

# 000 001 002 003 004 005 SOLVING IMPERFECT-RECALL GAMES VIA SUM-OF- 006 SQUARES OPTIMIZATION 007 008 009

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## ABSTRACT

Extensive-form games (EFGs) provide a powerful framework for modeling sequential decision making, capturing strategic interaction under imperfect information, chance events, and temporal structure. Most positive algorithmic and theoretical results for EFGs assume perfect recall, where players remember all past information and actions. We study the increasingly relevant setting of imperfect-recall EFGs (IREFGs), where players may forget parts of their history or previously acquired information, and where equilibrium computation is provably hard. We propose sum-of-squares (SOS) hierarchies for computing ex-ante optimal strategies in single-player IREFGs and Nash equilibria in multi-player IREFGs, working over behavioral strategies. Our theoretical results show that (i) these hierarchies converge asymptotically, (ii) under genericity assumptions, the convergence is finite, and (iii) in single-player non-absentminded IREFGs, convergence occurs at a finite level determined by the number of information sets. Finally, we introduce the new classes of (SOS)-concave and (SOS)-monotone IREFGs, and show that in the single-player setting the SOS hierarchy converges at the first level, enabling equilibrium computation with a single [semidefinite program \(SDP\)](#).

## 1 INTRODUCTION

Extensive-form games (EFGs) are a central framework for modelling sequential decision-making under imperfect information, with important applications in economics, operations research, and artificial intelligence (Osborne & Rubinstein, 1994; Fudenberg & Tirole, 1991). An EFG represents interactions on a tree in which players (or chance) take turns choosing actions. Each sequence of actions defines a history that either terminates at a leaf with associated payoffs or leads to another decision point. This representation is also able to capture imperfect information, where players cannot distinguish between different nodes. EFGs have been used extensively in modern AI, particularly in solving large-scale imperfect information games (Brown & Sandholm, 2018; Bowling et al., 2015; Moravčík et al., 2017; Bakhtin et al., 2022).

A key distinction between classes of EFGs is the ability of players to retain memory. In perfect-recall EFGs, players remember all past information sets and actions, whereas in imperfect-recall EFGs (IREFGs) some information may be lost, such as forgetting which action was taken or even whether a state has been visited before. Recent work has increasingly focused on IREFGs, since they capture a more realistic model of decision-making. IREFGs are used to abstractify large games, letting agents ignore strategically unimportant details (Waugh et al., 2009; Ganzfried & Sandholm, 2014; Brown et al., 2015; Čermák et al., 2017a). They also provide a natural representation for bounded rationality (Lambert et al., 2019), for describing teams of cooperating agents as a single imperfect-recall decision maker (Von Stengel & Koller, 1997; Celli & Gatti, 2018; Zhang et al., 2022), and for designing privacy-preserving or data-restricted agents (Conitzer, 2019; Conitzer & Oesterheld, 2023).

EFGs admit two principal strategy formalisms: mixed strategies, defined as distributions over pure strategies that select an action everywhere in the game tree, and behavioral strategies, which randomize between actions independently at each information set. The canonical solution concept is a Nash equilibrium (NE), a strategy profile from which no player can unilaterally deviate. Perfect recall plays a crucial role in the theoretical guarantees of EFGs: by Kuhn’s theorem (Kuhn, 1953), mixed and behavioral strategies are equivalent in perfect recall EFGs, implying the existence of behavioral NEs

054 and enabling efficient algorithms, such as polynomial-time methods for solving two-player zero-sum  
 055 EFGs (Koller & Megiddo, 1992).

056 The imperfect-recall setting is far more challenging. The equivalence between mixed and behavioral  
 057 strategies breaks down, and mixed NE still exist but may be unimplementable since they require  
 058 players to have perfect memory of the game states. Though behavioral NE are a more natural  
 059 solution concept, they might not exist, even in two-player zero-sum IREFGs (Wichardt, 2008).  
 060 Computationally, solving two-player zero-sum IREFGs is NP-hard (Koller & Megiddo, 1992),  
 061 deciding whether a single-player IREFG achieves a target value is hard (Gimbert et al., 2020; Tewolde  
 062 et al., 2023), even relaxed equilibrium notions such as EDT and CDT equilibria are hard to compute  
 063 (Tewolde et al., 2023), and deciding whether a behavioral NE exists is hard (Tewolde et al., 2024).

064 Many recent hardness results for IREFGs (Gimbert et al., 2020; Tewolde et al., 2023; 2024) rely on  
 065 the folklore observation that any IREFG over behavioral strategies can be expressed as a polynomial  
 066 game over a product of simplices (Piccione & Rubinstein, 1997). Importantly, the transformation  
 067 from an EFG to its polynomial game representation can be carried out in polynomial time. Polynomial  
 068 games form a key subclass of continuous games, originally studied by Dresher et al. (1950), in which  
 069 each player’s utility is a polynomial in the decision variables of all players.

070 On the other hand, there exists well-developed machinery for handling single- and multi-player  
 071 optimization problems where utilities are polynomial and strategy sets are semi-algebraic. In the  
 072 single-player case, the Moment-SOS hierarchy (Lasserre, 2001; Parrilo, 2000) applies to polynomial  
 073 optimization problems of the form  $\max f(x)$  subject to  $x \in \mathcal{X}$ , where  $f$  is a polynomial and  $\mathcal{X}$  is  
 074 defined by polynomial equalities and inequalities. Although such problems are NP-hard in general  
 075 (Motzkin & Straus, 1965), the hierarchy produces a sequence of semidefinite relaxations whose  
 076 optimal values converge to the true optimum under mild assumptions. At relaxation level  $d$ , it seeks  
 077 the smallest scalar  $t$  such that  $t - f(x)$  admits a sum-of-squares (SOS) certificate of nonnegativity  
 078 on  $\mathcal{X}$ , a condition expressible as a semidefinite program (SDP) of polynomial size. Extensions of  
 079 the Moment-SOS hierarchy have also been developed for the multi-player polynomial optimization  
 080 setting, applied to polynomially representable supersets of the equilibrium set (Nie & Tang, 2024).

081 Thus far, the link between IREFGs and polynomial optimization has mostly served to prove hardness  
 082 results, but we instead use it as a foundation for positive algorithmic results. Specifically, we ask:

084 *What guarantees does the Moment-SOS hierarchy provide for solving IREFGs?  
 085 Are there structured subclasses of IREFGs where it enables tractable computation?*

086 **Contributions.** By bringing these tools to imperfect-recall games, we obtain a general, provably  
 087 convergent framework for computing behavioral equilibria in IREFGs. Moreover, we identify  
 088 structural conditions under which convergence occurs at finite or low levels of the hierarchy, leading  
 089 to efficient algorithms for broad and natural subclasses. Specifically, our main contributions are:

- 091 • For *single-player IREFGs*, the Moment-SOS hierarchy converges asymptotically to the ex-ante  
 092 optimal value. Under a *genericity* assumption, we show that convergence is instead finite for  
 093 almost all games. Moreover, in non-absentminded IREFGs (NAM-IREFGs), we show that exact  
 094 convergence occurs at a level of the hierarchy depending on the number of infosets in the game.
- 095 • For *multi-player IREFGs*, we adapt an approach proposed in recent work by Nie & Tang (2024).  
 096 The method requires multiple instantiations of the Moment-SOS hierarchy and converges asymp-  
 097 to a behavioral NE, certifying non-existence otherwise. As with the single-player case,  
 098 for almost all games the convergence is finite. In NAM-IREFGs, we show that a variant of the  
 099 method generically converges in finite time, requiring only a single instantiation of the hierarchy.
- 100 • We define the tractable subclasses of concave and monotone IREFGs, and SOS-certifiable  
 101 counterparts thereof. In single-player **SOS-concave**/SOS-monotone IREFGs, we show that the  
 102 Moment-SOS hierarchy converges at the *first* level, enabling computation of ex-ante optima with  
 103 a single SDP.

104 **Related work in IREFGs.** Existing positive results for IREFGs typically rely on restrictions  
 105 that admit tractable perfect-recall refinements. A-loss recall games (Kaneko & Kline, 1995; Kline,  
 106 2002) limit forgetting to past actions, enabling sufficient conditions for behavioral NE to exist and  
 107 approximation methods in the two-player zero-sum case (Čermák, 2018). NAM-IREFGs, wherein

108 players always remember previously encountered decision points, can be transformed into equivalent  
 109 (though exponentially larger) A-loss recall games (Gimbert et al., 2025). Finally, chance-relaxed  
 110 skew well-formed games are a subclass of IREFGs where counterfactual regret minimization still  
 111 provably minimizes regret (Lanctot et al., 2012; Kroer & Sandholm, 2016).

## 113 2 PRELIMINARIES

### 115 2.1 IMPERFECT RECALL EXTENSIVE-FORM GAMES (IREFGs)

117 We first define extensive-form games of imperfect recall. For a more thorough review of standard  
 118 concepts in EFGs, the reader is referred to Fudenberg & Tirole (1991); Osborne & Rubinstein (1994).

119 An  $n$ -player extensive form game  $\mathcal{G}$  is a tuple  $\mathcal{G} := \langle \mathcal{H}, \mathcal{A}, \mathcal{Z}, \rho, \mathcal{I} \rangle$  where:

- 121 • The set  $\mathcal{H}$  denotes the states of the game which are decision points for the players. The states  
 $\pi \in \mathcal{H}$  form a tree rooted at an initial state  $r \in \mathcal{H}$ . We denote terminal nodes in  $\mathcal{H}$  by  $\mathcal{Z}$ . Each  
 122 nonterminal state  $\pi \in \mathcal{H} \setminus \mathcal{Z}$  is associated with a set of *available actions*  $\mathcal{A}_\pi$ .
- 124 • Given  $\mathcal{N} = \{1, \dots, n\}$ , the set  $\mathcal{N} \cup \{c\}$  denotes the  $n+1$  players of the game. Each state  $\pi \in \mathcal{H}$   
 125 admits a label  $\text{Label}(\pi) \in \mathcal{N} \cup \{c\}$  which denotes the *acting player* at state  $\pi$ . The letter  $c$   
 126 denotes a *chance player*, representing exogenous stochasticity.  $\mathcal{H}_i \subseteq \mathcal{H}$  denotes the states  $\pi \in \mathcal{H}$   
 127 with  $\text{Label}(\pi) = i$ . Each chance node  $\pi \in \mathcal{H}_c$  is associated with a fixed distribution  $\mathbb{P}_c(\cdot | \pi)$  over  
 128  $\mathcal{A}_\pi$ , denoting the distribution over actions chosen by the chance player at each node.
- 129 • For each  $i \in \mathcal{N}$ , payoff function  $\rho_i : \mathcal{Z} \rightarrow \mathbb{R}$  specifies the payoff that player  $i$  receives if the  
 130 game ends at terminal state  $z \in \mathcal{Z}$ .
- 131 • The game states  $\mathcal{H}$  are partitioned into *information sets* (also called infosets) ascribed to each  
 132 player, namely  $\mathcal{I}_i \in (\mathcal{I}_1, \dots, \mathcal{I}_n)$ . Each information set  $I \in \mathcal{I}_i$  encodes groups of nodes that the  
 133 acting player  $i$  cannot distinguish between, so if  $\pi_1, \pi_2 \in I$ , then  $\mathcal{A}_{\pi_1} = \mathcal{A}_{\pi_2}$ . We let  $\mathcal{A}_I$  denote  
 134 the shared action set of infoset  $I$ .
- 135 • For notational convenience, we ascribe a singleton information set to each chance node and define  
 136  $\mathcal{I}_c$  as the collection of these chance node infosets. For each non-terminal node  $\pi \in \mathcal{H} \setminus \mathcal{Z}$ , we  
 137 thus define  $I_{\pi \in (\mathcal{I}_1, \dots, \mathcal{I}_n) \cup \mathcal{I}_c}$  to be the infoset it belongs to.

138 **Memory.** There are two key distinctions regarding players’ memory. A game has *perfect recall* if  
 139 no player ever forgets their past history, namely, the sequence of information sets visited, actions  
 140 taken within those information sets, and any information acquired along the way. Formally, for any  
 141 information set  $I \in \mathcal{I}_i$  and any two nodes  $\pi_1, \pi_2 \in I$ , the sequence of Player  $i$ ’s actions from the  
 142 root  $r$  to  $\pi_1$  must coincide with the sequence from  $r$  to  $\pi_2$ ; otherwise, the player could distinguish  
 143 between the two nodes. A game is said to have perfect recall if this property holds for all players;  
 144 otherwise, it is said to have the *imperfect recall* property.

145 **Strategy formalism.** A *pure strategy* specifies a deterministic action at every information set  
 146 of a player. A *mixed strategy* is a probability distribution over pure strategies. However, mixed  
 147 strategies require players to coordinate their actions across information sets, which implicitly assumes  
 148 memory and therefore conflicts with the imperfect-recall setting. For this reason, IREFGs are  
 149 most naturally analyzed using *behavioral strategies*, which specify independent randomizations at  
 150 each information set. Formally, for any information set  $I \in \mathcal{I}_i$ , let  $\Delta(\mathcal{A}_I)$  denote the simplex of  
 151 probability distributions over the available actions  $\mathcal{A}_I$ . A behavioral strategy for player  $i$  is a mapping  
 152  $\mu_i : \mathcal{I}_i \rightarrow \bigcup_{I \in \mathcal{I}_i} \Delta(\mathcal{A}_I)$ , assigning to each information set  $I$  a distribution  $\mu_i(\cdot | I) \in \Delta(\mathcal{A}_I)$ . The  
 153 joint behavioral strategy profile for all players is denoted by  $\mu := (\mu_i)_{i \in \mathcal{N}}$ .

154 Kuhn’s theorem (Kuhn, 1953) establishes that, in games with perfect recall, behavioral and mixed  
 155 strategies are outcome-equivalent—that is, they induce the same probability distribution over terminal  
 156 histories. This equivalence no longer holds in the imperfect-recall setting, making behavioral  
 157 strategies the canonical choice for IREFGs.

158 Going forward, we establish some additional notational conventions regarding behavioral strategies.  
 159 We use  $(\mu_i, \mu_{-i})$  to describe the influence of player  $i$  on  $\mu$ , where  $\mu_{-i}$  collects all components except  
 160 player  $i$ . Let  $\ell_i$  denote the number of infosets of player  $i$ , i.e.  $\ell_i := |\mathcal{I}_i|$ , and  $m_i^j$  denote the number  
 161 of actions in a given infoset  $I_i^j \in \mathcal{I}_i$  of player  $i$ , i.e.  $m_i^j := |\mathcal{A}_{I_i^j}|$ . The strategy set of player  $i$  over all

162 their infosets can be written as a Cartesian product of simplices:  $\mathcal{S}_i := \bigtimes_{j=1}^{\ell_i} \Delta^{m_i^{j-1}}$ . Finally, the  
 163 strategy set over all players is  $\mathcal{S} := \bigtimes_{i=1}^n \mathcal{S}_i$ . In the single-player case, we set  $n = 1$  and drop the  
 164 player index  $i$  for clarity.  
 165

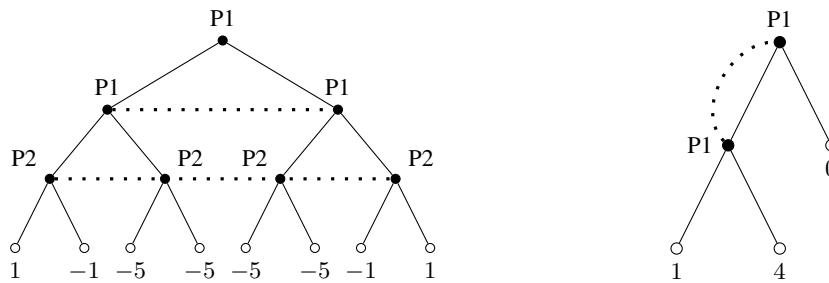
166 **Solution concepts.** The expected utility of player  $i \in \mathcal{N}$  following (joint) behavioral strategy  $\mu$  is  
 167  $u_i(\mu) := \sum_{z \in \mathcal{Z}} \mathbb{P}(z|\mu, r) \cdot \rho_i(z)$ , where  $\mathbb{P}(z|\mu, r)$  is the probability that leaf  $z \in \mathcal{Z}$  is reached from  
 168 root  $r$  following joint behavioral strategy  $\mu$ . In a single-player IREFG  $\mathcal{G}$ , a behavioral strategy  $\mu^*$  is  
 169 said to be *ex-ante optimal* if it is a solution to the following optimization problem:  
 170

$$\max_{\mu} u(\mu) \quad \text{s.t. } \mu \in \mathcal{S}. \quad (1)$$

171 In these games,  $\mathcal{S}$  is compact and  $u$  is continuous, so  $\arg \max_{\mathcal{S}} u \neq \emptyset$  (i.e. ex-ante optima always  
 172 exist). In a multi-player IREFG  $\mathcal{G}$ , a joint behavioral strategy  $\mu^* \in \mathcal{S}$  is a (behavioral) Nash  
 173 equilibrium if for every player  $i \in \mathcal{N}$ , we have that:  
 174

$$u_i(\mu^*) \geq u_i(\mu_i, \mu_{-i}^*), \quad \forall \mu_i \in \mathcal{S}_i, \quad (2)$$

175 i.e., no player can profitably deviate from  $\mu^*$  to any other behavioral strategy. Importantly, behav-  
 176 ioral Nash equilibria need not exist in IREFGs; a counterexample is given in Wicardt (2008) (cf.  
 177 Figure 2.1 (a)). Moreover, even deciding whether one exists is hard (Tewolde et al., 2024).  
 178



189 Figure 2.1: (a) A two-player zero-sum IREFG with no NE; (b) the single-player absentminded taxi  
 190 driver IREFG. Dotted lines denote infosets. In (b), P1 cannot distinguish between nodes in the same  
 191 history, so behavioral strategies (e.g., *Left* w.p.  $x$ , *Right* w.p.  $1 - x$ ) yield expected utility  
 192  $u(x) = x^2 + 4x(1 - x)$ , a polynomial in the behavioral strategy space.  
 193

## 2.2 THE MOMENT/SUM-OF-SQUARES HIERARCHY

194 We will utilize ideas from the polynomial optimization literature to solve IREFGs. We direct the  
 195 reader to Lasserre (2009b); Laurent (2009) for more thorough discussions of these techniques. [Here](#)  
 196 we give only a brief overview of the Moment-SOS hierarchy tailored to our purposes; a step-by-step  
 197 derivation in our setting is provided in Appendix D.  
 198

199 We now introduce the Moment-SOS hierarchy. Consider the polynomial optimization problem:  
 200

$$f^* := \max_x \{f(x) : x \in \mathcal{X}\}, \quad (3)$$

201 where the feasible region is a basic semi-algebraic set  $\mathcal{X} := \{x \in \mathbb{R}^m : g_j(x) \geq 0, j \in \llbracket m_g \rrbracket, h_k(x) = 0, k \in \llbracket m_h \rrbracket\}$  for two families of polynomials  $g_1, \dots, g_{m_g}, h_1, \dots, h_{m_h} \in \mathbb{R}[x]$ .  
 202 Let  $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$  be a multi-index, and write  $x^\alpha := \prod_{i=1}^m x_i^{\alpha_i}$ ,  $|\alpha| := \sum_{i=1}^m \alpha_i$ . For  
 203 two multi-indices  $\alpha, \beta$ , the sum  $\alpha + \beta$  is taken componentwise, i.e.,  $(\alpha + \beta)_i = \alpha_i + \beta_i$ . Denote  
 204 by  $\mathbb{R}[x]$  the ring of real polynomials in the variables  $x := (x_1, \dots, x_m)$ , and by  $\mathbb{R}[x]_d$  the vector  
 205 space of polynomials of degree at most  $d$  (whose dimension is  $s(m) := \binom{m+d}{m}$ ). A sequence  
 206  $y = (y_\alpha)_{\alpha \in \mathbb{N}^m} \subset \mathbb{R}$  defines a Riesz linear functional  $L_y : \mathbb{R}[x] \rightarrow \mathbb{R}$ ,  $L_y(x^\alpha) = y_\alpha$ . Given a  
 207 sequence  $y$  and an integer  $d \geq 0$ , the *degree- $d$  moment matrix*  $M_d(y)$  is the symmetric matrix with  
 208 rows and columns indexed by all multi-indices  $\alpha, \beta$  satisfying  $|\alpha|, |\beta| \leq d$ , and whose entry is  
 209  $M_d(y)_{\alpha, \beta} = y_{\alpha+\beta}$ . For any polynomial  $g(x) = \sum_\gamma g_\gamma x^\gamma$ , the *degree- $d$  localizing matrix* of  $y$  with  
 210 respect to  $g$  is  $M_d(g \star y)_{\alpha, \beta} := L_y(g(x)x^{\alpha+\beta}) = \sum_\gamma g_\gamma y_{\alpha+\beta+\gamma}$ ,  $|\alpha|, |\beta| \leq d - \lceil \deg(g)/2 \rceil$ .  
 211

212 Denote by  $\Sigma[x] := \{\sigma \in \mathbb{R}[x] : \sigma(x) = \sum_{k=1}^K q_k(x)^2, q_k \in \mathbb{R}[x]\}$  the set of SOS polynomials and  
 213 by  $\Sigma[x]_d$  the set of SOS polynomials of degree at most  $2d$ . The quadratic module of  $\mathcal{X}$  is  $Q(\mathcal{X}) :=$

216  $\{\sigma_0 + \sum_{j=1}^{m_g} \sigma_j g_j + \sum_{k=1}^{m_h} p_k h_k : \sigma_0, \sigma_j \in \Sigma[x], p_k \in \mathbb{R}[x]\}$ . Its truncation at degree  $d$ , denoted  
 217  $Q_d(\mathcal{X})$ , consists of those elements for which  $\deg(\sigma_0) \leq 2d$ ,  $\deg(\sigma_j g_j) \leq 2d$ ,  $\deg(p_k h_k) \leq 2d$ .  
 218 Note that simplex action sets (as are standard in IREFGs) are basic semi-algebraic sets.  
 219

220 Let  $d_f := \lceil \deg(f)/2 \rceil$ ,  $d_g := \max_j \lceil \deg(g_j)/2 \rceil$ ,  $d_h := \max_k \lceil \deg(h_k)/2 \rceil$ , and  $d_{\mathcal{X}} :=$   
 221  $\max\{d_g, d_h\}$ , then  $d_0 := \max\{d_f, d_{\mathcal{X}}\}$ . For  $d \geq d_0$ , the degree- $d$  SOS relaxation of Equation (3) is

$$\begin{aligned} f_d^{\text{sos}} &= \inf_{t, \sigma_0, \{\sigma_j\}, \{p_k\}} t \\ \text{s.t. } t - f(x) &\in Q_d(\mathcal{X}). \end{aligned} \quad (4)$$

225 Its dual is the degree- $d$  moment relaxation:  
 226

$$\begin{aligned} f_d^{\text{mom}} &= \sup_y L_y(f) \\ \text{s.t. } y_0 &= 1, \quad M_d(y) \succeq 0, \quad M_{d-d_{\mathcal{X}}}(g_j \star y) \succeq 0, \quad \forall j, \\ L_y(h_k q) &= 0, \quad \forall k, \quad \forall q \in \mathbb{R}[x], \quad \deg(h_k q) \leq 2d. \end{aligned} \quad (5)$$

231 It is immediate that  $f_d^{\text{mom}} \leq f_d^{\text{sos}}$  for all  $d$ , and both sequences  $\{f_d^{\text{sos}}\}$  and  $\{f_d^{\text{mom}}\}$  are nonincreasing  
 232 in  $d$ . Moreover, the hierarchy converges asymptotically under mild assumptions on the description of  
 233 the semi-algebraic set, e.g., see Lasserre (2001). In addition to approximating the optimal value, the  
 234 moment relaxation also allows one to extract maximizers under appropriate assumptions. It is well  
 235 known that if, at some order  $d$ , the flatness condition  $\text{rank } M_s(y_d) = \text{rank } M_{s-d_{\mathcal{X}}}(y_d)$  holds for  
 236 some  $s \leq d$ , then the relaxation is exact and an optimal strategy can be extracted from  $M_s(y_d)$ ; see  
 237 Henrion & Lasserre (2005) and Appendix D.4 for details about the extraction procedure.  
 238

239 **Computational considerations.** The degree- $d$  moment relaxation in Equation (5) is an SDP  
 240 with  $\binom{m+2d}{2d}$  moment variables  $y_{\alpha}$ ,  $1 + m_g$  PSD constraints where  $M_d(y)$  has size  $\binom{m+d}{d}$  and  
 241  $M_{d-d_{\mathcal{X}}}(g_j \star y)$  has size  $\binom{m+d-d_j}{d-d_j}$ ,  $1 + \sum_{k=1}^{m_h} \binom{m+2d-2d_k}{2d-2d_k}$  linear equality constraints (normalization  
 242 and the  $h_k$ -constraints). The dual degree- $d$  SOS relaxation Equation (4) is an SDP with on the order  
 243 of  $1 + \binom{m+2d}{2d} + \sum_{j=1}^{m_g} \binom{m+2(d-d_j)}{2(d-d_j)} + \sum_{k=1}^{m_h} \binom{m+2(d-d_k)}{2(d-d_k)}$  scalar variables (for  $t$ , the coefficients  
 244 of  $\sigma_0, \sigma_j$ , and  $p_k$ ),  $1 + m_g$  PSD constraints for the Gram matrices of  $\sigma_0, \sigma_j$ , of sizes  $\binom{m+d}{d}$  and  
 245  $\binom{m+d-d_j}{d-d_j}$  respectively,  $\binom{m+2d}{2d}$  linear equality constraints from coefficient matching.  
 246

247 The SDP size can grow quickly with  $m$  and  $d$ , which constrains the Moment-SOS hierarchy to  
 248 polynomial optimization problems of modest dimension. Since computational efficiency is not  
 249 a focus of our work, we briefly mention that several recent works have aimed to improve the  
 250 scalability of SDP solving. For instance, DSOS/SDSOS relaxations trade off solution quality for  
 251 better computational performance (Ahmadi & Majumdar, 2019), and recent low-rank SDP methods  
 252 reduce memory and runtime by exploiting approximate low-rank structure (Monteiro et al., 2024; Han  
 253 et al., 2024; 2025; Aguirre et al., 2025). Our polynomial IREFG formulation fits into this framework,  
 254 so these techniques could be combined with our degree bounds to handle larger imperfect-recall  
 255 benchmarks. Exploring these scalable SDP solvers is a key direction for future work.  
 256

### 257 3 THE LINK BETWEEN IREFGS AND POLYNOMIAL OPTIMIZATION

259 We begin this section with a folklore result that bridges the study of IREFGs with polynomial  
 260 optimization, originating in Piccione & Rubinstein (1997) and expanded further in Tewolde et al.  
 261 (2023; 2024). A thorough exposition of this connection is given in Appendix B.

262 **Theorem 3.1** (Folklore). *In any IREFG, the expected utility of a player  $i \in \mathcal{N}$  under a joint  
 263 behavioral strategy  $\mu$  is a polynomial in the entries of  $\mu$ . As a consequence:*

- 264 (i) *In the single-player case, computing an ex-ante optimal behavioral strategy reduces to a  
 265 polynomial optimization problem over products of simplices.*
- 266 (ii) *In the multi-player case, computing a behavioral Nash equilibrium  $\mu^* \in \mathcal{S}$  reduces to solving  
 267 coupled polynomial optimization problems over products of simplices, i.e.,*

$$\mu_i^* \in \arg \max_{\mu_i \in \mathcal{S}_i} u_i(\mu_i, \mu_{-i}^*), \quad \forall i \in \mathcal{N}.$$

270 Moreover, these reductions can be carried out in polynomial time.  
 271

272 In view of this, while the Moment-SOS hierarchy applies directly in the single-player case, this is not  
 273 the case for the multi-player setting. We next introduce two modifications that extend its applicability  
 274 to the multi-player case and provide stronger convergence guarantees.

275 **KKT-based Moment-SOS hierarchy.** In the multi-player case, the reduction yields a polynomial  
 276 game over a product of simplices. However, it is unclear how to use this, since the set of NEs admits  
 277 no explicit semi-algebraic description. To apply the Moment/SOS hierarchy in this setting, we follow  
 278 (Nie & Tang, 2024). At a high level, their approach defines a semi-algebraic superset of the Nash  
 279 equilibria and then applies the Moment/SOS hierarchy to this larger set. This set is obtained by  
 280 concatenating the KKT conditions of each individual player’s optimization problem. These conditions  
 281 are already polynomial in the primal and dual variables, but under appropriate conditions it is possible  
 282 to reduce variables by expressing the dual multipliers as polynomials of the primal variables.

283 To explain this approach, we introduce some additional notation. The action space of each player  $i$  is  
 284 a product of simplices over its information sets, which admits a simple semi-algebraic description  
 285  $\mathcal{S}_i = \{\mu_i : h_i^j(\mu) = 0, g_i^j(\mu) \geq 0, \forall j\}$ , where  $h_i^j(\mu) := \mathbf{1}^\top \mu_i^j - 1$ ,  $g_{i,a}^j(\mu) := \mu_{i,a}^j \forall a$ ,  
 286  $g_i^j := (g_{i,1}^j, \dots, g_{i,m_i^j}^j)^\top$ . For each player  $i$  and information set  $j$ , we define the block gradient  
 287  $w_i^j(\mu) := \nabla_{\mu_i^j} u_i(\mu) \in \mathbb{R}^{m_i^j}$ . Denote by  $\nu_i^j \in \mathbb{R}$  and  $\lambda_i^j \in \mathbb{R}_{\geq 0}^{m_i^j}$  the Lagrange multipliers associated  
 288 with the constraints  $h_i^j(\mu) = 0$  and  $g_i^j(\mu) \geq 0$ , respectively. The KKT stationarity condition at an  
 289 optimizer requires that  $w_i^j(\mu) - \nu_i^j \mathbf{1} - \lambda_i^j = 0$  for all  $j$ . Taking the inner product with  $g_i^j(\mu)$ , and using  
 290 both feasibility  $h_i^j(\mu) = 0$  and complementary slackness  $g_i^j(\mu)^\top \lambda_i^j = 0$ , yields the following explicit  
 291 (polynomial) expressions for the multipliers:  $\nu_i^j(\mu) = g_i^j(\mu)^\top w_i^j(\mu)$  and  $\lambda_i^j(\mu) = w_i^j(\mu) - \nu_i^j(\mu) \mathbf{1}$ .  
 292 Hence, all Lagrange multipliers are polynomials in  $\mu$ .  
 293

294 Furthermore, in our setting (i.e., product of simplices), linear independence constraint qualification  
 295 (LICQ) holds at every feasible point, and hence the KKT conditions are necessary for optimality.  
 296 Indeed, at any info-set block the constraints are  $\mathbf{1}^\top \mu = 1$  and  $\mu_a \geq 0$ . Thus, the gradients of the active  
 297 constraints are  $\mathbf{1}$  and  $e_a$  for all  $a$  with  $\mu_a = 0$ . Since  $\sum_a \mu_a = 1$  implies at least one  $\mu_a > 0$ , not  
 298 all  $e_a$  can be active, so  $\{1\} \cup \{e_a : \mu_a = 0\}$  is linearly independent and LICQ holds. Consequently,  
 299 every NE must satisfy the joint KKT system of polynomial equations:  
 300

$$\begin{cases} w_i^j(\mu) - \nu_i^j(\mu) \mathbf{1} - \lambda_i^j(\mu) = 0, & \lambda_i^j(\mu) \geq 0, \\ g_i^j(\mu) \geq 0, & h_i^j(\mu) = 0, \quad g_{i,a}^j(\mu) \lambda_{i,a}^j(\mu) = 0, \end{cases} \quad \forall i \in \llbracket n \rrbracket, j \in \llbracket \ell_i \rrbracket, a \in \llbracket m_i^j \rrbracket. \quad (\text{KKT})$$

301 Summarizing, in the multi-player case, and following Nie & Tang (2024), a behavioral NE can be  
 302 computed by applying the Moment-SOS hierarchy to the semi-algebraic set defined by the collection  
 303 of all coupled KKT conditions. We refer to the resulting hierarchy as the *KKT-based hierarchy*.  
 304

305 An important feature of this hierarchy is that it exhibits *finite convergence*, in contrast to the standard  
 306 asymptotic convergence guarantee. This is achieved under canonical *genericity* assumptions: a  
 307 property is called *generic* if it holds for all inputs outside a Lebesgue-measure-zero set in the  
 308 coefficient space. For a fixed (multi-)degree, a polynomial is generic if its coefficient vector is  
 309 generic. Nie & Tang (2024, Theorem A.1) show that for a polynomial game with generic utilities  
 310 and constraints, the KKT equations admit only finitely many complex solutions. Consequently, the  
 311 corresponding complex variety is finite, and existing results (Laurent, 2008; Lasserre et al., 2008)  
 312 imply finite convergence of the hierarchy. However, applying this result directly in our setting requires  
 313 care, since our constraints are not generic but correspond to simplices. Nevertheless, as shown earlier  
 314 the constraints satisfy LICQ, which turns out to be the only property required in the proof of Nie &  
 315 Tang (2024, Theorem A.1) (i.e., genericity is used there only to imply LICQ).  
 316

317 **Vertex-restricted Moment-SOS hierarchy.** An important structural feature of IREFGs is *absent-  
 318 mindedness*. Absentmindedness refers to the situation where a player cannot distinguish between  
 319 two nodes that lie along the same history. Formally, a player is absentminded if there exist nodes  
 320  $\pi_1, \pi_2 \in I$  with  $\pi_1 \neq \pi_2$  such that  $\pi_1$  lies on the unique path from  $r$  to  $\pi_2$ —that is, multiple nodes  
 321 along the same history belong to the same information set (cf. Figure 2.1 (b)). Whether a player is  
 322 absentminded has important implications for the corresponding polynomial utility.  
 323

324 **Proposition 3.2.** *In non-absentminded IREFGs (NAM-IREFGs), each player’s utility  $u_i(\mu)$  is multi-  
 325 affine in the blocks  $\{\mu_i^j = (\mu_{i,a}^j)_{a=1}^{m_i^j}\}_{j=1}^{\ell_i}$ , i.e., for any player  $i$  and infoset  $j$ , the map  $\mu_i^j \mapsto u_i(\mu)$  is  
 326 affine when all other blocks  $\{\mu_{i'}^{j'}\}_{(i',j') \neq (i,j)}$  are held fixed.*

328 The proof is given in Appendix F.1. Proposition 3.2 has an interesting consequence for the Moment-  
 329 SOS hierarchy in NAM-IREFG case. Indeed, since  $u_i$  is affine in each block  $\mu_i^j$ , and the feasible  
 330 region is a product of simplices, every blockwise maximization attains its value at an extreme point.  
 331 More precisely, fixing all variables except  $\mu_i^j$  (denote this by  $\mu^{-(i,j)}$ ), the map  $\mu_i^j \mapsto u_i(\mu)$  is  
 332 affine, hence its maximum over  $\Delta^{m_i^j}$  is achieved at a vertex  $e_{i,a}^j$ . Starting from any feasible profile,  
 333 successively replacing each block  $\mu_i^j$  by a maximizing vertex with respect to the current  $\mu^{-(i,j)}$  never  
 334 decreases the objective. After finitely many replacements we obtain a global maximizer whose blocks  
 335 are all vertices. Defining  $b_{i,a}^j(\mu) := \mu_{i,a}^j(\mu_{i,a}^j - 1)$ , we see that instead of working with the product  
 336 of simplices, we can equivalently run the hierarchy over the semi-algebraic vertex-restricted set  
 337

$$338 \quad \mathcal{S}_{i,\text{vr}} := \{\mu_i : h_i^j(\mu) = 0 \forall j, b_{i,a}^j(\mu) = 0 \forall j, a\}, \quad (6)$$

340 since  $\max_{\mu_i \in \mathcal{S}_i} u_i(\mu) = \max_{\mu_i \in \mathcal{S}_{i,\text{vr}}} u_i(\mu)$ . We refer to the resulting construction as the *vertex-  
 341 restricted hierarchy*. In subsequent sections, we explore how this construction enables finite conver-  
 342 gence guarantees in the single-player case, and yields computational savings in the multi-player case.  
 343

## 344 4 SINGLE-PLAYER IREFGS

346 In this section, we focus on *single-player* IREFGs. As explained in the previous section, this setting  
 347 corresponds to a polynomial optimization problem over a product of simplices, to which the Moment-  
 348 SOS hierarchy readily applies. We summarize our results for the single-player case below. In addition,  
 349 Appendix C provides a summary of known complexity results, together with a new hardness result  
 350 that leverages the connection to POPs.

351 **Theorem 4.1.** *Consider a single-player IREFG  $\mathcal{G}$  with utility function  $u$ . Let  $\ell$  be the number of in-  
 352 fosets,  $d_0 := \max_{j,a} \{\lceil \deg(u)/2 \rceil, \lceil \deg(g_a^j)/2 \rceil, \lceil \deg(h^j)/2 \rceil\}$ , and  $u^*$  be the ex-ante optimal value  
 353 of  $\mathcal{G}$ . Denote by  $u_d^{\text{sos}}$ ,  $u_d^{\text{sos,kkt}}$ ,  $u_d^{\text{sos,vr}}$  the values obtained from the SOS-Moment hierarchies applied  
 354 respectively to the vanilla product-of-simplices, KKT-based, and vertex-restricted formulations.  
 355 Similarly, we use the superscript mom to denote the moment hierarchy. Then we have the following:*

- 356 (i)  $\lim_{d \rightarrow \infty} u_d^{\text{sos}} = \lim_{d \rightarrow \infty} u_d^{\text{mom}} = u^*$ .
- 357 (ii) If  $u$  is generic, there exists  $d \geq d_0$  with  $u_d^{\text{mom,kkt}} = u_d^{\text{sos,kkt}} = u^*$ .
- 358 (iii) If  $\mathcal{G}$  is non-absentminded, the degree- $(\ell+1)$  moment relaxation of the vertex-restricted problem  
 359 is exact:  $u_{\ell+1}^{\text{mom,vr}} = u^*$ .

360 *Proof sketch.* (i) follows from the standard convergence result for Lasserre’s hierarchy; see, e.g.,  
 361 Putinar (1993); Lasserre (2001); Laurent (2009); Lasserre (2024). (ii) uses that, if  $u$  is generic, the  
 362 KKT system has finitely many solutions, so the feasible set of the augmented problem is a finite real  
 363 variety. On such finite varieties the Lasserre hierarchy is finitely exact (Laurent, 2008; Lasserre et al.,  
 364 2008). (iii) relies on the key observation that the ranks of the moment matrices stabilize once the  
 365 degree exceeds the number of infosets  $\ell$  (cf. Lemma F.1); by the flat extension theorem (Curto &  
 366 Fialkow, 2000), this rank stabilization at degree  $\ell + 1$  implies exactness at that level.

367 This result capitalizes on the connection between polynomial optimization and IREFGs. While  
 368 convergence is asymptotic in general, for *almost all* single-player IREFGs, convergence is instead  
 369 finite. Moreover, in the special case of NAM-IREFGs, we obtain finite convergence at an *explicit  
 370 level* of the hierarchy which depends on the number of infosets in the game. The full version of the  
 371 proof for this result is deferred to Appendix F.2.

372 Crucially, we can also *extract* ex-ante optima from the Moment-SOS hierarchies (the full procedure  
 373 is given in Appendix D.4). In the NAM setting, flatness occurs at a fixed order determined by the  
 374 number of infosets (??), so extraction is guaranteed at level  $\ell + 1$ . Under genericity, the KKT-based  
 375 formulation allows one to extract a solution once finite convergence occurs.

378 **5 MULTI-PLAYER IREGFs**  
 379

380 In this section, we turn our attention to the computation of behavioral NE in *multi-player* IREGFs.  
 381 Unlike the single-player case, NE are not guaranteed to exist (Wichardt, 2008), and indeed even  
 382 deciding if one exists is hard (Tewolde et al., 2024). We therefore adopt the *select-verify-cut (SVC)*  
 383 framework of Nie & Tang (2024), which searches for an NE (or certifies nonexistence) by solving a  
 384 sequence of polynomial subproblems with Moment-SOS relaxations. Specifically, the SVC method  
 385 searches for an NE by iterating over three steps:

386 (i) *Select*: Solve a selector program over the joint (KKT) system (plus any accumulated cuts) to  
 387 pick an NE candidate.  
 388 (ii) *Verify*: For each player, fix opponent behavior at the candidate NE and solve a unilateral best-  
 389 response problem. If no player can improve, the candidate is an NE; otherwise, extract a violated  
 390 valid inequality from a deviator’s best response.  
 391 (iii) *Cut*: Add new valid inequalities to the selector and re-solve. Each cut removes the non-NE  
 392 candidate while preserving all NEs.

393 The method is iterated until a candidate passes verification (NE found) or the selector becomes  
 394 infeasible (nonexistence certified). A full description of the method is given in Appendix E. Every  
 395 subproblem in the SVC loop is a polynomial optimization problem, so we can solve them using the  
 396 Moment-SOS hierarchy. See also Nie & Tang (2024, Section 4) for more details.

397 In general, SVC needs to iterate: each round solves one selector Moment-SOS relaxation and up to  
 398  $n$  verification relaxations (one per player). By contrast, in NAM-IREFGs we can dispense with the  
 399 verify/cut phases and solve a *single* vertex-restricted Moment-SOS relaxation to compute an NE or  
 400 certify nonexistence, yielding substantial computational savings. Indeed, since  $u_i(\cdot, \mu_{-i})$  is linear in  
 401  $\mu_i$  (Proposition 3.2), the “no profitable deviation by player  $i$ ” NE condition is equivalent to the family  
 402 of vertex inequalities  $u_i(\mu_i, \mu_{-i}) \geq u_i(v_i, \mu_{-i})$  for all  $v_i \in \mathcal{S}_{i,\text{vr}}$ . Embedding these inequalities  
 403 directly into the *Select* step gives the one-shot vertex-restricted program:

404

$$\min_{\mu} \varphi_{\Theta}(\mu) := [\mu]_1^{\top} \Theta [\mu]_1 \quad \text{s.t.} \quad \begin{cases} \mu \text{ satisfies (KKT),} \\ u_i(\mu_i, \mu_{-i}) - u_i(v_i, \mu_{-i}) \geq 0, \quad \forall v_i \in \mathcal{S}_{i,\text{vr}}, \quad \forall i, \end{cases} \quad (7)$$

405 where  $[\mu]_1 = (1, \mu^{\top})^{\top}$  and  $\Theta \succ 0$  is generic. Feasibility of Equation (7) is equivalent to the  
 406 existence of a behavioral NE; any feasible point is an NE, and the objective  $\varphi_{\Theta}$  merely selects one  
 407 among multiple solutions (if any). For instance, replacing  $\varphi_{\Theta}$  with  $\max_{\mu} \sum_{i=1}^n u_i(\mu)$  under the same  
 408 constraints yields a *welfare-maximizing* NE. Therefore, in the NAM setting, the verify and cut phases  
 409 are unnecessary.

410 **Theorem 5.1.** *Let  $\mathcal{G}$  be a multi-player IREGF with utility functions  $u_i$  for each player  $i$ . Throughout,  
 411 subproblems are solved by the KKT-based hierarchies of increasing order. Then, we have the  
 412 following:*

413 (i) *The SVC procedure is asymptotically exact: as the relaxation order and number of iterations  
 414 grows, it returns a behavioral NE when one exists, and otherwise a certificate of nonexistence.*  
 415 (ii) *If  $u_i$  are all generic, the KKT-based hierarchy has finite convergence for all SVC subproblems,  
 416 and the SVC loop terminates in finitely many iterations.*  
 417 (iii) *If  $\mathcal{G}$  is non-absentminded, the Verify/Cut phases in SVC are unnecessary: a single vertex-  
 418 restricted Select (Equation (7)) suffices to compute an NE or certify nonexistence. Its Moment-  
 419 SOS hierarchy is asymptotically exact; if  $u_i$  are generic, it attains exactness at a finite order.*

420 *Proof sketch.* (i) combines the standard asymptotic convergence of Lasserre’s hierarchy with the  
 421 SVC steps: each loop either eliminates a non-NE candidate via a violated deviation inequality or  
 422 certifies NE (or non-existence via selector infeasibility) (Nie & Tang, 2024). (ii) then follows from  
 423 the fact that, under generic utilities, the KKT system has finitely many solutions, so only finitely  
 424 many candidates can appear in SVC, and on such finite KKT sets the KKT-based SOS hierarchy is  
 425 known to converge in finite order. (iii) uses the fact that in the non-absentminded case, “no profitable  
 426 deviation” is equivalent to “no profitable vertex deviation”, so a single vertex-restricted Select already

432 encodes the NE conditions. Its Moment-SOS hierarchy is asymptotically exact, and under generic  
 433 utilities, finite exact.

434 The full version of proof for the above result is deferred to Appendix F.3. In analogy to the single-  
 435 player case, Theorem 5.1 shows that finite convergence guarantees can be obtained for *almost*  
 436 all IREFGs. NAM-IREFGs enjoy further computational improvements, needing only a single  
 437 Moment-SOS hierarchy per iteration. Unlike the single-player case, explicit convergence at a fixed  
 438 game-dependent level is not guaranteed, further highlighting the challenging nature of multi-player  
 439 IREFGs.

440

## 441 6 (SOS)-CONCAVE AND (SOS)-MONOTONE IREFGs

443

444 In this section, we focus on defining tractable subclasses of IREFGs, and show how our methods  
 445 obtain improved convergence guarantees in these subclasses. In the study of continuous games, the  
 446 seminal work of Rosen (1965) introduced concave and monotone games, which exhibit desirable  
 447 properties—in concave games, NE always exist, and in strictly monotone games, a unique NE  
 448 exists. Leveraging the connection between IREFGs and polynomials, the following definitions are  
 449 immediate:

450 An IREFG  $\mathcal{G}$  is *concave* if, for every  $i \in \mathcal{N}$ , the map  $\mu_i \mapsto u_i(\mu_i, \mu_{-i})$  is concave on  $\mathcal{S}_i$  for every  
 451 fixed  $\mu_{-i} \in \mathcal{S}_{-i}$ . Since  $\mathcal{G}$  is polynomial, this is equivalent to the block Hessians being negative  
 452 semidefinite:  $\mathbf{H}_{u_i}(\mu) := \nabla_{\mu_i}^2 u_i(\mu) \preceq 0, \forall \mu \in \mathcal{S}, \forall i \in \mathcal{N}$ .

453 Following Rosen (1965), we collect the block partial derivatives into the pseudo-gradient:  $v(\mu) :=$   
 454  $(\nabla_{\mu_1}^\top u_1(\mu), \dots, \nabla_{\mu_n}^\top u_n(\mu))^\top$ . Let  $\mathbf{J}(\mu) := \nabla v(\mu)$  denote its Jacobian, and define the symmetrized  
 455 Jacobian  $\mathbf{SJ}(\mu) := \frac{1}{2}(\mathbf{J}(\mu) + \mathbf{J}(\mu)^\top)$ . Then, an IREFG  $\mathcal{G}$  is *monotone* if  $\langle v(\mu) - v(\nu), \mu - \nu \rangle \leq$   
 456  $0, \forall \mu, \nu \in \mathcal{S}$ . Since  $\mathcal{G}$  is polynomial, monotonicity holds if and only if the symmetrized Jacobian of  
 457  $v$  is negative semidefinite (Rockafellar & Wets, 1998):  $\mathbf{SJ}(\mu) \preceq 0, \forall \mu \in \mathcal{S}$ .

458

459 While monotonicity immediately implies concavity, the converse does not hold. Moreover, we can  
 460 define strict notions of concavity and monotonicity where the block Hessians and symmetrized  
 461 Jacobian of a game are negative definite, respectively. With these notions in place, we recall the  
 462 classical existence/uniqueness guarantees (Rosen, 1965, Theorems 1 and 2) tailored to our setting.

463

**Proposition 6.1.** *Let  $\mathcal{S}$  be nonempty, convex, and compact, and let each  $u_i$  be continuous.*

464

*(i) If the IREFG  $\mathcal{G}$  is concave, then a behavioral Nash equilibrium exists.*

465

*(ii) If the IREFG  $\mathcal{G}$  is strictly monotone, then the behavioral Nash equilibrium is unique.*

467

468 With respect to the SVC method utilized in Section 5, when an IREFG is concave, a joint profile  $\mu$  is  
 469 a Nash equilibrium *if and only if* it is a (KKT) point. Since the strategy set  $\mathcal{S}$  is nonempty, convex,  
 470 and compact, Proposition 6.1 guarantees existence, so the *Select* step is always feasible and returns  
 471 an NE without any cuts.

472

**Corollary 6.2.** *If the IREFG  $\mathcal{G}$  is concave, then any feasible point of the Select step (without any  
 473 cuts) is an NE. Furthermore, if  $\mathcal{G}$  is strictly monotone, the NE is unique and the Select step returns  
 474 that unique equilibrium.*

475

476 While concavity and monotonicity are widely studied, it is in general hard to *verify* these properties  
 477 (Ahmadi et al., 2013; Leon et al., 2025). Towards improving the tractability of these classes, Hel-  
 478 ton & Nie (2010) introduced ‘effective’ SOS-variants thereof, which are verifiable using a single  
 479 SDP. In particular, we call an IREFG  $\mathcal{G}$  *SOS-concave* if, for every  $i \in \mathcal{N}$ , the negative block Hes-  
 480 sian is an SOS-matrix polynomial on  $\mathcal{S}$ , i.e., there exists a real matrix polynomial  $F_i(\mu)$  such that  
 481  $-\mathbf{H}_{u_i}(\mu) = F_i(\mu)F_i(\mu)^\top, \forall \mu \in \mathcal{S}, \forall i \in \mathcal{N}$ . Furthermore, we call  $\mathcal{G}$  *SOS-monotone* if its negative  
 482 symmetrized Jacobian is an SOS-matrix polynomial, i.e., there exists a real matrix polynomial  $F(\mu)$   
 483 such that  $-\mathbf{SJ}(\mu) = F(\mu)F(\mu)^\top, \forall \mu \in \mathcal{S}$ .

484

485 These are subclasses of concave and monotone IREFGs respectively, so SOS-concavity implies  
 486 concavity and SOS-monotonicity implies monotonicity. However, the converse does not hold  
 487 since there exist polynomials which are convex but not SOS-convex (Ahmadi & Parrilo, 2012).  
 488 Going forward, we explore how these notions can be used to further improve the convergence

486 guarantees of our methods in single-player IREFGs. We note that the definitions of concavity and  
 487 monotonicity coincide in the single-player case (cf. Proposition F.2). We show that for strict and  
 488 **SOS-concave**/monotone single-player IREFGs, our proposed methods obtain stronger convergence  
 489 guarantees:

490 **Theorem 6.3.** *Consider a single-player IREFG  $\mathcal{G}$  with utility function  $u$ . Let  $d_0 :=$   
 491  $\max_{j,a}\{\lceil \deg(u)/2 \rceil, \lceil \deg(g_a^j)/2 \rceil, \lceil \deg(h^j)/2 \rceil\}$ . Then, the following holds:*

492 (i) *If  $\mathcal{G}$  is strictly **concave**/monotone, then the Moment-SOS hierarchy has finite convergence: there  
 493 exists  $d \geq d_0$  such that  $u_d^{\text{sos}} = u_d^{\text{mom}} = u^*$ .*

494 (ii) *If  $\mathcal{G}$  is **SOS-concave**/SOS-monotone, the degree- $d_0$  Moment-SOS relaxations are exact:  $u_{d_0}^{\text{mom}} =$   
 495  $u_{d_0}^{\text{sos}} = u^*$ , i.e., the Moment-SOS hierarchy converges at the first level.*

496 The proof follows the same overall line of argument as (Lasserre, 2009a, Theorems 3.3 and 3.4),  
 497 but adapted to our IREFG setting; full details are given in Appendix F.4. In the case of strictly  
 498 **concave**/monotone single-player IREFGs, we obtain a sharper, finite-time guarantee of convergence  
 499 to the unique behavioral NE without needing to modify the vanilla Moment-SOS hierarchy. Moreover,  
 500 in **SOS-concave**/SOS-monotone single-player IREFGs, we show that convergence is possible at  
 501 the first level of the hierarchy, requiring only a single SDP. Furthermore, by leveraging known  
 502 results (Laurent, 2008; Lasserre, 2009a), once relaxation is exact, optimal solutions can be obtained  
 503 directly from the first moments. In particular, for any optimal moment vector  $y^*$ ,  $\mu^* = (y_{e_a^j}^*)_{j,a}$ .  
 504 Thus, the extraction procedure as outlined in Appendix D.4 is no longer necessary, enabling efficient  
 505 extraction of ex-ante optima.

## 506 7 DISCUSSION AND FUTURE WORK

507 In this paper, we applied Moment-SOS hierarchies to compute ex-ante optima in single-player  
 508 IREFGs, and Nash equilibria in multi-player IREFGs. We showed that almost all IREFGs enjoy finite  
 509 convergence. Moreover, in the case of NAM-IREFGs, the convergence is finite and dependent on  
 510 the number of infosets in the game. We also propose tractable subclasses of IREFGs, and efficiently  
 511 certifiable SOS-variants thereof. This leads to a subclass of **SOS-concave**/SOS-monotone single-  
 512 player IREFGs which can be solved with a single SDP. Our results constitute an initial step towards  
 513 better understanding the tractability of equilibrium computation in IREFGs, bringing together ideas  
 514 from both extensive-form game theory and polynomial optimization.

515 Future work includes finding and analyzing other tractable subclasses of IREFGs, and designing  
 516 faster algorithms that incorporate decentralized methods to compute KKT points. Moreover, while  
 517 Statement (i) of Theorem 6.3 holds for the Moment-SOS approach, first-order decentralized methods  
 518 are known to efficiently solve strictly monotone games (see e.g. Cai & Zheng (2023); Ba et al. (2025)  
 519 and references therein). A natural question is how these approaches compare, both theoretically  
 520 and empirically. Finally, while we have implemented our methods on some simple examples in Ap-  
 521 pendix G, implementing more tractable SDP solvers to improve scalability (Ahmadi & Majumdar,  
 522 2019; Monteiro et al., 2024; Han et al., 2025) remains a crucial direction to explore.

### 523 ETHICS STATEMENT

524 While our results are primarily theoretical, we acknowledge that there could be potential societal and  
 525 ethical consequences of our work, none of which we feel must be specifically highlighted.

### 526 REPRODUCIBILITY STATEMENT

527 The code used to generate the examples in our paper are given in a supplementary file, and use  
 528 standard Julia scientific computing packages, alongside the SumOfSquares package (Legat et al.,  
 529 2017; Weisser et al., 2019).

### 530 LARGE LANGUAGE MODEL USAGE STATEMENT

531 LLMs were used sparingly to polish writing and grammar. They were not used in the ideation and  
 532 analysis of our main theoretical results.

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756 **A ADDITIONAL RELATED WORK**

757  
 758 **Sum-of-Squares in Polynomial Games.** Outside of the works cited in the introduction, few other  
 759 works have studied IREFGs from the perspective of polynomial optimization. However, games with  
 760 polynomial utility functions have been proposed and studied in the past Dresher et al. (1950). Parrilo  
 761 (2006); Laraki & Lasserre (2012) used semidefinite programming methods to find the value of two-  
 762 player zero-sum polynomial games, and Stein et al. (2008) applied similar techniques to separable  
 763 games (where utilities take a sum-of-products form). Recently, Nie & Tang (2023; 2024) also used  
 764 semidefinite programming techniques to solve for Nash equilibria in  $n$ -player polynomial games under  
 765 certain genericity assumptions, or otherwise detect the nonexistence of equilibria. Moreover, Bach  
 766 (2025) studied sum-of-squares relaxations for min-max problems, deriving convergence results of  
 767 their proposed hierarchies. Finally, Leon et al. (2025) used sum-of-squares hierarchies to certify  
 768 concavity and monotonicity in polynomial games.

769  
 770 **Equilibrium Computation in IREFGs.** Despite the hardness of computing equilibria in IREFGs,  
 771 some work has studied the special case of two-player, zero-sum IREFGs and derived approximate  
 772 algorithms to solve them. In particular, Bosansky et al. (2016); Čermák et al. (2017b) used MILP  
 773 techniques to solve for minmax-optimal strategies. In the case of non-absentminded two-player  
 774 zero-sum IREFGs, Čermák et al. (2018) also used MILP-based methods to obtain scalable algorithms  
 775 that can approximate minmax strategies. In the case where the IREFGs are timeable and two-  
 776 player zero-sum (a subclass of non-absentminded two-player zero-sum games), Zhang & Sandholm  
 777 (2022); Zhang et al. (2023) utilized tree-decomposition based methods to obtain LP/CFR bounds for  
 778 computing team-correlated (mixed) equilibria.

779 **B FROM IREFGs TO POLYNOMIALS AND BACK**

780  
 781 The equivalence between IREFGs and polynomial optimization is crucial to our proposed methods.  
 782 In particular, the translation from IREFGs to polynomials is classical (Piccione & Rubinstein, 1997),  
 783 and we give a full description here for clarity. First, let  $P(h'|\mu, h)$  denote the realization probability  
 784 of reaching  $h'$  given that players using strategy  $\mu$  are at state  $h$ . Note that if  $h \notin \text{hist}(h')$  (i.e., if  
 785  $h'$  is not reachable from  $h$ ) then the probability is 0. Intuitively, the realization probability given a  
 786 behavioral strategy is just the product of choice probabilities along the path from  $h$  to  $h'$ . In order to  
 787 formally define  $P(h'|\mu, h)$ , we will need some additional notation. First, any node  $h \in \mathcal{H}$  uniquely  
 788 corresponds to a history  $\text{hist}(h)$  from root  $r$  to  $h$ .

789  
 790 • Function  $\delta(h) : \mathcal{H} \rightarrow \mathbb{N}$  denotes the depth of the game tree starting from node  $h \in \mathcal{H}$ .  
 791 • Function  $\nu(h, d) : \mathcal{H} \times \mathbb{N} \rightarrow \mathcal{H}$  identifies the node ancestor at depth  $d \leq \delta$  from node  $h$ .  
 792 • Function  $\alpha(h, d) : \mathcal{H} \times \mathbb{N} \rightarrow \cup_{h \in \mathcal{H}} \mathcal{A}(h)$  identifies the action ancestor at depth  $d \leq \delta$  from  
 793 node  $h$ .

794  
 795 Together, the sequence  $(\nu(h, 0), \nu(h, 1), \dots, \nu(h, \delta(h)))$  uniquely identifies the history of nodes  
 796 from  $r$  to  $h$ . Likewise, the sequence  $(\alpha(h, 0), \alpha(h, 1), \dots, \alpha(h, \delta(h) - 1))$  uniquely identifies the  
 797 history of actions taken from  $r$  to  $h$ . Then, the realization probability of node  $h'$  from  $h$  if the players  
 798 use joint strategy profile  $\sigma$  is given by:

799 **Definition B.1** (Realization Probability).

800  
 801 
$$P(h'|\mu, h) = \prod_{j=\delta(h')}^{\delta(h)-1} \mu(\alpha(h', j) | I_{\nu(h', j)}) \quad \text{if } h \in \text{hist}(h').$$

802  
 803  
 804 **Definition B.2** (Expected Utility for Player  $i$ ). For player  $i$  at node  $h \in \mathcal{H} \setminus \mathcal{Z}$ , if strategy profile  
 805  $\mu$  is played, their expected utility is given by  $u_i(\mu|h) := \sum_{z \in \mathcal{Z}} (P(z|\mu, h) \cdot \rho_i(z))$ . In its complete  
 806 form, we can write the expected utility for each player as follows:

807  
 808 
$$u_i(\mu) = \sum_{z \in \mathcal{Z}} \left( \prod_{j=0}^{\delta(z)-1} \mu(\alpha(z, j) | I_{\nu(z, j)}) \cdot \rho_i(z) \right)$$

With some abuse of notation, we can write  $P(h|\mu) := P(h|\mu, r)$  where  $r$  is the root node, and similarly  $u_i(\mu) := u_i(\mu|r)$ . Notice that by definition, the expected utility of each player is a polynomial function. In particular,  $P(z|\mu, h) \cdot \rho_i(z)$  is a monomial in  $\mu$  multiplied by a scalar.

Tewolde et al. (2023) also recently established that any polynomial can be transformed into a single-player IREFG. Subsequently, Tewolde et al. (2024) extended this result to a set of polynomials, and multi-player IREFGs. We report the single-player variant of the theorem below, and provide a concrete example to aid readability.

**Theorem B.3.** *Given a polynomial  $p : \bigtimes_{j=1}^{\ell} \mathbb{R}^{m^j} \rightarrow \mathbb{R}$ , we can construct a single-player extensive-form game with imperfect recall  $\mathcal{G}$  such that the expected utility function of  $\mathcal{G}$  satisfies  $u(\mu) = p(\mu)$ , with  $\mu \in \bigtimes_{j=1}^{\ell} \Delta^{m^j-1}$ . Moreover, the construction can be done in polynomial time.*

*Proof.* The proof follows the analysis of Tewolde et al. (2023), with minor modifications to notation. Let  $d_p = \deg(p)$  and write

$$p(x) = \sum_{D \in \text{MB}(d_p, \mathbf{m})} \lambda_D \prod_{j,a=1}^{\ell, m^j} (x_a^j)^{D_a^j},$$

where  $\mathbf{m} := (m^j)_{j=1}^{\ell}$ ,  $\lambda_D$  are rational, and

$$\text{MB}(d_p, \mathbf{m}) := \left\{ D = (D_a^j)_{j,a} \in \bigtimes_{j=1}^{\ell} \mathbb{N}_0^{m^j} : \sum_{j=1}^{\ell} \sum_{a=1}^{m^j} D_a^j \leq d_p \right\}.$$

Let  $\text{supp}(p) := \{D \in \text{MB}(d_p, \mathbf{m}) : \lambda_D \neq 0\}$  and  $|D| := \sum_{j,a} D_a^j$ . For each  $D$ , define the multiset  $\text{supp}(D)^{\text{ms}}$  that contains  $D_a^j$  copies of the pair  $(j, a)$  when  $D_a^j > 0$ . Then  $|\text{supp}(D)^{\text{ms}}| = |D|$ . The input encoding of  $p$  consists of  $\ell$ ,  $(m^j)_{j=1}^{\ell}$ , and the rational coefficients  $(\lambda_D)_{D \in \text{supp}(p)}$ .

Given such a polynomial function, we build a single-player extensive-form game  $\mathcal{G}$  with imperfect recall whose information sets are  $I^j$  ( $j \in [\ell]$ ), each with action set  $\mathcal{A}_{I^j} = \{\tau_1, \dots, \tau_{m^j}\}$ . The game  $\mathcal{G}$  has a chance root and depth at most  $d_p + 1$ . The chance node  $h_0$  has one outgoing edge to a node  $h_D$  for each  $D \in \text{supp}(p)$ , and chance selects each  $h_D$  with probability  $1/|\text{supp}(p)|$ .

Fix a deterministic ordering  $\prec$  of the multiset  $\text{supp}(D)^{\text{ms}}$  (e.g., lexicographic on pairs  $(j, a)$ , repeated  $D_a^j$  times). Initialize the *current edge* as the chance edge from  $h_0$  into  $h_D$ .

- If  $D = \mathbf{0}$  (the zero multi-index), make  $h_D$  terminal with payoff  $u(h_D) = \lambda_0 \cdot |\text{supp}(p)|$ .
- Otherwise, for each next element  $(j, a)$  of  $\text{supp}(D)^{\text{ms}}$  (in order  $\prec$ ), do:
  - Insert a nonterminal decision node  $h$  on the current edge and assign  $h$  to the information set  $I^j$ .
  - Create  $m^j$  outgoing edges from  $h$ , one for each action in  $\mathcal{A}_{I^j} = \{\tau_1, \dots, \tau_{m^j}\}$ .
  - For every edge labeled  $\tau_{a'}$  with  $a' \neq a$ , attach a terminal node with utility 0.
  - Update the *current edge* to be the unique edge labeled  $\tau_a$ .

After all elements of  $\text{supp}(D)^{\text{ms}}$  have been processed, terminate the current edge with a terminal node  $z_D$  and set its utility to  $u(z_D) = \lambda_D \cdot |\text{supp}(p)|$ . This procedure yields a subtree  $T_D$  of depth  $|\text{supp}(D)^{\text{ms}}| = \sum_{j,a} D_a^j = |D|$ .

In this reduction, any point  $x = (x_a^j)_{j,a} \in \bigtimes_{j=1}^{\ell} \Delta^{m^j-1}$  induces a behavioral strategy  $\mu$  in  $\mathcal{G}$  by  $\mu(a_j | I^j) = x_a^j$  for all  $j, a$ . Let  $z_D$  denote the terminal node associated with monomial index  $D$  (including  $D = \mathbf{0}$ ). At the chance root,  $\mathbb{P}(h_D | \mu) = 1/|\text{supp}(p)|$  for each  $D \in \text{supp}(p)$ . If  $D = \mathbf{0}$ , then  $z_D$  is reached immediately, so  $\mathbb{P}(z_D | \mu) = 1/|\text{supp}(p)|$ . If  $D \neq \mathbf{0}$ , the loop that builds  $T_D$  creates exactly  $|D|$  decision nodes along the designated branch. At each visit to information set  $I^j$ , the unique continuing edge is labeled  $\tau_a$  and is chosen with probability  $x_a^j$ . Since  $(j, a)$  appears  $D_a^j$

864 times in  $\text{supp}(D)^{\text{ms}}$ , we obtain  
 865

$$866 \quad \mathbb{P}(z_D \mid \mu) = \frac{1}{|\text{supp}(p)|} \prod_{j,a=1}^{\ell, m^j} (x_a^j)^{D_a^j}.$$

$$867$$

$$868$$

869 All sibling edges terminate with utility 0 and do not contribute.  
 870

871 Therefore the expected utility is  
 872

$$872 \quad u(\mu) = \sum_{D \in \text{supp}(p)} \mathbb{P}(z_D \mid \mu) \cdot \rho(z_D)$$

$$873$$

$$874 \quad = \sum_{D \in \text{supp}(p)} \left( \frac{1}{|\text{supp}(p)|} \prod_{j,a} (x_a^j)^{D_a^j} \right) \cdot \lambda_D \cdot |\text{supp}(p)|$$

$$875$$

$$876 \quad = \sum_{D \in \text{supp}(p)} \lambda_D \cdot \prod_{j,a} (x_a^j)^{D_a^j}$$

$$877$$

$$878 \quad = p(x).$$

$$879$$

$$880$$

881 This extends to  $u(\mu) = p(\mu)$  for all  $\mu \in \bigtimes_{j=1}^{\ell} \Delta^{m^j-1}$ .  
 882

883 For each  $D \in \text{supp}(p)$ ,  $T_D$  contributes a path of length  $|D| \leq d_p$  where, at each depth, at most  
 884  $\max_j m^j$  leaves are created. Hence the total number of nodes and edges is  
 885

$$886 \quad O\left(\sum_{D \in \text{supp}(p)} (|D| + |D| \cdot \max_j m^j)\right) \subseteq O(|\text{supp}(p)| \cdot d_p \cdot \max_j m^j).$$

$$887$$

888 All payoffs are rationals of the form  $\lambda_D \cdot |\text{supp}(p)|$ , and the chance probabilities are  $1/|\text{supp}(p)|$ , so  
 889 labels are computable with bit complexity polynomial in the input size. Thus the game  $\mathcal{G}$  is produced  
 890 in polynomial time in the Turing model.  $\square$   
 891

892 Clearly, this construction is not unique. Choosing a different total order on the multiset  $\text{supp}(D)^{\text{ms}}$   
 893 yields a (potentially) different game tree. All such variants are payoff-equivalent, since the reach  
 894 probability of  $z_D$  depends only on the multiplicities  $(D_a^j)$ , hence  $u(\mu) = p(\mu)$  in every case. For  
 895 concreteness we fix the lexicographic order. Moreover, we provide a concrete example constructing a  
 896 (single-player) IREFG from a polynomial.

897 **Example B.4.** *In this example, we index by (infoset, action) using subscripts to avoid ambiguity:  
 898 the first subscript denotes the infoset and the second denotes the action (e.g.,  $x_{12}$ ). Let  $\ell = 2$  with  
 899  $m_1 = m_2 = 2$ , and write  $x_{11}, x_{12}$  for  $I_1$  and  $x_{21}, x_{22}$  for  $I_2$ . Consider*

$$900 \quad p(x) = 2 + 3x_{11}x_{21} - 5x_{12}x_{22} + 4x_{21}^2.$$

$$901$$

902 Then  $\deg(p) = 2$ . Fix the variable order  $x_{11} \prec x_{12} \prec x_{21} \prec x_{22}$  and use the induced lexicographic  
 903 order on multi-indices  $D = (D_{11}, D_{12}, D_{21}, D_{22}) \in \mathbb{N}_0^4$ . For each  $D$ , order the multiset  $\text{supp}(D)^{\text{ms}}$   
 904 by listing the pairs  $(j, a)$  in lexicographic order with multiplicity. With this convention,  
 905

$$905 \quad \text{supp}(p) = \{D^{(0)} = \mathbf{0}, D^{(1)} = (1, 0, 1, 0), D^{(2)} = (0, 1, 0, 1), D^{(3)} = (0, 0, 2, 0)\}$$

$$906$$

907 so  $|\text{supp}(p)| = 4$  and the lexicographic order is  $D^{(0)} \prec D^{(1)} \prec D^{(2)} \prec D^{(3)}$ .  
 908

909 Then, we construct the game tree as follows:  
 910

910 **Root (chance).** Create a chance node  $h_0$  with four equiprobable edges (1/4 each) to  $h_{D^{(t)}}$ ,  $t =$   
 911  $0, 1, 2, 3$ .  
 912

913 **Subtree  $T_{D^{(0)}}$  (constant term).** Make  $h_{D^{(0)}}$  terminal with payoff  $\rho = \lambda_{D^{(0)}} \cdot |\text{supp}(p)| = 2 \cdot 4 = 8$ .  
 914

915 **Subtree  $T_{D^{(1)}}$  for  $x_{11}x_{21}$ .** Here  $\text{supp}(D^{(1)})^{\text{ms}} = \{(1, 1), (2, 1)\}$ . Process in order: (1) insert a  
 916 node  $h_1 \in I_1$ ; create two edges labeled  $x_{11}, x_{12}$ ; attach 0 to the  $x_{12}$  edge; move along  $x_{11}$ . (2)  
 917 insert a node  $h_2 \in I_2$ ; create two edges labeled  $x_{21}, x_{22}$ ; attach 0 to the  $x_{22}$  edge; move along  $x_{21}$ .  
 918 Terminate with  $z_{D^{(1)}}$  and payoff  $\rho(z_{D^{(1)}}) = \lambda_{D^{(1)}} \cdot 4 = 12$ .

**Subtree  $T_{D^{(2)}}$  for  $x_{22}$ .** Here  $\text{supp}(D^{(2)})^{\text{ms}} = \{(1, 2), (2, 2)\}$ . Process in order: (1) insert  $h_3 \in I_1$ ; create  $x_{11}, x_{12}$ ; attach 0 to  $x_{11}$ ; move along  $x_{12}$ . (2) insert  $h_4 \in I_2$ ; create  $x_{21}, x_{22}$ ; attach 0 to  $x_{21}$ ; move along  $x_{22}$ . Terminate with  $z_{D^{(2)}}$  and payoff  $\rho(z_{D^{(2)}}) = \lambda_{D^{(2)}} \cdot 4 = -20$ .

**Subtree  $T_{D^{(3)}}$  for  $x_2^{(2)}$ .** Here  $\text{supp}(D^{(3)})^{\text{ms}} = \{(2, 1), (2, 1)\}$  (two copies). Process in order: (1) insert  $h_5 \in I_2$ ; create  $x_{21}, x_{22}$ ; attach 0 to  $x_{22}$ ; move along  $x_{21}$ . (2) insert  $h_6 \in I_2$ ; again  $x_{21}$  continues,  $x_{22}$  gets 0. Terminate with  $z_{D^{(3)}}$  and payoff  $\rho(z_{D^{(3)}}) = \lambda_{D^{(3)}} \cdot 4 = 16$ .

The constructed game tree is shown in Figure B.1.

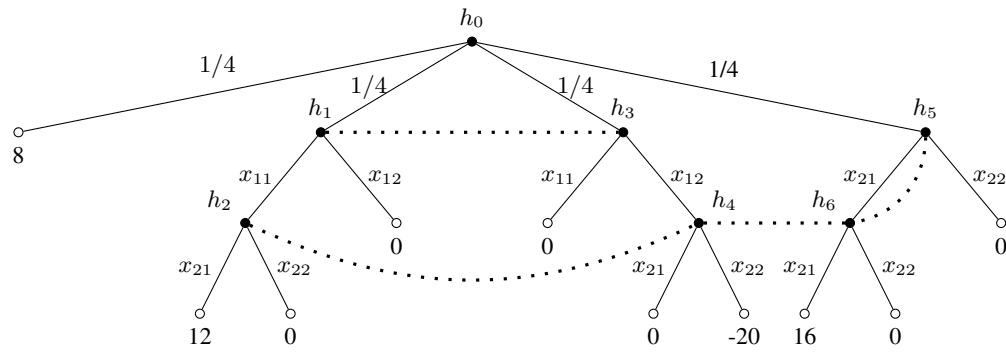


Figure B.1: Constructed Single-Player Imperfect-Recall Game for  $p(x) = 2 + 3x_{11}x_{21} - 5x_{12}x_{22} + 4x_{21}^2$ .

**Verification.** Given  $x \in \Delta^1 \times \Delta^1$ , define the behavioral strategy by  $\mu(a_j \mid I^j) = x_j^j$ . Then

$$\mathbb{P}(z_{D^{(0)}} \mid \mu) = \frac{1}{4}, \quad \mathbb{P}(z_{D^{(1)}} \mid \mu) = \frac{1}{4} x_{11} x_{21}, \quad \mathbb{P}(z_{D^{(2)}} \mid \mu) = \frac{1}{4} x_{12} x_{22}, \quad \mathbb{P}(z_{D^{(3)}} \mid \mu) = \frac{1}{4} x_{21}^2.$$

*Hence*

$$\begin{aligned}
u(\mu) &= \sum_{t=0}^3 \mathbb{P}(z_{D^{(t)}} \mid \mu) \rho(z_{D^{(t)}}) \\
&= \frac{1}{4} \cdot 8 + \frac{1}{4} x_{11} x_{21} \cdot 12 + \frac{1}{4} x_{12} x_{22} \cdot (-20) + \frac{1}{4} x_{12} x_{21}^2 \cdot 16 \\
&= 2 + 3x_{11}x_{21} - 5x_{12}x_{22} + 4x_{21}^2 \\
&= p(x).
\end{aligned}$$

## C ON THE COMPUTATIONAL COMPLEXITY OF SINGLE-PLAYER JEEGGS

While the computational hardness of the ex-ante problem (Equation (1)) has already been established in prior work (Tewolde et al., 2023), we note that one can extend recent results in the polynomial optimization literature to improve these complexity results. First, we introduce a hardness result for finding local optima over polytopes by Ahmadi & Zhang (2022):

**Lemma C.1** (Ahmadi & Zhang (2022), Theorem 2.6). *Unless  $P = NP$ , there is no polynomial-time algorithm that finds a point within Euclidean distance  $c^n$  (for any constant  $c \geq 0$ ) of a local minimizer of an  $n$ -variate quadratic function over a polytope.*

**Corollary C.2.** *Finding a local minimizer of a quadratic program over a simplex is NP-hard. Moreover, unless  $P = NP$ , there is no FPTAS for this problem.*

*Proof.* Specializing the argument in (Ahmadi & Zhang, 2022, Theorem 2.6, p. 7) by replacing  $\lceil 3c^n\sqrt{n} \rceil$  by 1 yields the following: unless  $P = NP$ , there is no polynomial-time algorithm that finds an  $\epsilon$ -approximate local minimizer (for any constant  $\epsilon \in [0, 0.5)$ ) of a quadratic program over a simplex. If an exact local minimizer could be computed in polynomial time, then, in particular, an  $\epsilon$ -approximate local minimizer (take  $\epsilon = 0$ ) could also be computed in polynomial time, contradicting Lemma C.1. Therefore, computing an exact local minimizer over a simplex is NP-hard, and no FPTAS exists unless  $P = NP$ .  $\square$

972 Hence, corresponding to single-player IREFGs, we have:  
 973

974 **Proposition C.3.** *Finding a local optimum of Equation (1) is NP-hard. Moreover, unless  $P = NP$ ,  
 975 there is no FPTAS for this problem. NP-hardness and conditional inapproximability hold even if the  
 976 game instance  $\mathcal{G}$  has no chance nodes, a tree depth of 2, and only one information set.*

977 Following this, we can improve the results from (Tewolde et al., 2023, Proposition 4):  
 978

979 **Proposition C.4.** *Finding the ex-ante optimal strategy  $\mu^*$  of a given extensive-form game instance  $\mathcal{G}$   
 980 is NP-hard. Specifically:*

- 981 1. *Unless  $P = NP$ , no FPTAS exists. NP-hardness and conditional inapproximability hold  
 982 even if the game has no chance nodes, a tree depth of 2, and a single information set.*
- 983 2. *Unless  $NP = ZPP$ , no FPTAS exists. NP-hardness and conditional inapproximability  
 984 hold even if the game has a tree depth of 3 and a single information set.*
- 985 3. *NP-hardness holds even without absentmindedness, with tree depth 4 and two actions per  
 986 information set.*
- 987 4. *NP-hardness holds even without absentmindedness, with tree depth 3 and three actions per  
 988 information set.*
- 989
- 990
- 991

992  
 993 *Proof.* To prove statement 1, we note that since any global optimum is in particular a local optimum,  
 994 finding a global maximum must be at least as hard as finding a local one. Thus, the same hardness  
 995 and inapproximability from Proposition C.3 immediately carry over to the problem of finding a global  
 996 optimum, i.e., ex-ante optimal strategy  $\mu^*$ .

997 The statements 2-4 are proven in (Tewolde et al., 2023, Proposition 4).  $\square$   
 998

1000  
 1001 Following Tewolde et al. (2023), we compile the core correspondences between the polynomial  
 1002 optimization formulation of Equation (1) and solution concepts in single-player IREFGs, together  
 1003 with the computational complexity of computing each notion, in Table C.1.

1004  
 1005 Table C.1: Correspondence between single-player IREFGs and POPs with complexity results.

1006 1007 POP Optimality Notion	1008 IREFG Equilibrium Notion		1009 Rel.	1010 Complexity
1008 1009 Global maximizer	1010 Ex-ante optimal strategy		1011 $\Leftrightarrow$	1012 NP-hard
	(EDT, GDH)	1013 with absentmindedness	1014 $\Rightarrow$	
KKT point	(CDT, GT)		$\Leftrightarrow$	1015 CLS-hard

1013  
 1014 **Remark C.5.** *The three equilibrium notions for single-player IREFGs form an inclusion chain  
 1015 in general (Tewolde et al., 2023, Lemma 17): ex-ante optimal  $\Rightarrow$  (EDT, GDH)-equilibrium  $\Rightarrow$   
 1016 (CDT, GT)-equilibrium. In games without absentmindedness, this strengthens to an equivalence  
 1017 (Tewolde et al., 2023, Lemma 13): a strategy is (CDT, GT)-equilibrium if and only if it is (EDT, GDH)-  
 1018 equilibrium. Moreover, when the game has a single information set, the blockwise argmax condition  
 1019 of (Tewolde et al., 2023, Lemma 15) reduces to a global argmax, so every (EDT, GDH)-equilibrium is  
 1020 also ex-ante optimal. Since ex-ante optimization is NP-hard, it follows that computing an (EDT, GDH)-  
 1021 equilibrium is already NP-hard in this one- infoset setting.*

## 1022 D THE MOMENT-SOS HIERARCHY

1023 In this section, we present a thorough derivation of the Moment-SOS hierarchy which we have  
 1024 specialized to single-player IREFGs.

1026 D.1 PROBLEM REFORMULATION  
10271028 Let  $m = \sum_{j=1}^{\ell} m^j$ . Denote the strategy variables collectively by  $\mu = (\mu_a^j) \in \mathbb{R}^m$ . The feasible set is  
1029

1030 
$$\mathcal{S} = \bigtimes_{j=1}^{\ell} \Delta(A_{I^j}) = \{\mu \in \mathbb{R}^m : \mu_a^j \geq 0 \forall j, a, \sum_{j=1}^{m^j} \mu_a^j = 1 \forall j\}. \quad (8)$$
  
1031  
1032

1033 Define constraint polynomials  
1034

1035 
$$g_a^j(\mu) = \mu_a^j, \quad h^j(\mu) = \sum_{j=1}^{m^j} \mu_a^j - 1. \quad (9)$$
  
1036  
1037

1038 By construction,  
1039

1040 
$$\mathcal{S} = \{\mu \in \mathbb{R}^m : g_a^j(\mu) \geq 0, j \in [\ell], a \in [m^j], h^j(\mu) = 0, j \in [\ell]\}. \quad (10)$$
  
1041

1042 Accordingly, the quadratic module of  $\mathcal{S}$  is  
1043

1044 
$$Q(\mathcal{S}) = \left\{ \sigma_0(\mu) + \sum_{j=1}^{\ell} \sum_{a=1}^{m^j} \sigma_a^j(\mu) g_a^j(\mu) + \sum_{j=1}^{\ell} p^j(\mu) h^j(\mu) : \sigma_0, \sigma_a^j \in \Sigma[\mu], p^j \in \mathbb{R}[\mu] \right\}. \quad (11)$$
  
1045  
1046

1047 An important property in our formulation is the Archimedean property, defined below:  
10481049 **Definition D.1** (Archimedean property). *The quadratic module  $Q(\mathcal{X})$  is Archimedean if there exists  
1050  $N > 0$  such that*

1051 
$$N - \|x\|^2 \in Q(\mathcal{X}).$$
  
1052

1053 This property guarantees that  $\mathcal{X}$  is compact.  
10541055 For any  $\mu \in \mathcal{S}$  one has  $0 \leq \mu_a^j \leq 1$ , hence  $\|\mu\|^2 = \sum_{j,a} (\mu_a^j)^2 \leq \sum_{j,a} \mu_a^j = \sum_{j=1}^{\ell} \sum_{a=1}^{m^j} \mu_a^j = \ell$ .  
1056 Therefore the quadratic polynomial  $g_0(\mu) := \ell - \|\mu\|^2$  is nonnegative on  $\mathcal{S}$ . Adding the redundant  
1057 inequality  $g_0(\mu) \geq 0$  to the description of  $\mathcal{S}$  gives  $\mathcal{S} = \{\mu : g_a^j(\mu) \geq 0, h^j(\mu) = 0, g_0(\mu) \geq 0\}$ , so  
1058 that  $N - \|\mu\|^2 \in Q(\mathcal{S})$  with  $N = \ell$ . Hence, the quadratic module  $Q(\mathcal{S})$  is Archimedean.1059 From Equation (1), the expected payoff of a single-player IREGF can be written as a polynomial  
1060  $u(\mu)$ . We seek  
1061

1062 
$$u^* = \max_{\mu \in \mathcal{S}} u(\mu). \quad (12)$$
  
1063

1064 D.2 MOMENT-SOS RELAXATION  
10651066 To approximate the nonconvex problem in Equation (12), we use Lasserre's Moment-SOS hierarchy.  
1067 We derive a pair of dual hierarchies—one in the space of SOS multipliers (primal) and one in the  
1068 space of moments (dual)—which yield provable upper bounds on  $u^*$ .1069 **(a) Primal (SOS-relaxation)**  
1070

1071 Observe that

1072 
$$u^* = \inf \{t : t - u(\mu) \geq 0 \quad \forall \mu \in \mathcal{S}\} = \inf \{t : t - u(\mu) > 0 \quad \forall \mu \in \mathcal{S}\}. \quad (13)$$
  
1073

1074 Since  $\mathcal{S} = \{\mu : g_a^j(\mu) \geq 0, h^j(\mu) = 0\}$  has Archimedean quadratic module  $Q(\mathcal{S})$ , Putinar's  
1075 Positivstellensatz (Putinar, 1993) gives the equivalent infinite-dimensional certificate  
1076

1077 
$$u^* = \inf \{t : t - u(\mu) \in Q(\mathcal{S})\}. \quad (14)$$
  
1078

1079 However, membership in  $Q(\mathcal{S})$  is still an infinite-dimensional constraint.

1080 **Truncation to finite SDP.** Let  $d_u := \lceil \deg(u)/2 \rceil$  and  $d_S :=$   
 1081  $\max_{j,a} \{\lceil \deg(g_a^j)/2 \rceil, \lceil \deg(h^j)/2 \rceil\} = 1$ , then  $d_0 := \max\{d_u, d_S\} = d_u$ . Fix any relax-  
 1082 ation order  $d \geq d_0$ . We truncate the sums-of-squares and polynomial multipliers to degree  $2d$ ,  
 1083 obtaining:  
 1084

$$1085 \quad u_d^{\text{sos}} = \inf_{t, \sigma_0, \{\sigma_a^j\}, \{p^j\}} t \\ 1086 \quad \text{s.t. } t - u(\mu) \in Q_d(\mathcal{S}). \quad (15)$$

1089 **(b) Dual (Moment-relaxation)**  
 1090

1091 We begin by expressing the original problem (12) in the space of Borel measures  $\phi$  supported on  
 1092  $\mathcal{S}$ . Since any admissible  $\phi$  must satisfy  $\phi \geq 0$  and  $\int d\phi = 1$ , one has the exact infinite-dimensional  
 1093 program

$$1094 \quad u^* = \sup_{\substack{\phi \in \mathcal{M}_+(\mathcal{S}) \\ \int d\phi = 1}} \int u(\mu) d\phi(\mu). \quad (16)$$

1097 A measure  $\phi$  is equivalently described by its full sequence of moments  
 1098

$$1099 \quad y_\alpha = \int \mu^\alpha d\phi(\mu), \quad \forall \alpha \in \mathbb{R}^m. \quad (17)$$

1102 Introduce the Riesz functional  $L_y : \mathbb{R}[\mu] \rightarrow \mathbb{R}$  by  $L_y(\mu^\alpha) = y_\alpha$ ,  $\forall \alpha$ . Expand  $u(\mu) = \sum_\alpha u_\alpha \mu^\alpha$ ,  
 1103 so that

$$1104 \quad \int u(\mu) d\phi(\mu) = \sum_\alpha u_\alpha \int \mu^\alpha d\phi = \sum_\alpha u_\alpha y_\alpha =: L_y(u). \quad (18)$$

1107 Requiring  $\phi \geq 0$  and  $\text{supp}(\phi) \subseteq \mathcal{S}$  is equivalent to the following linear matrix constraints on  $y$ :  
 1108

$$1109 \quad M_d(y) \succeq 0, \quad \forall d \in \mathbb{N} \iff \int v(\mu)^2 d\phi \geq 0, \quad \forall v \in \mathbb{R}[\mu], \quad (19)$$

$$1112 \quad M_d(g_a^j \star y) \succeq 0, \quad \forall d \in \mathbb{N} \iff \int g_a^j(\mu) v(\mu)^2 d\phi \geq 0, \quad \forall v \in \mathbb{R}[\mu], \quad (20)$$

$$1114 \quad L_y(h^j q) = 0, \quad \forall q \in \mathbb{R}[\mu], \iff \int h^j(\mu) q(\mu) d\phi = 0, \quad \forall q \in \mathbb{R}[\mu], \quad (21)$$

$$1116 \quad y_0 = 1 \iff \int 1 d\phi = 1. \quad (22)$$

1118 Thus Equation (16) can be rewritten as the (infinite-dimensional) moment program  
 1119

$$1120 \quad u^* = \sup_y L_y(u) \\ 1121 \quad \text{s.t. } (19), (20), (21), (22). \quad (23)$$

1124 **Truncation to finite SDP.** Fix any relaxation order  $d \geq d_0$ . In practice, we truncate Equation (23)  
 1125 to the degree- $d$  moment relaxation:  
 1126

$$1128 \quad u_d^{\text{mom}} = \sup_y L_y(u) \\ 1129 \quad \text{s.t. } M_d(y) \succeq 0, \\ 1130 \quad M_{d-1}(g_a^j \star y) \succeq 0, \quad \forall j, a, \\ 1131 \quad L_y(h^j q) = 0, \quad \forall j, \forall q \in \mathbb{R}[\mu], \deg(h^j q) \leq 2d, \\ 1132 \quad y_0 = 1. \quad (24)$$

1134 D.3 MOMENT-SOS HIERARCHY FOR NON-ABSENTMINDED GAMES  
11351136 Recall that because  $u$  is multi-affine, we have the following:  
1137

1138 
$$u^* = \max_{\mu \in \mathcal{S}} u(\mu) = \max_{\mu \in \mathcal{S}_{\text{vr}}} u(\mu). \quad (25)$$
  
1139

1140 Let  $d_0^{\text{vr}} := \max\{d_u, d_{\mathcal{S}_{\text{vr}}}\} = d_u$ , and fix any relaxation order  $d \geq d_0^{\text{vr}}$ . The degree- $d$  SOS relaxation  
1141 of Equation (25) is  
1142

1143 
$$\begin{aligned} u_d^{\text{sos,vr}} &= \inf t \\ 1144 \text{s.t. } t - u(\mu) &\in Q_d(\mathcal{S}_{\text{vr}}). \end{aligned} \quad (26)$$
  
1145

1146 The corresponding degree- $d$  moment relaxation reads  
1147

1148 
$$\begin{aligned} u_d^{\text{mom,vr}} &= \sup_y L_y(u) \\ 1149 \text{s.t. } M_d(y) &\succeq 0, \\ 1150 L_y(h^j q) &= 0 \quad \forall j, \forall q \in \mathbb{R}[\mu], \deg(h^j q) \leq 2d, \\ 1151 L_y(b_a^j q) &= 0 \quad \forall j, a, \forall q \in \mathbb{R}[\mu], \deg(b_a^j q) \leq 2d, \\ 1152 y_0 &= 1. \end{aligned} \quad (27)$$
  
1153

1154 Note that there are no localizing PSD constraints for  $\mu_a^j \geq 0$ , because the binomials  $b_a^j = 0$  already  
1155 enforce  $\mu_a^j \in \{0, 1\} \subset [0, 1]$ .  
11561157 D.4 PSEUDO-EXPECTATIONS AND EXTRACTING SOLUTIONS  
11581159 A feasible point  $y = (y_\alpha)_{|\alpha| \leq 2d}$  of the truncated moment SDP in Equation (24) defines a linear  
1160 functional  $\tilde{E}_d : \mathbb{R}[\mu]_{2d} \rightarrow \mathbb{R}$  via  $\tilde{E}_d[\mu^\alpha] = y_\alpha$ . This functional behaves like an expectation operator  
1161 up to degree  $2d$ : it is normalized ( $\tilde{E}_d[1] = 1$ ), positive on squares ( $\tilde{E}_d[q^2] \geq 0$  for all  $q$  with  
1162  $\deg q \leq d$ ), and it enforces feasibility through the linear identities induced by the constraints. Such a  
1163 functional is often called a degree- $2d$  pseudo-expectation.  
11641165 Expanding  $u(\mu) = \sum_\alpha u_\alpha \mu^\alpha$ , the moment objective is  $L_y(u) = \sum_\alpha u_\alpha y_\alpha = \tilde{E}_d[u(\mu)]$ . Hence the  
1166 truncated moment relaxation in Equation (24) can be viewed as:  
1167

1168 
$$f_d^{\text{mom}} = \sup \left\{ \tilde{E}_d[u(\mu)] : \tilde{E}_d \text{ is a degree-}2d \text{ pseudo-expectation consistent with } (g \geq 0, h=0) \right\}.$$
  
1169

1170 In words, the primal SDP maximizes the pseudo-expected ex-ante payoff over all degree- $2d$  “virtual  
1171 laws” that satisfy the polynomial feasibility conditions up to degree  $2d$ .  
11721173 A feasible  $y$  in Equation (24) need not come from any genuine probability measure on  $\mathcal{S}$ , and it  
1174 generally encodes only a pseudo-expectation. A fundamental exception is the flat extension condition  
1175 (Curto & Fialkow, 1996): if for some  $s \leq d$ ,  
1176

1177 
$$\text{rank } M_s(y) = \text{rank } M_{s-1}(y), \quad (28)$$
  
1178

1179 then there exist atoms  $\mu^{(1)}, \dots, \mu^{(r)} \in \mathcal{S}$  with  $r = \text{rank}(M_s(y))$  and weights  $\lambda_k \geq 0$  with  $\sum_k \lambda_k =$   
1180 1 such that  
1181

1182 
$$\tilde{E}_s[p] = L_y(p) = \sum_{k=1}^r \lambda_k p(\mu^{(k)}) \quad \forall p \in \mathbb{R}[\mu]_{2s}.$$
  
1183

1184 Thus a flat pseudo-expectation is the true expectation with respect to a finitely atomic probability  
1185 measure supported on  $\mathcal{S}$ . Consequently, in the flat regime the SDP objective  $\tilde{E}_s[u] = L_y(u)$  equals  
1186 the true ex-ante payoff under optimal strategies, and the atoms  $\{\mu^{(k)}\}$  (optimal solutions) can be  
1187 extracted from  $M_s(y)$  by standard linear-algebraic procedures (e.g., multiplication matrices).  
1188

1188 D.4.1 EXTRACTION PROCEDURE  
1189

1190 A standard sufficient flatness test certifying exactness and enabling solution extraction is  
1191  $\text{rank } M_s(y^*) = \text{rank } M_{s-d_K}(y^*)$ , where  $d_K = d_S$  in Equation (24), and  $d_K = d_{S_{\text{vr}}}$  in Equation  
1192 (27). Note that  $d_S = d_{S_{\text{vr}}} = 1$ , then in both cases the flatness condition specializes to  
1193 Equation (28).

1194 Let  $y^*$  be optimal for the degree- $d$  moment SDP and assume Equation (28) holds at some order  
1195  $s \leq d$ . Set  $r := \text{rank } M_s(y^*) = \text{rank } M_{s-1}(y^*)$ . Then  $y^*$  admits an  $r$ -atomic representing measure  
1196  $\sum_{k=1}^r \lambda_k \delta_{\mu^{(k)}}$  supported on the feasible set, with  $\lambda_k > 0$  and  $\sum_k \lambda_k = 1$ . Since  $L_{y^*}(u) =$   
1197  $\sum_k \lambda_k u(\mu^{(k)})$  equals the global optimum, every atom satisfies  $u(\mu^{(k)}) = u^*$  and hence is a global  
1198 maximizer of  $u$ .

1199 We now recover  $\{\mu^{(k)}\}_{k=1}^r$  directly from the optimal moment matrix  $M_s(y^*)$  by the standard  
1200 multiplication-matrix routine. The extraction steps are based on Henrion & Lasserre (2005).  
1201

1202 Let  $v_s(\mu)$  be the vector of all monomials in  $\mu$  of total degree  $\leq s$ , with length  $N_s = \binom{m+s}{s}$ . Then  
1203 the order- $s$  moment matrix can be represented as:

$$1204 \quad 1205 \quad M_s(y^*) = \sum_{k=1}^{r^*} \lambda_k v_s(\mu^{(k)}) v_s(\mu^{(k)})^\top = V^* (V^*)^\top, \quad (29)$$

1206 where  $\lambda_k \geq 0$ ,  $\sum_{k=1}^r \lambda_k = 1$ , and  $V^* \in \mathbb{R}^{N_s \times r}$  collects the columns  $\sqrt{\lambda_k} v_s(\mu^{(k)})$ .  
1207

1208 For computation, we form a rank factor of  $M_s(y^*)$  by retaining the  $r$  positive modes (e.g., via  
1209 eigendecomposition or a Cholesky-type factorization):  
1210

$$1212 \quad M_s(y^*) = V V^\top, \quad V \in \mathbb{R}^{N_s \times r}. \quad (30)$$

1213 By construction,  $\text{span}(V) = \text{span}(V^*)$ . Hence the columns of  $V$  are linear combinations of  
1214  $\{\sqrt{\lambda_j} v_s(\mu^{(k)})\}_{k=1}^{r^*}$ .  
1215

1216 To obtain an explicit monomial basis of that subspace, we reduce  $V$  to column-echelon form by  
1217 Gaussian elimination with column pivoting, and rescale so that the pivot block is the identity. This  
1218 gives  $\widehat{V} = V T$ ,  $\widehat{V}_{B,:} = I_{r^*}$ , with  $T \in \mathbb{R}^{r \times r}$  invertible and  $B = \{\beta_1, \dots, \beta_r\}$  the indices of the pivot  
1219 rows (monomials). The pivot indices select a monomial “generating basis”:

$$1220 \quad w(\mu) := [\mu^{\beta_1} \dots \mu^{\beta_r}]^T.$$

1221 The same elimination step simultaneously produces linear reduction rules for all monomials of degree  
1222  $\leq s$ : each  $\mu^\alpha$  is written, on the support, as a linear combination of the generators  $\{\mu^{\beta_k}\}_{k=1}^r$ . Stacking  
1223 these relations row-wise yields the rewriting matrix  $\mathcal{R} \in \mathbb{R}^{N_s \times r}$  of the form (identity in the pivot  
1224 rows and coefficients elsewhere)

$$1226 \quad v_s(\mu) = \mathcal{R} w(\mu), \quad \mathcal{R}_{B,:} = I_r, \quad (31)$$

1227 which is exactly the coordinate change from the standard monomial vector  $v_s$  to the generating basis  
1228  $w$  on the atoms  $\{\mu^{(k)}\}_{k=1}^r$ .  
1229

1230 In the basis  $w$ , multiplication by each coordinate  $\mu_\eta$  acts linearly. For  $\eta = 1, \dots, m$  we build the  
1231 multiplication matrices  $N_\eta \in \mathbb{R}^{r \times r}$  defined by  
1232

$$1233 \quad N_\eta w(\mu) = \mu_\eta w(\mu). \quad (32)$$

1234 Concretely, for the  $k$ th basis monomial  $\mu^{\beta_k}$ , form  $\gamma = \beta_k + e_\eta$ ; if  $\gamma \in B$  set  $N_\eta(:, k) = e_{\text{row}(\gamma)}$ ,  
1235 otherwise take  $N_\eta(:, k) = \mathcal{R}_{\gamma,:}^\top$  from Equation (31).  
1236

1237 The atoms appear as common eigenpairs of  $\{N_\eta\}$ . Indeed, with  $e_k := w(\mu^{(k)})$  one has  $N_\eta e_k =$   
1238  $\mu_\eta^{(k)} e_k$  for all  $\eta$ . For robust computation we form a random convex combination  
1239

$$1240 \quad 1241 \quad N = \sum_{\eta=1}^m \lambda_\eta N_\eta, \quad \lambda_\eta \geq 0, \quad \sum_\eta \lambda_\eta = 1,$$

1242

**Algorithm D.1:** Extraction for single-player IREFGs

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**Input:** Optimal solution  $y^*$ ; order  $s$  with rank  $M_s(y^*) = \text{rank } M_{s-1}(y^*) = r$ .

1244

**Output:**  $r$  optimal strategies  $\{\mu^{(k)}\}_{k=1}^r$ .

1245

1. Factor  $M_s = VV^\top$  as in Equation (30).

1246

2. Build  $\mathcal{R}$  so  $v_s = \mathcal{R}w$  as in Equation (31).

1247

3. For each  $\mu_\eta$ , build  $N_\eta$  so  $N_\eta w(\mu) = \mu_\eta w(\mu)$  as in Equation (32).

1248

4. Form  $N = \sum_\ell \lambda_\ell N_\ell$  with random  $\lambda$ ; compute  $N = QTQ^\top$  as in Equation (33).

1249

5. For each  $k$ , set  $\mu_\eta^{(k)} = q_k^\top N_\eta q_k$  for all  $\eta$  as in Equation (34).

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which generically has simple spectrum and shares the same eigenvectors. The ordered real Schur decomposition

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$$N = QTQ^\top, \quad Q = [q_1 \cdots q_r] \quad (33)$$

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returns orthonormal vectors  $q_k$  spanning the eigenvectors  $w(\mu^{(k)})$ . The coordinates of each atom are then read by Rayleigh quotients:

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1259

$$\mu_\eta^{(k)} = q_k^\top N_\eta q_k, \quad \eta = 1, \dots, m, \quad k = 1, \dots, r. \quad (34)$$

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The above extraction procedure is summarized in Algorithm D.1.

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Since both  $\mathcal{S}$  and  $\mathcal{S}_{\text{vr}}$  are nonempty, both Equation (24) and Equation (27) are always feasible (take, e.g., the Dirac measure at any  $\mu \in \mathcal{S}$  or  $\mathcal{S}_{\text{vr}}$ ). We can thus solve them using the standard procedure:

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**Moment-SOS loop.** Recall  $d_0 = d_0^{\text{vr}} = d_u$ . Initialize  $d := d_0$  and do:

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1. Solve the order- $d$  moment SDP; obtain optimal  $y_d$  and upper bound  $u_d^{\text{mom}} = L_{y_d}(u)$ .
2. For  $s = d_0, \dots, d$ , test the rank condition in Equation (28). If it holds for some  $s$ , *terminate*: the relaxation is exact and one can extract the global maximizers from  $M_s(y_d)$  using Algorithm D.1.
3. Otherwise, increase the relaxation order:  $d \leftarrow d + 1$  and go back to Step 1.

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For single-player NAM-IREFGs, Lemma ?? shows that  $\text{rank } M_s(y) = \text{rank } M_\ell(y) \forall s > \ell$ , and Statement (iii) of Theorem 4.1 gives exactness at degree  $\ell+1$ :  $u_{\ell+1}^{\text{mom,vr}} = u_{\ell+1}^{\text{sos,vr}} = u^*$ . Hence the flatness test in Step 2 necessarily succeeds at  $s = d = \ell+1$  (indeed for all  $s > \ell$ ). The loop is therefore guaranteed to terminate at this fixed order, determined solely by the number of infosets, and the extraction returns at least one optimal pure strategy.

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For single-player IREFGs with absentmindedness, the same loop produces a monotone sequence of upper bounds  $u_d^{\text{mom}} \downarrow u^*$  and terminates as soon as flatness is detected at some order  $s$ , which certifies exactness and enables extraction. In the absence of flatness, one increases  $d$  to tighten the bound, with asymptotic convergence to  $u^*$  guaranteed. If  $u$  is generic, Statement (ii) of Theorem 4.1 ensures finite termination of the loop, with extraction of at least one certified global maximizer for the KKT-based problem.

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**E THE SELECT-VERIFY-CUT PROCEDURE**

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Recall the joint KKT system

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$$\begin{cases} w_i^j(\mu) - \nu_i^j(\mu)\mathbf{1} - \lambda_i^j(\mu) = 0, & \lambda_i^j(\mu) \geq 0, \\ g_i^j(\mu) \geq 0, & h_i^j(\mu) = 0, \quad g_{i,a}^j(\mu) \lambda_{i,a}^j(\mu) = 0, \end{cases} \quad \forall i \in \llbracket n \rrbracket, j \in \llbracket \ell_i \rrbracket, a \in \llbracket m_i^j \rrbracket. \quad (35)$$

The following exposition is based on the method introduced in Nie &amp; Tang (2024).

1296 (i) **Select.** Let  $n_0 := \sum_{i=1}^n \sum_{j=1}^{\ell_i} m_i^j$  be the total dimension, set  $[\mu]_1 = (1, \mu^\top)^\top$ , and choose a  
 1297 generic positive definite matrix  $\Theta \in \mathbb{R}^{(n_0+1) \times (n_0+1)}$ . Then all NEs are feasible points of  
 1298

1299 
$$\min_{\mu} \varphi_{\Theta}(\mu) := [\mu]_1^\top \Theta [\mu]_1 \quad \text{s.t.} \quad \begin{cases} \mu \text{ satisfies (KKT),} \\ u_i(\mu_i, \mu_{-i}) - u_i(v_i, \mu_{-i}) \geq 0, \quad \forall v_i \in K_i, \forall i, \end{cases} \quad (36)$$
  
 1300

1301 where  $K_i$  is the current (finite) set of deviation profiles used as cuts (initially  $K_i = \emptyset$ ). If Equation (36)  
 1302 is infeasible, there is no NE. If it is feasible, a minimizer exists because the feasible set is compact  
 1303 and  $\varphi_{\Theta}$  is continuous.  
 1304

1305 (ii) **Verify.** Let  $\hat{\mu} \in \mathbb{R}^{n_0}$  be an optimizer of Equation (36). For each player  $i$ , evaluate the best-  
 1306 response improvement against  $\hat{\mu}_{-i}$  by solving the KKT-restricted POP (same value as the unrestricted  
 1307 best-response since LICQ holds on products of simplices):  
 1308

1309 
$$\begin{aligned} \omega_i &:= \max_{\mu_i} u_i(\mu_i, \hat{\mu}_{-i}) - u_i(\hat{\mu}_i, \hat{\mu}_{-i}) \\ 1310 \text{s.t. } &w_i^j(\mu) - \nu_i^j(\mu) \mathbf{1} - \lambda_i^j(\mu) = 0, \quad \lambda_i^j(\mu) \geq 0, \quad \forall j, \\ 1312 &g_i^j(\mu) \geq 0, \quad h_i^j(\mu) = 0, \quad g_{i,a}^j(\mu) \lambda_{i,a}^j(\mu) = 0, \quad \forall j, a. \end{aligned} \quad (37)$$
  
 1313

1314 If every  $\omega_i \leq 0$ , no player can profitably deviate and  $\hat{\mu}$  is an NE.  
 1315

1316 (iii) **Cut.** If some  $\omega_i > 0$ , take one or more maximizers  $v_i \in \arg \max u_i(\mu_i, \hat{\mu}_{-i}) - u_i(\hat{\mu}_i, \hat{\mu}_{-i})$   
 1317 and add the valid NE cuts  
 1318

1319 
$$u_i(\mu_i, \mu_{-i}) - u_i(v_i, \mu_{-i}) \geq 0 \quad (v_i \in K_i \leftarrow K_i \cup \{v_i\}), \quad (38)$$

1320 which every NE satisfies but  $\hat{\mu}$  violates; then resolve Equation (36) with the enlarged cut set. Each  
 1321 violated cut eliminates the current candidate while preserving the entire NE set. Repeat (select-verify-  
 1322 cut) until an NE is certified or nonexistence is proved.  
 1323

## 1324 F OMITTED PROOFS FROM MAIN TEXT

### 1325 F.1 PROOFS FROM SECTION 3

1326 **Proposition 3.2.** *In non-absentminded IREGGs (NAM-IREGGs), each player's utility  $u_i(\mu)$  is multi-  
 1327 affine in the blocks  $\{\mu_i^j = (\mu_{i,a}^j)_{a=1}^{m_i^j}\}_{j=1}^{\ell_i}$ , i.e., for any player  $i$  and infoset  $j$ , the map  $\mu_i^j \mapsto u_i(\mu)$  is  
 1328 affine when all other blocks  $\{\mu_{i'}^{j'}\}_{(i',j') \neq (i,j)}$  are held fixed.*

1329 *Proof.* Fix a terminal history  $z \in Z$ . For each player  $i$ , let  $\mathcal{I}_i(z) \subseteq [\ell_i]$  be the set of (distinct) infosets  
 1330 of player  $i$  visited on the unique path to  $z$ . By non-absentmindedness, each  $I_i^j$  is visited at most  
 1331 once. Let  $a_i^j(z)$  be the action taken at  $I_i^j \in \mathcal{I}_i(z)$ , and let  $c(z)$  denote the product of chance move  
 1332 probabilities (independent of  $\mu$ ). The reach probability factorizes as  
 1333

1334 
$$\mathbb{P}(z \mid \mu) = c(z) \prod_{i'=1}^n \prod_{j \in \mathcal{I}_{i'}(z)} \mu_{i', a_{i'}^j(z)}^j.$$
  
 1335

1336 Hence player  $i$ 's expected payoff is  
 1337

1338 
$$u_i(\mu) = \sum_{z \in Z} \rho_i(z) \mathbb{P}(z \mid \mu) = \sum_{z \in Z} (\rho_i(z) c(z)) \prod_{i'=1}^n \prod_{j \in \mathcal{I}_{i'}(z)} \mu_{i', a_{i'}^j(z)}^j.$$
  
 1339

1340 In each summand, the dependence on the block  $\mu_i^j$  is either absent (if  $j \notin \mathcal{I}_i(z)$ ) or linear through  
 1341 a single coordinate  $\mu_{i, a_i^j(z)}^j$  (if  $j \in \mathcal{I}_i(z)$ ); by non-absentmindedness, no monomial contains two  
 1342 coordinates from the same block. Therefore, with all other blocks  $\{\mu_{i'}^{j'}\}_{(i',j') \neq (i,j)}$  held fixed, the  
 1343 map  $\mu_i^j \mapsto u_i(\mu)$  is affine on the simplex  $\Delta^{m_i^j}$ . Since this holds for every  $(i, j)$ ,  $u_i$  is multi-affine in  
 1344 the blocks  $\{\mu_i^j\}_{j=1}^{\ell_i}$ .  $\square$   
 1345

1350 F.2 PROOFS FROM SECTION 4  
1351

1352 **Theorem 4.1.** Consider a single-player IREG  $\mathcal{G}$  with utility function  $u$ . Let  $\ell$  be the number of in-  
1353 fosets,  $d_0 := \max_{j,a} \{\lceil \deg(u)/2 \rceil, \lceil \deg(g_a^j)/2 \rceil, \lceil \deg(h^j)/2 \rceil\}$ , and  $u^*$  be the ex-ante optimal value  
1354 of  $\mathcal{G}$ . Denote by  $u_d^{\text{sos}}$ ,  $u_d^{\text{sos,kkt}}$ ,  $u_d^{\text{sos,vr}}$  the values obtained from the SOS-Moment hierarchies applied  
1355 respectively to the vanilla product-of-simplices, KKT-based, and vertex-restricted formulations.  
1356 Similarly, we use the superscript mom to denote the moment hierarchy. Then we have the following:  
1357

1358 (i)  $\lim_{d \rightarrow \infty} u_d^{\text{sos}} = \lim_{d \rightarrow \infty} u_d^{\text{mom}} = u^*$ .  
1359 (ii) If  $u$  is generic, there exists  $d \geq d_0$  with  $u_d^{\text{mom,kkt}} = u_d^{\text{sos,kkt}} = u^*$ .  
1360 (iii) If  $\mathcal{G}$  is non-absentminded, the degree- $(\ell+1)$  moment relaxation of the vertex-restricted problem  
1361 is exact:  $u_{\ell+1}^{\text{mom,vr}} = u^*$ .  
1362

1363  
1364 *Proof.* To prove Statement (i), note that because  $Q(\mathcal{S})$  is Archimedean, the asymptotic convergence  
1365 follows from Putinar's Positivstellensatz and Lasserre's hierarchy (Putinar, 1993; Lasserre, 2001;  
1366 Laurent, 2009; Lasserre, 2024).  
1367

1368 To prove Statement (ii), recall from Section 3 that if  $u$  is generic (a property which holds for almost  
1369 all single-player IREGs), the KKT set is finite. Augmenting Equation (1) with the polynomial  
1370 KKT system does not change the set of maximizers, but restricts feasibility to a finite real variety.  
1371 For such finite varieties, the Lasserre hierarchy has finite convergence: for some  $d$  large enough,  
1372  $u_d^{\text{mom,kkt}} = u_d^{\text{sos,kkt}} = u^*$  and the flat extension (rank) condition holds, allowing recovery of  $\mu^*$ ; this  
1373 is immediate from e.g. Laurent (2008, Thm. 6.15) and Lasserre et al. (2008, Prop. 4.6).  
1374

1375 In order to prove Statement (iii), we first establish a key rank-stabilization lemma:  
1376

1377 **Lemma F.1.** For every feasible solution  $y$  of the degree- $s$  moment relaxation in Equation (27) with  
1378  $s > \ell$ , it holds that  $\text{rank } M_s(y) = \text{rank } M_\ell(y)$ .  
1379

1380 *Proof of Lemma F.1.* Let  $v_s(\mu)$  collect all monomials of total degree  $\leq s$  and recall  $M_s(y) =  
1381 L_y(v_s v_s^\top)$ . Index the columns of  $M_s(y)$  by monomials and write the block decomposition  
1382

$$1383 M_s(y) = \begin{bmatrix} M_\ell(y) & B \\ B^\top & C \end{bmatrix},$$

1384 where  $M_\ell(y)$  is indexed by monomials of degree  $\leq \ell$  and  $B$  by monomials of degree  $> \ell$ .  
1385

1386 Fix a column of  $B$  indexed by  $m(\mu) = \prod_{j=1}^\ell \prod_{a=1}^{m^j} (\mu_a^j)^{\alpha_a^j}$  with  $\sum_{j,a} \alpha_a^j = \deg m > \ell$ . Define the  
1387 clipped monomial  
1388 
$$1389 \widehat{m}(\mu) := \prod_{j=1}^\ell \prod_{a=1}^{m^j} (\mu_a^j)^{\min\{1, \alpha_a^j\}},$$

1390 so each exponent  $\geq 1$  is replaced by 1.  
1391

1392 By repeatedly using  $L_y((\mu_a^j)^2 q) = L_y(\mu_a^j q)$  (i.e.,  $b_a^j = 0$ ), for any row index monomial  $r$  (degree  
1393  $\leq s$ ) we obtain  
1394

$$L_y(m(\mu) r(\mu)) = L_y(\widehat{m}(\mu) r(\mu)).$$

1395 Hence the column of  $M_s(y)$  indexed by  $m$  coincides with the column indexed by  $\widehat{m}$ .  
1396

1397 If  $\widehat{m}$  uses at most one variable per block, then  $\deg \widehat{m} \leq \ell$  and the column indexed by  $m$  is *identical*  
1398 to a column of  $M_\ell(y)$ . If, instead,  $\widehat{m}$  contains two distinct variables from the same block (say  $\mu_a^j$  and  
1399  $\mu_{a'}^j$  with  $a \neq a'$ ), then  $\widehat{m}$  vanishes on the vertex set  $\mathcal{S}_{\text{NAM}}$  (one-hot per block), so for all admissible  
1400 rows  $r$ ,  $L_y(\widehat{m} r) = 0$ , and the entire column indexed by  $m$  is the zero vector.  
1401

1402 Consequently, every column of  $B$  is either zero or identical to a column of  $M_\ell(y)$ . Applying the  
1403 same argument to the lower block, with  $A := \begin{bmatrix} M_\ell(y) \\ B^\top \end{bmatrix}$  and  $D := \begin{bmatrix} B \\ C \end{bmatrix}$ , shows that every column of  
1404  $D$  is either zero or identical to a column of  $A$ . Therefore the column space of  $M_s(y)$  is contained  
1405

1404 in the column space of  $A$ , hence  $\text{rank } M_s(y) \leq \text{rank } A = \text{rank } M_\ell(y)$ . The reverse inequality is  
 1405 obvious because  $M_\ell(y)$  is a principal submatrix of  $M_s(y)$ . Hence  $\text{rank } M_s(y) = \text{rank } M_\ell(y)$  for all  
 1406  $s > \ell$ .  $\square$

1407 Let  $s := \ell+1$  and let  $y^*$  be an optimal solution of the order- $s$  moment relaxation Equation (27). By  
 1408 Lemma F.1,  $\text{rank } M_s(y^*) = \text{rank } M_\ell(y^*)$ , so the flatness condition holds at order  $s$ .

1409 Set  $r := \text{rank } M_s(y^*)$ . By (Curto & Fialkow, 2000, Theorem 1.6),  $y^*$  admits an  $r$ -atomic representing  
 1410 measure  $\sum_{k=1}^r \lambda_j \delta_{\mu^{(k)}}$  supported on the feasible set, with  $\lambda_k > 0$  and  $\sum_k \lambda_k = 1$ . Moreover,  
 1411 the equalities  $L_{y^*}(h^j q) = L_{y^*}(b_a^j q) = 0$  in Equation (27) enforce  $\text{supp}(\mu^*) \subseteq \mathcal{S}_{\text{vr}}$ . Therefore,  
 1412

$$1414 \quad L_{y^*}(u) = \sum_{k=1}^r \lambda_k u(\mu^{(k)}) \leq \max_{\mu \in \mathcal{S}_{\text{vr}}} u(\mu) = u^*. \\ 1415 \\ 1416$$

1417 Because Equation (27) is a relaxation of the original problem,  $u_s^{\text{mom,vr}} \geq u^*$  for all  $s$ . At  $s = \ell+1$ ,  
 1418 we have

$$1419 \quad u^* \leq u_{\ell+1}^{\text{mom,vr}} = L_{y^*}(u) \leq u^*.$$

1420 Hence  $u_{\ell+1}^{\text{mom,vr}} = u^*$  and the optimum is attained at  $y^*$ .  $\square$

### 1422 F.3 PROOFS FROM SECTION 5

1424 **Theorem 5.1.** *Let  $\mathcal{G}$  be a multi-player IREFG with utility functions  $u_i$  for each player  $i$ . Throughout,  
 1425 subproblems are solved by the KKT-based hierarchies of increasing order. Then, we have the  
 1426 following:*

- 1427 (i) *The SVC procedure is asymptotically exact: as the relaxation order and number of iterations  
 1428 grows, it returns a behavioral NE when one exists, and otherwise a certificate of nonexistence.*
- 1429 (ii) *If  $u_i$  are all generic, the KKT-based hierarchy has finite convergence for all SVC subproblems,  
 1430 and the SVC loop terminates in finitely many iterations.*
- 1431 (iii) *If  $\mathcal{G}$  is non-absentminded, the Verify/Cut phases in SVC are unnecessary: a single vertex-  
 1432 restricted Select (Equation (7)) suffices to compute an NE or certify nonexistence. Its Moment-  
 1433 SOS hierarchy is asymptotically exact; if  $u_i$  are generic, it attains exactness at a finite order.*

1434 *Proof.* (i) Let  $\ell_0 := \sum_{i=1}^n \ell_i$  be the total number of infosets. For any feasible  $\mu$  we have, for each  
 1435 infoset block,  $\sum_{a=1}^{m_i^j} \mu_{i,a}^j = 1$  and  $0 \leq \mu_{i,a}^j \leq 1$ . Hence

$$1436 \quad \|\mu\|^2 = \sum_{i=1}^n \sum_{j=1}^{\ell_i} \sum_{a=1}^{m_i^j} (\mu_{i,a}^j)^2 \leq \sum_{i=1}^n \sum_{j=1}^{\ell_i} \sum_{a=1}^{m_i^j} \mu_{i,a}^j = \sum_{i=1}^n \sum_{j=1}^{\ell_i} 1 = \ell_0,$$

1437 so  $g_0(\mu) := \ell_0 - \|\mu\|^2 \geq 0$  on  $\mathcal{S}$ . Adding  $g_0 \geq 0$  yields an Archimedean quadratic module, and  
 1438 the same holds for each verification feasible set. By standard results for Lasserre's hierarchy on  
 1439 Archimedean sets (see, e.g., Lasserre (2001); Laurent (2009)), every fixed selector (with a fixed cut  
 1440 set) and every verification problem is asymptotically exact: the moment optimal values converge to  
 1441 the true optima as  $d \rightarrow \infty$ , flat truncation recovers optimizers, and infeasibility is detected at high  
 1442 order.

1443 At loop  $t$ , solve one selector and up to  $n$  verifications. If the selector becomes infeasible at some order,  
 1444 nonexistence is certified and the procedure stops. Otherwise, let  $\hat{\mu}^{(t)}$  be a selector optimizer recovered  
 1445 once flatness occurs. If all verification values are  $\leq 0$ , then  $\hat{\mu}^{(t)}$  is an NE and we stop. If some player  
 1446 gains ( $> 0$ ), extract one or more violated valid inequalities  $u_i(\mu_i, \mu_{-i}) - u_i(v_i, \mu_{-i}) \geq 0$  from the  
 1447 deviator  $v_i$  and add them to the selector. As relaxation orders increase across loops, subproblem  
 1448 solutions approach their true optima; any limit point of flat selector solutions satisfies all accumulated  
 1449 valid inequalities, i.e., the Nash conditions. Hence the method converges asymptotically to an NE, or  
 1450 certifies nonexistence if the selector turns infeasible.

1451 (ii) Under generic utilities, which hold for almost all IREFGs, the joint KKT set is finite (cf. Nie &  
 1452 Tang (2024)). Then the selector's feasible set (joint KKT plus cuts) is finite. Each failed candidate is

1458 removed by the new valid inequalities without excluding any NE, so only finitely many candidates can  
 1459 be visited before selecting an NE or proving infeasibility. On finite feasible sets, the Moment-SOS  
 1460 hierarchy attains finite convergence and yields atomic solutions (see, e.g., Laurent (2008); Lasserre  
 1461 et al. (2008)). Therefore both the select-verify-cut loop and its SDP subproblems terminate in finitely  
 1462 many steps.

1463 (iii) In the NAM case,  $u_i(\cdot, \mu_{-i})$  is linear in each block  $\mu_i^j$  (Proposition 3.2), so “no profitable  
 1464 deviation by  $i$ ”  $\iff u_i(\mu_i, \mu_{-i}) \geq u_i(v_i, \mu_{-i}) \forall v_i \in S_{i,\text{rm}}$ . Thus feasibility of the single vertex-  
 1465 restricted selector Equation (7) is equivalent to the existence of a behavioral NE, and the verify/cut  
 1466 phases are unnecessary.

1467 To see asymptotic exactness, note that the feasible region of Equation (7) is contained in the product  
 1468 of simplices (per-player KKT equalities are imposed together with vertex deviation inequalities).  
 1469 Similar to (i), adding  $g_0(\mu) = \ell_0 - \|\mu\|^2 \geq 0$  makes the quadratic module Archimedean. By standard  
 1470 results for Lasserre’s hierarchy on Archimedean sets, the Moment-SOS relaxation of Equation (7) is  
 1471 asymptotically exact; flatness yields extraction, and infeasibility is detected at sufficiently high order.

1472 If, moreover, the utilities  $u_i$  are generic, the joint KKT variety over the product of simplices is finite.  
 1473 Since Equation (7) enforces these KKT equalities and further filters candidates by the vertex deviation  
 1474 inequalities, its feasible set is a finite real variety. On finite varieties, the Moment-SOS hierarchy  
 1475 attains exactness at some finite order, hence Equation (7) has finite convergence (returning an NE  
 1476 when feasible, and otherwise certifying nonexistence).  $\square$

1477

#### 1478 F.4 PROOFS FROM SECTION 6

1479

1480 **Single-player IREFGs.** First, we note that single-player IREFGs can be viewed as continuous  
 1481 identical-interest games (see e.g. Von Stengel & Koller (1997)), so the existence of ex-ante optima  
 1482 does not require concavity:  $\mathcal{S}$  is compact and  $u$  is continuous, hence  $\arg \max_{\mathcal{S}} u \neq \emptyset$ . We establish  
 1483 that in this setting, there is an equivalence between the definitions of concave and monotone games.

1484 **Proposition F.2.** *A single-player IREFG  $\mathcal{G}$  is monotone if and only if the expected utility  $u$  is concave  
 1485 on  $\mathcal{S}$ . Moreover,  $\mathcal{G}$  is strictly monotone if and only if  $u$  is strictly concave on  $\mathcal{S}$ .*

1486

1487 *Proof.* (Concavity  $\Rightarrow$  Monotonicity). Assume  $u$  is concave. The first-order concavity inequality  
 1488 gives, for all  $\mu, \nu$ ,

$$1489 \quad u(\mu) \leq u(\nu) + \nabla u(\nu)^\top (\mu - \nu), \quad u(\nu) \leq u(\mu) + \nabla u(\mu)^\top (\nu - \mu).$$

1490 Adding the two inequalities yields

$$1491 \quad (\nabla u(\mu) - \nabla u(\nu))^\top (\mu - \nu) \leq 0,$$

1492 which is exactly the definition of monotonicity. with  $v = \nabla u$ .

1493 (Monotonicity  $\Rightarrow$  Concavity). Assume  $\langle v(\mu) - v(\nu), \mu - \nu \rangle \leq 0, \forall \mu, \nu \in \mathcal{S}$  (i.e. the pseudogradient  
 1494 is monotone). Fix  $\mu, \nu \in \mathcal{S}$  and set  $\gamma(t) = \nu + t(\mu - \nu)$  for  $t \in [0, 1]$ . Define  $g(t) := u(\gamma(t))$ . Then  
 1495  $g'(t) = \nabla u(\gamma(t))^\top (\mu - \nu)$ . For  $0 \leq s < t \leq 1$ ,

$$1496 \quad g'(t) - g'(s) = (\nabla u(\gamma(t)) - \nabla u(\gamma(s)))^\top (\mu - \nu) = \langle v(\gamma(t)) - v(\gamma(s)), \gamma(t) - \gamma(s) \rangle \leq 0,$$

1497 so  $g'$  is nonincreasing on  $[0, 1]$ . Therefore

$$1498 \quad u(\mu) - u(\nu) = \int_0^1 g'(t) dt \leq \int_0^1 g'(0) dt = \nabla u(\nu)^\top (\mu - \nu),$$

1499 which is the first-order characterization of concavity, hence  $u$  is concave on  $\mathcal{S}$ .

1500 (Strict case). If equality in  $\langle v(\mu) - v(\nu), \mu - \nu \rangle \leq 0, \forall \mu, \nu \in \mathcal{S}$  holds only for  $\mu = \nu$ , then for any  
 1501  $\mu \neq \nu$  and any  $t \in (0, 1]$  we have

$$1502 \quad g'(t) - g'(0) = \langle v(\gamma(t)) - v(\gamma(0)), \gamma(t) - \gamma(0) \rangle < 0,$$

1503 so  $g'$  is strictly decreasing and  $u(\mu) - u(\nu) = \int_0^1 g'(t) dt < \nabla u(\nu)^\top (\mu - \nu)$ . This is the strict  
 1504 first-order concavity inequality, hence  $u$  is strictly concave. The converse (strict concavity  $\Rightarrow$  strict  
 1505 monotonicity for  $\mu \neq \nu$ ) follows by repeating the first part with inequalities strict.  $\square$

1512 **Theorem 6.3.** Consider a single-player IREFG  $\mathcal{G}$  with utility function  $u$ . Let  $d_0 :=$   
 1513  $\max_{j,a}\{\lceil \deg(u)/2 \rceil, \lceil \deg(g_a^j)/2 \rceil, \lceil \deg(h^j)/2 \rceil\}$ . Then, the following holds:

1514 (i) If  $\mathcal{G}$  is strictly **concave**/monotone, then the Moment-SOS hierarchy has finite convergence: there  
 1515 exists  $d \geq d_0$  such that  $u_d^{\text{sos}} = u_d^{\text{mom}} = u^*$ .

1516 (ii) If  $\mathcal{G}$  is **SOS-concave**/SOS-monotone, the degree- $d_0$  Moment-SOS relaxations are exact:  $u_{d_0}^{\text{mom}} =$   
 1517  $u_{d_0}^{\text{sos}} = u^*$ , i.e., the Moment-SOS hierarchy converges at the first level.

1518 *Proof.* (i) Let  $\mu^* \in \mathcal{S}$  be a global maximizer, so  $u^* = u(\mu^*)$ . Since  $\mathcal{S}$  is a nonempty polyhedron  
 1519 and by Proposition F.2  $u$  is (strictly) concave, there exist KKT multipliers  $\{\lambda_a^j\}_{j,a}$  with  $\lambda_a^j \geq 0$  and  
 1520  $\{\nu^j\}_j$  such that

$$1521 \nabla u(\mu^*) + \sum_{j,a} \lambda_a^j \nabla g_a^j(\mu^*) + \sum_i \nu^j \nabla h^j(\mu^*) = 0, \quad \lambda_a^j g_a^j(\mu^*) = 0.$$

1522 Let  $I_m$  be the  $m \times m$  identity. Since  $-\nabla^2 u \succ 0$  on  $\mathcal{S}$ , the (strictly positive) smallest eigenvalue of  
 1523  $-\nabla^2 u(\mu)$  is continuous in  $\mu$ , and the compactness of  $\mathcal{S}$  implies that there exists  $\delta > 0$  such that  
 1524  $-\nabla^2 u(\mu) \succeq \delta I_m$  for all  $\mu \in \mathcal{S}$ . Define the (convex) Lagrangian-type polynomial

$$1525 G(\mu) := u(\mu^*) - u(\mu) - \sum_{j,a} \lambda_a^j g_a^j(\mu) - \sum_j \nu^j h^j(\mu).$$

1526 Then  $G(\mu^*) = 0$ ,  $\nabla G(\mu^*) = 0$ . Define

$$1527 F(\mu - \mu^*) := \int_0^1 \left( \int_0^t \nabla^2 G(\mu^* + s(\mu - \mu^*)) ds \right) dt,$$

1528 so that the identity holds (Helton & Nie, 2010):

$$1529 G(\mu) = G(\mu^*) + \nabla G(\mu^*)(\mu - \mu^*) + (\mu - \mu^*)^\top F(\mu, \mu^*)(\mu - \mu^*) \\ 1530 = \langle \mu - \mu^*, F(\mu - \mu^*)(\mu - \mu^*) \rangle.$$

1531 Since  $\nabla^2 G(\mu) = -\nabla^2 u(\mu) \succeq \delta I_m$  on  $\mathcal{S}$ , for any  $\xi \in \mathbb{R}^n$  we have

$$1532 \xi^\top F(\mu, \mu^*) \xi \geq \delta \int_0^1 \int_0^t \xi^\top \xi ds dt = \frac{\delta}{2} \xi^\top \xi.$$

1533 Hence  $F(\mu, \mu^*) \succeq \frac{\delta}{2} I_n$  for all  $\mu \in \mathcal{S}$ . Since  $F(\mu, \mu^*)$  is a symmetric polynomial matrix that is  
 1534 positive definite on  $\mathcal{S}$ , the matrix polynomial version of Putinar's Positivstellensatz yields SOS-matrix  
 1535 polynomials  $F_0$ ,  $\{F_a^j\}$  and polynomial matrices  $\{H^j\}$  such that

$$1536 F(\mu, \mu^*) = F_0(\mu) + \sum_{j,a} F_a^j(\mu) g_a^j(\mu) + \sum_j H^j(\mu) h^j(\mu).$$

1537 Multiply it on both sides by  $(\mu - \mu^*)$  to obtain

$$1538 G(\mu) = \sigma_0(\mu) + \sum_{j,a} \sigma_a^j(\mu) g_a^j(\mu) + \sum_j p^j(\mu) h^j(\mu),$$

1539 where  $\sigma_0(\mu) := \langle \mu - \mu^*, F_0(\mu)(\mu - \mu^*) \rangle \in \Sigma[\mu]$ ,  $\sigma_a^j(\mu) := \langle \mu - \mu^*, F_a^j(\mu)(\mu - \mu^*) \rangle \in \Sigma[\mu]$ ,  
 1540  $p^j(\mu) := \langle \mu - \mu^*, H^j(\mu)(\mu - \mu^*) \rangle \in \mathbb{R}[\mu]$ . Recalling the definition of  $G$  and rearranging,

$$1541 u(\mu^*) - u(\mu) = \sigma_0(\mu) + \sum_{j,a} \underbrace{(\sigma_a^j(\mu) + \lambda_a^j)}_{\in \Sigma[\mu]} g_a^j(\mu) + \sum_j \underbrace{(p^j(\mu) + \nu^j)}_{\in \mathbb{R}[\mu]} h^j(\mu).$$

1542 Let  $d = \max\{\lceil \deg(\sigma_0)/2 \rceil, \lceil \deg(\sigma_a^j)/2 \rceil, \lceil \deg(p^j)/2 \rceil\} + 1$ . Then with  $u^* = u(\mu^*)$ , the tuple  
 1543  $(u^*, \sigma_0, \{\sigma_a^j + \lambda_a^j\}, \{p^j + \nu^j\})$  is feasible for the degree- $d_0$  SOS program in Equation (15), so  
 1544  $u_d^{\text{sos}} \leq u^*$ . By weak duality, we have  $u_d^{\text{sos}} \geq u_d^{\text{mom}} \geq u^*$ . Therefore,  $u_d^{\text{sos}} = u_d^{\text{mom}} = u^*$ .  
 1545 Conversely, choosing  $y$  as the Dirac moments of  $\delta_{\mu^*}$  in the moment SDP of Equation (24) gives a  
 1546 feasible point with value  $L_y(u) = u^*$ .

1547 (ii) Recall  $d_0 = \max\{d_u, d_S\} = d_u$ . Let  $\mu^* \in \arg \max_{\mu \in \mathcal{S}} u(\mu)$  and set  $u^* := u(\mu^*)$ . Because  $\mathcal{G}$   
 1548 is (SOS-)concave/monotone, Proposition F.2 implies that  $u$  is concave on  $\mathcal{S}$ . Since  $\mathcal{S}$  is a nonempty

polyhedron, the KKT conditions are necessary and sufficient for optimality. Hence, for any optimal solution  $\mu^*$  there exist Lagrange multipliers  $\{\lambda_a^j\}_{j,a}$  with  $\lambda_a^j \geq 0$  and  $\{\nu^j\}_j$  such that:

$$\begin{aligned} \nabla(u)(\mu^*) + \sum_{j,a} \lambda_a^j \nabla g_a^j(\mu^*) + \sum_j \nu^j \nabla h^j(\mu^*) &= 0, \\ \lambda_a^j g_a^j(\mu^*) &= 0, \quad g_a^j(\mu^*) \geq 0, \quad \lambda_a^j \geq 0, \quad h^j(\mu^*) = 0. \end{aligned}$$

Since  $-\nabla^2 u$  is SOS-matrix and  $\deg(-u) \leq 2d_0$ , by (Lasserre, 2024, Theorem 3.9):

$$(-u)(\mu) - (-u)(\mu^*) - \nabla(-u)(\mu^*)^T(\mu - \mu^*) = \sigma_0(\mu),$$

with  $\sigma_0 \in \Sigma[\mu]_{d_0}$ . Using stationarity and linearity of  $g_a^j, h^j$  (their gradients are constant), we obtain from the KKT condition:

$$\begin{aligned} \nabla(u)(\mu^*)^T(\mu - \mu^*) &= - \sum_{j,a} \lambda_a^j \nabla(g_a^j)^T(\mu - \mu^*) - \sum_j \nu^j \nabla(h^j)^T(\mu - \mu^*) \\ &= - \sum_{j,a} \lambda_a^j (g_a^j(\mu) - g_a^j(\mu^*)) - \sum_j \nu^j h^j(\mu), \end{aligned}$$

where we used  $h^j(\mu^*) = 0$ . Plugging this back, we have

$$\begin{aligned} u^* - u(\mu) &= \sigma_0(\mu) + \sum_{j,a} \lambda_a^j (g_a^j(\mu) - g_a^j(\mu^*)) + \sum_j \nu^j h^j(\mu) \\ &= \underbrace{\sigma_0(\mu)}_{\in \Sigma[\mu]} + \underbrace{\sum_{j,a} \lambda_a^j g_a^j(\mu)}_{\in \Sigma[\mu]} + \sum_j \nu^j h^j(\mu). \end{aligned}$$

since  $\sum_{j,a} \lambda_a^j g_a^j(\mu^*) = 0$ . Thus we have the original-domain SOS certificate

$$u^* - u(\mu) = \sigma_0(\mu) + \sum_{j,a} \sigma_a^j(\mu) g_a^j(\mu) + \sum_j p^j(\mu) h^j(\mu),$$

with  $\sigma_a^j(\mu) \equiv \lambda_a^j$  (nonnegative constants are SOS) and  $p^j(\mu) \equiv \nu^j$ . Degree bounds:  $\deg(\sigma_0) \leq 2d_0$ ,  $\deg(\sigma_a^j g_a^j) \leq 1$ ,  $\deg(p^j h^j) \leq 1$ . Thus  $(u^*, \sigma_0, \{\sigma_a^j\}, \{p^j\})$  is feasible for the SOS dual Equation (15) at order  $d_0$ , yielding  $u_{d_0}^{\text{sos}} \leq u^*$ . By weak duality, we have  $u_{d_0}^{\text{sos}} \geq u_{d_0}^{\text{mom}} \geq u^*$ . Therefore  $u_{d_0}^{\text{mom}} = u_{d_0}^{\text{sos}} = u^*$ . Conversely, choosing  $y$  as the Dirac moments of  $\delta_{\mu^*}$  in the moment SDP Equation (24) gives a feasible point with value  $L_y(u) = u^*$ .  $\square$

## G EMPIRICAL EXAMPLES

In this section, we show some illustrative examples for how our proposed methods can be used to compute ex-ante optima in single-player IREFGs. We remark that we use only standard scientific computing packages in Julia, alongside an off-the-shelf SumOfSquares package (Legat et al., 2017; Weisser et al., 2019). The code is run on a PC with an AMD Ryzen 5 5600 processor and 16 GB of RAM running a 64-bit version of Windows 11, and is provided in a supplementary file.

**Example B.4.** As a running example, we revisit Example B.4. Since the game is not multilinear (i.e. the player is absentminded), we use the standard Moment-SOS hierarchies. In particular, we run the SOS hierarchy in the Moment-SOS loop, testing the rank condition at each level until an atomic measure (i.e. a feasible maximizer) can be extracted. The program converges at truncation degree  $d = 4$ , returning optimal solution  $(x_{11}^*, x_{12}^*) = (1, 0)$  and  $(x_{21}^*, x_{22}^*) = (1, 0)$ . This gives objective value  $p(x^*) = 9$ . The compute time required to solve this example was 0.02 seconds.

**Randomly Generated NAM-IREFG.** We also create a procedure to randomly generate single-player IREFGs. Specifically, we seek to validate Statement (iii) of Theorem 4.1, that convergence occurs at a structure dependent level of the Moment-SOS hierarchy. For example, consider a (randomly generated) non-absentminded game  $\mathcal{G}_1$  with 3 infosets and two actions per infoset, resulting in variables  $x_1, x_2$  for  $I_1$ ,  $y_1, y_2$  for  $I_2$ , and  $z_1, z_2$  for  $I_3$ . The payoff function for  $\mathcal{G}_1$  is given by:

$$u_{\mathcal{G}_1}(x, y, z) = -4z_1 + x_2 y_2 + x_2 y_2 z_1 - 3x_2 y_1 z_2 - 3x_2 y_1 z_1. \quad (39)$$

1620 Due to Theorem 4.1, we expect convergence and extraction to be possible at level  $d = 4$  of the  
 1621 hierarchy, since there are 3 infosets. Moreover, since the game is a NAM-IREFG, we can further  
 1622 restrict the feasible region to vertex set  $\mathcal{S}_{\text{vr}}$ . Setting  $d = 4$  in the hierarchy, we find ex-ante optimal  
 1623 solution  $(x_1^*, x_2^*) = (0, 1)$ ,  $(y_1^*, y_2^*) = (0, 1)$ , and  $(z_1^*, z_2^*) = (0, 1)$  with optimal value 1. The  
 1624 compute time required to solve this game was 0.06 seconds.  
 1625

1626 **Randomly Generated Absentminded IREFG.** As another example, we show that in a randomly  
 1627 generated absentminded game, the hierarchies empirically converge at ‘reasonable’ levels. Consider  
 1628 game  $\mathcal{G}_2$  with two infosets  $I_1$  and  $I_2$ , where the player chooses between 3 actions in each infoset.  
 1629 This gives variables  $x_1, x_2, x_3$  in  $I_1$  and  $y_1, y_2, y_3$  in  $I_2$ . The payoff function for  $\mathcal{G}_2$  is given by

$$u_{\mathcal{G}_2}(x, y) = 4x_1x_3 + 2x_2x_3y_3 - 5x_1x_2y_3 + x_1x_2y_1 - 4x_2x_3y_2y_3. \quad (40)$$

1630  
 1631  
 1632 Running the hierarchies, we obtain convergence at level  $d = 6$ , with ex-ante optimal solution  
 1633  $(x_1^*, x_2^*, x_3^*) = (0.5, 0, 0.5)$  and  $(y_1^*, y_2^*, y_3^*) = (0.134, 0.594, 0.272)$ , giving optimal value 1. Notice  
 1634 that unlike the NAM case, the optimal solution is not a vertex. The total compute time required to  
 1635 solve this example was 0.41 seconds.  
 1636

1637 **SOS-Monotone Example.** We show experimental corroboration for Statement (ii) of Theorem 6.3.  
 1638 Using a technique established in Ahmadi et al. (2013), we construct a game  $\mathcal{G}_{\text{SOS}}$  with degree-4  
 1639 polynomial utility which is SOS-convex. The polynomial is given below:  
 1640

$$\begin{aligned} u_{\mathcal{G}_{\text{SOS}}}(x, y) = & 9.37y_2^4 + 9.37y_1^2y_2^2 + 9.37y_1^4 + 1.17x_2^2y_2^2 - 0.09x_2^2y_1y_2 + 0.94x_2^2y_1^2 \\ & + 9.37x_2^4 - 0.78x_1x_2y_2^2 - 0.52x_1x_2y_1y_2 + 0.55x_1x_2y_1^2 + 0.13x_1^2y_2^2 \\ & + 0.16x_1^2y_1y_2 + 0.13x_1^2y_1^2 + 9.37x_1^2x_2^2 + 9.37x_1^4 \end{aligned} \quad (41)$$

1641 Even though this polynomial is quartic, we need only run the SOS hierarchy at level  $d = 4$  to  
 1642 obtain the optimal value and extract a solution. We obtain the solution  $(x_1^*, x_2^*) = (0, 1)$  and  
 1643  $(y_1^*, y_2^*) = (0, 1)$ , with value 19.9. The compute time was < 0.001 seconds.  
 1644

1645 **Comparison With A Local Method.** To further illustrate the gap between local methods and SOS,  
 1646 we consider a small but nontrivial randomly generated absentminded game  $\mathcal{G}_3$  with two infosets  $I_1$   
 1647 and  $I_2$ , where  $I_1$  has 2 actions and  $I_3$  has 3 actions. This gives variables  $x_1, x_2$  in  $I_1$  and  $y_1, y_2, y_3$  in  
 1648  $I_2$ . The payoff function for  $\mathcal{G}_3$  is given by  
 1649

$$u_{\mathcal{G}_3}(x, y) = 6x_1^2y_3 + 8x_1x_2y_2 - 3y_3^2 + x_1^3 + 4x_2y_1y_2 - 4x_2^2y_1^2. \quad (42)$$

1650 Using our Moment-SOS implementation, the degree-4 relaxation is already flat and certifies the  
 1651 global optimum at  $(x_1^*, x_2^*) = (1, 0)$ ,  $(y_1^*, y_2^*, y_3^*) = (0, 0, 1)$ , with optimal value 4. The compute  
 1652 time was 0.03 seconds.  
 1653

1654 As a baseline, we apply projected gradient descent (PGD) directly to the same objective over the  
 1655 feasible region. We define the concatenated variable  $z = (x, y) \in \mathbb{R}^5$ , perform gradient steps  
 1656  $z_{t+1} = z_t + \eta \nabla f(z_t)$  with step size  $\eta = 0.02$ , and after each step project the  $x$ - and  $y$ -coordinates  
 1657 onto their respective simplices using the standard Euclidean simplex projection. We stop when  
 1658  $\|z_{t+1} - z_t\|_2 < 10^{-8}$  or after 5000 iterations. Running PGD from 100 random interior initializations,  
 1659 66/100 runs converge to the global optimum, while the remaining runs converge to a distinct  
 1660 stationary point with payoff  $\approx 2.15$ . Thus, first-order methods can get trapped at suboptimal KKT  
 1661 points with nontrivial probability, whereas SOS returns the global solution together with a certificate  
 1662 at a modest relaxation degree.  
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