A Novel Continuous-Time Algorithm for Nonsmooth Convex Resource Allocation Optimization*

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***Abstract*—This paper develops a novel continuous-time algorithm modeled by a differential inclusion for solving a resource allocation optimization with nonsmooth objective functions and constraints over multi-agent network. It is proved that the state solution of the proposed algorithm enters feasible region of resource allocation problem in finite time and lies in the feasible region after that. Furthermore, it is shown that the state solution is global bounded and finally converges to an optimal solution to the nonsmooth convex resource allocation problem. Compared with the existing algorithms, the continuous-time algorithm proposed in this paper does not rely on the strong convexity or strict convexity of objective functions and has a simple structure with a low amount of state variables. Moreover, the algorithm avoids employing the projection operator. To show the effectiveness and practicability of the presented algorithm, numerical simulations and an application in power system are presented.**

***Index Terms*—Continuous-time algorithm, multi-agent network, resource allocation, distributed nonsmooth optimization.**

I. INTRODUCTION

Distributed optimization with the global objective function through summation by local objective functions has become a hot research topic due to the rapid development of modern network and communication technology. Many distributed optimization algorithms based on multi-agent systems have been proposed and the local objective function and constraint on each agent can not be shared with others because of communication failure, privacy concern and computational burden. Resource allocation problem, as one of an important category of distributed optimization, where different agents possess different local decision variables, is widely used in many applications such as robot networks, transportation systems and power systems (see [1], [2]). For resource allocation problem, after the first study by Ibaraki et.al in centralized form ([3]), various centralized policies have been established ([4]). However, when taking complicated network structures, environmental uncertainties and heavy communication burden into account, this method may fail to solve the resource allocation optimization in large-scale networks. Therefore, the distributed optimization algorithms are desirable (see [5]–[7]). For example, in ([7]), a novel fully distributed algorithm based on a relaxation of the primal problem and duality-based methods was proposed to solve constraint-coupled convex optimization problems. A completely distributed fast gradient-based algorithm by virtue of distributed back-tracking step size rule was proposed in ([8]) for resource allocation optimization with strong concave objective functions. A fully distributed algorithm was presented in ([9]) for resource allocation problem with continuously differentiable objective functions over time-varying directed topology.

Recently, the distributed continuous-time dynamics for resource allocation optimization has received great attention due to the advantages of excellent parallel computing ability and advanced implementation in physical systems. And plenty of significant achievements have been developed ([10]–[13]). For instance, in ([13]), an initialization-free distributed continuous-time algorithm based on projection was designed to solve smooth resource allocation optimization over undirected network. In ([11]), a distributed event-triggered continuous-time algorithm based on undirected graph was proposed to solve resource allocation optimization with smooth cost functions and resource constraint. Note that all aforementioned resource allocation algorithms are implemented to deal with the distributed smooth optimization and may fail to solve the resource allocation optimization with nonsmooth objective functions, which frequently appears in engineering applications.

Therefore, the investigation of nonsmooth distributed optimization has great significance. As far as we know, there are a few continuous-time resource allocation algorithms which take nonsmooth objective functions into consideration ([14]–[16]). For instance, a nonsmooth resource allocation optimization with strongly convex objective functions over weight-balanced digraphs was solved by a continuous-time subgradient-based algorithm in ([14]). A distributed continuous-time algorithm

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was established to handle the nonsmooth resource allocation optimization with strictly convex objective functions over undirected graph in ([15]). However, in these results, the strong convexity or strict convexity of the objective function is required. Thus, it's still a challenging task to investigate the resource allocation optimization with convex and nonsmooth objective functions.

In recent years, for solving constrained resource allocation optimization, various continuous-time algorithms based on the projected method have been proposed (see [13], [14], [17]–[19]). Specifically, in ([13]), two fully distributed continuous-time algorithms as projected dynamics were presented to investigate the resource allocation problem with an initialization-free over undirected graph. A non-differentiable resource allocation problem over weight-balanced digraphs was considered in ([14]) by projection-based nonsmooth algorithm in the form of differential inclusion. A nonsmooth resource allocation optimization with network resource constraint and heterogeneous constraints was investigated by the distributed projection subgradient algorithm over connected undirected graphs in ([17]). Based on modified Lagrangian functions, a novel distributed continuous-time projection-based algorithm was designed to solve the nonsmooth resource allocation optimization with general constraints over undirected graph ([18]). A distributed continuous-time algorithm by virtue of subgradient projections was proposed in ([20]) to solve the general monotropic optimization problem with differentiable objective function and constraints over undirected connected graph. As we all known, projection method is generally applicable to some special constrained regions. When considering the complexity of the constructed algorithms, computational accuracy as well as technical difficulties, it's difficult to compute the projection operator in general situations. In order to improve the computational accuracy as well as avoid technical difficulties, a simple algorithm with lower complexity is proposed in this paper.

Motivated by above discussions, a novel continuous-time algorithm modeled by a differential inclusion is proposed in this paper to investigate the nonsmooth convex resource allocation constraints optimization over undirected networks. The main advantages of this paper list below.

- Furthermore, the proposed algorithm has a simple structure with a low amount of solution space dimension. In fact, compared with the continuous-time algorithm in ([19]) with $(nd+n)$-dimensional solution space and the algorithm in ([16]) with $(nd+nm+2nq)$-dimension, the algorithm proposed herein only has the nd-dimensional solution space, which is lower than those in ([16], [19]). Therefore, when investigating the optimization problems with high dimensional constraint set, the algorithm presented in this paper enjoys lower model complexity and computational cost than algorithms in ([16], [19]).
- In general, when solving optimization problems with set constraints, the algorithms with projection operator are usually used. However, when the constraints have complex structure, it is not an easy task to calculate the projection operator considering the model complexity and computational accuracy. The algorithm proposed in this paper avoids using the projection-based method to solve constrained optimization problems. Therefore, the algorithm proposed in this paper reduces the computational load and improves the practicability in engineering application.

Notation: Let $\mathbb{R}$ be the real number set. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be the vectors of n-dimension and $n\times m$ matrix, respectively. $\mathfrak{B}(\mathbb{R}^d)$ is the collection of all subsets of $\mathbb{R}^d$. $(\cdot)^{\mathrm{T}}$ and $\|\cdot\|$ indicate the transpose and the Euclidean norm, respectively. I_n is the $n\times n$ identity matrix. Furthermore, $\mathbf{1}_d=(1,1,\ldots,1)^{\mathrm{T}}\in\mathbb{R}^d$ and $\mathbf{0}_d=(0,0,\ldots,0)^{\mathrm{T}}\in\mathbb{R}^d$. Besides, $\otimes$ is Kronecker product. Let $P\subseteq\mathbb{R}^d$ be a subset of $\mathbb{R}^d$, then $\mathrm{int}(P)$ represents the interior of P in $\mathbb{R}^d$. The closure ball which the center is p and the radius is $\epsilon>0$ is given by $\bar{B}(p,\epsilon)=\{x|\|x-p\|\leq\epsilon\}$.

II. Preliminaries and Problem Formulation

Next, some relevant definitions and propositions about nonsmooth analysis ([21]), graph theory ([22]) are introduced, and then the problem is formulated.

A. Nonsmooth Analysis

A map $\mathcal{F}:K\subseteq\mathbb{R}^n\to\mathfrak{B}(\mathbb{R}^d)$ is called as a set-valued map if for each point $x\in K$, there correspondingly exists a nonempty set $\mathcal{F}(x)\in\mathfrak{B}(\mathbb{R}^d)$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is Lipschitz continuous near x_0, if there are $l,\delta>0$ satisfying $|f(\mu)-f(\nu)|\leq l\|\mu-\nu\|$, for any $\mu,\nu\in B(x_0,\delta)$. If f is Lipschitz near each $x\in\mathbb{R}^n$, then f is locally Lipschitz continuous on $\mathbb{R}^n$.

Proposition 2.1: ([21]) If $f_i:\mathbb{R}^n\to\mathbb{R}$ is convex $(i=1,2,\cdots,n)$, then

1) the subdifferential of f_i at x is defined as $\partial f_i(x)=\{\xi\in\mathbb{R}^n: f_i(x)-f_i(y)\leq\langle\xi,x-y\rangle,\forall y\in\mathbb{R}^n\}$.
2) $\partial(\sum_{i=1}^n f_i)(x)=\sum_{i=1}^n\partial f_i(x)$.

Lemma 2.1: ([21]) If $f:\mathbb{R}^n\to\mathbb{R}$ is convex, then the following properties hold:

1) $\partial f(\cdot)$ is upper semicontinuous.
2) $\partial f(\cdot)$ is maximal monotone, i.e., $\langle v-v_0,x-x_0\rangle\geq 0$ for any $v\in\partial f(x)$ and $v_0\in\partial f(x_0)$.
3) $\partial f(x)$ is a convex and compact nonempty set of $\mathbb{R}^n$.

Definition 2.1: ([21]) For a nonempty convex set $E\subseteq\mathbb{R}^n$, the normal cone to E at $\bar{x}\in E$ is defined as

$$N_E(\bar{x})=\{\xi\in\mathbb{R}^n|\langle\xi,x-\bar{x}\rangle\leq 0\ for\ all\ x\in E\}. \tag{1}$$

Lemma 2.2: ([21]) Let f be convex on the nonempty convex set $E\subseteq\mathbb{R}^n$. f attains a minimun over E at $\bar{x}\in E$ if and only if

$$0\in\partial f(\bar{x})+N_E(\bar{x}).$$

Lemma 2.3: ([23]) If closed convex sets $G_1,G_2\subseteq\mathbb{R}^n$ satisfy $0\in\mathrm{int}(G_1-G_2)$, then for any $x\in G_1\cap G_2$, $N_{G_1\cap G_2}(x)=N_{G_1}(x)+N_{G_2}(x)$.

Lemma 2.4: ([21]) Let $G = G_1 \times G_2$, where $G_1, G_2 \in \mathbb{R}^n$, and let $x = (x_1, x_2) \in C_1 \times C_2$, where C_1, C_2 are subsets of G_1, G_2, respectively. Then

$$N_{C_1 \times C_2}(x) = N_{C_1} \times N_{C_2}.$$

B. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \Upsilon, \mathcal{A})$ denote a weighted undirected graph, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ is a vertex set, $\Upsilon \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set and $\mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ represents an adjacency matrix. $(i,j) \in \Upsilon$ is an edge of $\mathcal{G}$ if vertex j sends some information to vertex i. Let $N_i = \{j|(i,j) \in \Upsilon\}$ be the set of neighbors of node i. The adjacency matrix $\mathcal{A}$ is defined as $a_{ij} > 0$ if $(i,j) \in \Upsilon$ and $a_{ij} = 0$ otherwise. An edge which consists of a sequence of vertices is called a path. If for all i, j, $(i,j) \in \Upsilon$ implies $(j,i) \in \Upsilon$, then graph $\mathcal{G}$ is called undirected. Moreover, for any pair of vertices in $\mathcal{G}$, if there is a path, then, the graph $\mathcal{G}$ is connected. The Laplacian matrix is $L = D - \mathcal{A}$, where $D = \mathrm{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$ with $d_i = \sum_{j=1}^n a_{ij}$. For connected undirected graph $\mathcal{G}$, one has $L\mathbf{1}_n = \mathbf{0}_n$ with $\mathbf{0}_n$ is the simple eigenvalue.

For graph $\mathcal{G}$, the out-degree of v_i is defined as $d^w_{\mathrm{out}}(v_i) = \sum_{j=1}^n a_{ij}$, and the in-degree of v_i is $d^w_{\mathrm{in}}(v_i) = \sum_{j=1}^n a_{ji}$. $D_{out} = \mathrm{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$ denotes the out-degree matrix of $\mathcal{G}$ with $d_i = d^w_{out}(v_i)$. and

Lemma 2.5: ([19]) For the Laplacian matrix $L \in \mathbb{R}^{n\times n}$ of connected undirected graph $\mathcal{G}$, there exists an orthogonal matrix $R = [q, Q] \in \mathbb{R}^{n\times n}$ with $Q \in \mathbb{R}^{n\times(n-1)}$ and $q = \frac{1}{\sqrt{n}}\mathbf{1}_N$ such that

1) $L = \begin{bmatrix} q & Q \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_1 \end{bmatrix} \begin{bmatrix} q^{\mathrm{T}} \\ Q^{\mathrm{T}} \end{bmatrix}$, where $\Lambda_1 = \mathrm{diag}\{\lambda_2, \cdots, \lambda_n\}$ with $\lambda_2 \le \cdots \le \lambda_n$ being the positive eigenvalues of Laplacian matrix L.
2) For any $\xi, \omega \in \mathbb{R}^{nd}$, if $(\mathbf{1}_n \otimes I_d)^{\mathrm{T}}\xi = 0$, then, it has $\xi^{\mathrm{T}} S(L \otimes I_d)\omega = \xi^{\mathrm{T}}\omega$, where $S = E \otimes I_d$ and $E = \begin{bmatrix} q & Q \end{bmatrix} \begin{bmatrix} \varrho & 0 \\ 0 & (\Lambda_1)^{-1} \end{bmatrix} \begin{bmatrix} q^{\mathrm{T}} \\ Q^{\mathrm{T}} \end{bmatrix}$ with any positive constant $\varrho > 0$ and $(\Lambda_1)^{-1} = \mathrm{diag}\{\lambda_2^{-1}, \cdots, \lambda_n^{-1}\}$.

C. Problem Formulation

Consider a multi-agent network with n agents, interacting over a connected undirected graph G. For each agent i, there are a local state variable $x_i(t)$ and one particular objective function f^i for $i \in \{1, 2, \cdots, n\}$ and each agent shares the information with its neighbors through local communications. In this paper, the following resource allocation optimization is considered:

$$\begin{aligned} \text{minimize} \quad & f(x) = \sum_{i=1}^n f^i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^n x_i = b, \\ & g_i(x_i) \le 0, i = 1, 2, \cdots, n \end{aligned} \tag{2}$$

where $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nd}, f : \mathbb{R}^{nd} \to \mathbb{R}$ is the global objective function, $g_i(x_i) = [g_{i1}(x_i), g_{i2}(x_i), \cdots, g_{im}(x_i)]^{\mathrm{T}} : \mathbb{R}^d \to \mathbb{R}^m$ is local inequality constraint about x_i for all $i \in \{1, 2, \cdots, n\}$ and $b \in \mathbb{R}^d$ is a known constant vector. Without loss of generality, we assume that there exists an optimal solution to the problem (2).

Define $g(x) = [g_1^{\mathrm{T}}(x_1), g_2^{\mathrm{T}}(x_2), \cdots, g_n^{\mathrm{T}}(x_n)]^{\mathrm{T}}$. Then, the inequality constraint can be written as $g(x) \le 0$. Denote

$$S_1 = \{x \in \mathbb{R}^{nd} : (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$$

as equality constraint set and

$$S_2 = \{x \in \mathbb{R}^{nd} : g(x) \le 0\}$$

as the inequality constraint set. Let

$$\Omega = S_1 \cap S_2 = \{x \in \mathbb{R}^{nd} : (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b, g(x) \le 0\}$$

be the feasible region of problem (2).

Next, the following assumptions are set up throughout this paper.

Assumption 2.1:

(i) The weighted graph $\mathcal{G}$ is undirected and connected.
(ii) For $i = 1, 2, \cdots, n$, function f^i and the components of g_i are convex but not necessarily smooth.
(iii) (Slater's condition) For resource allocation problem (2), there exists $\hat{x} = (\hat{x}_1^{\mathrm{T}}, \hat{x}_2^{\mathrm{T}}, \cdots, \hat{x}_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nd}$ satisfying $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)\hat{x} = b$ and $g(\hat{x}) < 0$.
(iv) For $i = 1, 2, \cdots, n$, the inequality constraint set $\{x_i \in \mathbb{R}^d : g_i(x_i) \le 0\}$ is bounded, i.e., there exists $r_i > 0$ such that
$$\{x_i \in \mathbb{R}^d : g_i(x_i) \le 0\} \subset B(\hat{x}_i, r_i).$$

III. A NOVEL CONTINUOUS-TIME ALGORITHM

In this section, a novel distributed continuous-time algorithm for the resource allocation problem (2) is proposed. Moreover, the existence and boundedness of the solution to the algorithm are investigated. Next, some notations and a lemma are provided.

Define

$$\begin{aligned} J_i(x_i) &\triangleq \sum_{k=1}^m \max\{0, g_{ik}(x_i)\}, \\ J(x) &\triangleq \sum_{i=1}^n J_i(x_i). \end{aligned}$$

Obviously, $J(x)$ is convex on $\mathbb{R}^{nd}$.

For solving the resource allocation problem (2), a novel distributed continuous-time algorithm in the form of differential inclusion for each agent $i \in \{1, 2, \cdots, n\}$ is designed as:

$$\begin{cases} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \in -\sum_{j\in N_i} a_{ij}\Big(\partial f^i(x_i(t)) - \partial f^j(x_j(t)) \\ \qquad + (t+1)^2\big(\partial J_i(x_i(t)) - \partial J_j(x_j(t))\big)\Big), \\ x_i(0) = x_{i_0}. \end{cases} \tag{3}$$

The compact form of continuous-time algorithm (3) can be written as:

$$\begin{cases} \dot{x}(t) \in -(L \otimes I_d)\left(\partial f(x(t)) + (t+1)^2 \partial J(x(t))\right), \\ x(0) = x_0. \end{cases} \tag{4}$$

Next, the global existence and uniqueness of the solution to the continuous-time algorithm (4) are given.

Theorem 3.1: Under the Assumption 2.1, for the initial point $x(0) \in S_1 = \{x \in \mathbb{R}^{nd} | (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$, the continuous-time algorithm (4) has the global unique state solution $x(t) \in \mathbb{R}^{nd}$ defined on $[0, +\infty)$. Moreover, $x(t) \in S_1, \ \forall t \geq 0$.

Theorem 3.2: Letting Assumption 2.1 hold and $x(t) \in \mathbb{R}^{nd}$ be the state solution to the continuous-time algorithm (4) starting from initial point $x(0) \in S_1 = \{x \in \mathbb{R}^{nd} | (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$, then the state solution $x(t)$ is bounded for $t \in [0, +\infty)$.

Proof: Assume that $x^* \in \mathbb{R}^{nd}$ is an optimal solution to the resource allocation problem (2). From (4), there are measurable functions $v(t) \in \partial f(x(t))$ and $\eta(t) \in \partial J(x(t))$ satisfying

$$\dot{x}(t) = -(L \otimes I_d)\left(v(t) + (t+1)^2 \eta(t)\right), \tag{5}$$

for a.e. $t \in [0, +\infty)$. From $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)(L \otimes I_d) = 0$ and (5), we have $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)\dot{x}(t) = 0$. According to $x(0) \in \{x \in \mathbb{R}^{nd} | (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$, one has $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x(t) = (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x(0) = b$.

Define

$$W(x) = \frac{1}{2}(x - x^*)^{\mathrm{T}} S (x - x^*).$$

Then, by the definition of S, we have

$$W(x) \geq \frac{\kappa}{2}\|x - x^*\|, \tag{6}$$

where $\kappa = \min\{\varrho, \lambda_2^{-1}, \cdots, \lambda_n^{-1}\}$ and $\varrho, \lambda_i (i = 2, 3, \cdots, n)$ are defined in Lemma 2.5. Differentiating $W(x)$ along with the state of continuous-time algorithm (5) and combined with the Lemma 2.5, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W(x(t)) =& (x(t) - x^*)^{\mathrm{T}} S \dot{x}(t) \\
=& -(x(t) - x^*)^{\mathrm{T}} S (L \otimes I_d)\left(v(t) + (t+1)^2 \eta(t)\right) \\
=& -(x(t) - x^*)^{\mathrm{T}}\left(v(t) + (t+1)^2 \eta(t)\right),
\end{aligned} \tag{7}$$

for a.e. $t \in [0, +\infty)$. Based on the convexity of f and J, it follows that

$$\begin{aligned}
&-(x(t) - x^*)^{\mathrm{T}} v(t) \leq f(x^*) - f(x(t)), \\
&-(x(t) - x^*)^{\mathrm{T}} \eta(t) \leq J(x^*) - J(x(t)).
\end{aligned} \tag{8}$$

Combining (7) with (8), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W(x(t)) \leq& \left(f(x^*) + (t+1)^2 J(x^*)\right) \\
&- \left(f(x(t)) + (t+1)^2 J(x(t))\right),
\end{aligned} \tag{9}$$

for a.e. $t \in [0, +\infty)$. According to (iv) of Assumption 2.1, we have

$$\{x_i \in \mathbb{R}^d | g_i(x_i) \leq 0\} \subseteq B(\hat{x}_i, r_i),$$

where $\hat{x} = (\hat{x}_1^{\mathrm{T}}, \hat{x}_2^{\mathrm{T}}, \cdots, \hat{x}_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nd}$ is defined in (iii) of Assumption 2.1. Denote

$$r = \left(\sum_{i=1}^{n} r_i^2\right)^{\frac{1}{2}} \quad \text{and} \quad \hat{g}_i = \max_{k=1,2,\cdots,m}\{g_{ik}(\hat{x}_i)\}.$$

Let L_i^* be the Lipschitz constant of f^i near x_i^*. Taking

$$\vartheta = \sum_{i=1}^{n} L_i^* \quad \text{and} \quad T_1 = \max_{i \in \{1,2,\cdots,n\}}\left\{\frac{\vartheta r}{-\hat{g}_i}\right\},$$

then, we claim that

$$f(x(t)) + (t+1)^2 J(x(t)) \geq f(x^*) + (t+1)^2 J(x^*), \forall t \geq T_1.$$

In fact, considering that $x^* \in \mathbb{R}^{nd}$ is an optimal solution to the problem (2), we have

$$0 \in \partial f(x^*) + N_{\Omega}(x^*) = \partial f(x^*) + N_{S_1}(x^*) + N_{S_2}(x^*). \tag{10}$$

Combined with ([24]), we know that there exist $\sigma_{ik} \in [0, +\infty)$, $\gamma \in [0, +\infty)$ and $\xi \in \mathbb{R}^{nd}$ satisfying $\|\xi\| \leq 1$ such that

$$0 \in \prod_{i=1}^{n} \partial f^i(x_i^*) + \prod_{i=1}^{n} \sum_{k \in I_i^0(x_i^*)} \sigma_{ik} \partial g_{ik}(x_i^*) + \gamma(\mathbf{1}_n \otimes I_d)\xi. \tag{11}$$

In the following, we show that $\sigma_{ik} \leq T_1, k \in I_i^0(x_i^*), i = 1, 2, \cdots, n$. If it does not hold, then there exist $i_0 \in \{1, 2, \cdots, n\}$ and $k_0 \in I_{i_0}^0(x_{i_0}^*)$ such that $\sigma_{i_0 k_0} > T_1$. From (10), there exists $\mu^* \in N_{\Omega}(x^*)$ such that $-\mu^* \in \prod_{i=1}^{n} \partial f^i(x_i^*)$. Therefore, we have $\|\mu^*\| \leq \vartheta = \sum_{i=1}^{n} L_i^*$ and

$$|\langle \mu^*, x^* - \hat{x} \rangle| \leq \|\mu^*\| \|x^* - \hat{x}\| \leq \vartheta r. \tag{12}$$

On the other hand, according to (iii) of Assumption 2.1 and considering the definition of x^*, we have $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)\hat{x} = (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x^* = b$. Thus

$$\langle (\mathbf{1}_n \otimes I_d), x^* - \hat{x} \rangle = 0. \tag{13}$$

Combining the convexity of g_{ik} with (13), one has

$$\begin{aligned}
\langle \mu^*, x^* - \hat{x} \rangle =& \sum_{i=1}^{n} \sum_{k \in I_i^0(x_i^*)} \sigma_{ik} \langle \partial g_{ik}(x_i^*), x_i^* - \hat{x}_i \rangle \\
&+ \langle \gamma(\mathbf{1}_n \otimes I_d)\xi, x^* - \hat{x} \rangle \\
\geq& \sum_{i=1}^{n} \sum_{k \in I_i^0(x_i^*)} \sigma_{ik}\left(g_{ik}(x_i^*) - g_{ik}(\hat{x}_i)\right).
\end{aligned}$$

Since $k \in I_i^0$, it is clear that $g_{ik}(x_i^*) = 0$. Then, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{k \in I_i^0(x_i^*)} \sigma_{ik}\left(g_{ik}(x_i^*) - g_{ik}(\hat{x}_i)\right) \\
\geq& \sigma_{i_0 k_0}\left(-g_{i_0 k_0}(\hat{x}_{i_0})\right) \geq \sigma_{i_0 k_0}(-\hat{g}_{i_0}) > \frac{\vartheta r}{-\hat{g}_{i_0}}(-\hat{g}_{i_0}).
\end{aligned}$$

Thus, it can be deduced that $\|\mu^*\| \|x^* - \hat{x}\| > \vartheta r$, which is contradiction with (12). Hence, we have $\sigma_{ik} \leq T_1, k \in I_i^0(x_i^*), i = 1, 2, \cdots, n$, which implies that there exist $\kappa_{ik} \in [0, 1]$ such that $\sigma_{ik} = T_1 \kappa_{ik}, k \in I_i^0(x_i^*), i = 1, 2, \cdots, n$. Moreover, it can be obtained that

$$\begin{aligned}
\sum_{k \in I_i^0(x_i^*)} \sigma_{ik} \partial g_{ik}(x_i^*) =& T_1 \sum_{k \in I_i^0(x_i^*)} \kappa_{ik} \partial g_{ik}(x_i^*) \\
\subseteq& T_1 \sum_{k \in I_i^0(x_i^*)} [0, 1] \partial g_{ik}(x_i^*).
\end{aligned}$$

On the other hand, it is easy to get

$$\partial J_i(x_i^*) = \sum_{k\in I_i^0(x_i^*)} [0,1]\partial g_{ik}(x_i^*).$$

Therefore, we have

$$\sum_{k\in I_i^0(x_i^*)} \sigma_{ik}\partial g_{ik}(x_i^*) \subseteq T_1\partial J_i(x_i^*).$$

Moreover, we can further obtain

$$\sum_{k\in I_i^0(x_i^*)} \sigma_{ik}\partial g_{ik}(x_i^*) \subseteq (t+1)^2\partial J_i(x_i^*), \forall t \geq T_1.$$

Combining with (11), it follows that

$$0 \in \partial f(x^*) + (t+1)^2\partial J(x^*) + \gamma(\mathbf{1}_n \otimes I_d)\xi, \forall t \geq T_1. \tag{14}$$

According to $\gamma\xi^{\mathrm{T}}(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)(x(t) - x^*) = 0$ and (14), we have

$$0 \in \left(\partial f(x^*) + (t+1)^2\partial J(x^*)\right)^{\mathrm{T}}(x(t) - x^*), \forall t \geq T_1.$$

Considering the convexity of $f + (t+1)^2 J$, when $t \geq T_1$, one has

$$f(x(t)) + (t+1)^2 J(x(t)) - f(x^*) - (t+1)^2 J(x^*) \geq 0. \tag{15}$$

From (9) and (15), we can get

$$\frac{\mathrm{d}}{\mathrm{dt}}W(x(t)) \leq 0,$$

for a.e. $t \in [T_1, +\infty)$. Therefore, one has

$$\frac{\kappa}{2}\|x(t) - x^*\| \leq W(x(t)) \leq W(x(T_1)), \forall t \geq T_1,$$

which means that $x(t)$ is bounded for $\forall t \geq T_1$. $x(t)$ is bounded on $[0, T_1]$ because the global state solution $x(t)$ to the algorithm (4) exists for a.e. $t \in [0, +\infty)$. Therefore, $x(t)$ is bounded on $[0, T_1] \cup [T_1, +\infty) = [0, +\infty)$. ∎

IV. Convergence analysis

The convergence of continuous-time algorithm (4) is analyzed in this section. That is, we prove that the state solution $x(t) \in \mathbb{R}^{nd}$ to the continuous-time algorithm (4) enters the inequality constraint set S_2 in finite time and converges to an optimal solution to the resource allocation problem (2).

Lemma 4.1: ([25]) Under the Assumption 2.1, there is a constant $\alpha > 0$ such that for any $x \notin S_2$, one has

$$\left\langle \eta, x - \hat{x}\right\rangle > \alpha, \forall \eta \in \partial J(x),$$

where $\hat{x} \in \mathbb{R}^{nd}$ is defined in (iii) of Assumption 2.1.

Theorem 4.1: Under the Assumption 2.1, let $x(t) \in \mathbb{R}^{nd}$ be the state solution to the continuous-time algorithm (4) starting from any initial value $x(0) \in S_1 = \{x \in \mathbb{R}^{nd} | (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$. Then there exists $T_2 \geq 0$ such that the state solution $x(t) \in S_2$, for any $t \geq T_2$.

Proof: From Theorems 3.1 and 3.2, we know that the continuous-time algorithm (4) has a unique bounded state solution $x(t) \in \mathbb{R}^{nd}$ defined on $[0, +\infty)$. That is, there exist measurable functions $v(t) \in \partial f(x(t))$ and $\eta(t) \in \partial J(x(t))$ satisfying

$$\dot{x}(t) = -(L \otimes I_d)\left(v(t) + (t+1)^2\eta(t)\right), \tag{16}$$

for a.e. $t \geq 0$. Differentiating $J(x(t))$ along with the state of continuous-time algorithm (16), one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(x(t)) =& \left\langle \eta(t), \dot{x}(t)\right\rangle \\ =& -\left\langle \eta(t), (L \otimes I_d)\left(v(t) + (t+1)^2\eta(t)\right)\right\rangle,\end{aligned} \tag{17}$$

for a.e. $t \in [0, +\infty)$. According to Lemma 4.1, when $x(t) \notin S_2$, one has

$$\left\langle \eta(t), x(t) - \hat{x}\right\rangle > \alpha,$$

where $\hat{x} \in \mathbb{R}^{nd}$ is defined in (iii) of Assumption 2.1. Based on Lemma 2.5 and $(\mathbf{1}_n \otimes I_d)^{\mathrm{T}}(x(t) - \hat{x}) = 0$, when $x(t) \notin S_2$, we have

$$(x(t) - \hat{x})^{\mathrm{T}} S (L \otimes I_d)\eta(t) = (x(t) - \hat{x})^{\mathrm{T}}\eta(t) > \alpha.$$

Thus, we can deduce that

$$\|(L^{\frac{1}{2}} \otimes I_d)\eta(t)\| \geq \frac{\alpha}{r\|S\|\|L^{\frac{1}{2}} \otimes I_d\|} \triangleq \beta, \tag{18}$$

for any $x(t) \notin S_2$. In the following, we demonstrate that there is $T_2 > 0$ such that $x(T_2) \in S_2$. If it does not hold, we have

$$x(t) \notin S_2, \forall t > 0.$$

By Theorem 3.2, $x(t)$ is bounded. Therefore, combining the upper semi-continuity of $\partial f(\cdot)$ with the definition of $v(t)$ in (16), we know $v(t)$ is also bounded, which means that there exists $M_2 > 0$ such that $\|(L^{\frac{1}{2}} \otimes I_d)v(t)\| \leq M_2$. Consequently,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}J(x(t)) \\ \leq& \|(L^{\frac{1}{2}} \otimes I_d)\eta(t)\|\left(M_2 - (t+1)^2\|(L^{\frac{1}{2}} \otimes I_d)\eta(t)\|\right) \\ \leq& \|(L^{\frac{1}{2}} \otimes I_d)\eta(t)\|\left(M_2 - (t+1)^2\beta\right),\end{aligned} \tag{19}$$

for a.e. $t \in [0, +\infty)$. It's obviously that there exists $T_e > 0$ such that $\left(M_2 - (t+1)^2\beta\right) < 0$ for $t \geq T_e$. Then, for any $t \geq T_e$, from (18), it can get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(x(t)) \leq& \|(L^{\frac{1}{2}} \otimes I_d)\eta(t)\|\left(M_2 - (t+1)^2\beta\right) \\ \leq& \beta\left(M_2 - (t+1)^2\beta\right),\end{aligned} \tag{20}$$

for a.e. $t \geq T_e$. Taking $T_0 = \beta^2 + \sqrt{\beta^4 - \beta^2 + \beta M_2} + 1$ and according to (20), for a.e. $t \geq \max\{T_0, T_e\}$, there is a constant $\rho_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}J(x(t)) \leq -\rho_1. \tag{21}$$

Taking the integral of (21) from T_0 to $t (\geq T_0)$, we can get

$$J(x(t)) \leq J(x(T_0)) - \rho_1(t - T_0) \to -\infty, \quad \text{as } t \to \infty,$$

which leads to the contradiction with the fact that $J(x(t)) \geq 0$. Therefore, there is $T_2 > 0$ such that $x(T_2) \in S_2$.

Next, we claim that the state solution $x(t)$ will always be in S_2 once it enters S_2 after T_2. If not, there exist t_1 and t_2 satisfying $t_2 > t_1 \geq T_2$ and

$$x(t_1) \in S_2, x(t) \notin S_2, \forall t \in (t_1, t_2],$$

which means that $J(x(t_1)) = 0$. And from the above analysis, we know that the inequality (21) still holds for $\forall t \in (t_1, t_2]$. By integrating of (21) from t_1 to t_2, one has

$$J(x(t_2)) \leq J(x(t_1)) - \rho_1(t_2 - t_1) = -\rho_1(t_2 - t_1) < 0,$$

which contradicts with $J(x(t_2)) \geq 0$. Therefore, the state solution $x(t)$ will always be in S_2 once it enters S_2 after T_2. ■

In the following, the optimal solution of resource allocation problem (2) can be obtained by the convergence of continuous-time algorithm (4).

Theorem 4.2: Suppose Assumption 2.1 holds. Then, the state solution $x(t) \in \mathbb{R}^{nd}$ to continuous-time algorithm (4) starting from any initial value $x(0) \in S_1 = \{x \in \mathbb{R}^{nd} | (\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x = b\}$ globally converges to an optimal solution to the resource allocation problem (2).

Proof: Assume that $x^* \in \mathbb{R}^{nd}$ is an optimal solution to the resource allocation problem (2) and (16) holds. Therefore, there exist measurable functions $v(t) \in \partial f(x(t))$ and $\eta(t) \in \partial J(x(t))$ satisfying

$$\dot{x}(t) = -(L \otimes I_d) \left(v(t) + (t+1)^2 \eta(t) \right), \tag{22}$$

for a.e. $t \geq 0$. From Theorem 3.1, we have $(\mathbf{1}_n^{\mathrm{T}} \otimes I_d)x(t) = b$, $\forall t \geq 0$.

Define a energy function as

$$\begin{aligned} V(t) =& f(x(t)) - f(x^*) + (t+1)^2 (J(x(t)) - J(x^*)) \\ &+ \frac{1}{2}(x(t) - x^*)^{\mathrm{T}} S (x(t) - x^*). \end{aligned}$$

Based on Theorem 4.1, we know that $x(t) \in S_2, \forall t \geq T_2$, consequently, $J(x(t)) = J(x^*) = 0$, $t \geq T_2$. On the other hand, considering x^* is an optimal solution to the resource allocation problem (2), therefore, it has

$$V(t) \geq 0,$$

for any $t \geq T_2$. Differentiating $V(t)$ along with the state of continuous-time algorithm (22) and combining Lemma 2.5 with the convexity of f and J, we can deduce that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} V(t) \\ =& 2(t+1) \left(J(x(t)) - J(x^*) \right) + \left(v(t) + (t+1)^2 \eta(t) \right) \dot{x}(t) \\ &+ (x(t) - x^*)^{\mathrm{T}} S \dot{x}(t) \\ =& - \left(v(t) + (t+1)^2 \eta(t) \right)^{\mathrm{T}} (L \otimes I_d) \left(v(t) + (t+1)^2 \eta(t) \right) \\ &- (x(t) - x^*)^{\mathrm{T}} S (L \otimes I_d) \left(v(t) + (t+1)^2 \eta(t) \right) \qquad (23) \\ \leq& - \left(v(t) + (t+1)^2 \eta(t) \right)^{\mathrm{T}} (L \otimes I_d) \left(v(t) + (t+1)^2 \eta(t) \right) \\ &+ f(x^*) - f(x(t)) + (t+1)^2 \left(J(x^*) - J(x(t)) \right) \\ \leq& - \left(v(t) + (t+1)^2 \eta(t) \right)^{\mathrm{T}} (L \otimes I_d) \left(v(t) + (t+1)^2 \eta(t) \right), \end{aligned}$$

for a.e. $t \geq T_2$. Then, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq - \inf \Big\{ &- \left(v(t) + (t+1)^2 \eta(t) \right)^{\mathrm{T}} (L \otimes I_d) \\ &\cdot \left(v(t) + (t+1)^2 \eta(t) \right) : \eta(t) \in \partial J(x(t)), \\ &v(t) \in \partial f(x(t)) \Big\}, \end{aligned} \tag{24}$$

for a.e. $t \geq T_2$. Define a function H as follows:

$$\begin{aligned} &H(t, x) \\ =& \inf \Big\{ \left\| \left(v + (t+1)^2 \eta \right)^{\mathrm{T}} (L \otimes I_d) \left(v + (t+1)^2 \eta \right) \right\|^2 : \\ &\eta \in \partial J(x), v \in \partial f(x) \Big\}. \end{aligned}$$

Next, we certify that there is an increasing sequence $\{t_k\}$, such that

$$\lim_{k \to +\infty} H\left(t_k, x(t_k)\right) = 0. \tag{25}$$

If it does not hold, there is $\varepsilon > 0$ such that

$$\liminf_{t \to +\infty} H\left(t, x(t)\right) = \varepsilon > 0.$$

That is, there is $T > 0$ such that $H(t, x(t)) \geq \frac{\varepsilon}{2}, \forall t \in [T, +\infty)$. By (24), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -\frac{\varepsilon}{2}, \text{ for a.e. } t \in [T, +\infty). \tag{26}$$

Taking integration of inequality (26) from T to $t (\geq T)$, one has

$$V(t) \leq V(T) - \frac{\varepsilon}{2}(t - T).$$

Taking the limit on both sides of above inequality, we have $\lim_{t \to +\infty} V(t) = -\infty$. Obviously, it contradicts with $V(t) \geq 0, \forall t \geq T_2$. Thus, the equality (25) holds. That is, there exist measure functions $\eta(t_k) \in \partial J(x(t_k))$ and $v(t_k) \in \partial f(x(t_k))$ satisfying

$$\begin{aligned} \lim_{k \to +\infty} &\left(v(t_k) + (t_k+1)^2 \eta(t_k) \right)^{\mathrm{T}} (L \otimes I_d) \\ &\cdot \left(v(t_k) + (t_k+1)^2 \eta(t_k) \right) = 0. \end{aligned} \tag{27}$$

According to (27), we have

$$\lim_{k \to +\infty} (L \otimes I_d) \left(v(t_k) + (t_k+1)^2 \eta(t_k) \right) = 0.$$

It is easy to derive that

$$\lim_{k \to +\infty} S (L \otimes I_d) \left(v(t_k) + (t_k+1)^2 \eta(t_k) \right) = 0. \tag{28}$$

On the other hand, based on the Theorem 3.2, we know $x(t)$ is bounded. Therefore, there is a convergent subsequence of $\{x(t_k)\}$ which is still denoted as $\{x(t_k)\}$ such that

$$\lim_{k \to +\infty} x(t_k) = \bar{x}. \tag{29}$$

From Theorem 4.1, we have $\bar{x} \in \Omega$.

Next, we prove that $\bar{x}$ is an optimal solution of resource allocation problem (2). In fact, combining with the compactness of ∂J and ∂f, we have

$$\lim_{k \to +\infty} v(t_k) = \bar{v} \in \partial f(\bar{x}), \lim_{k \to +\infty} \eta(t_k) = \bar{\eta} \in \partial J(\bar{x}). \tag{30}$$

Meanwhile, from (28), for any $y \in \Omega$, one has

$$\begin{aligned} 0 &= \lim_{k\to+\infty} (y - x(t_k))^{\mathrm{T}} S(L \otimes I_d) \left(v(t_k) + (t_k+1)^2\eta(t_k)\right) \\ &= \lim_{k\to+\infty} (y - x(t_k))^{\mathrm{T}} \left(v(t_k) + (t_k+1)^2\eta(t_k)\right). \end{aligned} \tag{31}$$

Based on the convexity of J, it follows

$$\begin{aligned} &(t_k+1)^2 (y - x(t_k))^{\mathrm{T}} \eta(t_k) \\ \le &(t_k+1)^2 (J(y) - J(x(t_k))) = 0. \end{aligned} \tag{32}$$

Therefore, from (29)-(32), one has

$$\begin{aligned} 0 &= \lim_{k\to+\infty} (y - x(t_k))^{\mathrm{T}} S(L \otimes I_d) \left(v(t_k) + (t_k+1)^2\eta(t_k)\right) \\ &= \lim_{k\to+\infty} (y - x(t_k))^{\mathrm{T}} \left(v(t_k) + (t_k+1)^2\eta(t_k)\right) \\ &\le \lim_{k\to+\infty} (y - x(t_k))^{\mathrm{T}} v(t_k) \\ &= (y - \bar{x})^{\mathrm{T}} \bar{v}. \end{aligned}$$

Then, from the convexity of f on Ω, we have

$$f(y) - f(\bar{x}) \ge 0, \forall y \in \Omega, \tag{33}$$

which shows that $\bar{x}$ is an optimal solution of resource allocation problem (2).

Finally, we show that $\lim_{t\to+\infty} x(t) = \bar{x}$. Define another energy function as

$$\tilde{V}(t) = \frac{1}{2}(x(t) - \bar{x})^{\mathrm{T}} S (x(t) - \bar{x}).$$

By similar analyses of (7) and (9), one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}(t) \le &\left(f(\bar{x}) + (t+1)^2 J(\bar{x})\right) \\ &- \left(f(x(t)) + (t+1)^2 J(x(t))\right), \end{aligned}$$

for a.e. $t \ge 0$. Considering that $\bar{x}$ is an optimal solution, we have $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}(t) \le 0$ for a.e. $t \ge T_2$. Therefore, $\lim_{t\to+\infty} \tilde{V}(t)$ exists. On account of $\lim_{k\to\infty} \tilde{V}(t_k) = 0$, $\lim_{t\to+\infty} \tilde{V}(t) = 0$. Since $\frac{\kappa}{2}\|x(t) - \bar{x}\|^2 \le \tilde{V}(t)$, consequently,

$$\lim_{t\to+\infty} x(t) = \bar{x}.$$

Thus, for any initial value $x_0 \in S_1$, the state solution $x(t)$ of the continuous-time algorithm (4) converges to an optimal solution of the resource allocation problem (2). ■

V. Simulations and Application

In this section, a numerical simulation and an application in power grid are given to verify the effectiveness and practicality of the proposed algorithm.

A. Numerical Simulation

Example 1: ([19]) A multi-agent network with four agents is given, whose communication topology is an undirected graph illustrated in Fig.2. And each agent exchanges information only with its neighbors. The objective functions of four agents are defined as

$$\begin{aligned} f_1(x_1) &= \| x_1 - (3,3) \| + x_{11}^2 + x_{12}^2, \\ f_2(x_2) &= \frac{x_{21}^2}{5\sqrt{x_{21}^2+1}} + \frac{x_{22}^2}{5\sqrt{x_{22}^2+1}}, \\ f_3(x_3) &= | x_{31} - 3 | + | x_{32} - 4 |, \\ f_4(x_4) &= \ln\left(\exp(-0.05x_{41}) + \exp(0.05x_{41})\right) + \| x_4 \|^2 \\ &\quad + \ln\left(\exp(-0.05x_{42}) + \exp(0.05x_{42})\right), \end{aligned} \tag{34}$$

where $x_i = (x_{i1}, x_{i2})^{\mathrm{T}} \in \mathbb{R}^2, i = 1,2,3,4$. And the local constraint functions for different agents are given as: $g_1(x_1) = (x_{11}-2)^2 + (x_{12}-2)^2 - 5$, $g_2(x_2) = (g_{21}(x_2), g_{22}(x_2))^{\mathrm{T}}$ with $g_{21}(x_2) = (x_{21}-3)(x_{21}-1)$ and $g_{22}(x_2) = x_{22}(x_{22}-1)$, $g_3(x_3) = (g_{31}(x_3), g_{32}(x_3))^{\mathrm{T}}$ with $g_{31}(x_3) = -x_{31}+1$ and $g_{32}(x_3) = -x_{32}+1$ and $g_4(x_4) = (x_{41}-2)^2 + (x_{42}-2)^2 - 4$. The constant vector is $b = (5, 20)^{\mathrm{T}}$.

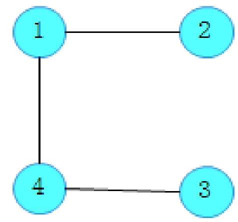

Fig. 1. Communication topology with 4 agents.

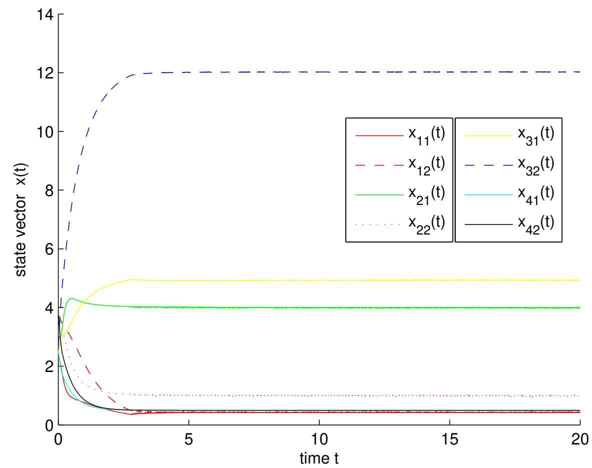

Fig. 2. Trajectory of the state solution $x(t)$ to the continuous-time algorithm (4).

From the simulation results in Fig. 3, we can see that the state solutions converge to the optimal solution: $x_1^* = (0.49, 0.49)^{\mathrm{T}}, x_2^* = (4,1)^{\mathrm{T}}, x_3^* = (4.9, 12)^{\mathrm{T}}, x_4^* = (0.41, 0.41)^{\mathrm{T}}$. And all of them meet the conditions of theorems. Therefore, the proposed algorithm can solve the problem (2) and the simulation results verify the effectiveness.

TABLE I
GENERATOR PARAMETERS

Generator	α_i	β_i	γ_i	ϵ_i	$P_{Gi}^{\min}$	$P_{Gi}^{\max}$	P_{di}
1	0.5	3	2	30	20	40	45
2	1.5	4	1	28	25	35	40
3	3	5	0.5	45	35	50	25
4	1	2	1.5	35	25	50	35
5	2.5	3.5	1	40	30	47	30
6	2	4.5	1.5	35	28	42	30

VI. CONCLUSIONS

In this paper, a novel continuous-time algorithm over multi-agent network model by a differential inclusion is presented for solving a nonsmooth constrained resource allocation optimization. By using nonsmooth analysis and stability theory, the existence and boundedness of solution to the continuous-time algorithm are obtained. For the resource allocation optimization with convex and nonsmooth objective functions, the algorithm can guarantee the state solutions of all agents converge to the optimal solutions. Compared with the existing continuous-time algorithms, the proposed algorithm has a simple structure with lower dimensional state variables. Moreover, the algorithm presented in this paper avoids the use of projection operators. Finally, the effectiveness and practicability have been illustrated by numerical simulations and an application in power system.

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