mathcal{P}_{exp}$ and $\hat{\alpha} \leq \alpha < 2\hat{\alpha}$. Since α-exp-concave functions are also $\hat{\alpha}$-exp-concave, we can directly utilize the theoretical guarantee of ONS expert with $\hat{\alpha}$ to attain

$$\begin{aligned} & \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_{t,s_p}^k) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\ \leq & \frac{d}{2\hat{\beta}}\log\left(\frac{8H\hat{\beta}^2D^2}{d}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \frac{4H\hat{\beta}D^2}{d} + 4H\hat{\beta}D^2\log\frac{4H\hat{\beta}D^2}{e} + 2\right) + \frac{1}{2\hat{\beta}} \\ \leq & \frac{d}{\beta}\log\left(\frac{8H\beta^2D^2}{d}\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \frac{4H\beta D^2}{d} + 4H\beta D^2\log\frac{4H\beta D^2}{e} + 2\right) + \frac{1}{\beta}. \end{aligned}$$

We combine the meta-regret and expert-regret to achieve the final adaptive regret bound,

$$\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \mathrm{MR}(s_q - 1) + \mathrm{ER}(s_p, s_q - 1) + \Xi \tag{45}$$

where

$$\mathrm{MR}(s_q-1)=\left(4GD+\frac{1}{2\beta}\right)\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1}f_t(\mathbf{w})\right)$$

$$\mathrm{ER}(s_p,s_q-1)=\frac{d}{\beta}\log\left(\frac{8H\beta^2D^2}{d}\sum_{t=s_p}^{s_q-1}f_t(\mathbf{w})+\frac{4H\beta D^2}{d}+4H\beta D^2\log\frac{4H\beta D^2}{e}+2\right)$$

$$\Xi=\left(4GD+\frac{4}{\beta}\right)\epsilon(T)+\frac{2}{\beta}\epsilon^2(T)+\frac{1}{\beta}.$$

Finally, we extend the above regret bound to any interval $[r,s]\subseteq[T]$ by repeating the analysis of Theorem 1, since they enjoy the same order of meta-regret and expert-regret. We highlight the only difference in the analysis that we bound the regret over interval $[r,s_a-1]$,

$$\sum_{t=r}^{s_a-1}f_t(\mathbf{w}_t)-\sum_{t=r}^{s_a-1}f_t(\mathbf{w})\leq\sum_{t=r}^{s_a-1}f_t(\mathbf{w}_t)\leq\sum_{t=s_{a-1}}^{s_a-1}f_t(\mathbf{w}_t)\leq\mathcal{G}(a)+F\leq O\left(\log\sum_{t=1}^{s_a-1}f_t(\mathbf{w})\right).$$

To proceed, we omit the analysis for λ-strongly convex functions because strongly convex functions with bounded gradients are also exp-concave.

Small-loss bound for general convex functions. To deal with general convex functions, we go back to the first inequality in (42),

$$\begin{aligned}&\sum_{t=s_p}^{s_q-1}f_t(\mathbf{w}_t)-\sum_{t=s_p}^{s_q-1}f_t(\mathbf{w})\\ \leq&2GD\Gamma(m)+4HD^2\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}}+\sqrt{2}\right)^2\\ &+\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}}+\sqrt{2}\right)\sqrt{4HD^2}\sqrt{\frac{G^2}{H}+2GD\Gamma(m)+\sum_{t=s_p}^{s_q-1}f_t(\mathbf{w})}\\ \leq&\mathrm{A}(s_q-1)+\mathrm{B}(s_q-1)\sqrt{\sum_{t=s_p}^{s_q-1}f_t(\mathbf{w})}\end{aligned}\tag{46}$$

where

$$\begin{aligned}\mathrm{A}(s_q-1)=&4HD^2\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1}f_t(\mathbf{w})\right)+(8\sqrt{2}HD^2+2GD+1)\sqrt{\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1}f_t(\mathbf{w})\right)}\\ &+\left(16HD^2(\sqrt{2}+1)+4GD+2\right)\epsilon(T)+16HD^2\epsilon^2(T)+24HD^2+2\sqrt{2}GD\\ \mathrm{B}(s_q-1)=&\sqrt{4HD^2}\sqrt{\left(\ln\left(3+\frac{8}{\mathcal{G}(1)}\sum_{t=1}^{s_q-1}f_t(\mathbf{w})\right)+4(\sqrt{2}+1)\epsilon(T)+4\epsilon^2(T)\right)}\\ \epsilon(T)=&\ln\left(3+2\log_2 T\right)\end{aligned}$$

Finally, we extend the above regret bound to any interval $[r,s]\subseteq[T]$ which is similar to the analysis of FLHS. Let s_a be the smallest marker that is larger than r, and s_b be the largest marker that is not larger than s. We bound the regret over $[r,s_a-1]$,

$$\sum_{t=r}^{s_a-1}f_t(\mathbf{w}_t)-\sum_{t=r}^{s_a-1}f_t(\mathbf{w})\leq\sum_{t=r}^{s_a-1}f_t(\mathbf{w}_t)\leq\sum_{t=s_{a-1}}^{s_a-1}f_t(\mathbf{w}_t)\leq\mathcal{G}(a)+F\leq O\left(\log\sum_{t=1}^{s_a-1}f_t(\mathbf{w})\right).\tag{47}$$

Then we bound the regret over $[s_a, s]$ by repeating the analysis of Theorem 3.8, for the first $v-1$ intervals, (46) implies

$$\sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w}_t) - \sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w}) \leq \mathrm{A}(s_{i_{k+1}}-1) + \mathrm{B}(s_{i_{k+1}}-1)\sqrt{\sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w})}, \forall k \in [v-1].$$

And for the last interval, we have

$$\sum_{t=s_{i_v}}^{s} f_t(\mathbf{w}_t) - \sum_{t=s_{i_v}}^{s} f_t(\mathbf{w}) \leq \mathrm{A}(s) + \mathrm{B}(s)\sqrt{\sum_{t=s_{i_v}}^{s} f_t(\mathbf{w})}.$$

By adding them together, we attain

$$\begin{aligned}\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) &\leq vA(s) + B(s)\left(\sum_{k=1}^{v-1}\sqrt{\sum_{t=s_{i_k}}^{s_{i_{k+1}}-1} f_t(\mathbf{w})} + \sqrt{\sum_{t=s_{i_v}}^{s} f_t(\mathbf{w})}\right)\\ &\leq vA(s) + B(s)\sqrt{v\sum_{t=r}^{s} f_t(\mathbf{w}) + \mathcal{G}(a) + F}\end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality. Finally, we use the upper bound of v by summing (43) over $i = a, \cdots, b-1$ and finish the proof.

Adaptive regret bounds for non-smooth functions. We start with α-exp-concave functions to show that USIA can also attain an adaptive regret bound for α-exp-concave and non-smooth functions. From the construction of markers, we have

$$\sum_{t=s_i}^{s_{i+1}-1} f_t(\mathbf{w}_t) \geq \mathcal{G}(i) \geq \mathcal{G}(1).$$

Summing the above bound over $i = 1, \ldots, m-1$, we attain

$$m \leq 1 + \frac{1}{\mathcal{G}(1)}\sum_{t=s_1}^{s_m-1} f_t(\mathbf{w}_t) \leq 1 + \frac{1}{\mathcal{G}(1)}\sum_{t=s_1}^{s_m-1} F \leq 1 + \frac{F}{\mathcal{G}(1)}(s_q - 1).$$

Thus, (39) implies that

$$\begin{aligned}&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}^k_{t,s_p})\\ \leq& 4GD\left(\ln(2m+1) + \ln(3 + 2\log_2 T)\right) + \frac{1}{2\beta}\left(\ln(2m+1) + 8\ln(3+\log_2 T) + \frac{4\ln^2(3+2\log_2 T)}{\ln(2m+1)}\right)\\ \leq& \left(4GD + \frac{1}{2\beta}\right)\ln\left(3 + \frac{2F}{\mathcal{G}(1)}(s_q-1)\right) + 4\left(1+\frac{1}{\beta}\right)\ln(3+\log_2 T) + \frac{2}{\beta}\ln^2(3+\log_2 T).\end{aligned}$$

Then, we combine the meta-regret with the expert-regret of ONS. Finally, we extend to $[r, s] \subseteq [T]$ by repeating the above analysis. Similarly, we can also have the same guarantee for for λ-strongly convex and non-smooth functions.

For general convex and non-smooth, we modify (40) to achieve

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t,s_p}^k \rangle \\
\leq & 2GD\Gamma(m) + \frac{\Gamma(m)D}{\sqrt{\ln(2m+1)}} \sqrt{4G^2 + \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2} \\
\leq & 2GD\Gamma(m) + \frac{\Gamma(m)GD}{\sqrt{\ln(2m+1)}} \sqrt{4 + (s_q - s_p)} \\
\leq & 2GD\left(\mathrm{MR}^c(s_q) + \epsilon(T)\right) + GD\left(\epsilon(T) + \sqrt{\mathrm{MR}^c(s_q)}\right)\sqrt{s_q - s_p + 4}
\end{aligned} \tag{48}$$

where $\mathrm{MR}^c(s_q) = \ln\left(3 + \frac{2F}{\mathcal{G}(1)}(s_q - 1)\right)$ and $\epsilon(T) = 2\ln(3 + \log_2 T)$. Combining te above bound with expert-regret in (41), we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle \\
\leq & 2GD\left(\mathrm{MR}^c(s_q) + \epsilon(T)\right) + GD\left(\sqrt{2} + \epsilon(T) + \sqrt{\mathrm{MR}^c(s_q)}\right)\sqrt{s_q - s_p + 4}
\end{aligned}$$

Then, we extend this bound to any interval $[r, s] \subseteq [T]$ which is similar to the above analysis.

D.4. Proof of Lemma D.1

The analysis is similar to the proofs of Corollary 4 of Gaillard et al. (2014). We first introduce the following lemma.

Lemma D.4. *Under Assumptions 3.1 and 3.2, for any interval $[s_p, s_q - 1] \in \widetilde{\mathcal{C}}$, the meta-regret of USIA with respect to $E_{s_p}^k$ satisfies*

$$\begin{aligned}
\sum_{t=s_p}^{s_q-1} \ell_t - \sum_{t=s_p}^{s_q-1} \ell_{t,s_p}^k \leq & \frac{1}{\eta_{s_p-1,s_p}^k} \ln \frac{1}{x_{s_p-1,s_p}^k} + \underbrace{\sum_{t=s_p}^{s_q-1} \eta_{t,s_p}^k (\ell_t - \ell_{t,s_p}^k)^2}_{\text{first term}} \\
& + \frac{1}{\eta_{s_q-1,s_p}^k} \ln(1 + \frac{1}{e} \underbrace{\sum_{t=1}^{s_q-1} \sum_{E_j^k \in \mathcal{A}_t} (\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k} - 1))}_{\text{second term}}
\end{aligned} \tag{49}$$

where $\eta_{t,s_p}^k = \min\left\{\frac{1}{2}, \sqrt{\frac{\gamma_{s_p}^k}{1+L_{t,s_p}^k}}\right\}$.

The proof of Lemma D.4 could be found below. Our goal of proof is attain the bounds of two terms in (49). The following lemma (Gaillard et al., 2014, Lemma 14) will be useful.

Lemma D.5. *Let $a_0 > 0$ and $a_1, \cdots, a_m \in [0, 1]$ be real numbers and let $f : (0, +\infty) \to [0, +\infty]$ be a nonincreasing function. Then*

$$\sum_{i=1}^{m} a_i f(a_0 + \cdots + a_{i-1}) \leq f(a_0) + \int_{a_0}^{a_0 + a_1 + \cdots + a_m} f(u) \mathrm{d}u.$$

For the first term in (49), we utilize the definition of learning rate and arrive at

$$\sum_{t=s_p}^{s_q-1} \eta_{t,s_p} (\ell_t - \ell_{t,s_p}^k)^2 \leq \sqrt{\gamma_{s_p}^k} \sum_{t=s_p}^{s_q-1} \frac{(\ell_t - \ell_{t,s_p}^k)^2}{\sqrt{1 + L_{t-1,s_p}^k}}.$$

Then we apply Lemma D.5 with $f(x) = 1/\sqrt{x}$, and get

$$\begin{aligned}\sum_{t=s_p}^{s_q-1} \frac{(\ell_t - \ell_{t,s_p}^k)^2}{\sqrt{1+L_{t-1,s_p}^k}} &\leq \frac{1}{\sqrt{1+L_{s_p-1,s_p}^k}} + \int_{L_{s_p-1,s_p}^k}^{L_{s_q-1,s_p}^k} \frac{1}{\sqrt{1+u}}\mathrm{d}u \\ &\leq \underbrace{1-2\sqrt{1}}_{<0} + 2\sqrt{1+\sum_{t=s_p}^{s_q-1}(\ell_t - \ell_{t,s_p}^k)^2}.\end{aligned} \tag{50}$$

For the second term in (49), we have

$$\begin{aligned}\sum_{t=1}^{s_q-1}\sum_{E_j^k\in\mathcal{A}_t}\left(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1\right) &\leq \sum_{t=1}^{s_q-1}\sum_{E_j^k\in\mathcal{A}_t}\left(\sqrt{\frac{1+L_{t,j}^k}{1+L_{t-1,j}^k}}-1\right) \\ &= \sum_{t=1}^{s_q-1}\sum_{E_j\in\mathcal{A}_t}\left(\sqrt{1+\frac{(\ell_t-\ell_{t,j}^k)^2}{1+L_{t-1,j}^k}}-1\right) \\ &\leq \frac{1}{2}\sum_{t=1}^{s_q-1}\sum_{E_j^k\in\mathcal{A}_t}\frac{(\ell_t-\ell_{t,j}^k)^2}{1+L_{t-1,j}^k} \\ &= \frac{1}{2}\sum_{E_j^k\in\bigcup_{i=1}^{s_q-1}\mathcal{A}_i}\ \sum_{t=j}^{(s_q-1)\wedge e_j}\frac{(\ell_t-\ell_{t,j}^k)^2}{1+L_{t-1,j}^k} \\ &\leq \frac{1}{2}\sum_{E_j^k\in\bigcup_{i=1}^{s_q-1}\mathcal{A}_i}\left(1+\ln\left(1+\sum_{t=j}^{(s_q-1)\wedge e_j}L_{t-1,j}^k\right)-\ln(1)\right) \\ &\leq \frac{1}{2}\sum_{E_j^k\in\bigcup_{i=1}^{s_q-1}\mathcal{A}_i}(1+\ln s_q)\end{aligned}$$

where e_j denotes the ending time of an expert E_j^k and $(s_q-1)\wedge e_j = \min\{s_q-1, e_j\}$. In the above derivation, the second inequality is attained by $g(1+z) \leq g(1) + zg'(1), z \geq 0$ for any concave function g. For the third inequality, we apply Lemma D.5 with $f(x) = 1/x$. From the structure of CPGC intervals, we have

$$\ln\left(1+\frac{1}{e}\sum_{t=1}^{s_q-1}\sum_{E_j^k\in\mathcal{A}_t}\left(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1\right)\right) \leq \ln\left(1+\frac{1}{2e}n(1+\ln s_q)\right).$$

where n denotes the number of experts until round $s_q - 1$. According to our algorithm USIA, $[s_p, s_q-1] \in \widetilde{\mathcal{C}}$ denotes the m-th interval we created. It is also easy to verify that $m \leq m' \leq 2m+1$, where m' denotes the number of intervals created until round $s_q - 1$, due to the construction of CPGC intervals. Because we create $1 + 2\lceil\log_2 T\rceil$ experts on each interval, thus $n \leq (2m+1)(1+2\lceil\log_2 T\rceil)$. Then we arrive at

$$\ln\left(1+\frac{1}{e}\sum_{t=1}^{s_q-1}\sum_{E_j^k\in\mathcal{A}_t}\left(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1\right)\right) \leq \ln(1+2m) + 2\ln(3+2\log_2 T) = Q(m).$$

After obtaining two bounds for the first term and second term, we now get back to (49). Because of the choice of nonincreasing learning rates, we use $\eta_{s_p-1,s_p}^k \geq \eta_{s_q-1,s_p}^k$ and attain

$$\sum_{t=s_p}^{s_q-1}\ell_t - \sum_{t=s_p}^{s_q-1}\ell_{t,s_p}^k \leq \frac{1}{\eta_{s_q-1,s_p}^k}\left(\ln\frac{1}{x_{s_p-1,s_p}^k} + Q(m)\right) + 2\sqrt{\gamma_{s_p}^k\left(1+\sum_{t=s_p}^{s_q-1}(\ell_t-\ell_{t,s_p}^k)^2\right)}. \tag{51}$$

Now if $\sqrt{1+\sum_{t=s_p}^{s_q-1}(\ell_t-\ell_{t,s_p}^k)^2} > 2\sqrt{\gamma_{s_p}^k}$ then $\eta_{s_q-1,s_p}^k \leq 1/2$, (51) is bounded by

$$\sum_{t=s_p}^{s_q-1}\ell_t - \sum_{t=s_p}^{s_q-1}\ell_{t,s_p}^k \leq \sqrt{1+\sum_{t=s_p}^{s_q-1}(\ell_t-\ell_{t,s_p}^k)^2}\left(2\sqrt{\gamma_{s_p}^k} + \frac{\ln\frac{1}{x_{s_p-1,s_p}^k}+Q(m)}{\sqrt{\gamma_{s_p}^k}}\right). \tag{52}$$

Alternatively, if $\sqrt{1+\sum_{t=s_p}^{s_q-1}(\ell_t-\ell_{t,s_p}^k)^2} \leq 2\sqrt{\gamma_{s_p}^k}$ then $\eta_{s_q-1,s_p}^k = 1/2$, (51) is bounded by

$$\sum_{t=s_p}^{s_q-1}\ell_t - \sum_{t=s_p}^{s_q-1}\ell_{t,s_p}^k \leq 2\ln\frac{1}{x_{s_p-1,s_p}^k} + 2Q(m) + 4\gamma_{s_p}^k. \tag{53}$$

Combining the bound in (52) and (53), we have

$$\sum_{t=s_p}^{s_q-1}\ell_t - \sum_{t=s_p}^{s_q-1}\ell_{t,s_p}^k \leq \frac{2\gamma_{s_p}^k+Q(m)}{\sqrt{\gamma_{s_p}^k}}\sqrt{1+\sum_{t=s_p}^{s_q-1}(\ell_t-\ell_{t,s_p}^k)^2} + 2Q(m) + 4\gamma_{s_p}^k.$$

Then we set $\gamma_{s_p}^k = \ln(2m+1)$ and finish the proof.

D.5. Proof of Lemma D.4

This lemma is an extension of Gaillard et al. (2014, Theorem 3) to sleeping experts. The analysis will rely on the following lemma (Gaillard et al., 2014, Lemma 13).

Lemma D.6. *For all $x > 0$ and all $\alpha \geq 1$, we have $x \leq x^\alpha + (\alpha-1)/e$.*

Following the proof of Gaillard et al. (2014), we start to analyze the meta regret for any interval $[r,s] \in \widetilde{\mathcal{C}}$. Let $X_s = \sum_{E_j^k \in \mathcal{A}_s} x_{s,j}^k$, we bound $\ln X_s$ from below and above. For the lower bound, we start with $\ln X_s \geq \ln x_{s,j}^k$ and arrive at

$$\begin{aligned}
\ln x_{s,j}^k &= \frac{\eta_{s,j}^k}{\eta_{s-1,j}^k}\ln x_{s-1,j}^k + \frac{\eta_{s,j}^k}{\eta_{s-1,j}^k}\ln\left(1+\eta_{s-1,j}^k(\ell_s-\ell_{s,j})\right) \\
&\geq \frac{\eta_{s,j}^k}{\eta_{r-1,j}^k}\ln x_{r-1,j}^k + \sum_{t=r}^{s}\eta_{t,j}^k\left((\ell_t-\ell_{t,j}) - \eta_{t-1,j}^k(\ell_t-\ell_{t,j})^2\right) \\
&\geq \frac{\eta_{s,j}^k}{\eta_{r-1,j}^k}\ln x_{r-1,j}^k + \eta_{s,j}^k\left(\sum_{t=r}^{s}(\ell_t-\ell_{t,j}) - \eta_{t-1,j}^k(\ell_t-\ell_{t,j})^2\right)
\end{aligned} \tag{54}$$

where the first inequality uses the inequality $\ln(1+x) \geq x - x^2$ for all $x \geq -1/2$ (Cesa-Bianchi et al., 2005, Lemma 1) and the second inequality utilizes the property of nonincreasing in learning rate.

We now bound from above. According to the definition of weight update, we have

$$\begin{aligned}
\sum_{E_j\in\mathcal{A}_t}(x_{t,j}^k)^{\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}} &= \sum_{E_j\in\mathcal{A}_t} x_{t-1,j}^k(1+\eta_{t-1,j}^k(\ell_t-\ell_{t,j})) \\
&= \sum_{E_j\in\mathcal{A}_t} x_{t-1,j}^k + (\sum_{E_j\in\mathcal{A}_t}\eta_{t-1,j}^k x_{t-1,j}^k)\ell_t - \sum_{E_j\in\mathcal{A}_t}\eta_{t-1,j}^k x_{t-1,j}^k\ell_{t,j} \\
&= \sum_{E_j\in\mathcal{A}_t} x_{t-1,j}^k
\end{aligned} \tag{55}$$

where the last equality is obtained by (8). Because we always have $x_{t,j}^k > 0$ and $\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k} \geq 1$, applying Lemma D.6 in (55) can make us obtain

$$\begin{aligned}\sum_{E_j\in\mathcal{A}_t} x_{t,j}^k &\leq \sum_{E_j\in\mathcal{A}_t} (x_{t,j}^k)^{\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}} + \frac{1}{e}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1)\\ &= \sum_{E_j\in\mathcal{A}_t}(x_{t-1,j}^k) + \frac{1}{e}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1).\end{aligned} \tag{56}$$

Similar to the proof of Lemma, we sum (56) over $t = 1, \cdots, s$ and arrive at

$$\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t} x_{t,j}^k \leq \sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(x_{t-1,j}^k) + \frac{1}{e}\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1)$$

which can be rewritten as

$$\begin{aligned}&\sum_{E_j\in\mathcal{A}_s} x_{s,j}^k + \sum_{t=1}^{s-1}\left(\sum_{E_j\in\mathcal{A}_t\setminus\mathcal{A}_{t+1}} x_{t,j}^k + \sum_{E_j\in\mathcal{A}_t\cap\mathcal{A}_{t+1}} x_{t,j}^k\right)\\ \leq &\sum_{E_j\in\mathcal{A}_1} x_{0,j}^k + \sum_{t=2}^{s}\left(\sum_{E_j\in\mathcal{A}_t\setminus\mathcal{A}_{t-1}} x_{t-1,j}^k + \sum_{E_j\in\mathcal{A}_t\cap\mathcal{A}_{t-1}} x_{t-1,j}^k\right) + \frac{1}{e}\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1)\end{aligned}$$

implying

$$\begin{aligned}&\sum_{E_j\in\mathcal{A}_s} x_{s,j}^k + \sum_{t=1}^{s-1}\sum_{E_j\in\mathcal{A}_t\setminus\mathcal{A}_{t+1}} x_{t,j}^k\\ \leq &\sum_{E_j\in\mathcal{A}_1} x_{0,j}^k + \sum_{t=2}^{s}\sum_{E_j\in\mathcal{A}_t\setminus\mathcal{A}_{t-1}} x_{t-1,j}^k + \frac{1}{e}\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1)\\ \leq &1 + \frac{1}{e}\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1).\end{aligned}$$

By the definition that weights are always non-negative, we can get the upper bound

$$X_s = \sum_{E_j\in\mathcal{A}_s} x_{s,j}^k \leq 1 + \frac{1}{e}\sum_{t=1}^{s}\sum_{E_j\in\mathcal{A}_t}(\frac{\eta_{t-1,j}^k}{\eta_{t,j}^k}-1). \tag{57}$$

We finish the proof by combing the upper bound and lower bound in (57) and (54).

E. Improved Implementation of USIA Algorithm

To equip the USIA algorithm with dual adaptivity to function types and changing environments, it maintains $O(\log^2 T)$ expert-algorithms which is the same as that of existing universal algorithm for adaptive regret (Zhang et al., 2021). Within the meta-expert framework, each expert-algorithm needs to query the function gradient once and evaluate the function value once per round, thus leading to significant concerns about computational efficiency. In this section, we introduce an improved implementation of USIA with *one* gradient query and *one* value estimation per round.

E.1. Key Ideas

Note that the USIA algorithm constructs three types of expert-algorithms to address the uncertainty of function types, i.e., SOGD, S^{2}OGD, and ONS, with each expert running the original loss function. To reduce the number of gradient queries, the basic idea is to construct suitable surrogate losses for these expert-algorithms. For general convex functions, we can construct the following linearized loss for each SOGD expert,

$$\hat{\ell}(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}\rangle.$$

The challenge lies in developing appropriate surrogate losses for the S^2OGD and ONS expert, as they must meet the following criteria: (i) inherently enjoying strong convexity or exponential concavity, (ii) enabling the expert-algorithms to obtain small-loss bounds, and (iii) ensuring their compatibility with the linearized loss of the meta-algorithm within the meta-expert framework. Next, we take strongly convex functions as an example to describe our proposed method. For simplicity, we assume that λ is known and construct the following surrogate loss for the S^2OGD expert E^k_{t,s_p},

$$\ell^{\text{str}}_{t,k}(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t \rangle + \frac{\lambda}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_t - \mathbf{w}\|^2. \tag{58}$$

According to our derived Lemma E.3, the above surrogate loss is a $\frac{\lambda}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2$-strongly convex functions with bounded gradients, i.e., $\|\nabla \ell^{\text{str}}_{t,k}(\mathbf{w})\|^2 \leq (1 + D/G)^2\|\nabla f_t(\mathbf{w}_t)\|^2$. Therefore, the S^2OGD expert E^k_{t,s_p} can obtain the squared gradient-norm bound according to Lemma E.4, i.e.,

$$\mathrm{ER}(r, s) = \sum_{t=r}^{s}[\ell^{\text{str}}_{t,k}(\mathbf{w}^k_{t,s_p}) - \ell^{\text{str}}_{t,k}(\mathbf{w})] = O\left(\frac{1}{\lambda}\log\left(\sum_{t=r}^{s}\|\nabla f_t(\mathbf{w}_t)\|^2\right)\right), \tag{59}$$

which can be directly converted to a small-loss bound. Subsequently, we prove that our proposed surrogate loss is compatible with the meta-algorithm within the meta-expert framework. For any interval $[r, s] \subseteq [T]$ and any $\mathbf{w} \in \mathcal{W}$, we have

$$\begin{aligned}
&\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \\
\leq &\sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}\rangle - \sum_{t=r}^{s}\frac{\lambda}{2}\|\mathbf{w}_t - \mathbf{w}\|^2 \\
\leq &\sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle + \sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}^k_{t,s_p} - \mathbf{w}\rangle - \sum_{t=r}^{s}\frac{\lambda}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_t - \mathbf{w}\|^2 \\
\overset{(58)}{=} &\underbrace{\sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle}_{\texttt{meta-regret}} + \underbrace{\sum_{t=r}^{s}[\ell^{\text{str}}_{t,k}(\mathbf{w}^k_{t,s_p}) - \ell^{\text{str}}_{t,k}(\mathbf{w})]}_{\texttt{expert-regret}} - \underbrace{\sum_{t=r}^{s}\frac{\lambda}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}^k_{t,s_p} - \mathbf{w}_t\|^2}_{:=V_r^s}
\end{aligned} \tag{60}$$

where the first inequality is due to strong convexity, and the second inequality is because the gradients are bounded, and the equality is according to the definition of surrogate losses in (58).

As outlined in the above analysis, the expert-regret can be bounded by (59). Next, we turn our focus to bounding the meta-regret. According to our extended theoretical guarantee of Adapt-ML-Prod in Lemma D.1, the meta-regret enjoys the second-order bound, i.e.,

$$\begin{aligned}
\sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle &\leq O\left(\sqrt{\sum_{t=r}^{s}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle^2}\right) \\
&\leq O\left(\frac{G^2}{\lambda}\right) + \sum_{t=r}^{s}\frac{\lambda}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_t - \mathbf{w}^k_{t,s_p}\|^2.
\end{aligned}$$

Substituting the meta-regret into (60), we arrive at

$$\sum_{t=r}^{s} f_t(\mathbf{w}_t) - \sum_{t=r}^{s} f_t(\mathbf{w}) \leq O\left(\frac{G^2}{\lambda}\right) + \mathrm{ER}(r, s).$$

It is worth noting that the negative term V_r^s in (60) is cancelled by the last term in meta-regret, implying that the optimality of the algorithm only depends on the expert-regret because $O(G^2/\lambda)$ is small. For exp-concave functions, we construct the following surrogate loss,

$$\ell^{\exp}_{t,k}(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t\rangle + \frac{\beta}{2}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}\rangle^2,$$

and the analysis is similar. From the formulations of our proposed surrogate losses, it is evident that the expert-algorithms only accesses $f_t(\cdot)$ through $\nabla f_t(\cdot)$. Moreover, the only function value estimation occurs during the construction of problem-dependent intervals. Therefore, equipped with our proposed surrogate losses, the USIA algorithm reduces the number of gradient queries and value estimations per round from $O(\log^2 T)$ to 1. The theoretical guarantee of the improved implementation of USIA is as follows.

Theorem E.1. *Under the same assumptions as Theorem 3.11, by setting appropriate surrogate loss functions and parameters, the improved implementation of USIA algorithm possesses the same theoretical guarantee as Theorem 3.11.*

Remark Theorem E.1 demonstrates that our proposed surrogate losses can effectively reduce the gradient complexity of USIA algorithm without sacrificing the optimality of adaptive regret bounds.

E.2. Algorithm Description: Improved Implementation of USIA

Algorithm 7 An Improved Implementation of USIA

Initialize indicator *NewInterval=true*, the total number of intervals $m = 0$
for $t = 1$ **to** T **do**
  if *NewInterval* is *true* **then**
    Create multiple experts by running Algorithm 8 and add all the experts to the set active experts: $\mathcal{A}_t = \mathcal{A}_{t-1} \cup \{E_t^k\}$
    Reset the indicator *NewInterval=false* and update the total number of intervals $m = m + 1$
    Initialize $\gamma_t^k = \ln(2m + 1)$, $x_{t-1,t}^k = 1$ and $L_{t-1,t}^k = 0$ for all the experts
    Set $g_t = j$ such that $[m, j - 1] \in \mathcal{C}$ and $\hat{L}_{t-1} = 0$
  end if
  Set $\eta_{t,i}^k$ and calculate the weight $p_{t,i}^k$ by (8) for each expert $E_i^k \in \mathcal{A}_t$
  Receive output $\mathbf{w}_{t,i}^k$ from each expert $E_i^k \in \mathcal{A}_t$
  Calculate $\mathbf{w}_t = \sum_{E_j^k \in \mathcal{A}_t} p_{t,j}^k \mathbf{w}_{t,j}^k$ and evaluate the gradient $\nabla f_t(\mathbf{w}_t)$
  Update the cumulative loss of the latest expert: $\hat{L}_t = \hat{L}_{t-1} + f_t(\mathbf{w}_t)$
  if $\hat{L}_t > \mathcal{G}(m)$ **then**
    Set the indicator *NewInterval=true*
    Remove experts whose ending times are $t + 1$: $\mathcal{A}_t = \mathcal{A}_t \setminus \{E_i^k | g_i = m + 1\}$
  end if
  Observe the normalized linearized loss $\ell_{t,i}^k$ of each expert $E_i^k \in \mathcal{A}_t$ by (9)
  Observe the meta loss $\ell_t = \sum_{E_j^k \in \mathcal{A}_t} p_{t,j}^k \ell_{t,j}^k$
  Update $L_{t,i}^k$ and $x_{t,i}^k$ of each expert $E_i^k \in \mathcal{A}_t$ by (10)
end for

Algorithm 8 Expert-algorithms for improved USIA

Input: Initialize the number index of the expert-algorithm $k = 1$
Create an expert E_t^1 by running the algorithm of SOGD to minimize $\hat{\ell}_t(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} \rangle$
for all $\hat{\lambda} \in \mathcal{P}_{str}$ and $k = k + 1$ **do**
  Create an expert E_t^k by running the algorithm of S^{2}OGD to minimize $\ell_{t,k}^{\text{str}}(\cdot)$ in (62)
end for
for all $\hat{\alpha} \in \mathcal{P}_{exp}$ and $k = k + 1$ **do**
  Create an expert E_t^k by running the algorithm of ONS to minimize $\ell_{t,k}^{\text{exp}}(\cdot)$ in (61)
end for

Our efficient implementation of the USIA algorithm is summarized in Algorithm 7 and the procedure of the expert-algorithms is summarized in Algorithm 8. For expert-algorithms, the difference between USIA and its efficient version is that the ONS expert and the S^{2}OGD expert employ the different loss functions. Specifically, the ONS expert runs the following surrogate loss,

$$\ell_{t,k}^{\text{exp}}(\mathbf{w}) = \frac{\hat{\beta}}{2} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle^2 + \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t \rangle, \tag{61}$$

where $\hat{\beta} = \frac{1}{2}\min\{\frac{1}{4GD}, \hat{\alpha}\}$. And, the S^{2}OGD expert runs the following surrogate loss,

$$\ell_{t,k}^{\text{str}}(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t \rangle + \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2 \|\mathbf{w}_t - \mathbf{w}\|^2. \tag{62}$$

Other descriptions of meta-algorithm in improved USIA is same as that in USIA.

E.3. Proof of Theorem E.1

Adaptive regret bound for α-exp-concave functions. Let E_{t,s_p}^k be the ONS expert with $\hat{\alpha} \in \mathcal{P}_{exp}$. To achieve one-gradient evaluation, we need to construct the following surrogate loss for each ONS expert,

$$\ell_{t,k}^{\text{exp}}(\mathbf{w}) = \frac{\hat{\beta}}{2}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle^2 + \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t \rangle, \tag{63}$$

where $\hat{\beta} = \frac{1}{2}\min\{\frac{1}{4GD}, \hat{\alpha}\}$. Our proposed surrogate loss in (63) has the following property.

Lemma E.2. *Under Assumptions 3.1 and 3.2, $\ell_{t,k}^{exp}(\cdot)$ in (63) is $\frac{\hat{\beta}}{2}$-exp-concave, and*

$$\|\nabla \ell_{t,k}^{exp}(\mathbf{w})\|^2 \le (1 + \hat{\beta}GD)^2 \|\nabla f_t(\mathbf{w}_t)\|^2. \tag{64}$$

When we employ ONS algorithm to minimize the above surrogate loss, we set the parameter to be $\hat{\beta}' = \frac{1}{2}\min\{\frac{1}{4G'D}, \frac{\hat{\beta}}{2}\}$ where $G' = 4G^2$ denote the new upper bound of gradients. For expert E_{t,s_p}^k equipped with $\hat{\alpha} \le \alpha < 2\hat{\alpha}$, we can use the theoretical guarantee of ONS to arrive at

$$\begin{aligned}
\sum_{t=s_p}^{s_q-1} \ell_{t,k}^{\text{exp}}(\mathbf{w}_{t,s_p}^k) - \sum_{t=s_p}^{s_q-1} \ell_{t,k}^{\text{exp}}(\mathbf{w}) &\le \frac{d}{2\hat{\beta}'}\log\left(\frac{\hat{\beta}'^2 D^2}{d}\sum_{t=s_p}^{s_q-1}\|\nabla \ell_{t,k}^{\text{exp}}(\mathbf{w}_{t,s_p}^k)\|^2 + 1\right) + \frac{1}{2\hat{\beta}} \\
&\overset{(64)}{\le} \frac{d}{2\hat{\beta}'}\log\left(\frac{\hat{\beta}'^2 D^2 (1+\hat{\beta}GD)^2}{d}\sum_{t=s_p}^{s_q-1}\|\nabla f_t(\mathbf{w}_t)\|^2 + 1\right) + \frac{1}{2\hat{\beta}'} \\
&\le \frac{d}{\tilde{\beta}}\log\left(\frac{\beta^2 D^2 (1+\beta GD)^2}{d}\sum_{t=s_p}^{s_q-1}\|\nabla f_t(\mathbf{w}_t)\|^2 + 1\right) + \frac{1}{\tilde{\beta}},
\end{aligned}$$

where $\beta = \frac{1}{2}\min\{\frac{1}{4GD}, \alpha\}$ and $\tilde{\beta} = \frac{1}{2}\min\{\frac{1}{16G^2D}, \frac{1}{4GD}, \alpha\}$. The last inequality is due to $\hat{\alpha} \le \alpha < 2\hat{\alpha} \Rightarrow \hat{\beta} \le \beta < 2\hat{\beta} \Rightarrow \hat{\beta}' \le \beta' < 2\hat{\beta}'$ and definition of β'. According to the definition of surrogate loss, we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \ell_t^{\text{exp}}(\mathbf{w}_{t,s_p}^k) - \sum_{t=s_p}^{s_q-1} \ell_t^{\text{exp}}(\mathbf{w}) \\
=& \sum_{t=s_p}^{s_q-1}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_{t,s_p}^k - \mathbf{w}\rangle + \frac{\hat{\beta}}{2}\sum_{t=s_p}^{s_q-1}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t,s_p}^k\rangle^2 - \frac{\hat{\beta}}{2}\sum_{t=s_p}^{s_q-1}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}\rangle^2 \\
\le& \frac{d}{\tilde{\beta}}\log\left(\frac{\beta^2 D^2 (1+\beta GD)^2}{d}\sum_{t=s_p}^{s_q-1}\|\nabla f_t(\mathbf{w}_t)\|^2 + 1\right) + \frac{1}{\tilde{\beta}}.
\end{aligned} \tag{65}$$

Next, we formulate the meta-regret of USIA in (38) as,

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t,s_p}^k\rangle \\
\le& 2GD\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m)\right) + \frac{1}{2\hat{\beta}}\cdot\frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{\hat{\beta}}{2}\sum_{t=s_p}^{s_q-1}\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t,s_p}^k\rangle^2,
\end{aligned} \tag{66}$$

where the last step we set $\beta = \hat{\beta}$. Combining (65) with (66), we attain

$$
\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle = \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t,s_p}^k \rangle + \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_{t,s_p}^k - \mathbf{w} \rangle \\
\leq & 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{2\hat{\beta}} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} \\
&+ \frac{d}{\tilde{\beta}} \log \left(\frac{\beta^2 D^2 (1+\beta GD)^2}{d} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 + 1 \right) + \frac{1}{\tilde{\beta}} + \frac{\hat{\beta}}{2} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle^2 \\
\leq & 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} \\
&+ \frac{d}{\tilde{\beta}} \log \left(\frac{\beta^2 D^2 (1+\beta GD)^2}{d} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 + 1 \right) + \frac{1}{\tilde{\beta}} + \frac{\beta}{2} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle^2.
\end{aligned}
$$

Thus, for α-exp-concave functions, we have

$$
\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \leq \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle - \frac{\beta}{2} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w} \rangle^2 \\
\leq & 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} \\
&+ \frac{d}{\tilde{\beta}} \log \left(\frac{\beta^2 D^2 (1+\beta GD)^2}{d} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 + 1 \right) + \frac{1}{\tilde{\beta}}.
\end{aligned}
$$

Next, we utilize Lemma B.4 and B.5 to convert the above bound to obtain

$$
\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\
\leq & 2GD \left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m) \right) + \frac{1}{\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{1}{\tilde{\beta}} \\
&+ \frac{d}{\tilde{\beta}} \log \left(\frac{8H\beta^2 D^2 (1+\beta GD)^2}{d} \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) + \tilde{\epsilon}(T) \right)
\end{aligned} \tag{67}
$$

where

$$
\begin{aligned}
&\tilde{\epsilon}(T) \\
=& \frac{8H\beta^2 D^2 (1+\beta GD)^2}{d} \left(\frac{2GD\Gamma(m)}{\sqrt{\ln(2m+1)}} + 2GD\Gamma(m) + \frac{1}{\beta} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{1}{\beta} \right) \\
&+ 8H\beta D^2 (1+\beta GD)^2 \log(8H\beta D^2 (1+\beta GD)^2) + 2.
\end{aligned}
$$

Note that $\tilde{\epsilon}(T) = O(\log T)$ in the worst-case which can be treated as constant in double logarithmic term. Next, we utilize small-loss upper bound for m in (44) to arrive at the small-loss bound for α-exp-concave functions, which is similar to (45).

Adaptive regret bound for λ-strongly convex functions. Similar to α-exp-concave functions, we also construct the following surrogate loss for each S^2OGD expert,

$$
\ell_{t,k}^{\text{str}}(\mathbf{w}) = \langle \nabla f_t(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t \rangle + \frac{\hat{\lambda}}{2G^2} \|\nabla f_t(\mathbf{w}_t)\|^2 \|\mathbf{w}_t - \mathbf{w}\|^2, \tag{68}
$$

where E^k_{t,s_p} be the S^{2}OGD expert with $\hat{\lambda} \in \mathcal{P}_{str}$ where $\hat{\lambda} \leq \lambda < 2\hat{\lambda}$. Our proposed surrogate loss in (68) has the following property.

Lemma E.3. *Under Assumptions 3.1 and 3.2, the loss function $\ell^{str}_{t,k}(\cdot)$ is $\frac{\hat{\lambda}}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2$-strongly convex, and*

$$\|\nabla \ell^{str}_{t,k}(\mathbf{w})\|^2 \leq \left(1 + \frac{D}{G}\right)^2 \|\nabla f_t(\mathbf{w}_t)\|^2. \tag{69}$$

From the upper bound of $\nabla \ell^{\text{str}}_{t,k}(\cdot)$ according to Lemma E.3, we can see that the new upper bound of the gradients becomes $G' = G + D$. Thus, we need to refine the theoretical guarantee of S^{2}OGD.

Lemma E.4. *Under Assumptions 3.1 and 3.2, for the S^{2}OGD expert E^k_{t,s_p} with $\hat{\lambda} \in \mathcal{P}_{str}$, we have*

$$\sum_{t=s_p}^{s_q-1} \ell^{str}_{t,k}(\mathbf{w}^k_{t,s_p}) - \sum_{t=s_p}^{s_q-1} \ell^{str}_{t,k}(\mathbf{w}) \leq 1 + \frac{(G+D)^2}{2\hat{\lambda}} \log\left(\frac{\hat{\lambda}}{(G+D)^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 + 1\right).$$

According to the definition of the surrogate loss, we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \ell^{\text{str}}_t(\mathbf{w}^k_{t,s_p}) - \sum_{t=s_p}^{s_q-1} \ell^{\text{str}}_t(\mathbf{w}) \\
=&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}^k_{t,s_p} - \mathbf{w}\rangle + \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_t - \mathbf{w}^k_{t,s_p}\|^2 - \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_t - \mathbf{w}\|^2 \\
\leq&1 + \frac{(G+D)^2}{\lambda} \log\left(\frac{\lambda}{(G+D)^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 + 1\right).
\end{aligned} \tag{70}$$

The meta-regret of USIA in (38) is formulated as,

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle \\
\leq&2GD\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m)\right) + \frac{G^2}{2\hat{\lambda}} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{\hat{\lambda}}{2G^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 \|\mathbf{w}_t - \mathbf{w}^k_{t,s_p}\|^2,
\end{aligned} \tag{71}$$

where the last step we set $\beta = \hat{\lambda}$. Combining (70) and (71), we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}\rangle = \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}^k_{t,s_p}\rangle + \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}^k_{t,s_p} - \mathbf{w}\rangle \\
\leq&2GD\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m)\right) + \frac{G^2}{2\hat{\lambda}} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{\hat{\lambda}}{2G^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2 \|\mathbf{w}_t - \mathbf{w}\|^2 \\
&+1 + \frac{(G+D)^2}{\lambda} \log\left(\frac{\lambda}{(G+D)^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2\right) \\
\leq&2GD\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m)\right) + \frac{G^2}{\lambda} \cdot \frac{\Gamma^2(m)}{\ln(2m+1)} + \frac{\lambda}{2} \sum_{t=s_p}^{s_q-1} \|\mathbf{w}_t - \mathbf{w}\|^2 \\
&+1 + \frac{(G+D)^2}{\lambda} \log\left(\frac{\lambda}{(G+D)^2} \sum_{t=s_p}^{s_q-1} \|\nabla f_t(\mathbf{w}_t)\|^2\right).
\end{aligned} \tag{72}$$

Thus, for λ-strongly convex functions, we have

$$\begin{aligned}
&\sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}_t) - \sum_{t=s_p}^{s_q-1} f_t(\mathbf{w}) \\
\le& \sum_{t=s_p}^{s_q-1} \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}\rangle - \frac{\lambda}{2}\sum_{t=s_p}^{s_q-1} \|\mathbf{w}_t - \mathbf{w}\|^2 \\
\le& 2GD\left(\frac{\Gamma(m)}{\sqrt{\ln(2m+1)}} + \Gamma(m)\right) + \frac{G^2}{\lambda}\cdot\frac{\Gamma^2(m)}{\ln(2m+1)} \\
&+1 + \frac{(G+D)^2}{\lambda}\log\left(\frac{\lambda}{(G+D)^2}\sum_{t=s_p}^{s_q-1}\|\nabla f_t(\mathbf{w}_t)\|^2\right).
\end{aligned} \tag{73}$$

Similar to α-exp-concave functions, we can utilize Lemma B.4 and B.5 to convert the above bound to obtain small-loss adaptive regret bound for λ-strongly convex functions.

F. Supporting Lemmas

F.1. Proof of Lemma E.2

According to the definition of $\ell_t^{\exp}(\mathbf{w})$ in (63), we have

$$\begin{aligned}
&\nabla \ell_t^{\exp}(\mathbf{w})\nabla \ell_t^{\exp}(\mathbf{w})^\top \\
=&\nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top + 2\hat{\beta}\nabla f_t(\mathbf{w}_t)(\mathbf{w}-\mathbf{w}_t)^\top \nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top \\
+&\hat{\beta}^2 \nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top(\mathbf{w}-\mathbf{w}_t)(\mathbf{w}-\mathbf{w}_t)^\top\nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top \\
=&(1 + 2\hat{\beta}\langle\nabla f_t(\mathbf{w}_t), \mathbf{w}-\mathbf{w}_t\rangle + \hat{\beta}^2\langle\nabla f_t(\mathbf{w}_t), \mathbf{w}-\mathbf{w}_t\rangle^2)\nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top \\
\preceq& 2\nabla f_t(\mathbf{w}_t)\nabla f_t(\mathbf{w}_t)^\top = \frac{2}{\hat{\beta}}\nabla^2 \ell_t^{\exp}(\mathbf{w})
\end{aligned}$$

where $\nabla^2 \ell_t^{\exp}(\mathbf{w})$ denote the Hessian matrix of $\ell_t^{\exp}(\cdot)$ and the inequality is due to $\hat{\beta} \le 1/(8GD)$. Therefore, $\ell_t^{\exp}(\cdot)$ is $\frac{\hat{\beta}}{2}$-exp-concave (Hazan, 2016, Lemma 4.1). Then we provide the upper bound of $\ell_t^{\exp}(\cdot)$ as follows:

$$\|\nabla \ell_t^{\exp}(\mathbf{w})\|^2 \le \left\|\nabla f_t(\mathbf{w}_t)\left(1 + \hat{\beta}\langle\nabla f_t(\mathbf{w}_t), \mathbf{w}-\mathbf{w}_t\rangle\right)\right\|^2 \le (1+\hat{\beta}GD)^2\|\nabla f_t(\mathbf{w}_t)\|^2.$$

F.2. Proof of Lemma E.3

According to the definition of $\ell_{t,k}^{\mathrm{str}}(\mathbf{w})$ in (68), we have

$$\ell_{t,k}^{\mathrm{str}}(\mathbf{w}') \ge \ell_{t,k}^{\mathrm{str}}(\mathbf{w}) + \langle\nabla \ell_{t,k}^{\mathrm{str}}(\mathbf{w}), \mathbf{w}'-\mathbf{w}\rangle + \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}'-\mathbf{w}\|^2$$

where $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$. According to Definition 3.5, it can be seen that $\ell_{t,k}^{\mathrm{str}}(\cdot)$ is $\frac{\hat{\lambda}}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2$-strongly convex. Next, we provide the upper bound of the gradient of $\ell_{t,k}^{\mathrm{str}}(\cdot)$ as follows:

$$\begin{aligned}
\|\nabla \ell_{t,k}^{\mathrm{str}}(\mathbf{w})\|^2 \le& \left\|\nabla f_t(\mathbf{w}_t) + \frac{\hat{\lambda}}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2(\mathbf{w}-\mathbf{w}_t)\right\|^2 \\
\le& \|\nabla f_t(\mathbf{w}_t)\|^2 + 2\frac{\hat{\lambda}}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\langle\nabla f_t(\mathbf{w}_t), \mathbf{w}-\mathbf{w}_t\rangle + \frac{\hat{\lambda}^2}{G^2}D^2\|\nabla f_t(\mathbf{w}_t)\|^2 \\
\le& (1+\frac{\hat{\lambda}D}{G})^2\|\nabla f_t(\mathbf{w}_t)\|^2.
\end{aligned}$$

F.3. Proof of Lemma E.4

Let $\mathbf{w}'_{t+1,k} = \mathbf{w}_{t,k} - \frac{1}{\alpha_t}\nabla \ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k})$. According to Lemma E.3, we have

$$\begin{aligned}\ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k}) - \ell^{\text{str}}_{t,k}(\mathbf{w}) &\le \langle \nabla \ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k}), \mathbf{w}_{t,k} - \mathbf{w}\rangle - \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 \\ &= \alpha_t \langle \mathbf{w}_{t,k} - \mathbf{w}'_{t+1,k}, \mathbf{w}_{t,k} - \mathbf{w}\rangle - \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_{t,k} - \mathbf{w}\|^2.\end{aligned}$$

For the first term, we have

$$\begin{aligned}&\langle \mathbf{w}_{t,k} - \mathbf{w}'_{t+1,k}, \mathbf{w}_{t,k} - \mathbf{w}\rangle \\ =&\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 + \langle \mathbf{w} - \mathbf{w}'_{t+1,k}, \mathbf{w}_{t,k} - \mathbf{w}\rangle \\ =&\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 - \|\mathbf{w}'_{t+1,k} - \mathbf{w}\|^2 - \langle \mathbf{w}_{t,k} - \mathbf{w}'_{t+1,k}, \mathbf{w}'_{t+1,k} - \mathbf{w}\rangle \\ =&\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 - \|\mathbf{w}'_{t+1,k} - \mathbf{w}\|^2 + \|\mathbf{w}'_{t+1,k} - \mathbf{w}_{t,k}\|^2 + \langle \mathbf{w}'_{t+1,k} - \mathbf{w}_{t,k}, \mathbf{w}_{t,k} - \mathbf{w}\rangle\end{aligned}$$

which implies that

$$\langle \mathbf{w}_{t,k} - \mathbf{w}'_{t+1,k}, \mathbf{w}_{t,k} - \mathbf{w}\rangle = \frac{1}{2}\left(\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 - \|\mathbf{w}'_{t+1,k} - \mathbf{w}\|^2 + \|\mathbf{w}'_{t+1,k} - \mathbf{w}_{t,k}\|^2\right).$$

Thus,

$$\begin{aligned}\ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k}) - \ell^{\text{str}}_{t,k}(\mathbf{w}) \le& \frac{\alpha_t}{2}\left(\|\mathbf{w}_{t,k} - \mathbf{w}\|^2 - \|\mathbf{w}'_{t+1,k} - \mathbf{w}\|^2\right) \\ &+ \frac{1}{2\alpha_t}\|\nabla \ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k})\|^2 - \frac{\hat{\lambda}}{2G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\|\mathbf{w}_{t,k} - \mathbf{w}\|^2.\end{aligned}$$

Summing the above bound up over $t = s_p$ to $s_q - 1$, we attain

$$\begin{aligned}&\sum_{t=s_p}^{s_q-1}\ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k}) - \sum_{t=s_p}^{s_q-1}\ell^{\text{str}}_{t,k}(\mathbf{w}) \\ \le& \frac{\alpha_{s_p}}{2}\|\mathbf{w}_{s_p,k} - \mathbf{w}\|^2 + \sum_{t=s_p}^{s_q-1}\left(\alpha_t - \alpha_{t-1} - \frac{\hat{\lambda}}{G^2}\|\nabla f_t(\mathbf{w}_t)\|^2\right)\frac{\|\mathbf{w}_{t,k} - \mathbf{w}\|^2}{2} \\ &+ \sum_{t=s_p}^{s_q-1}\frac{1}{2\alpha_t}\|\nabla \ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k})\|^2 \\ \le& 1 + \sum_{t=s_p}^{s_q-1}\frac{1}{2\alpha_t}\|\nabla \ell^{\text{str}}_{t,k}(\mathbf{w}_{t,k})\|^2 \le 1 + \frac{(G+D)^2}{2\hat{\lambda}}\sum_{t=s_p}^{s_q-1}\frac{\|\nabla f_t(\mathbf{w}_t)\|^2}{(G+D)^2/\hat{\lambda} + \sum_{i=s_p}^{t}\|\nabla f_i(\mathbf{w}_i)\|^2}.\end{aligned}$$

where the last two inequalities is due to $\alpha_t = (1 + D/G)^2 + \frac{\hat{\lambda}}{G^2}\sum_{i=s_p}^{t}\|\nabla f_i(\mathbf{w}_i)\|^2$ which is specifically set for new surrogate loss. Finally, we utilize the following lemma to finish the proof (Hazan et al., 2007, Lemma 11).

Lemma F.1. *Let $l_1,\cdots,l_T$ and δ be non-negative real numbers. Then*

$$\sum_{t=1}^{T}\frac{l_t^2}{\sum_{i=1}^{t} l_i^2 + \delta} \le \log\left(\frac{1}{\delta}\sum_{t=1}^{T} l_t^2 + 1\right).$$

G. More Discussions

Bounded Modulus For α-exp-concave and λ-strongly convex functions, we assume that the modulus of the functions are both upper bounded and lower bounded, i.e., $\lambda \in [1/T, 1]$. We remark that this assumption is generally acceptable in most cases because it is unnecessary to explicitly consider the cases in which $\lambda < 1/T$ and $\lambda > 1$. (i) When $\lambda < 1/T$, the regret bound will become at least $\Omega(T)$ due to the inverse dependence of the regret bound on λ. Thus, it is more appropriate to treat these functions as general convex functions. (ii) When $\lambda > 1$, we observe that λ-strongly convex functions can also be viewed as 1-strongly convex functions according to Definition 3.5. By treating these functions as 1-strongly convex functions, we can establish the regret bound which remains optimal up to a constant factor, i.e., λ.

Circumvent the Assumption on Bounded Modulus For the modulus of strong convexity λ, there exists a method to remove the assumption that $\lambda \leq 1$. The basic idea is to provide an upper bound for the modulus of strongly convexity under Assumptions 3.1 (bounded gradients) and 3.2 (bounded domain). If the function $f(\cdot)$ is λ-strongly convex, then it satisfies

$$f(\mathbf{w}) - f(\mathbf{w}^*) \geq \frac{\lambda}{2}\|\mathbf{w} - \mathbf{w}^*\|^2,$$

for all $\mathbf{w} \in \mathcal{W}$, where $\mathbf{w}^* \in \mathcal{W}$ is the minimizer of function $f(\cdot)$ (this property can be found in Eq. (2) of Hazan & Kale (2014)). According to Assumption 3.1, we have

$$f(\mathbf{w}) - f(\mathbf{w}^*) \leq G\|\mathbf{w} - \mathbf{w}^*\|.$$

We combine the above two inequality to attain

$$\frac{\lambda}{2}\|\mathbf{w} - \mathbf{w}^*\| \leq G,$$

which holds for all $\mathbf{w} \in \mathcal{W}$. According to Assumption 3.2 that the diameter of decision domain is bounded by D, we can always find a point $\mathbf{w} \in \mathcal{W}$ such that $\|\mathbf{w} - \mathbf{w}^*\| = \frac{D}{2}$ (otherwise, the diameter will be strictly smaller than D). Therefore, we provide an upper bound for modulus of strongly convexity that $\lambda \leq \frac{4G}{D}$.

Adapting to Non-smooth Functions when functions are non-smooth, we can set H be any constant, e.g., 1. Note that the modulus of the smoothness H is only used to set parameter C in our algorithm. As a result, as long as H is a constant, C would be a constant. In this way, the number of the intervals always satisfies $m \leq O(T)$, which does not affect the optimality of the regret bound according to our analysis of Theorem 3.8.