Oblique Dual Frame Completion

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***Abstract*—We study the oblique dual frame completion problem, where the oblique dual frame contains certain prescribed vectors. We establish conditions for the existence and uniqueness of such oblique dual frames and provide explicit constructions using several approaches, including a product matrix approach and a singular value decomposition approach.**

***Index Terms*—Frames, Oblique dual frames, Completion**

I. Introduction

Frames are a generalization of orthonormal bases that provide redundancy in signal representation, making them well suited for signal processing and data transmission [2], [3]. Unlike bases, frames allow for multiple reconstruction formulas, ensuring stability even in the presence of missing or corrupted coefficients. Exact reconstruction of signals is possible via dual frames. In some applications, the dual frame elements lie in a subspace different from the original frame elements, leading to the concept of oblique dual frames [4], [5], [7].

Motivated by the problem of signal recovery when frame coefficient erasures are present (e.g. [6], [8]), the authors of [1] studied the dual frame completion problem: When given a frame F for a finite-dimensional Hilbert space, and a set H of vectors which is assumed to be a subset of a dual frame of F, they presented conditions under which a dual frame G for F contains the vectors from H.

In this paper, we study the *oblique dual frame completion problem*: Given a frame $F = \{f_1, \dots, f_k\}$ of a subspace W of a finite-dimensional Hilbert space and a set $H = \{h_1, \dots, h_s\}$, $s \leq k$ of vectors in another subspace V of the Hilbert space, we explore the possibility of finding an oblique dual frame $G = \{g_1, \dots, g_k\}$ in V of the fixed frame F such that $g_1 = h_1$, $g_2 = h_2, \dots, g_s = h_s$. When such an oblique dual frame G exists, we say that the oblique dual frame completion is possible. We note that the results from [1] can be interpreted as special cases of the results in this paper, with W and V both equal to the given Hilbert space.

In Section II, we review some known results on frames and oblique dual frames. In Section III, we explore methods for addressing the oblique dual frame completion problem, including techniques based on product matrices and singular value decomposition. Finally, Section IV presents the conclusion and a future direction.

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II. Preliminaries and known results

By $\mathbb{F}^n$ we denote a finite-dimensional Hilbert space ($\mathbb{R}^n$ or $\mathbb{C}^n$). If $F \subseteq \mathbb{F}^n$, then $|F|$ denotes the cardinality of F. A set of vectors $F = \{f_i\}_{i=1}^k$ in $\mathbb{F}^n$, $k \geq n$, is called a *frame* for $\mathbb{F}^n$ if it spans the space. Equivalently, there exist constants $0 < \alpha \leq \beta < +\infty$ such that for every $f \in \mathbb{F}^n$, the following inequalities hold:

$$\alpha \|f\|^2 \leq \sum_{i=1}^{k} |\langle f, f_i \rangle|^2 \leq \beta \|f\|^2. \tag{1}$$

Here, α is called a lower frame bound for F, and β is called an upper frame bound for F. For simplicity, we label the matrix $F = [f_1 \dots f_k]$, the set $F = \{f_i\}_{i=1}^k$, and the synthesis operator of the frame with the same label, F. We will say that a matrix is a frame for a subspace if its columns span the said subspace.

Given a frame $F = \{f_i\}_{i=1}^k$ for $\mathbb{F}^n$, a *dual frame* for F is a frame $G = \{g_i\}_{i=1}^k$ such that for every $f \in \mathbb{F}^n$,

$$f = \sum_{i=1}^{k} \langle f, g_i \rangle f_i = \sum_{i=1}^{k} \langle f, f_i \rangle g_i = FG^* f = GF^* f. \tag{2}$$

Note that the dual frames F and G are frames for the same space. In applications [5], it is sometimes beneficial to consider a type of duality where the paired frames are frames for different subspaces. For this, we discuss oblique duality, which makes use of the oblique projection.

Let V and W be subspaces of $\mathbb{F}^n$ satisfying

$$\mathbb{F}^n = V \oplus W^\perp. \tag{3}$$

Note that it follows from (3) that $\dim V = \dim W$. Given a vector $f \in \mathbb{F}^n = V \oplus W^\perp$, we can write f uniquely as the sum of a vector in V and a vector in $W^\perp$: $f = f_V + f_{W^\perp}$. Then the *oblique projection onto* V *along* $W^\perp$ is the unique linear operator $\pi_{V,W^\perp}$ defined as $\pi_{V,W^\perp} f = f_V$. Equivalently, $\pi_{V,W^\perp}$ maps all elements in V to itself, while it maps all elements in $W^\perp$ to 0. If $W = V$, then $\pi_{W,W^\perp} =: \pi_W$, the orthogonal projection onto W.

Lemma 1: Let V and W be subspaces of $\mathbb{F}^n$ satisfying (3). Then the following properties hold:

1) $\pi^*_{V,W^\perp} = \pi_{W,V^\perp}$.
2) $\pi_{V,W^\perp} + \pi_{W^\perp,V} = I_n$.
3) $\pi_{V,W^\perp}\pi_W = \pi_{V,W^\perp}$.
4) $\pi_W \pi_{V,W^\perp} = \pi_W$.

Let V and W be as above. Let F be a frame for W, and let G be a frame for V, with $|F| = |G|$. We say that G is an *oblique dual frame of* F *on* V if

$$GF^* = \pi_{V,W^\perp}. \tag{4}$$

Since (3) is equivalent to $W \oplus V^\perp = \mathbb{F}^n$ [4, Lemma 2.1], we can also say that F is an *oblique dual frame of* G *on* W.

A frame for a subspace W of $\mathbb{F}^n$ can have infinitely many oblique dual frames (on appropriately selected subspaces V):

Example 1: Let $[h_1\ h_2\ f_1\ f_2\ f_3] = \begin{bmatrix} x & a & 0 & 0 & 1 \\ y & b & 1 & 0 & 0 \\ z & c & 0 & 1 & 0 \end{bmatrix}$.

Let $F = [f_1, f_2]$, $G = [h_1, h_2]$. We want: $\mathbb{R}^3 = V \oplus W^\perp$, where $V = \operatorname{span} G$, and $W = \operatorname{span} F$; also, $W^\perp = \operatorname{span}\{f_3\}$. We want F and G to form an oblique dual frame pair for the subspaces W, V, thus we require $GF^T = \pi_{V,W^\perp}$. For the choice of $w = bz - cy$, we have

$$GF^T = \begin{bmatrix} 0 & x & a \\ 0 & y & b \\ 0 & z & c \end{bmatrix},\ \pi_{V,W^\perp} = \frac{1}{w}\begin{bmatrix} 0 & az-cx & bx-ay \\ 0 & w & 0 \\ 0 & 0 & w \end{bmatrix}.$$

It must be that $b = z = 0$, and $y = c = 1$. So, for any $a, x \in \mathbb{R}$, we have that for

$$G = \begin{bmatrix} x & a \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{ it holds }\ GF^T = \pi_{V,W^\perp} = \begin{bmatrix} 0 & x & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further on, $\dim G = 2$, and $f_3 \notin \operatorname{span} G$, therefore $\mathbb{R}^3 = V \oplus W^\perp$ for any $a, x \in \mathbb{R}$. Observe that for a fixed value of x, we have infinitely many dual frame completions for F (with a varying choice of the subspaces V).

In Section III-B we will utilize the Singular Value Decomposition (SVD) of a frame. Recall that given a $k \times n$ matrix A, its SVD is $A = U\Sigma B^*$ where U, B are unitary and Σ is a diagonal $k \times n$ matrix whose diagonal entries are the singular values of A, listed in decreasing order. Furthermore, if $\operatorname{rank} A = r$, then the first r columns of U form an orthonormal basis for the column space of A.

III. Finding an oblique dual frame

In this section, we explore several approaches towards solving the oblique dual frame completion problem. We begin with an example.

Example 2: Consider the 3×2 matrices

$$F_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ G_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and let } g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let $V = \operatorname{span} G_0$, $W = \operatorname{span} F_0$; observe that V and W are two-dimensional subspaces of $\mathbb{R}^3$, and $W^\perp = \operatorname{span} g$. In addition, $V \oplus W^\perp = \mathbb{R}^3$, with oblique projection operator

$$\pi_{V,W^\perp} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{while } \pi_V = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $G_0 F_0^T = \pi_{V,W^\perp}$, thus G_0 is an oblique dual frame of F_0 on V.

Let $F_1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$, and set $F = [F_0\ F_1]$.

Note that $\ker F_1 = \operatorname{span}[4\ -1\ 2]^T$. Now, set

$$G_1 = [x\ y\ z]^T\,[4\ -1\ 2].$$

With the choice $G = [G_0\ \pi_V G_1]$, we have $GF^T = \pi_{V,W^\perp}$ and $F_1(\pi_V G_1)^T = O$, where O is the zero matrix. Thus, G is also an oblique dual frame for F on V.

Motivated by Example 2, we reach the following conclusion:

Theorem 2: Let V and W be subspaces of $\mathbb{F}^n$ satisfying equation (3). Let F_0 and G_0 be frames for W and V, respectively, and let G_0 be an oblique dual frame for F_0 on V, where $|F_0| = |G_0| \geq s = \dim W$. Let $F = [F_0\ F_1]_{n\times k}$ be a frame for W. Given any $G_1 \in \mathbb{F}^{n\times(k-s)}$, we have that $G = [G_0\ \pi_V G_1]$ is an oblique dual frame for F on V if and only if

$$(\ker\ \pi_V G_1)^\perp \subseteq \ker\ F_1. \tag{5}$$

Furthermore,

i. if the columns of F_1 are linearly independent, then the only possible choice for the oblique dual frame completion is $G = [G_0\ O]$.
ii. if the columns of F_1 are linearly dependent, then there exist infinitely many oblique dual frame completions of F on V of type $G = [G_0\ G_1]$.

Proof 1:

We want to show that G is an oblique dual frame for F on V, which is equivalent to $GF^* = \pi_{V,W^\perp}$. So, $G_0F_0^* + \pi_V G_1 F_1^* = \pi_{V,W^\perp}$. By assumption, $G_0 F_0^* = \pi_{V,W^\perp}$, thus $\pi_V G_1 F_1^* = O$, that is $F_1(\pi_V G_1)^* = O$. We have that G is an oblique dual frame for F on V if and only if $\operatorname{im}(\pi_V G_1)^* \subseteq \ker F_1$. The conclusion follows. □

Remark 1: In Example 1, dual frame completion is possible when $b = z = 0$, $y = c = 1$. Fixing $x = 1$, we can say that $h_1 = [1\ 1\ 0]^T$ can always be completed to an oblique dual frame $\{h_1, h_2\}$ of $F = \{f_1, f_2\}$ on $V = \operatorname{span}\{h_1, h_2\}$, with $h_2 = [a\ 0\ 1]^T$, where a is any real number. In this case, the singular values of G indicate that the frame bounds are 1 and $a^2 + 2$, so the lower frame bound is always 1, while the upper bound is minimized for $a = 0$. Note that if instead we are given $h_1 = [x\ 0\ 0]^T$ we see that $\{h_1\}$ can never be completed to an oblique dual frame of F on any subspace V, as (5) would not be satisfied.

Theorem 3: Let V and W be m-dimensional subspaces of $\mathbb{F}^n$, satisfying (3). Let $F = \{f_1, \ldots, f_k\}$ be a frame for W, with $k > m$ and $F = F_0 \cup F_1$, where $F_0 = \{f_1, \ldots, f_s\}$ and $F_1 = \{f_{s+1}, \ldots, f_k\}$ such that $1 \leq s \leq k - m$. Let $H = \{h_1, \ldots, h_s\} \subseteq V$.

There exists an oblique dual frame $G = H \cup G_1$, with $G_1 = \{g_{s+1}, \dots, g_k\}$, of F on V if and only if the columns of $\pi_{W,V^\perp} - F_0 H^*$ are in the span of the columns of F_1.

In addition, if

i. $s = k - m$, then the completion is unique if and only if $f_{k-m+1}, \dots, f_k$ form a basis for $\mathbb{F}^n$.

ii. $s < k - m$, if F_1 is a frame for W then there exist infinitely many dual frame completions of F on V.

A. Oblique dual frame completion via product matrices

For the next result, we use the fact that for a frame F of a subspace W of $\mathbb{F}^n$, we can always find an invertible P such that

$$PF^* = \begin{bmatrix} \pi_W \\ O \end{bmatrix}. \tag{6}$$

Indeed, if F is a frame for W, then F^* and π_W generate the same row space, that is, they are row equivalent and P is the product of matrices corresponding to the row operations.

Example 3: Let V and W be as in Example 2. Let

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & -1/3 & 1 \end{bmatrix}.$$

Note that $\pi_W = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and observe that F_1 spans W, $\det P = 1/3$, and $PF_1^T = \begin{bmatrix} \pi_W \\ O \end{bmatrix}$, where $O = [0\ 0\ 0]$.

Then for any 3×1 matrix A, we have

$$\left([\pi_{V,W^\perp}\ \pi_V A]P\right) F_1^T = [\pi_{V,W^\perp}\ \pi_V A](PF_1^T)$$
$$= [\pi_{V,W^\perp}\ \pi_V A] \begin{bmatrix} \pi_W \\ O \end{bmatrix} = \pi_{V,W^\perp}\pi_W + O = \pi_{V,W^\perp}.$$

Thus $G_1 = [\pi_{V,W^\perp}\ \pi_V A]P$ is an oblique dual frame for F_1.

Theorem 4: Let V, W be m-dimensional subspaces of $\mathbb{F}^n$ satisfying (3). Let $F = \{f_1, \dots, f_k\}$ be a frame for W, with $k > n$. Then there exists an invertible $k \times k$ matrix P such that (6) holds. Write

$$P = \begin{bmatrix} P_{n\times s} & P_{n\times(k-s)} \\ P_{(k-n)\times s} & P_{(k-n)\times(k-s)} \end{bmatrix}, \tag{7}$$

and let $H = [h_1 \dots h_s]$ be any given matrix whose columns are in V, $1 \le s \le k - n$. Then the following hold:

i. There exists an oblique dual frame $G = \{h_1, \dots, h_s, g_{s+1}, \dots, g_k\}$ for F if and only if there exists a matrix A whose columns are in V satisfying

$$AP_{(k-n)\times s} = H - \pi_{V,W^\perp} P_{n\times s}. \tag{8}$$

Here,

$$G = \begin{bmatrix} \pi_{V,W^\perp} & A \end{bmatrix} P. \tag{9}$$

ii. If in addition $s = k - n$ and $P_{(k-n)\times(k-n)}$ is an invertible matrix, then the oblique dual frame completion is unique.

Proof 2: Suppose that there exists an $n \times (k - n)$ matrix A satisfying (8) or equivalently,

$$H = \pi_{V,W^\perp} P_{n\times s} + AP_{(k-n)\times s}. \tag{10}$$

Let $[g_{s+1} \dots g_k] = \pi_{V,W^\perp} P_{n\times(k-s)} + AP_{(k-n)\times(k-s)}$; then we obtain an oblique dual frame $G = \begin{bmatrix} \pi_{V,W^\perp} & A \end{bmatrix} P$ of frame F on subspace V. Note that since the columns of A are in V, it is guaranteed that G is a frame for V.

Conversely, we note that G is an oblique dual frame of F on V if and only if G can be written as $\begin{bmatrix} \pi_{V,W^\perp} & A \end{bmatrix} P$ for some matrix A whose columns are in V. So if there exists an oblique dual $G = \{h_1, \dots, h_s, g_{s+1}, \dots, g_k\}$, then (8) immediately follows.

If $s = k-n$ and $P_{(k-n)\times(k-n)}$ is invertible, then $A = (H - \pi_{V,W^\perp} P_{n\times s}) P^{-1}_{(k-n)\times(k-n)}$ and so the dual frame completion is unique. □

Remark 2: Theorem 4 requires the number of frame elements for W to be larger than the dimension of the whole space. It is natural to ask if we can still apply Theorem 4 even when $F = \{f_1, \dots, f_k\}$ where $k \le n$. Let $1 \le s \le k$ and let $H = [h_1 \dots h_s]$ be a matrix whose columns are in V. Consider $\tilde{F} = F \cup \{0, \dots, 0\}$ such that $|\tilde{F}| = n + s > n$. By Theorem 4, there exists an oblique dual frame $\tilde{G}$ of $\tilde{F}$, where $\tilde{G} = H \cup G_1$, $G_1 = \{g_{s+1}, \dots, g_{n+s}\}$, if and only if there is a matrix A such that $\pi_V A = A$ and $AP_{s\times s} = H - \pi_{V,W^\perp} P_{n\times s}$. Moreover, if $P_{s\times s}$ is invertible, then the oblique dual frame is unique. Now, if there exists an oblique dual $\tilde{G} = H \cup G_1$ of $\tilde{F}$, we must have

$$\pi_{W,V^\perp} = \tilde{F}\tilde{G}^* = [F\ O][H\ G_1]^* = [F\ O][G\ G_2]^* = FG^*,$$

where $G = [H\ \ g_{s+1}\ \ \dots\ \ g_k]$, and $G_2 = [g_{k+1}\ \ \dots\ \ g_{n+s}]$. Thus, G is an oblique dual frame for F.

B. Oblique dual frame completion via SVD

Let V, W be subspaces of $\mathbb{F}^n$ such that (3) holds. Suppose $F = \begin{bmatrix} f_1 & \cdots & f_k \end{bmatrix}$ is a frame for the m-dimensional subspace W and suppose $k \ge m$. Then rank $F = m$. Let $F = U\Sigma B^*$ be the SVD of F, where $U = \begin{bmatrix} U_{n\times m} & U_{n\times(n-m)} \end{bmatrix}$ and B are unitary matrices with

$$U_{n\times m} U^*_{n\times m} = \pi_W, \text{ and } \pi_W U_{n\times m} = U_{n\times m}. \tag{11}$$

In addition, $\Sigma_m = \text{diag}\{\sigma_1, \dots, \sigma_m\}$, and

$$\Sigma = \begin{bmatrix} \Sigma_m & 0_{m\times(k-m)} \\ 0_{(n-m)\times m} & 0_{(n-m)\times(k-m)} \end{bmatrix}. \tag{12}$$

Theorem 5: Let V, W be subspaces of $\mathbb{F}^n = V \oplus W^\perp$. Suppose F is a frame for the m-dimensional subspace W and let G be a frame for V. Let $F = U\Sigma B^*$ be the full SVD of F satisfying (11) and (12). Then G is an oblique dual frame of F on V if and only if there exist a matrix M_G satisfying

$$M_G = \begin{bmatrix} \Sigma_m^{-1} & X_{m\times(k-m)} \\ X_{(n-m)\times m} & X_{(n-m)\times(k-m)} \end{bmatrix}, \tag{13}$$

and $G = \pi_{V,W^\perp} U M_G B^*$ for some matrices $X_{m\times(k-m)}$, $X_{(n-m)\times m}$, $X_{(n-m)\times(k-m)}$.

Proof 3: Let G be a frame for V. Then the columns of GB are in V. Now since U is unitary and its columns form an ONB for $\mathbb{F}^n$, the columns of $\pi_{V,W^\perp}U$ span V. Hence, there exists a matrix M_G such that

$$\pi_{V,W^\perp}UM_G = GB, \tag{14}$$

and $G = \pi_{V,W^\perp}UM_GB^*$.

Now we show that (13) holds, with $G = \pi_{V,W^\perp}UM_GB^*$. We observe

$$\begin{aligned}\pi_{V,W^\perp} = GF^* &= \pi_{V,W^\perp}UM_GB^*(U\Sigma B^*)^* \\ &= \pi_{V,W^\perp}UM_G\Sigma^*U^*\end{aligned}$$

which is equivalent to $\pi_{V,W^\perp}U(I_n - M_G\Sigma^*) = 0$.

Now, the first m columns of $\pi_{V,W^\perp}U$ are linearly independent, which forces the first m rows of $I_n - M_G\Sigma^*$ to be 0. Thus, the upper left block in M_G is Σ_m^{-1}. We have shown that if G is an oblique dual frame of F on V, then there exists a matrix M_G such that (13) holds, with $G = \pi_{V,W^\perp}UM_GB^*$.

Conversely, if M_G is as in (13), and $G = \pi_{V,W^\perp}UM_GB^*$, then it follows that $GF^* = \pi_{V,W^\perp}$ (use the fact that $\ker(\pi_{V,W^\perp}) = W^\perp$ and the columns of $U_{n\times(n-m)}$ span $W^\perp$). $\square$

In the next corollary we will treat the unitary matrix B from the SVD of a frame as a block matrix,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{bmatrix}, \tag{15}$$

where B_{11} is an $s\times m$ matrix, B_{12} is an $s\times(k-m)$ matrix, B_{21} is a $(k-s)\times m$ matrix, and B_{22} is a $(k-s)\times(k-m)$ matrix.

Corollary 6: Let V, W be subspaces of $\mathbb{F}^n = V \oplus W^\perp$, with $\dim W = m < k$, and $W = \operatorname{span} F$, for some $F = \{f_1, \dots, f_k\}$. Let $F = U\Sigma B^*$ be the full SVD of F with (11), (12) and (15) satisfied. Let $H = \{h_1, \dots, h_s\} \subset V$, $1 \le s \le k$. Then the following hold:

i. There exists a set $G_1 \subset V$ such that $G = [H\, G_1]$ is an oblique dual frame for F if and only if there exists an $n \times (k-m)$ matrix Y such that

$$YB_{12}^* = H - \pi_{V,W^\perp}U_{n\times m}\Sigma_m^{-1}B_{11}^*. \tag{16}$$

ii. If $s = k - m$ and B_{12} is invertible, then the oblique dual frame completion is unique.

iii. The vectors in H are the first s vectors in the canonical oblique dual frame if and only if for all $j \in \{1, \dots, s\}$, we have

$$h_j = \pi_{V,W^\perp}U_{n\times m}\begin{bmatrix}\overline{b_{j,1}}/\sigma_1 \\ \vdots \\ \overline{b_{j,m}}/\sigma_m\end{bmatrix}.$$

Remark 3: When i. in Corollary 6 holds true, we have $G = \pi_{V,W^\perp}UM_GB^*$, where M_G is as in (13) and

$$Y = \pi_{V,W^\perp}U_{n\times m}X_{m\times(k-m)}$$

Example 4: Let F_0, G_0, V, W be as in Example 2. Let

$$F_1 = \begin{bmatrix}0 & 0\\ 1 & -1 \\ 1 & 1\end{bmatrix}, U = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, S = \begin{bmatrix}\sqrt{3} & 0 & 0 & 0\\ 0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$H = \begin{bmatrix}1 & -1\\ 1 & -1\\ 1 & 0\end{bmatrix} \text{ and } B = \frac{\sqrt{3}}{3}\begin{bmatrix}0 & 1 & \sqrt{2} & 0\\ 1 & 0 & 0 & -\sqrt{2}\\ 1 & 1 & -\sqrt{2}/2 & \sqrt{2}/2\\ 1 & -1 & \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}.$$

Set $F = [F_0\ F_1]$, a frame for a subspace W of $\mathbb{R}^3$; then the SVD of F is given by $F = USB^*$. Here $n = 3$, $m = 2$, $k = 4$, $s = 2$, thus $s = k - m$, and

$$B_{12} = \frac{\sqrt{6}}{3}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

which is invertible. Hence, by Corollary 6 ii., equation (16) has a unique solution Y, and the oblique dual frame of F on V is G, where

$$Y = \sqrt{6}\begin{bmatrix}1/3 & 1/2\\ 1/3 & 1/2\\ 1/2 & 1/6\end{bmatrix}, \ G = \begin{bmatrix}1 & -1 & 1/2 & 1/2\\ 1 & -1 & 1/2 & 1/2\\ 1 & 0 & 0 & 1\end{bmatrix}.$$

IV. Conclusion

In this paper, we investigated the existence of an oblique dual frame G of a frame F of a finite-dimensional Hilbert space under the assumption that certain vectors are elements of frame G. In Theorems 2 and 3, we established equivalency conditions for the solvability of the oblique dual frame completion problem. In Theorem 4, we constructed an oblique dual frame using a product matrix satisfying (6). Finally, in Corollary 6, we demonstrated that solving the oblique dual frame completion problem is possible via the SVD of the given frame matrix F. A direction for future work is to determine the oblique dual frame G with optimal bounds under the assumption that G contains certain prescribed vectors.

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