

# 000 001 002 003 004 005 006 007 008 009 010 011 012 013 014 015 016 017 018 019 020 021 022 023 024 025 026 027 028 029 030 031 032 033 034 035 036 037 038 039 040 041 042 043 044 045 046 047 048 049 050 051 052 053 TOWARDS EFFICIENT CHAIN-OF-THOUGHT REASON- ING VIA ADAPTIVE-BUDGETING BASED POLICY OP- TIMIZATION

006  
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## 011 ABSTRACT

013 Recently, *Chain-of-Thought* (CoT) reasoning has become a key problem-solving  
014 capability for advanced *large language models* (LLMs) to address difficult tasks  
015 such as the mathematical ones. However, balancing the efficiency and perfor-  
016 mance of long CoTs still remains an intractable challenge. In this paper, we ob-  
017 serve that assigning adaptive token budgets for different examples during training  
018 is a viable way to tackle with the above issue. Motivated by this, we propose  
019 a novel reinforcement learning scheme, termed *Adaptive-Budgeting based Pol-*  
020 *icy Optimization* (ABPO). Based on the popular GRPO, our ABPO redefines the  
021 RL training as an adaptive curriculum learning process, where example pools are  
022 curated to categorize training examples into three types, namely the mastered,  
023 learning and hard ones, respectively. As the training progresses, ABPO will adap-  
024 tively schedule the examples with proper length budgets, and the example pools  
025 will also be dynamically updated based on the model status. In this way, we can  
026 assign adaptive token lengths for different examples during RL training, achieving  
027 a good balance between efficiency and performance of CoTs. To validate ABPO,  
028 we apply it to three representative LLMs, and conduct extensive experiments on  
029 a bunch of CoT reasoning benchmarks. The experimental results not only show  
030 the substantial efficiency improvements with minimal performance loss, *e.g.*, re-  
031 ducing token length by 78.3% while improving 2.0% performance of DeepSeek-  
032 R1-Distill-Qwen-1.5B on average, but also show our obvious advantages over  
033 the compared methods, *e.g.*, reducing 59.4% length and increasing 8.3% perfor-  
034 mance on average than HAPO, respectively. Our code is anonymously released at  
035 <https://anonymous.4open.science/r/AnonymizeABPO-5380/>

## 036 1 INTRODUCTION

037 Recently, *test time scaling* Muennighoff et al. (2025) has become the *de facto* paradigm of ad-  
038 vanced *large language models* (LLMs) Grattafiori et al. (2024); Yang et al. (2024); Bai et al. (2023);  
039 Achiam et al. (2023). Following this paradigm, advanced LLMs DeepSeek-AI (2025); Team (2025)  
040 are trained in reinforced or supervised manners to generate long *chain-of-thoughts* (CoTs), thereby  
041 handling difficult tasks, such as mathematical problem solving of America (2024); Gao et al. (2024)  
042 and logical reasoning Xie et al. (2025). Despite the great success, “*over-thinking*” Chen et al. (2025)  
043 becomes another notable issue for these LLMs. For instance, Qwen-QwQ Team (2025) will gen-  
044 erate CoTs of more than 2k tokens to answer a simple question, *e.g.*, “*The product of integers 240*  
045 *and k is a perfect cube. What is the smallest possible positive value of k?*” This case will become  
046 more prominent for smaller LLMs, such as DeepSeek-R1-Distill-Qwen-1.5B and DeepScaleR-1.5B-  
047 Preview, which strongly contradicts their goals about efficient applications.

048 To address this problem, recent efforts are devoted to the research of efficient CoT reason-  
049 ing Muennighoff et al. (2025); Aggarwal & Welleck (2025); Ma et al. (2025); Luo et al.  
050 (2025a). Among these progresses, one plausible strategy is to imply length-related rewards to  
051 the RL training of LLMs Aggarwal & Welleck (2025); Yeo et al. (2025); Arora & Zanette  
052 (2025); Luo et al. (2025a). In particular, LLMs are assigned by a fixed-length token budget  
053 during its training. And when the output CoT exceeds the token budget, a negative reward  
will be returned to guide the RL optimization Aggarwal & Welleck (2025); Yeo et al. (2025).

054 In this way, LLMs are forced to learn to solve  
 055 the tasks using limited-length CoTs.  
 056

057 However, this fixed-budget based solution still  
 058 encounters several limitations. Above all, the  
 059 example difficulties for LLMs vary greatly,  
 060 even for the mathematical tasks. And fixing  
 061 the token budget is hard to meet the trade-off  
 062 between efficiency and performance. To ex-  
 063 plain, as shown in Fig.1, when the token budget  
 064 is small, LLMs are prone to under-cutting the  
 065 hard examples while over-supplying the easy  
 066 ones. For instance, under a budget of  $2k$  tokens,  
 067 DeepSeek-R1-Distill-Qwen-1.5B can perform  
 068 well on the simple tasks like MATH500 but re-  
 069 ceives obvious performance drops on the more  
 070 difficult ones, *e.g.*, -25% on AIME of America  
 071 (2024). Moreover, simply increasing the to-  
 072 ken budget will makes the efficiency benefit be-  
 073 come limited, *e.g.*,  $4k$  tokens, and the tuning  
 074 cost also increases substantially. In this case,  
 075 we argue that allocating appropriate token bud-  
 076 getts to different examples based on the model  
 077 status is the key to addressing this shortcoming.  
 078

079 However, achieving the adaptive budgeting is  
 080 notoriously challenging in existing RL training scheme. Although the example difficulty for LLMs  
 081 can be measured via accuracy, entropy or prediction confidence Su & Cardie (2025); AlKhuzay  
 082 et al. (2024), the model’s assessment of difficulty is a dynamic process that changes as training  
 083 progresses, and how to assign proper token budgets is still intractable. Besides, the training scheduler  
 084 also requires in-depth exploration. Gradually increasing the token budget is a natural solution, *e.g.*,  
 085 *curriculum learning* Bengio et al. (2009), but it also causes the LLM to continuously traverse the  
 086 simple examples, resulting additional issues of training efficiency and expenditure, especially con-  
 087 sidering the expensive cost of the widely used GRPO training Shao et al. (2024). Therefore, how to  
 088 effectively and dynamically adjust proper token budgets still remains an open problem.  
 089

090 To address these issues, we propose a novel training scheme termed *Adaptive-Budgeting based Pol-  
 091 icy Optimization* (ABPO), which extends the popular *Group Relative Policy Optimization* (GRPO)  
 092 Shao et al. (2024) with adaptive and budget-aware rewards in a dynamic training process. Con-  
 093 cretely, ABPO will first let the LLM to quickly read training examples, and then categorize the  
 094 examples into three pools, namely the *mastered*, the *learning* and the *hard* ones, respectively. In  
 095 practice, ABPO starts from a low token budget, which is then gradually increased. During train-  
 096 ing, the hard examples are not directly used, but they will be adaptively updated to the *learning*  
 097 pool based on the model status. Meanwhile, we also introduce an efficient review mechanism for  
 098 the *mastered* examples, ensuring the knowledge retention and avoiding catastrophic. This dynamic  
 099 scheduling can help to assign model-aware token budgets for different examples and avoid the redun-  
 100 dant training of examples. In this case, ABPO can achieve a good trade-off among CoT efficiency  
 101 and performance as well as training expenditure.  
 102

103 To validate ABPO, we apply it to a set of advanced LLMs, including DeepSeek-R1-Distill-Qwen-  
 104 1.5B, 7B DeepSeek-AI (2025) and DeepScaleR-1.5B-Preview Luo et al. (2025b), on four represen-  
 105 tative reasoning benchmarks of varying difficulties, which are MATH500 Hendrycks et al. (2021),  
 106 AIME of America (2024), AMC AMC (2025) and OlympiadBench He et al. (2024). The experiment  
 107 results show that ABPO can significantly reduce the token cost of CoT reasoning while main-  
 108 taining competitive performance across various benchmarks. For instance, ABPO reduces the average  
 109 CoT length of DeepSeek-R1-Distill-Qwen-1.5B by 78.3% while improving 2.0% of performance  
 110 on average. Compared with prior CoT optimization methods, our ABPO can also obtain better CoT  
 111 efficiency while achieving overall performance improvement on all benchmarks, *e.g.*, 2,217 v.s.,  
 112 5,466 length and 60.2 v.s. 55.6 accuracy compared to HAPO Huang et al. (2025). These results well  
 113 confirm the effectiveness of our the motivation and designs of ABPO towards efficient CoT.  
 114

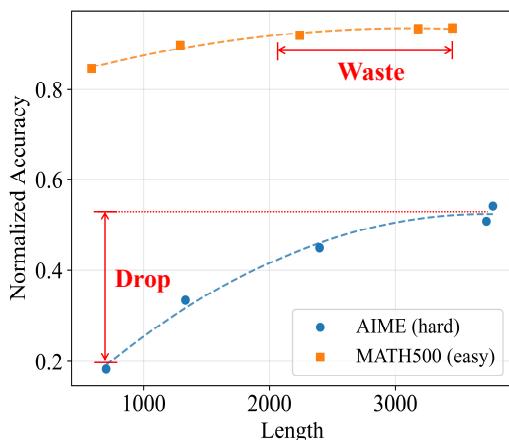


Figure 1: The performance-length curve of L1-Exact 1.5B Aggarwal & Welleck (2025) on two datasets. For easy questions, CoT will result in an obvious waste of computation. In contrast, for the hard ones, the limited token budgets of CoT will lead to obvious performance drop. Thus, assigning proper token budgets is the key to achieve the balance between efficiency and performance.

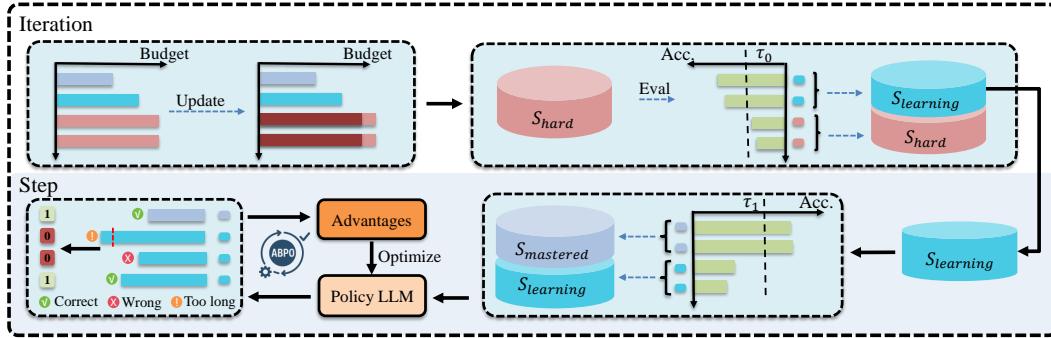


Figure 2: Illustration of *Adaptive Budgeting based Policy Optimization* (ABPO). During training, ABPO evaluates the difficulty of the training samples and groups them into three pools, *i.e.*, the mastered, learning and hard ones, respectively. As the training progresses, ABPO will adaptively allocate proper token budgets to the learning examples, and also actively review the mastered ones to avoid knowledge forgetting. In addition, these examples pools will be dynamically updated based on the LLM’s status during training, thereby balancing efficiency, performance and training costs.

Overall, our contributions are three-fold:

- We show the limitation of using fixed-budget CoT optimization schemes in terms of balancing CoT efficiency and performance, and also identify the key challenges of implementing adaptive budget rewards in existing RL training.
- We propose a novel training scheme termed *Adaptive-Budgeting based Policy Optimization* (ABPO), which can adaptively assign token budgets and schedule the RL training.
- On a set of LLMs and benchmarks, ABPO shows better capability of balancing CoT efficiency and performance than the compared methods.

## 2 RELATED WORK

*Chain-of-Thought* (CoT) refers to generating step-wise reasoning traces before answering the given question Wei et al. (2022), which has become a critical capability of advanced LLMs DeepSeek-AI (2025); Luo et al. (2025b); Team (2025); Team et al. (2025). However, these reasoning LLMs often produce excessively long CoT that commonly exhibit too many branching, verification, and backtracking. To address this issue, several works have recently been proposed for efficient CoT Aggarwal & Welleck (2025); Hou et al. (2025); Yeo et al. (2025); Hammoud et al. (2025). Specifically, one popular solution is using budget-based rewards in RL training Aggarwal & Welleck (2025); Yeo et al. (2025); Hou et al. (2025); Hammoud et al. (2025). For example, L1 Aggarwal & Welleck (2025) introduces a penalty term to enforce length budgets specified in the prompt. ThinkPrune Hou et al. (2025) introduces a uniform budget limit for all training samples. Curriculum GRPO Hammoud et al. (2025) gradually reduce the number of tokens a LLM is allowed to use, which however ignores the differences among questions. However, this static method often hard to meet the trade-off between efficiency and performance. In this case, recent works Luo et al. (2025a); Huang et al. (2025); Arora & Zanette (2025); Su & Cardie (2025); Yi et al. (2025) start exploring dynamic-length rewards for LLMs. O1-Pruner Luo et al. (2025a) and Training Efficiently Arora & Zanette (2025) apply a length penalty based on the within-group comparison. ShorterBetter Yi et al. (2025) ties its reward function to the length of the shortest and correct response for each sample. Although these methods incorporate dynamic reward signals, they still need to repeatedly train on the mastered samples, which significantly reduces training efficiency. In contrast, our approach considers the budget and training schedule for different samples, achieving a balance between performance and efficiency.

162 **3 METHOD**  
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164 **3.1 PRELIMINARY**  
 165

166 *Chain-of-thought* (CoT) Wei et al. (2022) is a key capability of advanced LLMs in terms of long and  
 167 complex task solving. The inference process of an LLM with CoT can be defined by

$$168 \quad \hat{y}_i = LLM(o_i, q_i), \quad \text{where } o_i = LLM(q_i). \quad (1)$$

169 Here  $q_i$  denotes the input question,  $o_i$  is the generated reasoning trace, *i.e.*, CoT, and  $\hat{y}_i$  is the  
 170 predicted answer. Under the popular *Group Relative Policy Optimization* (GPRO) Shao et al. (2024),  
 171 its objective can be defined by

$$172 \quad \mathcal{J}_{GPO}(\theta) = \mathbb{E}[q \sim D, \{o_i\}_{i=1}^G \sim \pi_{\theta_{old}}(O|q)] \\ 173 \quad \frac{1}{G} \sum_{i=1}^G \frac{1}{|o_i|} \sum_{t=1}^{|o_i|} \left[ \frac{\pi_{\theta}(o_{i,t}|q, o_{i,<t})}{\pi_{\theta_{old}}(o_{i,t}|q, o_{i,<t})} \hat{A}_{i,t} - \beta L_{KL}(\pi_{\theta}, \pi_{ref}) \right], \quad (2) \\ 174 \quad \text{where } \hat{A}_{i,t} = \frac{\mathbb{I}[\hat{y}_i = y_i] - \text{mean}(R)}{\text{std}(R)}.$$

175 Here,  $q$  denotes the sampled query from the dataset  $D$ ,  $o_i$  is the  $i$ -th generated output sampled from  
 176 the old policy  $\pi_{\theta_{old}}$ ,  $\pi_{\theta}$  is the updated policy for optimization, and  $\pi_{ref}$  is the reference model used  
 177 for KL-regularization.  $\hat{A}_{i,t}$  is the standardized reward advantage for token  $t$  with  $R$  denoting the set  
 178 of rewards.  $\beta$  controls the strength of the *KL-regularization*  $L_{KL}(\pi_{\theta}, \pi_{ref})$ . In terms of the default  
 179 GPRO, the reward function does not consider the length of CoT outputs, so the training mainly  
 180 focuses on the prediction accuracy, which makes the model prone to *over-thinking*. And this issue  
 181 will become more obvious in the smaller distilled LLMs Cuadron et al. (2025).

182 In this case, some recent works Hou et al. (2025); Aggarwal & Welleck (2025); Yeo et al. (2025)  
 183 implement length-related reward to RL training. When the output CoT exceeds the max number of  
 184 tokens, the LLM will receive a negative reward to penalize its optimization. As discussed above, the  
 185 length-aware reward can encourage the model to learn to answer the questions with limited length  
 186 of CoTs. However, the fixed budget commonly used Aggarwal & Welleck (2025); Hou et al. (2025)  
 187 will also make the model under-cutting the hard examples while over-supplying the easy ones. In  
 188 this case, we aim to achieve dynamic RL training with adaptive token budget scheduling.

189 **3.2 ADAPTIVE-BUDGETING BASED POLICY OPTIMIZATION**  
 190

191 In this paper, we propose a novel training scheme to balance the efficiency and performance of  
 192 LLM’s CoT reasoning, termed *Adaptive-Budgeting based Policy Optimization* (ABPO), of which  
 193 illustration is depicted in Fig. 2. Concretely, ABPO introduces adaptive token budgets to the training  
 194 of GPRO, and then the objective can be defined by

$$195 \quad \mathcal{J}_{ABPO}(\theta) = \mathbb{E}[q \sim D, \{o_i\}_{i=1}^G \sim \pi_{\theta_{old}}(O|q)] \\ 196 \quad \frac{1}{G} \sum_{i=1}^G \frac{1}{\min(|o_i|, b(q_i))} \sum_{t=1}^{\min(|o_i|, b(q_i))} \left[ \frac{\pi_{\theta}(o_{i,t}|q, o_{i,<t})}{\pi_{\theta_{old}}(o_{i,t}|q, o_{i,<t})} \hat{A}_{i,t} - \beta L_{KL}(\pi_{\theta}, \pi_{ref}) \right], \quad (3) \\ 197 \quad \text{where } \hat{A}_{i,t} = \frac{\mathbb{I}[\hat{y}_i = y_i \wedge |o_i| \leq b(q_i)] - \text{mean}(R)}{\text{std}(R)}.$$

198 Here  $b(q_i)$  is the dynamic token budget that LLM can use to answer the question  $q_i$ . And the budget-  
 199 aware function can be defined by

$$200 \quad r(o_i, \hat{y}_i) = \begin{cases} 1, & \text{if } \hat{y}_i = y_i \text{ and } |o_i| \leq b(q_i), \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

201 where  $o_i$  and  $\hat{y}_i$  are the reasoning trace and the predicted answer for the input question  $q_i$ . Notably,  
 202  $b(q_i)$  will be adaptively adjusted according to the difficulties of examples to the LLM.

203 Before training, ABPO first considers that all examples are unknown (hard) to the model, and eval-  
 204 uates their accuracies for the current model  $\pi_{\theta}$ . In practice, we will first generate  $n$  CoT traces  $O_i$   
 205 for each question  $q_i$ :

$$206 \quad O_i = \{(o_{i,j}, \hat{y}_{i,j}) \mid (o_{i,j}, \hat{y}_{i,j}) = \pi_{\theta,j}(q_i)\}, \quad (5)$$

216 where  $\pi_{\theta,j}$  represents the  $j$ -th sampling based on the policy model  $\pi_\theta$ . Notably, these traces can also  
 217 be used as  $\{o_i\}_{i=1}^G$  in Eq.3, avoiding the waste of computation. Then, the average accuracy of the  
 218 model  $\pi_{\theta_{old}}$  for the question  $q_i$  under the length budget  $b(q_i)$  can be obtained by  
 219

$$220 \quad 221 \quad 222 \quad 223 \quad 224 \quad 225 \quad 226 \quad 227 \quad 228 \quad 229 \quad 230 \quad 231 \quad 232 \quad 233 \quad 234 \quad 235 \quad 236 \quad 237 \quad 238 \quad 239 \quad 240 \quad 241 \quad 242 \quad 243 \quad 244 \quad 245 \quad 246 \quad 247 \quad 248 \quad 249 \quad 250 \quad 251 \quad 252 \quad 253 \quad 254 \quad 255 \quad 256 \quad 257 \quad 258 \quad 259 \quad 260 \quad 261 \quad 262 \quad 263 \quad 264 \quad 265 \quad 266 \quad 267 \quad 268 \quad 269$$

$$Acc(q_i) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}[\hat{y}_{ij} = y_i \wedge |o_{i,j}| \leq b(q_i)], \quad (6)$$

where  $\mathbb{I}[\cdot]$  is an indicator function. Initially, all samples are given the same budget  $b(q_i) = t_0$ . Via Eq. 6, we can categorize these examples into three pools, defined by

$$\mathcal{S}_{\text{mastered}} = \{q_i \mid Acc(q_i) \geq \tau_1\}, \quad (7)$$

$$\mathcal{S}_{\text{learning}} = \{q_i \mid \tau_1 > Acc(q_i) > \tau_0\}, \quad (8)$$

$$\mathcal{S}_{\text{hard}} = \{q_i \mid \tau_0 \geq Acc(q_i)\}. \quad (9)$$

Here,  $\tau_0$  and the  $\tau_1$  are the lower and upper thresholds, respectively. Thus, the mastered pool  $\mathcal{S}_{\text{mastered}}$  contains examples that have been solved under existing token budget  $b(q_i)$ . The learning pool  $\mathcal{S}_{\text{learning}}$  have samples with a certain probability of being learned. In contrast, the hard pool  $\mathcal{S}_{\text{hard}}$  contains difficult examples that are not suitable to learn by LLMs, and they will not be used.

During ABPO training, the examples used are mainly composed of the *learning* ones, but we will still review a certain number of the *mastered* examples to avoid knowledge forgetting, with a review rate denoted by  $\alpha$ . After optimizing the LLM with the current batch, we calculate their accuracies according to Eq.6. Based on the accuracies, we can update the examples pools  $\mathcal{S}_{\text{learning}}$  and  $\mathcal{S}_{\text{mastered}}$ . When the number of  $\mathcal{S}_{\text{learning}}$  is reduced to a certain proportion  $\lambda$ , we will increase the length budgets  $b(q_i)$  for the samples in  $\mathcal{S}_{\text{hard}}$ , and then start a new difficulty evaluation for these examples, which will accordingly update the pools of both hard and learning.

Specifically, the new budget for the questions  $q_i \in \mathcal{S}_{\text{hard}}$  can be calculated by

$$b(q_i) = t_0 + (k - 1) \cdot d, \quad \text{where } q_i \in \mathcal{S}_{\text{hard}}. \quad (10)$$

Here  $k$  is the number of the current iteration and  $d$  denotes the increased length budget. Then we evaluate the accuracy of samples from  $\mathcal{S}_{\text{hard}}$  under the new budget and move some examples into  $\mathcal{S}_{\text{learning}}$  according to Eq.8 and Eq.9. In this way, easy examples will be added to the learning and mastered pools, which cost only a small budget, and the more difficult examples will receive a larger budget. Due to page limit, the detailed algorithm of ABPO is given in Appendix A.2.

## 4 EXPERIMENTS

### 4.1 DATASET AND METRIC

The experiments are conducted on four widely-used benchmarks, including MATH500 Hendrycks et al. (2021), AIME of America (2024), AMC AMC (2025), and Olympiad-Bench He et al. (2024). Among these benchmarks, MATH500 is often regarded as the easiest one, which is a subset of MATH Hendrycks et al. (2021). AMC and AIME are moderately difficult, which has examples from mathematics competitions. In particular, AIME is regarded as the most difficult one, of which examples are from American Invitational Mathematics Examination. Following the previous works Arora & Zanette (2025), the accuracy of an example is defined by the average performance across  $K$  runs. During the evaluation, we set  $K = 3$  for MATH500,  $K = 5$  for Olympiad-Bench and  $K = 10$  for both AIME and AMC. Additionally, we use the *Accuracy–Efficiency Score (AES)* Luo et al. (2025a) to evaluate the trade-off between accuracy and computation overhead. A higher AES indicates a better balance between accuracy and efficiency.

### 4.2 IMPLEMENTATION DETAILS

We train ABPO on three base models, *i.e.*, DeepSeek-R1-Distill-Qwen-1.5B, DeepSeek-R1-Distill-Qwen-7B DeepSeek-AI (2025), and DeepScaleR-1.5B-Preview Luo et al. (2025b). The main experiments are conducted on  $2.4k$  training samples from the *AIME-AMC* subset of the Prime dataset Cui et al. (2025). To further validate the effectiveness of ABPO, we additionally train DeepSeek-R1-Distill-Qwen-1.5B on  $32k$  samples from the AM dataset Zhao et al. (2025). For training steps,

Table 1: Comparison with existing efficient CoT optimization methods on three LLMs for four mathematical benchmarks. “Acc.”, “Len.” and “AES” denote accuracy (higher is better), length (lower is better) and Accuracy–Efficiency Score (higher is better), respectively. The best and second best results are marked in **bold** and underline, respectively.

Train Data	MATH500			AMC			Olympiad			AIME			Average			
	Acc.	↑ Len.	↓ AES	↑ Acc.	↑ Len.	↓ AES	↑ Acc.	↑ Len.	↓ AES	↑ Acc.	↑ Len.	↓ AES	↑ Acc.	↑ Len.	↓ AES	
<i>Base model: DeepSeek-R1-Distill-Qwen-1.5B</i>																
Baseline	—	83.3	5278	—	70.3	9314	—	50.3	10820	—	32.0	15476	—	59.0	10222	—
ShorterBetter	40k	68.9	<b>864</b>	-0.03	48.3	<b>1194</b>	-0.69	36.2	<b>2072</b>	-0.60	19.7	<b>4286</b>	-1.20	43.2	<b>2104</b>	-0.54
Training Efficient	3.2k	<u>81.9</u>	2536	0.44	70.3	5183	0.44	49.2	6818	0.27	<u>29.0</u>	10552	-0.15	57.6	6272	0.27
HAPO	2k	81.5	2633	0.39	69.0	4507	0.43	49.2	5458	0.38	22.7	9268	-1.06	55.6	5466	0.18
<b>ABPO</b> (Ours)	2.4k	80.9	1647	0.54	71.3	2537	0.77	49.7	3311	0.63	<b>29.3</b>	6028	0.19	<b>57.8</b>	3371	<b>0.57</b>
<b>ABPO</b> (Ours)	32k	<b>85.2</b>	<u>1434</u>	<b>0.80</b>	<b>74.3</b>	<u>2068</u>	<b>0.95</b>	<b>52.7</b>	<u>2141</u>	<b>0.95</b>	28.7	<b>3223</b>	<b>0.28</b>	<b>60.2</b>	<u>2217</u>	<b>0.85</b>
<i>Base model: DeepSeek-R1-Distill-Qwen-7B</i>																
Baseline	—	93.2	4086	—	89	6484	—	66.1	8686	—	56.3	12837	—	71.9	8536	—
ShorterBetter	40k	85.6	<b>1233</b>	<u>0.29</u>	82.3	<b>2443</b>	0.24	56.7	<b>2990</b>	-0.06	<b>49.0</b>	<b>5673</b>	<b>-0.09</b>	64.5	3910	<u>0.03</u>
Training Efficient	3.2k	<b>89.1</b>	<u>2121</u>	0.26	<b>87.5</b>	4300	<u>0.25</u>	<u>61.9</u>	5626	0.04	<b>52.3</b>	9410	<b>-0.09</b>	65.0	5112	-0.08
<b>ABPO</b> (Ours)	2.4k	<b>92.2</b>	2243	<b>0.40</b>	<b>87.3</b>	<u>3511</u>	<b>0.36</b>	<b>63.5</b>	4852	<b>0.25</b>	48.7	8493	-0.34	<b>66.0</b>	4354	<b>0.08</b>
<i>Base model: DeepScaleR-1.5B-Preview</i>																
Baseline	—	88.2	3155	—	81.3	5067	—	58.3	5718	—	40.0	9041	—	66.9	5745	—
L1-Exact	40k	82.1	3189	-0.35	71.0	3067	-0.24	50.7	3196	-0.22	22.3	<b>3715</b>	-1.62	56.5	<b>3292</b>	-0.35
L1-Max	40k	84.5	<b>1527</b>	<b>0.31</b>	75.5	<b>1768</b>	<b>0.30</b>	51.1	<b>1733</b>	0.07	27.0	<b>2290</b>	-0.88	59.5	<b>1830</b>	<b>0.13</b>
HAPO	2k	81.9	2537	-0.16	70.5	4119	-0.47	53.6	3976	-0.10	31.3	6974	-0.86	59.3	4401	-0.33
ThinkPrune	2.4k	<u>85.5</u>	2029	0.21	<u>78.5</u>	3125	0.21	<b>55.0</b>	3532	<u>0.10</u>	<u>34.7</u>	5743	-0.30	<u>63.4</u>	3607	0.11
<b>ABPO</b> (Ours)	2.4k	<b>85.6</b>	1874	<u>0.26</u>	<b>79.8</b>	<u>2845</u>	<b>0.35</b>	<b>55.0</b>	<u>3302</u>	<b>0.14</b>	<b>37.3</b>	5736	<b>0.03</b>	<b>64.4</b>	3439	<b>0.21</b>

Table 2: Comparison between fixed-budget baselines and ABPO on DeepSeek-R1-Distill-Qwen-1.5B on AMC and AIME.

	AMC			AIME			Average			Training Time (h)
	Acc. $\uparrow$	Len. $\downarrow$	AES	Acc. $\uparrow$	Len. $\downarrow$	AES	Acc. $\uparrow$	Len. $\downarrow$	AES	
Base	70.3	9314	—	32.0	15476	—	51.1	12395	—	—
Fixed-Budget 2000	66.3	<b>2320</b>	0.47	24.0	<b>5576</b>	-0.61	45.1	<b>3948</b>	0.09	<b>15.5</b>
Fixed-Budget 3000	68.8	2850	0.59	23.3	6309	-0.76	46.0	4980	0.13	22.2
Fixed-Budget 4000	<b>71.8</b>	3600	0.68	<u>25.7</u>	6376	-0.40	<u>48.7</u>	4988	<u>0.36</u>	28.0
<b>ABPO</b> (Ours)	71.3	<u>2537</u>	<b>0.77</b>	<b>29.3</b>	<u>6028</u>	<b>0.19</b>	<b>50.3</b>	<u>4283</u>	<b>0.57</b>	<u>20.3</u>

DeepSeek-R1-Distill-Qwen-1.5B is trained for 350 steps under both the standard and extended-data settings, while DeepSeek-R1-Distill-Qwen-7B and DeepScaleR-1.5B-Preview are each trained for 50 steps. The compared methods are **ShorterBetter** Yi et al. (2025), **Training Efficient** Arora & Zanette (2025), **HAPO** Huang et al. (2025), **L1-Exact**, **L1-Max** Aggarwal & Welleck (2025) and **ThinkPrune** Hou et al. (2025). The detailed experimental settings are provided in Appendix A.3.

### 4.3 QUANTITATIVE ANALYSIS

**Comparison with existing methods** In Tab.1, we first compare the performance and efficiency of ABPO with existing CoT optimization methods on three representative LLMs. From this table, we can first observe that the default LLMs (*Baseline*) often requires excessive tokens to generate CoTs on all benchmarks, *e.g.*, 5,278 tokens on MATH500 by DeepSeek-R1-Distill-Qwen-1.5B, showing notable over-thinking issues. ABPO and the compared methods can well alleviate this problem. For instance, via taking the shortest reasoning length within a sampled group, ShorterBetter can reduce the CoT length of DeepSeek-R1-Distill-Qwen-1.5B to 864 tokens on MATH500. However, we can also see that the compared methods have obvious compromises in performance, *e.g.*, -38.4% and -29.1% by ShorterBetter and HAPO on the difficult AIME. In addition, it can be seen that the efficiency benefits of some methods become marginal on the difficult tasks in order to reduce performance loss. For instance, the SOTA method *Training Efficient* requires about 10k and 6.8k token budgets on AIME and Olympiad for DeepSeek-R1-Distill-Qwen-1.5B, respectively, resulting worse *Accuracy-Efficiency Score* (AES). In terms of fixed-budget methods, *e.g.*, ThinkPrune Hou et al. (2025), it can maintain the CoT length at a relatively short output, but it obviously drops the performance by 13.2% on the difficult AIME. In contrast, ABPO can achieve a good balance

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 325 Table 3: Ablation study of ABPO on DeepSeek-R1-Distill-Qwen-1.5B. We progressively add Cur-  
 326 riculum Learning, Adaptive Budget and Example Review to ABPO. \* denotes the full setting of  
 327 ABPO.

	AMC			AIME			Average		
	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$
Base	70.3	9314	–	32.0	15476	–	51.1	12395	–
+ Curriculum Learning	<u>68.0</u>	2652	0.56	24.7	<u>5250</u>	-0.48	46.3	<u>3951</u>	0.21
+ Adaptive Budget	<u>68.0</u>	<b>2232</b>	0.60	<u>26.3</u>	<b>5219</b>	-0.22	<u>47.2</u>	<b>3726</b>	0.31
+ Review*	<b>71.3</b>	<u>2537</u>	<b>0.77</b>	<b>29.3</b>	6028	<b>0.19</b>	<b>50.3</b>	4283	<b>0.57</b>

334 Table 4: Comparison between ABPO and the alternative data scheduling strategies on DeepSeek-  
 335 R1-Distill-Qwen-1.5B.  $Mix(\cdot)$  denotes the mixing of different types of examples for sampling.

	AMC			AIME			Average		
	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$	Acc. $\uparrow$	Len. $\downarrow$	AES $\uparrow$
Base	70.3	9314	–	32.0	15476	–	51.1	12395	–
$Mix(\mathcal{S}_{learning}, \mathcal{S}_{hard})$	62.8	<b>1942</b>	0.26	16.3	<b>3954</b>	-1.70	39.5	<b>2948</b>	-0.37
$Mix(\mathcal{S}_{learning}, \mathcal{S}_{mastered})$	<u>69.8</u>	2768	0.67	<u>28.7</u>	6180	0.08	49.2	4474	0.45
<b>ABPO (Ours)</b>	<b>71.3</b>	<u>2537</u>	<b>0.77</b>	<b>29.3</b>	6028	<b>0.19</b>	<b>50.3</b>	<u>4283</u>	<b>0.57</b>

343 between efficiency and performance. When increasing the amount of training data, *i.e.*, 32k, ABPO  
 344 can help DeepSeek-R1-Distill-Qwen-1.5B reduce about 78.3% CoT length while achieving 2.0%  
 345 performance gains on average. These results well confirm the effectiveness of our ABPO.

346 To better show our advantages, we compare ABPO with length-control approaches in Fig. 3 (a),  
 347 *i.e.*, L1-Exact and L1-Max Aggarwal & Welleck (2025). We plot the *length-accuracy* of the  
 348 DeepScaleR-1.5B-Preview under different budget settings. Specifically, L1-Max controls the maxi-  
 349 mum output length of the model through prompting, whereas L1-Exact enforces the model to gener-  
 350 ate outputs with an exact length budget specified in the prompt. We show the average performance  
 351 on the four widely used benchmarks, *i.e.*, AMC, MATH500, AIME and Olympiad. From this figure,  
 352 we can first observe that the original LLM takes a very large number of tokens to achieve better  
 353 performance. Specifically, the original LLM achieves only 3.9% performance improvement while  
 354 consuming nearly twice the token overhead of our ABPO. We can also observe that both L1-Exact  
 355 and L1-Max can reduce the length of CoTs. When extending the CoT lengths, the performance  
 356 of these two methods slightly increases but eventually reaches the ceiling. Compared with these  
 357 two methods, the proposed ABPO achieves better performance with the same budget. As shown in  
 358 the figure, ABPO obviously occupies a more upper left position in the length-performance curve.  
 359 Overall, these results show that the proposed ABPO method can achieve a better balance between  
 360 efficiency and effect and significantly improve the LLMs’ reasoning ability.

361 **Comparison with fixed-budget Baselines** In Tab. 2, we compare ABPO to the fixed-budget meth-  
 362 ods under the same experimental settings. For the fixed-budget baselines, we assign a fixed token  
 363 budget to all training samples. All samples are trained together, and this setting can serve as a di-  
 364 rect counterpart to our adaptive scheduling strategy. From Tab. 2, we can first observe that this  
 365 fixed-budget baselines will impose a negative impact on difficult examples. Specifically, training  
 366 with a budget of 3,000 tokens on AIME reduces the output length by 59.2% but leads to 27.2%  
 367 performance drop. In comparison, ABPO requires 8.6% less training time and reduces the output  
 368 length by 61.0%, while resulting in an 8.4% performance drop. This performance is higher than the  
 369 result under the fixed budget of 3,000. Another observation is that for medium-difficulty examples,  
 370 ABPO achieves a better balance between accuracy and length. Although its accuracy is close to  
 371 the baseline of a higher fixed budget, the output length is much shorter. For instance, on the AMC  
 372 dataset, ABPO’s accuracy is only 0.6% lower than that of Fixed-budget 4,000, but its length is re-  
 373 duced by 29.5%, while obtaining a 13.2% improvement in AES. Overall, these results demonstrate  
 374 that ABPO can provide reasonable CoT budgets for examples of varying difficulties.

375 **Ablation study** In Tab. 3, we ablate the key designs of ABPO on DeepSeek-R1-Distill-Qwen-  
 376 1.5B. In this table, “+ Curriculum Learning” denotes training with curriculum scheduling and train-  
 377 ing batches are constructed by the examples from  $\mathcal{S}_{learning}$  with CoT length limited by 4,000 tokens.  
 “+ Adaptive Budget” further introduces dynamic budget scheduling, where their initial budget is set

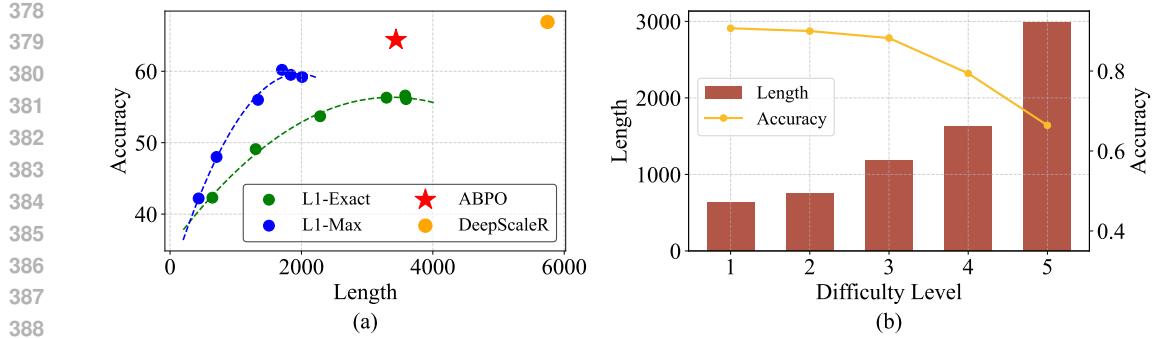


Figure 3: (a) Comparison between ABPO and efficient CoT optimization methods under different length budgets on DeepScaleR-1.5B-Preview. The accuracy is averaged over AMC, MATH500, AIME and Olympiad. (b) The average CoT length and accuracy of ABPO on examples of MATH500 at different difficulty levels. It can be seen that the proposed ABPO can generate CoT with different lengths according to the example difficulty.

to  $t_0=2,000$  and gradually expanded with  $d=500$  per iteration. Finally, “+ *Review*” corresponds to the full settings of ABPO, which additionally adds the review mechanism by mixing a fraction of the mastered samples. From Tab. 3, we can first observe that using curriculum learning can reduce the CoT length substantially. For instance, on AMC benchmark, “+ *Curriculum Learning*” reduces the reasoning trace by 71.5% with 3.3% performance drop. After adding adaptive budgeting, *i.e.*, “+ *Adaptive Budget*”, the output length is further shortened while improving accuracy. For instance, the length of CoT is shorten by 76.0%, while keeping the same performance on AIME benchmark. With “+ *Review*”, ABPO achieves the best trade-off, and it restores the accuracy close to the original LLMs while significantly reducing CoT token length. For instance, on AMC, the proposed ABPO method not only reduces the CoT length by 72.8%, but also improve the performance by 1.4%. These results well confirm the designs of the adaptive budget assignments and the example review.

In Tab. 4, we also examine the roles of three example pools in ABPO. We conduct two experiments with different sampling strategies while keeping all other configurations identical to ABPO. Here, “ $Mix(\mathcal{S}_{learning}, \mathcal{S}_{hard})$ ” combing the examples of  $\mathcal{S}_{learning}$  with  $\mathcal{S}_{hard}$  for sampling, *i.e.*, not considering the difficulty of examples for training. “ $Mix(\mathcal{S}_{learning}, \mathcal{S}_{mastered})$ ” denotes the mixing of  $\mathcal{S}_{learning}$  and  $\mathcal{S}_{mastered}$ , *i.e.*, all examples are either under-learning or too hard for training. In contrast, ABPO will assess examples via three types while reviewing partial examples from  $\mathcal{S}_{mastered}$ , as described in Sec. 3.2. From these results, we can observe that “ $Mix(\mathcal{S}_{learning}, \mathcal{S}_{hard})$ ” significantly reduces the CoT length of the LLM. For instance, it reduces CoT length by 74.4% on AIME. However, under this scheme, the longer CoTs for difficult examples always give the LLM a penalty signal, causing the LLM to mistakenly tend to output shorter CoT. On the other hand, we can observe that “ $Mix(\mathcal{S}_{learning}, \mathcal{S}_{mastered})$ ” significantly reduces the CoT length while obtaining better performance. For example, “ $Mix(\mathcal{S}_{learning}, \mathcal{S}_{mastered})$ ” reduces the inference length by 60.1% with 10.3% performance drop on AMC. To explain, after identifying the hard examples, this scheme can help the LLM achieve better optimizations during training. However, its performance is still limited due to the excessive use of easy examples, especially on the difficult ones. In contrast, via categorizing examples into three pools with a review design, ABPO can help the LLM better compensate the performance drops while keeping low CoT budgets. Overall, these ablation studies well confirm the designs of the example pools and the review operation in ABPO.

**Inference cost w.r.t. example difficulty.** In Fig. 3 (b), we show the relationships between inference cost and performance on the examples of MATH500 Hendrycks et al. (2021). These examples have the corresponding difficulty-level annotations. We apply ABPO to DeepSeek-R1-Distill-Qwen-1.5B and evaluate its inference cost and performance on the subsets of different difficulty levels. From the figure, we observe that our ABPO can help the LLM generate CoT with proper lengths related to the example difficulties. This case shows that our ABPO can adaptively allocate more suitable budgets to more challenging examples during training, thereby achieving a good trade-off between performance and efficiency.

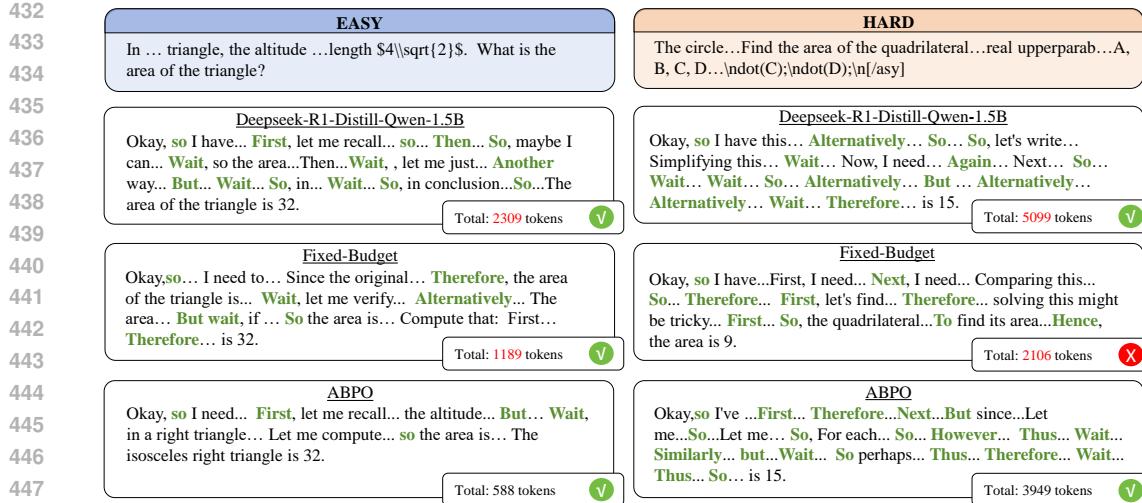


Figure 4: Visualization of CoTs and answers on both easy and hard examples by the default LLM, fixed-budget CoT optimization ( $2k$  tokens) and our ABPO. These examples show that ABPO can help the LLM generate more concise and compact CoTs for simple questions and more complex and longer CoTs for the hard ones, thereby balancing the efficiency and performance of CoT reasoning.

#### 4.4 QUALITATIVE ANALYSIS

To gain insights into the proposed ABPO, we visualize its outputs on both easy and hard examples, and compare them with the default LLM and the fixed-budget baselines ( $2k$ ), as shown in Fig. 4. The details of these examples can be found in Appendix A.4. From these examples, we can see that ABPO can generate concise and compact CoT traces on easy examples, helping the LLM quickly arrive at the correct answer without unnecessary elaboration. In contrast, the baseline model DeepSeek-R1-Distill-Qwen-1.5B tends to generate lengthy and redundant CoTs even for easy examples, leading to inefficient reasoning traces. For more difficult examples, ABPO adaptively generates longer CoTs that involve multiple attempts, self-reflection and alternative solution paths before giving the correct answer. Meanwhile, when training the LLM under fixed-budget settings, the reasoning process for the harder examples is too tightly to correctly answer the question. In other words, producing shorter CoTs will fail to contain the necessary reasoning steps, and also results in incorrect answers. This case demonstrates that ABPO is capable of adaptively performing test time scaling according to example difficulty, validating its ability to achieve difficulty-aware and efficient reasoning. Overall, these findings demonstrate the effectiveness of ABPO in adjusting token budgets according to example difficulty and its adaptive design principle.

## 5 CONCLUSION

In this paper, we focus on addressing the challenges of efficient and adaptive chain-of-thought (CoT) reasoning for *large language models* (LLMs), and propose a novel reinforcement learning scheme termed *Adaptive-Budgeting based Policy Optimization (ABPO)*. To balance the efficiency and performance of CoT reasoning, ABPO introduces dynamic example pools to categorize training examples into three types based on the model’s status, and then adopts an adaptive scheduling mechanism to assign proper token budgets for different examples during RL training. In this way, ABPO helps the LLM to be capable of solving different tasks with adaptive CoT lengths, achieving a remarkable balance between the efficiency and performance. Extensive experiments on three representative LLMs and a bunch of CoT reasoning benchmarks are conducted, of which the results demonstrate that our ABPO method can achieve substantial efficiency improvements with minimal performance loss. Moreover, we also show obvious advantages over the compared methods in balancing the efficiency and performance of CoT reasoning. For instance, ABPO reduces token length by 78.3% while improving accuracy by 2.3% on DeepSeek-R1-Distill-Qwen-1.5B on average, and achieves 59.4% shorter reasoning length with 8.3% higher accuracy compared to HAPO. These results well confirm the effectiveness of ABPO towards efficient CoT reasoning for LLMs.

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## A APPENDIX

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### A.1 THE USE OF LARGE LANGUAGE MODELS (LLMs)

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We employed a large language model (LLM) to assist with language polishing and proofreading.  
The use of the LLM was limited to improving clarity and readability of the manuscript, without  
influencing the design of experiments, analyses, or conclusions.601  
602  

### A.2 PSEUDO-CODE OF ADAPTIVE-BUDGETING BASED POLICY OPTIMIZATION (ABPO)

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In this subsection, we present the pseudo-code of ABPO.605  
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**Algorithm 1:** Adaptive-Budgeting based Policy Optimization (ABPO)607  
608  
**Input:**  $D = \{(q_i, y_i)\}_{i=1}^N$   
**Output:** LLM  $M$  with compact CoT  
1 **Init:**  $\mathcal{S}_{\text{hard}} \leftarrow D$ ,  $\mathcal{S}_{\text{learning}} \leftarrow \emptyset$ ,  $\mathcal{S}_{\text{mastered}} \leftarrow \emptyset$ ;  $\text{budget}[q] \leftarrow 0$ ;  $t \leftarrow 0$ ;  $k \leftarrow 1$ .  
2 **while**  $t < S_{\text{max}}$  **and**  $\mathcal{S}_{\text{mastered}} \neq D$  **do**  
3   **foreach**  $q \in \mathcal{S}_{\text{hard}}$  **do**  
4      $\text{budget}[q] \leftarrow t_0 + (k-1)d$ ;  
5     **if**  $\text{Acc}(q; n) > \tau_0$  **then** move  $q$  to tail of  $\mathcal{S}_{\text{learning}}$ ;  
6      $L_0 \leftarrow |\mathcal{S}_{\text{learning}}|$ ;  
7     **while**  $|\mathcal{S}_{\text{learning}}| \geq \lambda L_0$  **and**  $t < S_{\text{max}}$  **do**  
8       Sample batch  $\mathcal{B} := \text{mix}(\mathcal{S}_{\text{mastered}}, \mathcal{S}_{\text{learning}}; \alpha)$ ;  
9       GRPO\_UPDATE( $M, \mathcal{B}, n$ );;  
10       **foreach**  $q \in \mathcal{B}$  **do**  
11         **if**  $\text{Acc}(q; n) \geq \tau_1$  **then** move  $q$  to tail of  $\mathcal{S}_{\text{mastered}}$ ;  
12         **else** move  $q$  to tail of  $\mathcal{S}_{\text{learning}}$ ;  
13          $t \leftarrow t+1$ ;  
14      $k \leftarrow k+1$ ;624  
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### A.3 HYPERPARAMETER SETTINGS

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Table 5 summarizes the detailed hyperparameter settings used in all ABPO experiments, including  
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Table 5: Hyperparameter settings of ABPO experiments across different models and data scales.

Model	Steps	$t_0$	$d$	$\tau_0$	$\tau_1$	Rollouts	Batch Size
DeepSeek-R1-Distill-Qwen-1.5B	350	2000	500	0.0	0.85	16	128
DeepScaleR-1.5B-Preview	50	2000	500	0.2	0.85	16	128
DeepSeek-R1-Distill-Qwen-7B	50	3000	500	0.3	0.85	16	128
DeepSeek-R1-Distill-Qwen-1.5B (32k data)	350	2000	500	0.2	0.85	8	1024

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## A.4 ADDITIONAL CASE STUDIES OF ABPO

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We provide more detailed outputs on both easy and hard problems in Fig.4. The transition words are highlighted in red.

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## EASY

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In a certain isosceles right triangle, the altitude to the hypotenuse has length  $4\sqrt{2}$ . What is the area of the triangle?

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## DeepSeek-R1-Distill-Owen-1.5B

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Okay, **so** I have this problem here: In a certain isosceles right triangle, the altitude to the hypotenuse has length  $4\sqrt{2}$ . I need to find the area of the triangle. Hmm, let's see. I remember that in an isosceles right triangle, the two legs are equal, and the angles are  $45^\circ$ ,  $45^\circ$ , and  $90^\circ$ . **So**, maybe I can use some properties of right triangles here.

First, let me visualize the triangle. It's an isosceles right triangle, **so** the legs are equal. Let me denote the length of each leg as  $x$ . Then, the hypotenuse can be found using the Pythagorean theorem. Since it's a right-angled triangle, the hypotenuse  $c$  is  $x\sqrt{2}$ . That makes sense because in a 45-45-90 triangle, the hypotenuse is leg length times  $\sqrt{2}$ .

Now, the problem mentions the altitude to the hypotenuse. In a right triangle, the altitude to the hypotenuse relates to the two segments it creates on the hypotenuse. I remember that the area can be calculated in two ways: one as half the product of the legs, and another using the hypotenuse and the altitude. **So**, maybe I can set up an equation using these two expressions for the area.

Let me denote the altitude to the hypotenuse as  $h$ , which is given as  $4\sqrt{2}$ . **So**, the area of the triangle is also equal to  $\frac{1}{2} \times \text{hypotenuse} \times h$ . Since I know the hypotenuse is  $x\sqrt{2}$ , substituting in the values, the area is  $\frac{1}{2} \times x\sqrt{2} \times 4\sqrt{2}$ .

Let me compute that:  $\frac{1}{2} \times x\sqrt{2} \times 4\sqrt{2}$ . Multiplying the constants first:  $\frac{1}{2} \times 4 = 2$ . Then,  $\sqrt{2} \times \sqrt{2} = 2$ . **So**, putting it together, the area becomes  $2 \times x \times 2 = 4x$ .

**Wait, so** the area is  $4x$ . **But** I also know that the area is  $\frac{1}{2} \times \text{leg} \times \text{leg}$ , which is  $\frac{1}{2} \times x^2$ . **So**, setting these two expressions equal:  $4x = \frac{1}{2} \times x^2$ . Hmm, let me solve for  $x$ .

Multiplying both sides by 2 to eliminate the fraction:  $8x = x^2$ . Then, bringing all terms to one side:  $x^2 - 8x = 0$ . Factoring out an  $x$ :  $x(x - 8) = 0$ . **So**,  $x = 0$  or  $x = 8$ . Since a triangle can't have a side of length 0,  $x = 8$ .

Okay, **so** each leg is 8 units long. Then, to find the area, I can use  $\frac{1}{2} \times x^2$  which is  $\frac{1}{2} \times 8^2 = \frac{1}{2} \times 64 = 32$ .

Alternatively, using the other area expression,  $4x$ , **so**  $4 \times 8 = 32$ . Either way, I get the same result. **So**, the area is 32.

**Wait**, let me just make sure I didn't skip any steps or make a mistake. Starting from the beginning: it's an isosceles right triangle, legs equal, hypotenuse  $x\sqrt{2}$ . Altitude to hypotenuse is  $4\sqrt{2}$ . Area is  $\frac{1}{2} \times \text{hypotenuse} \times \text{altitude}$ , which is  $\frac{1}{2} \times x\sqrt{2} \times 4\sqrt{2}$ . Simplify:  $\frac{1}{2} \times x\sqrt{2} \times 4\sqrt{2} = 2x\sqrt{2}$ . **so**  $2x\sqrt{2} = 4x$ . Area is also  $\frac{1}{2} \times x^2$ . **So**,  $4x = \frac{1}{2} \times x^2$ . Multiply both sides by 2:  $8x = x^2$ . **so**,  $x^2 - 8x = 0$ .  $x(x - 8) = 0$ . **So**,  $x = 8$ . Then, area is  $\frac{1}{2} \times x^2 = 32$ . Yep, that seems right.

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**Wait**, let me just make sure I didn't skip any steps or make a mistake. Starting from the beginning: it's an isosceles right triangle, legs equal, hypotenuse  $\langle x \sqrt{2} \rangle$ . Altitude to hypotenuse is  $\langle 4\sqrt{2} \rangle$ . Area is  $\langle \frac{1}{2} \times \text{hypotenuse} \times \text{altitude} \rangle$ , which is  $\langle \frac{1}{2} \times x \sqrt{2} \times 4\sqrt{2} \rangle$ . Simplify:  $\langle \frac{1}{2} \times 4 = 2 \rangle$ ,  $\langle \sqrt{2} \times \sqrt{2} = 2 \rangle$ , **so**  $2 * x * 2 = 4x$ . Area is also  $\langle \frac{1}{2} x^2 \rangle$ . **So**,  $4x = \langle \frac{1}{2} x^2 \rangle$ . Multiply both sides by 2:  $8x = x^2$ . **So**,  $x^2 - 8x = 0$ .  $x(x - 8) = 0$ . **So**,  $x = 8$ . Then, area is  $\langle \frac{1}{2} x^2 = 32 \rangle$ . Yep, that seems right.

Another way to think about it: In an isosceles right triangle, the altitude to the hypotenuse also acts as a median and an angle bisector. Moreover, in such triangles, the altitude is equal to half the hypotenuse times  $\langle \sqrt{2} \rangle$ . **Wait**, is that correct? Let me see.

If the legs are each  $\langle x \rangle$ , then the hypotenuse is  $\langle x \sqrt{2} \rangle$ . The altitude to the hypotenuse in a right triangle is given by  $\langle h = \frac{\text{product of legs}}{\text{hypotenuse}} \rangle = \frac{x \times x \sqrt{2}}{\langle x \sqrt{2} \rangle} = \frac{x}{\sqrt{2}}$ . **So**,  $\langle h = \frac{x}{\sqrt{2}} \rangle$ .

**But** we are given  $\langle h = 4\sqrt{2} \rangle$ , **so** setting them equal:  $\langle \frac{x}{\sqrt{2}} = 4\sqrt{2} \rangle$ . Multiplying both sides by  $\langle \sqrt{2} \rangle$ :  $\langle x = 4\sqrt{2} \times \sqrt{2} = 4 \times 2 = 8 \rangle$ . Yep, same result. **So**, legs are 8, hypotenuse is  $\langle 8\sqrt{2} \rangle$ , area is  $\langle 32 \rangle$ . **So**, either way, the area is 32.

**Wait**, maybe I can also think about similar triangles here. In an isosceles right triangle, when you draw the altitude to the hypotenuse, it creates two smaller isosceles right triangles, each similar to the original triangle. **So**, each of these smaller triangles also has legs equal to each other. Let me try to confirm.

Let me denote the original triangle as ABC, right-angled at C, with legs AC and BC each of length  $\langle x \rangle$ , and hypotenuse AB of length  $\langle x \sqrt{2} \rangle$ . The altitude from C to AB is CD, which is given as  $\langle 4\sqrt{2} \rangle$ . Then, triangles ACD and BCD are both isosceles right triangles. **So**, in triangle ACD, legs AD and CD are equal. Similarly, in triangle BCD, legs BD and CD are equal.

**So**, in triangle ACD, legs AD and CD are equal, **so**  $AD = CD = \langle 4\sqrt{2} \rangle$ . Similarly,  $BD = CD = \langle 4\sqrt{2} \rangle$ . **Therefore**, the hypotenuse AB is  $AD + BD = \langle 4\sqrt{2} + 4\sqrt{2} = 8\sqrt{2} \rangle$ , which matches our earlier calculation that the hypotenuse is  $\langle x \sqrt{2} \rangle$ , **so**  $\langle x = 8 \rangle$ . **So**, that's another verification.

**Alternatively**, using proportions: Since triangle ACD is similar to triangle ABC, the ratio of corresponding sides is the same. **So**, in triangle ACD, legs are  $AD = \langle 4\sqrt{2} \rangle$ , hypotenuse  $AC = 8$  (since AC is a leg of the original triangle, **but wait**, no: AC is a leg of the original triangle, **but** CD is a leg in triangle ACD).

**Wait**, maybe this approach is getting a bit convoluted. Maybe better to stick with the two methods we already used, which both lead to the same answer.

**So**, in conclusion, the legs are each 8, hypotenuse is  $\langle 8\sqrt{2} \rangle$ , area is  $\langle \frac{1}{2} \times 8 \times 8 = 32 \rangle$ . **Alternatively**, using the altitude,  $\langle \frac{1}{2} \times 8\sqrt{2} \times 4\sqrt{2} = \frac{1}{2} \times 8 \times 4 \times 2 = 32 \rangle$ . **So**, both methods confirm the area is 32.

I think that's solid. I don't see any mistakes in my reasoning. **So**, I feel confident that the area is 32.

**Final Answer\*\***

The area of the triangle is  $\boxed{32}$ .

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761Fixed-Budget762 Okay, **so** I have this problem about an isosceles right triangle where the altitude to the hypotenuse is  
763  $\sqrt{4}$ . I need to find the area of the triangle. Let me think about how to approach this.764  
765 First, let me recall what an isosceles right triangle looks like. It has two legs of equal length and a  
766 hypotenuse. The two non-right angles are each 45 degrees.767  
768 In such a triangle, the altitude to the hypotenuse is a line segment from the right angle to the  
769 hypotenuse, dividing it into two smaller congruent triangles. Since the original triangle is isosceles, I  
770 can use the properties of similar triangles here.771 I remember that in an isosceles right triangle, the legs are equal, let's say each leg has length  $a$ .  
772 Then the hypotenuse will be  $\sqrt{2}a$ , because for a right triangle, the hypotenuse squared is  
773 equal to the sum of the squares of the other two sides.774  
775 Now, the altitude to the hypotenuse in a right triangle relates to the legs. I think the formula for the  
776 altitude  $h$  is  $h = \frac{a \cdot b}{c}$ , where  $a$  and  $b$  are the legs, and  $c$  is the  
777 hypotenuse. Since it's an isosceles triangle,  $a = b$ , **so** the formula simplifies to  $h = \frac{a^2}{c}$ .778  
779 Given  $h = 4\sqrt{2}$ , **so** substituting:780  
781  $4\sqrt{2} = \frac{a^2}{c}$ 782  
783 **But**  $c = a\sqrt{2}$ , **so** substitute that in:784  
785  $4\sqrt{2} = \frac{a^2}{a\sqrt{2}}$ 

786 Simplify the denominator:

787  
788  $4\sqrt{2} = \frac{a^2}{\sqrt{2}} = a\sqrt{2}$ 789  
790 Hence:791  
792  $4\sqrt{2} = a\sqrt{2}$ 793 Multiply both sides by  $\sqrt{2}$ :794  
795  $4\sqrt{2} \cdot \sqrt{2} = a\sqrt{2} \cdot \sqrt{2}$ 

796 Simplify the left side:

797  
798  $4 \cdot 2 = a\sqrt{2} \cdot \sqrt{2}$ , **so**:800  
801  $4 \cdot 2 = a\sqrt{2} \cdot \sqrt{2}$ 802  
803  $4 \cdot 2 = a\sqrt{2} \cdot \sqrt{2}$ 804 So each leg is 8 units long. **Therefore**, the area of the triangle is  $\frac{1}{2} \cdot a \cdot b$ ,  
805 which is  $\frac{1}{2} \cdot 8 \cdot 8$ .806  
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815 Calculate that:

816  $\frac{1}{2} \times 64 = 32$ 

818 So the area is 32.

820 Wait, let me verify if my reasoning is correct.

822 Alternatively, I can think about the area in terms of the altitude. The area of the triangle can also be  
823 expressed as  $\frac{1}{2} \times c \times h$ , where  $(c)$  is the hypotenuse and  $(h)$  is the altitude.825 We found that  $(c = a\sqrt{2})$ , which is  $(8\sqrt{2})$ .826 But wait, if  $(a = 8)$ , then  $(c = 8\sqrt{2})$ , which is correct.828 So the area is  $\frac{1}{2} \times 8\sqrt{2} \times 4\sqrt{2}$ .

830 Compute that:

832 First, multiply the constants:  $\frac{1}{2} \times 8 \times 4 = \frac{1}{2} \times 32 = 16$ 833 Then multiply the radicals:  $\sqrt{2} \times \sqrt{2} = 2$ 835 So overall:  $(16 \times 2 = 32)$ . Perfect, same answer.

837 Therefore, the area of the triangle is 32.

&lt;/think&gt;

839 The problem involves an isosceles right triangle with an altitude to the hypotenuse of length  
840  $(4\sqrt{2})$ . We need to find the area of the triangle.

842 1. \*\*Identify the properties of the isosceles right triangle:\*\*

- Two legs of equal length, say  $(a)$ .
- Hypotenuse  $(c = a\sqrt{2})$ .

846 2. \*\*Formula for the altitude to the hypotenuse:\*\*

- The altitude  $(h)$  in a right triangle is given by  $(h = \frac{a \cdot b}{c})$ .
- Since  $(a = b)$ , this simplifies to  $(h = \frac{a^2}{c})$ .

849 3. \*\*Substitute the known values:\*\*

- Given  $(h = 4\sqrt{2})$  and  $(c = a\sqrt{2})$ .
- So,  $(4\sqrt{2}) = \frac{a^2}{a\sqrt{2}}$ .
- Simplify:  $(4\sqrt{2}) = \frac{a}{\sqrt{2}}$ .
- Multiply both sides by  $(\sqrt{2})$ :  $(4 \times 2 = a)$ , so  $(a = 8)$ .

854 4. \*\*Calculate the area:\*\*

- The area is  $\frac{1}{2} \times 8 \times 8 = 32$ .

857 Thus, the area of the triangle is  $(\boxed{32})$ 858  
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ABPO

870 Okay, **so** I need to find the area of an isosceles right triangle where the altitude to the hypotenuse is  
 871  $4\sqrt{2}$ . Hmm, let's see. First, let me recall some properties of isosceles right triangles.

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873 An isosceles right triangle has two legs of equal length and a hypotenuse. The angles are  $45^\circ$ ,  $45^\circ$ ,  
 874 and  $90^\circ$ . The legs are equal, let's call each leg 'a'. The hypotenuse in such a triangle is  $a\sqrt{2}$ .

875

876 Now, the altitude to the hypotenuse is given as  $4\sqrt{2}$ . I remember that in a right triangle, the altitude  
 877 to the hypotenuse relates to the lengths of the segments it creates on the hypotenuse. **But** in an  
 878 isosceles right triangle, the altitude might have a special relationship.

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880 **Wait**, in a right triangle, the length of the altitude to the hypotenuse can be calculated by the  
 881 formula: (product of the legs) divided by the hypotenuse. **So**, in this case, the altitude  $h$  is  $(a * a) / (a\sqrt{2})$ . That  
 882 simplifies to  $a^2 / (a\sqrt{2})$  which is  $a / \sqrt{2}$ .

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Given that this altitude is equal to  $4\sqrt{2}$ , **so**:

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$$a / \sqrt{2} = 4\sqrt{2}$$

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Let me solve for 'a' here. Multiply both sides by  $\sqrt{2}$ :

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$$a = 4\sqrt{2} * \sqrt{2}$$

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$\sqrt{2} * \sqrt{2}$  is 2, **so**:

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$$a = 4 * 2 = 8$$

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**So** each leg of the triangle is 8 units long.

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Now, the area of the triangle is  $(\text{base} * \text{height}) / 2$ . Since it's an isosceles right triangle, both legs are  
 891 equal, **so** the area is  $(a * a) / 2$ . Plugging in  $a = 8$ :

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$$\text{Area} = (8 * 8) / 2 = 64 / 2 = 32$$

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**So**, the area of the triangle is 32.

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901 **Alternatively**, I can think in terms of the hypotenuse. The hypotenuse of the triangle is  $a\sqrt{2} = 8\sqrt{2}$ .  
 902 The altitude is given as  $4\sqrt{2}$ . The area can also be expressed as  $(\text{hypotenuse} * \text{altitude}) / 2$ . **So**:

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$$\text{Area} = (8\sqrt{2} * 4\sqrt{2}) / 2$$

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Multiply the terms:

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$$(8 * 4) * (\sqrt{2} * \sqrt{2}) / 2 = 32 * 2 / 2 = 32. \text{ Same result.}$$

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**So**, I'm confident the area is 32.

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</think>

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The area of the isosceles right triangle is  $\boxed{32}$ .

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933**HARD**

The circle  $x^2 + y^2 = 2$  and the parabola  $y^2 = 8x$  have two common tangents, forming four points of tangency. Find the area of the quadrilateral formed by the four points of tangency.

```
[asy] unitsize(0.8 cm);
real upperparab (real x) {return (sqrt(8*x));} real lowerparab (real x) {return (-sqrt(8*x));}
pair A, B, C, D; A = (-1,1); B = (2,4); C = (-1,-1); D = (2,-4);
draw(graph(upperparab,0,3)); draw(graph(lowerparab,0,3)); draw(Circle((0,0),sqrt(2)));
draw(interp(A,B,-0.2)--interp(A,B,1.2)); draw(interp(C,D,-0.2)--interp(C,D,1.2));
draw(A--C); draw(B--D); dot(A); dot(B); dot(C); dot(D); [/asy]
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938DeepSeek-R1-Distill-Qwen-1.5B

Okay, **so** I have this problem where I need to find the area of a quadrilateral formed by the points where two common tangents to a circle and a parabola touch each curve. The circle is given by the equation  $x^2 + y^2 = 2$  and the parabola is  $y^2 = 8x$ . The figure provided in the Asymptote code shows the circle and the parabola, and then the tangents which intersect at four points forming a quadrilateral.

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First, I need to understand the problem. We have a circle and a parabola, and we're supposed to find their common tangents. Each tangent will touch both the circle and the parabola at exactly one point. Since there are two common tangents, each will have two points of tangency, and these four points will form a quadrilateral. The task is to find the area of this quadrilateral.

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I know that a common tangent to two curves is a line that touches each curve at exactly one point. For a circle and a parabola, the equations of the common tangents can be found by solving certain conditions. The approach is likely to involve setting up equations for the tangent lines and then solving for the points where these lines are tangent to both the circle and the parabola.

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Let me start by recalling that the equation of a tangent to a circle  $x^2 + y^2 = r^2$  at a point  $(x_1, y_1)$  is  $xx_1 + yy_1 = r^2$ . In this case, the circle has radius  $\sqrt{2}$ , **so** the tangent equation would be  $xx_1 + yy_1 = 2$ .

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For the parabola  $y^2 = 8x$ , which opens to the right, the equation of a tangent can be written in the form  $y = mx + \frac{1}{2}m$ , where  $m$  is the slope of the tangent. **Alternatively**, the tangent can be expressed as  $y = mx + c$ , and for the parabola  $y^2 = 4ax$ , the condition for the line  $y = mx + c$  to be tangent is  $c = \frac{1}{4}a$ . In this case, since  $4a = 8$ ,  $a = 2$ , **so**  $c = \frac{1}{4}a$ . That seems right.

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**So**, for the parabola  $y^2 = 8x$ , the tangent equations will be  $y = mx + \frac{1}{2}m$ .

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Now, since these lines are also tangent to the circle  $x^2 + y^2 = 2$ , the distance from the center of the circle (which is at  $(0,0)$ ) to the tangent line must be equal to the radius of the circle, which is  $\sqrt{2}$ .

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The formula for the distance from a point  $(x_0, y_0)$  to the line  $ax + by + c = 0$  is  $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$ . **So**, let's write the equation of the tangent line in standard form.

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Given  $y = mx + \frac{2}{m}$ , we can rearrange it to  $mx - y + \frac{2}{m} = 0$ . **So**,  $a = m$ ,  $b = -1$ , and  $c = \frac{2}{m}$ .

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The distance from  $(0,0)$  to this line is  $\sqrt{0 + 0 + \frac{2}{m^2}} = \sqrt{m^2 + 1}$ .

Since this distance must equal the radius  $\sqrt{2}$ , we have:

$$\sqrt{m^2 + 1} = \sqrt{2}$$

Simplifying this, we can write:

$$\sqrt{m^2 + 1} = \sqrt{2}$$

Multiply both sides by  $|m|$ :

$$2 = \sqrt{2} |m| \sqrt{m^2 + 1}$$

Divide both sides by  $\sqrt{2}$ :

$$\sqrt{2} = |m| \sqrt{m^2 + 1}$$

Square both sides to eliminate the square roots:

$$2 = m^2 (m^2 + 1)$$

Expanding the right side:

$$2 = m^4 + m^2$$

Bring all terms to one side:

$$m^4 + m^2 - 2 = 0$$

This is a quadratic in terms of  $m^2$ . Let me set  $u = m^2$ , **so** the equation becomes:

$$u^2 + u - 2 = 0$$

Solving this quadratic equation using the quadratic formula:

$$u = \frac{-1 \pm \sqrt{1 + 8}}{2} = \frac{-1 \pm 3}{2}$$

**So**,  $u = 1$  or  $u = -2$ . Since  $u = m^2$  must be non-negative, we discard  $u = -2$ . Thus,  $u = 1$ , which means  $m^2 = 1$ , **so**  $m = \pm 1$ .

**So**, the slopes of the common tangents are  $m = 1$  and  $m = -1$ .

**Therefore**, the equations of the tangent lines are:

$$\text{For } m = 1: y = x + 1$$

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For  $(m = -1)$ :  $(y = -x + \frac{2}{m}(-1) = -x - 2)$

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**Wait**, hold on, earlier I wrote the tangent equations as  $(y = mx + \frac{2}{m}m)$ . **So**, for  $(m = 1)$ , it's  $(y = x + 2)$ , and for  $(m = -1)$ , it's  $(y = -x - 2)$ .

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**But** let's verify this because sometimes the signs can be tricky.

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**Wait**, when  $(m = -1)$ , the tangent equation would be  $(y = -x + \frac{2}{m}(-1) = -x - 2)$ . That seems correct.

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Now, I need to find the points where these tangent lines touch the circle and the parabola.

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First, let's find the points of tangency on the parabola  $(y^2 = 8x)$ .

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For the line  $(y = x + 2)$ :

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Substitute into the parabola equation:

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$(x + 2)^2 = 8x$

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Expanding:

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$(x^2 + 4x + 4 = 8x)$

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Bring all terms to one side:

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$(x^2 - 4x + 4 = 0)$

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This factors as:

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**So**,  $(x = 2)$ . Then,  $(y = 2 + 2 = 4)$ . **So**, the point of tangency on the parabola is  $(2, 4)$ .

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Similarly, for the line  $(y = -x - 2)$ :

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Substitute into the parabola equation:

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$((-x - 2)^2 = 8x)$

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Expanding:

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$(x^2 + 4x + 4 = 8x)$

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Bring all terms to one side:

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$(x^2 - 4x + 4 = 0)$

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**Again**, this factors as:

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 1084  $\backslash( (x - 2)^2 = 0 \backslash)$   
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 1086 Thus,  $\backslash( x = 2 \backslash)$ , and  $\backslash( y = -2 - 2 = -4 \backslash)$ . **So**, the point of tangency on the parabola is  $(2, -4)$ .  
 1087  
 1088 Now, let's find the points of tangency on the circle  $\backslash( x^2 + y^2 = 2 \backslash)$ .  
 1089  
 1090 For the line  $\backslash( y = x + 2 \backslash)$ :  
 1091  
 1092 Substitute into the circle equation:  
 1093  $\backslash( x^2 + (x + 2)^2 = 2 \backslash)$   
 1094  
 1095 Expanding:  
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 1097  $\backslash( x^2 + x^2 + 4x + 4 = 2 \backslash)$   
 1098  
 1099 Combine like terms:  
 1100  $\backslash( 2x^2 + 4x + 2 = 0 \backslash)$   
 1101  
 1102 Divide by 2:  
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 1104  $\backslash( x^2 + 2x + 1 = 0 \backslash)$   
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 1106 This factors as:  
 1107  $\backslash( (x + 1)^2 = 0 \backslash)$   
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 1109 **So**,  $\backslash( x = -1 \backslash)$ . Then,  $\backslash( y = -1 + 2 = 1 \backslash)$ . **Therefore**, the point of tangency on the circle is  $(-1, 1)$ .  
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 1111 Similarly, for the line  $\backslash( y = -x - 2 \backslash)$ :  
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 1113 Substitute into the circle equation:  
 1114  $\backslash( x^2 + (-x - 2)^2 = 2 \backslash)$   
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 1116 Expanding:  
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 1118  $\backslash( x^2 + x^2 + 4x + 4 = 2 \backslash)$   
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 1120 Combine like terms:  
 1121  
 1122  $\backslash( 2x^2 + 4x + 2 = 0 \backslash)$   
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 1124 Divide by 2:  
 1125  $\backslash( x^2 + 2x + 1 = 0 \backslash)$   
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 1127 Factor as:  
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 1129  $\backslash( (x + 1)^2 = 0 \backslash)$   
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11371138     **So**,  $\{x = -1\}$ . Then,  $\{y = 1 - 2 = -1\}$ . **Therefore**, the point of tangency on the circle is  $(-1, -1)$ .  
11391140     **So**, the four points of tangency are:  
11411142     1.  $(-1, 1)$  on the circle  
1143     2.  $(2, 4)$  on the parabola  
1144     3.  $(-1, -1)$  on the circle  
1145     4.  $(2, -4)$  on the parabola1146     Now, I need to find the area of the quadrilateral formed by connecting these four points:  $(-1, 1)$ ,  $(2, 4)$ ,  
1147      $(-1, -1)$ , and  $(2, -4)$ .  
11481149     Let me plot these points mentally.  $(-1, 1)$  is in the second quadrant,  $(2, 4)$  is in the first quadrant,  $(-1, -1)$  is in the third quadrant, and  $(2, -4)$  is in the fourth quadrant.  
11501151     Connecting these points in order would form a quadrilateral. To find its area, I can use the shoelace  
1152     formula, which is a method for finding the area of a polygon when the coordinates of the vertices  
1153     are known.  
11541155     The shoelace formula states that for a polygon with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ , ...,  $(x_n, y_n)$ , the area  
1156     is:  
11571158      $\left( \frac{1}{2} \left| \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i) \right| \right)$   
11591160     where  $(x_{n+1}, y_{n+1})$  is  $(x_1, y_1)$ .  
11611162     **So**, let's list the coordinates in order. Let's go around the quadrilateral clockwise or counterclockwise.  
1163     Let me choose the order:  $(-1, 1)$ ,  $(2, 4)$ ,  $(2, -4)$ ,  $(-1, -1)$ , and back to  $(-1, 1)$ .  
11641165     Let me write them down:  
11661167     1.  $(-1, 1)$   
1168     2.  $(2, 4)$   
1169     3.  $(2, -4)$   
1170     4.  $(-1, -1)$   
1171     5.  $(-1, 1)$  // Closing the polygon1172     Now, applying the shoelace formula:  
11731174     First, compute the sum of  $\{x_i y_{i+1}\}$ :  
1175

1176     
$$\begin{aligned} -(-1) * 4 &= -4 \\ -2 * (-4) &= -8 \\ -2 * (-1) &= -2 \\ -(-1) * 1 &= -1 \end{aligned}$$
  
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1178     Sum:  $(-4) + (-8) + (-2) + (-1) = -15$   
11791180     Next, compute the sum of  $\{y_i x_{i+1}\}$ :  
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$-1 * 2 = 2$   
 $-4 * 2 = 8$   
 $-(-4) * (-1) = 4$   
 $-(-1) * (-1) = 1$

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Sum:  $2 + 8 + 4 + 1 = 15$

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Now, subtract the two sums:

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$-15 - 15 = -30$

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Take the absolute value and divide by 2:

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$\left| \frac{1}{2}(-30) \right| = \frac{1}{2} * 30 = 15$

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**So**, the area is 15.

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**Wait**, that seems straightforward, **but** let me double-check my calculations because sometimes the order of the points can affect the result.

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**Alternatively**, I can plot the points and see if the quadrilateral is a trapezoid or some other shape.

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Looking at the coordinates:

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$(-1, 1), (2, 4), (2, -4), (-1, -1)$

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Plotting these, we see that two points are at  $(-1, \pm 1)$  and  $(2, \pm 4)$ . **So**, the quadrilateral is symmetric with respect to both the x-axis and y-axis. **Wait**, is it symmetric?

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**Wait**,  $(-1, 1)$  and  $(-1, -1)$  are symmetric over the x-axis. Similarly,  $(2, 4)$  and  $(2, -4)$  are symmetric over the x-axis. Similarly, the points  $(-1, 1)$  and  $(2, 4)$  are not symmetric, as are  $(-1, -1)$  and  $(2, -4)$ . **So**, **perhaps** the quadrilateral is symmetric across both axes.

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**But** regardless, the shoelace formula should work as long as the points are ordered correctly, either clockwise or counterclockwise.

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Let me try a different order to double-check.

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Let me arrange the points in a different order, say:  $(-1, 1), (-1, -1), (2, -4), (2, 4)$ , and back to  $(-1, 1)$ .

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Computing shoelace formula **again**:

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First sum:

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$-(-1) * (-1) = 1$   
 $-(-1) * (-4) = 4$   
 $-2 * 4 = 8$   
 $-2 * 1 = 2$

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Sum:  $1 + 4 + 8 + 2 = 15$

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Second sum:

$$\begin{aligned} -1 * (-1) &= -1 \\ -(-1) * 2 &= -2 \\ -(-4) * 2 &= -8 \\ -4 * (-1) &= -4 \end{aligned}$$

$$\text{Sum: } (-1) + (-2) + (-8) + (-4) = -15$$

$$\text{Subtract: } 15 - (-15) = 30$$

Take absolute value and divide by 2: 15

Same result, **so** that's consistent.

**Wait**, actually, the shoelace formula requires the points to be ordered either clockwise or counterclockwise without crossing. **So**, the first ordering I did was  $(-1,1), (2,4), (2,-4), (-1,-1)$ , which goes around the quadrilateral in clockwise order, and the second was  $(-1,1), (-1,-1), (2,-4), (2,4)$ , which is also clockwise.

Either way, the area is 15.

**Alternatively**, maybe we can think of the quadrilateral as a trapezoid. Let me see.

**Wait**, looking at the coordinates, the quadrilateral is symmetric across the x-axis. **So**, if I consider the upper half and the lower half separately.

**But perhaps** that complicates things because it's a trapezoid.

**Alternatively**, think of the quadrilateral as a polygon with vertices at  $(-1,1), (2,4), (2,-4), (-1,-1)$ . If I connect these points, I can divide it into two triangles.

**Alternatively**, think of it as a kite-shaped quadrilateral because two pairs of adjacent sides are equal. **Wait**, is that the case?

Compute the distances between consecutive points:

1. From  $(-1,1)$  to  $(2,4)$ :

$$\text{Distance squared: } (2 - (-1))^2 + (4 - 1)^2 = (3)^2 + (3)^2 = 9 + 9 = 18$$

2. From  $(2,4)$  to  $(2,-4)$ :

$$\text{Distance squared: } (2 - 2)^2 + (-4 - 4)^2 = 0 + (-8)^2 = 64$$

3. From  $(2,-4)$  to  $(-1,-1)$ :

$$\text{Distance squared: } (-1 - 2)^2 + (-1 - (-4))^2 = (-3)^2 + (3)^2 = 9 + 9 = 18$$

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4. From  $(-1, -1)$  to  $(-1, 1)$ :Distance squared:  $(-1 - (-1))^2 + (1 - (-1))^2 = 0 + (2)^2 = 4$ **So**, sides are  $\sqrt{18}$ , 8,  $\sqrt{18}$ , and 2.**So**, the sides are not equal, **so** it's not a kite. Maybe it's a trapezoid? Let me check if any sides are parallel.

Compute the slopes of the sides:

1. From  $(-1, 1)$  to  $(2, 4)$ : slope is  $(4 - 1)/(2 - (-1)) = 3/3 = 1$ .2. From  $(2, 4)$  to  $(2, -4)$ : slope is undefined (vertical line).3. From  $(2, -4)$  to  $(-1, -1)$ : slope is  $(-1 - (-4))/(-1 - 2) = (3)/(-3) = -1$ .4. From  $(-1, -1)$  to  $(-1, 1)$ : slope is undefined (vertical line).**So**, sides 2 and 4 are both vertical, hence parallel. Sides 1 and 3 have slopes 1 and -1, which are not equal, **so** they are not parallel. **So**, it's a trapezoid with two sides parallel.**Therefore**, the area can be found by the formula for the area of a trapezoid:  $\frac{1}{2} (b_1 + b_2) h$ , where  $b_1$  and  $b_2$  are the lengths of the two parallel sides, and  $h$  is the height (distance between them).Looking at the trapezoid, the two parallel sides are the vertical sides from  $(2, 4)$  to  $(2, -4)$  and from  $(-1, 1)$  to  $(-1, -1)$ . Their lengths are:- From  $(2, 4)$  to  $(2, -4)$ : distance is  $|4 - (-4)| = 8$  units.- From  $(-1, 1)$  to  $(-1, -1)$ : distance is  $|1 - (-1)| = 2$  units.**So**,  $b_1 = 8$ ,  $b_2 = 2$ .Now, the height  $h$  is the horizontal distance between the two parallel sides. Since the sides are vertical at  $x=2$  and  $x=-1$ , the distance between them is  $|2 - (-1)| = 3$  units.**Therefore**, area is  $\frac{1}{2} (8 + 2) * 3 = \frac{1}{2} * 10 * 3 = 5 * 3 = 15$ .

Same result as before.

**So**, regardless of the method, the area is 15. **Therefore**, I can be confident that 15 is the correct area of the quadrilateral.**But wait**, before I conclude, let me just visualize the points **again**. The quadrilateral has two vertical sides at  $x=2$  and  $x=-1$ , connecting  $(2, 4)$  to  $(2, -4)$  and  $(-1, 1)$  to  $(-1, -1)$ . Then, the other two sides are the lines connecting  $(2, 4)$  to  $(-1, -1)$  and  $(2, -4)$  to  $(-1, 1)$ . These cross each other, forming a quadrilateral.**Alternatively**, I can compute the area using vectors or coordinate geometry. **But** since both methods gave me 15, and my reasoning is consistent, I think it's safe.

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**Therefore**, the area of the quadrilateral formed by the four points of tangency is 15.

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\*\*Final Answer\*\*

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$\boxed{15}$

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$</\text{think}>$

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To find the area of the quadrilateral formed by the points of tangency of the common tangents to the circle  $(x^2 + y^2 = 2)$  and the parabola  $(y^2 = 8x)$ , we start by determining the equations of the common tangents.

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1. The circle  $(x^2 + y^2 = 2)$  has radius  $(\sqrt{2})$ .

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2. The parabola  $(y^2 = 8x)$  has the tangent equation  $(y = mx + \frac{2}{m})$ .

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Using the condition that the distance from the center of the circle  $(0,0)$  to the tangent line must equal the radius  $(\sqrt{2})$ , we find the slopes  $(m)$ :

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$$\left| \frac{\left| m \right|}{\sqrt{m^2 + 1}} \right| = \sqrt{2}$$

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Solving this, we get  $(m = \pm 1)$ . Thus, the equations of the tangent lines are  $(y = x + 2)$  and  $(y = -x - 2)$ .

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Next, we find the points of tangency on the parabola and the circle:

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- For the line  $(y = x + 2)$ :

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- Tangency on the parabola:  $((2, 4))$

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- Tangency on the circle:  $((-1, 1))$

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- For the line  $(y = -x - 2)$ :

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- Tangency on the parabola:  $((2, -4))$

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- Tangency on the circle:  $((-1, -1))$

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The four points of tangency are:  $(-1, 1)$ ,  $(2, 4)$ ,  $(-1, -1)$ , and  $(2, -4)$ .

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Using the shoelace formula to find the area of the quadrilateral formed by these points:

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1. List the coordinates in order:  $(-1, 1)$ ,  $(2, 4)$ ,  $(2, -4)$ ,  $(-1, -1)$ .

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2. Apply the shoelace formula:

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$$\text{Area} = \frac{1}{2} \left| (-1 \cdot 4 + 2 \cdot -4 + 2 \cdot -1 + -1 \cdot 1) - (1 \cdot 2 + 2 \cdot -1 + -4 \cdot -1 + 1 \cdot 2) \right|$$

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Calculating the sums:

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- First sum:  $(-4 - 8 - 2 - 1 = -15)$

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- Second sum:  $(2 + 8 + 4 + 1 = 15)$

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1408     Subtracting and taking the absolute value:  
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$$|\left| -15 - 15 \right| = 30$$
  
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$$\frac{1}{2} \times 30 = 15$$
  
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1414     Dividing by 2:  
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1416     
$$\frac{1}{2} \times 15 = 15$$
  
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1419     Thus, the area of the quadrilateral is:  
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$$\boxed{15}$$
  
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Okay, so I have this problem where I need to find the area of a quadrilateral formed by the points of tangency between the circle  $(x^2 + y^2 = 2)$  and the parabola  $(y^2 = 8x)$ . The figure provided is an Asymptote drawing, which shows the circle and the parabola, and some tangents with points of contact labeled A, B, C, D.

First, I need to recall how to find common tangents between a circle and a parabola. Since both are symmetric with respect to the x-axis, the points of tangency will also be symmetric. So, if point A is at  $(a, b)$ , point C will be at  $(a, -b)$ , and similarly for points B and D.

Next, I need an equation for the tangent lines to both the circle and the parabola. For a parabola  $(y^2 = 4ax)$ , the equation of the tangent at point  $(at^2, 2at)$  is  $(ty = x + at^2)$ . Comparing this with our given parabola  $(y^2 = 8x)$ , which can be seen as  $(y^2 = 4a x)$  with  $(a = 2)$ , so  $(4a = 8)$ .

So, the tangent equation becomes  $(ty = x + 2t^2)$ .

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Now, this line should also be tangent to the circle  $(x^2 + y^2 = 2)$ . The condition for a line  $(y = mx + c)$  to be tangent to a circle  $(x^2 + y^2 = r^2)$  is that  $(c^2 = r^2(1 + m^2))$ .

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Let me rearrange the tangent line equation:

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$$(ty = x + 2t^2) \rightarrow (x - ty + 2t^2 = 0)$$

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So, it's in the form  $(Ax + By + C = 0)$ , where  $(A = 1)$ ,  $(B = -t)$ ,  $(C = 2t^2)$ .

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Therefore, the condition for tangency is:

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$$(C^2 = r^2(1 + m^2))$$

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But I need to express this in terms of t.

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First, let's find the slope m of the tangent line.

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From the equation  $(x - ty + 2t^2 = 0)$ , we can write it as  $(y = \frac{1}{t}x + \frac{2t}{1})$ .

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$$(So, (m = \frac{1}{t}))$$

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$$(Therefore, (m^2 = \frac{1}{t^2}))$$

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Now, plugging back into the tangency condition:

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$$((2t^2)^2 = 2^2 \left(1 + \frac{1}{t^2}\right))$$

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Simplify:

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$$((4t^4 = 4 \left(1 + \frac{1}{t^2}\right)))$$

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Divide both sides by 4:

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### Fixed-Budget

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$$\backslash(t^4 = 1 + \frac{1}{t^2}).$$

Multiply both sides by  $\backslash(t^2)$ :

$$\backslash(t^6 = t^2 + 1).$$

This is a cubic equation in terms of  $\backslash(z = t^2)$ :

$$\backslash(z^3 - z - 1 = 0).$$

Hmm, solving this might be tricky. Maybe I can find rational roots. Trying  $z=1: \backslash(1 - 1 - 1 = -1 \neq 0)$ .  $z=-1$ : negative, **so** no. **So**, probably no rational roots. Maybe I can use some approximation method or use the cubic formula, **but** that might be complicated.

**Alternatively**, maybe I can use substitution or numerical methods. **Alternatively, perhaps** there's a substitution.

Let me consider  $\backslash(z = t^2)$ , **so** the equation is  $\backslash(z^3 - z - 1 = 0)$ .

**Perhaps** I can use the method for solving cubics.

The general formula for a cubic  $\backslash(z^3 + az^2 + bz + c = 0)$  is:

$$\backslash(z = \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}).$$

In our case,  $\backslash(a = 0)$ ,  $\backslash(b = -1)$ ,  $\backslash(c = -1)$ .

**So**,

$$\backslash(z = \sqrt[3]{\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 0}} + \sqrt[3]{\frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 + 0}}).$$

Simplify:

$$\backslash(z = \sqrt[3]{\frac{1}{2} + \frac{1}{2}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}}).$$

$$\backslash(z = \sqrt[3]{1} + \sqrt[3]{0} = 1 + 0 = 1).$$

**So**, one real root is  $\backslash(z = 1)$ , **so**  $\backslash(t^2 = 1)$ , **so**  $\backslash(t = 1)$  or  $\backslash(t = -1)$ .

**Therefore**, we can take  $t=1$  or  $t=-1$ .

**So**, the tangent lines are:

For  $t=1$ :

$$\backslash(y = \frac{1}{2}x + \frac{2}{2}) = x + 2.$$

For  $t=-1$ :

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 1570  $\backslash ( y = \frac{1}{-1}x + \frac{2(-1)^2}{1} = -x + 2 ).$   
 1571  
 1572 **So**, the two tangent lines are  $\backslash ( y = x + 2 )$  and  $\backslash ( y = -x + 2 )$ .  
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 1574 Now, let's find the points of contact on both the circle and the parabola.  
 1575  
 1576 For the parabola  $\backslash ( y^2 = 8x )$ , the points of tangency are  $\backslash ( (2t^2, 4t) )$ .  
 1577  
 1578 **So** for  $t=1$ , point is  $\backslash ( (2(1)^2, 4(1)) = (2, 4) )$ .  
 1579  
 1580 For  $t=-1$ , point is  $\backslash ( (2(-1)^2, 4(-1)) = (2, -4) )$ .  
 1581  
 1582 **So**, the two points of tangency on the parabola are A(2,4) and B(2,-4).  
 1583  
 1584 Similarly, for the circle  $\backslash ( x^2 + y^2 = 2 )$ , the points of tangency can be found by solving the  
 1585 system of the circle and the tangent line.  
 1586  
 1587 For line  $\backslash ( y = x + 2 )$ :  
 1588  $\backslash ( x^2 + (x + 2)^2 = 2 ).$   
 1589  
 1590  $\backslash ( x^2 + x^2 + 4x + 4 = 2 ).$   
 1591  
 1592  $\backslash ( 2x^2 + 4x + 2 = 0 ).$   
 1593  
 1594 Divide by 2:  
 1595  
 1596  $\backslash ( x^2 + 2x + 1 = 0 ).$   
 1597  
 1598 **So**,  $\backslash ( x = -1 )$ .  
 1599  
 1600 Then,  $\backslash ( y = -1 + 2 = 1 )$ .  
 1601  
 1602 Thus, the point of tangency is (-1,1).  
 1603  
 1604 Similarly, for line  $\backslash ( y = -x + 2 )$ :  
 1605  
 1606  $\backslash ( x^2 + (-x + 2)^2 = 2 ).$   
 1607  
 1608  $\backslash ( x^2 + x^2 - 4x + 4 = 2 ).$   
 1609  
 1610  $\backslash ( 2x^2 - 4x + 2 = 0 ).$   
 1611  
 1612 Divide by 2:  
 1613  
 1614  $\backslash ( x^2 - 2x + 1 = 0 ).$   
 1615  
 1616  $\backslash ( (x-1)^2 = 0 ).$   
 1617  
 1618  
 1619

1620  
 1621  
 1622  
 1623  
 1624 Thus,  $\{x = 1\}$ ,  $\{y = -1 + 2 = 1\}$ .  
 1625  
 1626 Thus, the point of tangency is  $(1,1)$ .  
 1627  
 1628 **So**, the four points of tangency are:  
 1629  
 1630  $A(2,4)$ ,  $B(2,-4)$ ,  $C(-1,1)$ ,  $D(1,1)$ .  
 1631  
 1632 Now, we need to find the area of quadrilateral ABCD.  
 1633  
 1634 Given the points  $A(2,4)$ ,  $B(2,-4)$ ,  $C(-1,1)$ ,  $D(1,1)$ .  
 1635  
 1636 First, I can plot these points:  
 1637  
 1638  $A(2,4)$  is in the upper right quadrant.  
 1639  
 1640  $B(2,-4)$  is in the lower right quadrant.  
 1641  
 1642  $C(-1,1)$  is in the upper left quadrant.  
 1643  
 1644  $D(1,1)$  is in the upper right quadrant.  
 1645  
 1646 **So**, the quadrilateral is symmetric with respect to the y-axis.  
 1647  
 1648 To find its area, I can use the shoelace formula.  
 1649  
 1650 The coordinates in order are  $A(2,4)$ ,  $B(2,-4)$ ,  $C(-1,1)$ ,  $D(1,1)$ , back to  $A(2,4)$ .  
 1651  
 1652 Compute shoelace sum:  
 1653  
 1654  $Sum1 = (2)(-4) + (2)(1) + (-1)(1) + (1)(4) = (-8) + 2 + (-1) + 4 = (-8+2) + (-1+4) = (-6) + 3 = -3$ .  
 1655  
 1656  $Sum2 = (4)(2) + (-4)(-1) + (1)(1) + (1)(2) = 8 + 4 + 1 + 2 = 15$ .  
 1657  
 1658  $Area = (|Sum1 - Sum2|)/2 = |-3 - 15|/2 = |-18|/2 = 9$ .  
 1659  
 1660 Hence, the area is 9.  
 1661  
 1662  
 1663  
 1664  
 1665  
 1666  
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1678  
1679

### ABPO

1680 Okay, **so** I've got this problem where I need to find the area of a quadrilateral formed by the points  
1681 of tangency of two common tangents to a circle and a parabola. The circle is given by the equation  
1682  $x^2 + y^2 = 2$ , and the parabola is  $y^2 = 8x$ . The points of tangency are four in total, and they form a  
1683 quadrilateral. I need to find the area of this quadrilateral.

1684 First, I remember that common tangents to a circle and a parabola are lines that touch both the circle  
1685 and the parabola at exactly one point each. Since there are two common tangents, each will touch  
1686 both the circle and the parabola, resulting in four points of tangency.

1687  
1688 I think the Asymptote code provided might help me visualize the figure. It shows the circle centered  
1689 at the origin with radius  $\sqrt{2}$ , and the parabola  $y^2 = 8x$  opens to the right. The points A, B, C, D are  
1690 given as  $(-1, 1)$ ,  $(2, 4)$ ,  $(-1, -1)$ , and  $(2, -4)$ . **So**, the quadrilateral is formed by connecting these points.

1691 **But** I need to confirm this. Maybe I should derive the equations of the common tangents and then  
1692 find their points of contact with the circle and parabola.

1693  
1694 Let me start by recalling some properties. For a parabola  $y^2 = 4ax$ , the general equation of a tangent  
1695 line is  $y = tx + a/t$ , where  $t$  is a parameter. Comparing this with  $y^2 = 8x$ , which is  $y^2 = 4a x$  with  $4a =$   
1696 8, **so**  $a = 2$ . Thus, the tangent equation becomes  $y = tx + 2/t$ .

1697  
1698 Similarly, the circle  $x^2 + y^2 = r^2$  has a general tangent equation. The tangent to a circle  $x^2 + y^2 = r^2$  at  
1699 point  $(x_1, y_1)$  is  $xx_1 + yy_1 = r^2$ . **So**, if I have a tangent line  $y = mx + c$ , then the condition for it to be  
1700 tangent to the circle is  $c^2 = r^2(1 + m^2)$ . Here,  $r^2 = 2$ , **so**  $c^2 = 2(1 + m^2)$ .

1701 **So**, the tangent lines to the circle will have the form  $y = mx \pm \sqrt{2(1 + m^2)}$ . These lines should also  
1702 be tangent to the parabola  $y^2 = 8x$ .

1703 Substituting  $y = mx \pm \sqrt{2(1 + m^2)}$  into  $y^2 = 8x$ :

1704  
1705  $(mx \pm \sqrt{2(1 + m^2)})^2 = 8x$

1706  
1707 Expanding the left side:

1708  
1709  $m^2x^2 \pm 2mx\sqrt{2(1 + m^2)} + 2(1 + m^2) = 8x$

1710 This is a quadratic in  $x$ . For the line to be tangent to the parabola, the discriminant of this quadratic  
1711 should be zero.

1712  
1713 **So**, let me write it as:

1714  
1715  $m^2x^2 \pm 2mx\sqrt{2(1 + m^2)} + 2(1 + m^2) - 8x = 0$

1716  
1717 Arrange terms:

1718  
1719  $m^2x^2 + (\pm 2m\sqrt{2(1 + m^2)} - 8)x + 2(1 + m^2) = 0$

1720  
1721 For this quadratic equation, the discriminant D is:

1722  
1723  
1724  
1725  
1726  
1727

1728  
 1729  
 1730  
 1731  
 1732  $D = [\pm 2m\sqrt{2(1 + m^2)} - 8]^2 - 4 * m^2 * 2(1 + m^2) = 0$   
 1733

1734 This seems a bit complicated, **but** let's compute it step by step.  
 1735

1736 First, compute the discriminant:  
 1737

1738  $D = [\pm 2m\sqrt{2(1 + m^2)} - 8]^2 - 8m^2(1 + m^2) = 0$   
 1739

1740 Let me compute the first square term:  
 1741

1742  $[\pm 2m\sqrt{2(1 + m^2)} - 8]^2 = (\pm 2m\sqrt{2(1 + m^2)})^2 - 2 * 2m\sqrt{2(1 + m^2)} * 8 + 8^2$   
 1743

1744 **Wait**, actually, expanding  $(a - b)^2 = a^2 - 2ab + b^2$ .  
 1745

1746 **So**,  
 1747

1748  $= (2m\sqrt{2(1 + m^2)})^2 - 2 * 2m\sqrt{2(1 + m^2)} * 8 + 8^2$   
 1749  
 $= 4m^2 * 2(1 + m^2) - 32m\sqrt{2(1 + m^2)} + 64$   
 1750  
 $= 8m^2(1 + m^2) - 32m\sqrt{2(1 + m^2)} + 64$   
 1751

1752 Then subtract  $8m^2(1 + m^2)$ :  
 1753

1754 **So**,  
 1755

1756  $D = [8m^2(1 + m^2) - 32m\sqrt{2(1 + m^2)} + 64] - 8m^2(1 + m^2) = 0$   
 1757

1758 Simplify:  
 1759

1760  $8m^2(1 + m^2)$  cancels with  $-8m^2(1 + m^2)$ , leaving:  
 1761

1762  $-32m\sqrt{2(1 + m^2)} + 64 = 0$   
 1763

1764 **So**,  
 1765

1766  $-32m\sqrt{2(1 + m^2)} + 64 = 0$   
 1767

1768 Move 64 to the other side:  
 1769

1770  $-32m\sqrt{2(1 + m^2)} = -64$   
 1771

1772 Divide both sides by -32:  
 1773

1774  $m\sqrt{2(1 + m^2)} = 2$   
 1775

1776 Square both sides to eliminate the square root:  
 1777

1778  $m^2 * 2(1 + m^2) = 4$   
 1779

1780 Simplify:  
 1781

1782  
 1783  
 1784  
 1785  
 1786  $2m^2(1 + m^2) = 4$   
 1787  
 1788 Divide both sides by 2:  
 1789  
 1790  $m^2(1 + m^2) = 2$   
 1791  
 1792 This is a quartic equation:  
 1793  
 1794  $m^4 + m^2 - 2 = 0$   
 1795  
 1796 Let me set  $z = m^2$ , **so** equation becomes:  
 1797  
 1798  $z^2 + z - 2 = 0$   
 1799  
 1800 Solve for  $z$ :  
 1801  
 1802  $z = [-1 \pm \sqrt{(1 + 8)}] / 2 = [-1 \pm 3]/2$   
 1803  
 1804 **So**,  $z = (2)/2 = 1$  or  $z = (-4)/2 = -2$   
 1805  
 1806 Since  $z = m^2$  cannot be negative,  $z = 1$ .  
 1807  
 1808 Thus,  $m^2 = 1 \Rightarrow m = \pm 1$   
 1809  
 1810 **Therefore**, the slopes of the common tangents are  $m = 1$  and  $m = -1$ .  
 1811  
 1812 **So**, the equations of the common tangents are:  
 1813  
 1814 For  $m = 1$ :  $y = x \pm \sqrt{2(1 + 1)} = x \pm \sqrt{4} = x \pm 2$   
 1815  
 1816 For  $m = -1$ :  $y = -x \pm \sqrt{2(1 + 1)} = -x \pm 2$   
 1817  
 1818 **So**, the four lines are  $y = x + 2$ ,  $y = x - 2$ ,  $y = -x + 2$ , and  $y = -x - 2$ .  
 1819  
 1820 Now, I need to find the points of tangency on both the circle and the parabola.  
 1821  
 1822 Let's start with the circle  $x^2 + y^2 = 2$ .  
 1823  
 1824 For each tangent line, solve for the point of contact.  
 1825  
 1826 Take  $y = x + 2$ .  
 1827  
 1828 Substitute into circle:  
 1829  
 1830  $x^2 + (x + 2)^2 = 2$   
 1831  
 1832  $x^2 + x^2 + 4x + 4 = 2$   
 1833  
 1834  $2x^2 + 4x + 2 = 0$   
 1835  
 Divide by 2:

1836  
 1837  
 1838  
 1839  
 1840      $x^2 + 2x + 1 = 0$   
 1841  
 1842     This factors as  $(x + 1)^2 = 0 \Rightarrow x = -1$   
 1843  
 1844     Then  $y = -1 + 2 = 1$   
 1845  
 1846     **So**, one point is  $(-1, 1)$ .  
 1847  
 1848     Similarly, for  $y = x - 2$ :  
 1849  
 1850     Substitute into circle:  
 1851      $x^2 + (x - 2)^2 = 2$   
 1852  
 1853      $x^2 + x^2 - 4x + 4 = 2$   
 1854  
 1855      $2x^2 - 4x + 2 = 0$   
 1856  
 1857     Divide by 2:  
 1858      $x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1$   
 1859  
 1860     Then  $y = 1 - 2 = -1$   
 1861  
 1862     **So**, the point is  $(1, -1)$ .  
 1863  
 1864     Similarly, for  $y = -x + 2$ :  
 1865  
 1866     Substitute into circle:  
 1867      $x^2 + (-x + 2)^2 = 2$   
 1868  
 1869      $x^2 + x^2 - 4x + 4 = 2$   
 1870  
 1871      $2x^2 - 4x + 2 = 0$   
 1872  
 1873     Divide by 2:  
 1874      $x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1$   
 1875  
 1876     Then  $y = -1 + 2 = 1$   
 1877  
 1878     Point is  $(1, 1)$ .  
 1879  
 1880     For  $y = -x - 2$ :  
 1881  
 1882     Substitute into circle:  
 1883      $x^2 + (-x - 2)^2 = 2$   
 1884  
 1885  
 1886  
 1887  
 1888  
 1889

1890  
 1891  
 1892  
 1893  
 1894  $x^2 + x^2 + 4x + 4 = 2$   
 1895  
 1896  $2x^2 + 4x + 2 = 0$   
 1897  
 1898 Divide by 2:  
 1899  
 1900  $x^2 + 2x + 1 = 0 \Rightarrow (x + 1)^2 = 0 \Rightarrow x = -1$   
 1901  
 1902 Then  $y = 1 - 2 = -1$   
 1903  
 1904 Point is  $(-1, -1)$ .  
 1905 **So**, the four points of tangency on the circle are  $(-1, 1), (1, -1), (1, 1), (-1, -1)$ .  
 1906  
 1907 Now, let's find the points of tangency on the parabola  $y^2 = 8x$ .  
 1908  
 1909 For each tangent line, solve for the point of contact.  
 1910  
 1911 Take  $y = x + 2$ .  
 1912  
 1913 Substitute into parabola:  
 1914  $(x + 2)^2 = 8x$   
 1915  
 1916  $x^2 + 4x + 4 = 8x$   
 1917  
 1918  $x^2 - 4x + 4 = 0$   
 1919  
 1920 This factors as  $(x - 2)^2 = 0 \Rightarrow x = 2$   
 1921  
 1922 Then  $y = 2 + 2 = 4$ . **So**, point is  $(2, 4)$ .  
 1923  
 1924 Similarly,  $y = x - 2$ :  
 1925  
 1926  $(x - 2)^2 = 8x$   
 1927  
 1928  $x^2 - 4x + 4 = 8x$   
 1929  
 1930  $x^2 - 12x + 4 = 0$   
 1931  
 1932 Solve using quadratic formula:  

$$x = [12 \pm \sqrt{(144 - 16)}]/2 = [12 \pm \sqrt{128}]/2 = [12 \pm 8\sqrt{2}]/2 = 6 \pm 4\sqrt{2}$$
  
 1933  
 1934 **Wait, but** then  $y = x - 2$ , **so**  $y$  would be  $(6 \pm 4\sqrt{2}) - 2 = 4 \pm 4\sqrt{2}$ .  
 1935  
 1936 **But** since the parabola is  $y^2 = 8x$ , which opens to the right, the points  $(6 + 4\sqrt{2}, 4 + 4\sqrt{2})$  and  $(6 - 4\sqrt{2}, 4 - 4\sqrt{2})$  are valid.  
 1937  
 1938 **Wait, but** maybe I made a mistake because the tangent lines are  $y = x \pm 2$ , **so** when solving, the  
 1939 points should satisfy both the line and the parabola.  
 1940  
 1941  
 1942  
 1943

1944  
1945  
1946  
1947

1948 **Wait, perhaps** I should approach it differently. Let's use the condition that the line  $y = mx + c$  is tangent to the parabola  $y^2 = 4ax$ , which has the condition  $c^2 = a(1 + m^2)$ .

1949 For our parabola  $y^2 = 8x$ ,  $a=2$ , **so**  $c^2 = 2(1 + m^2)$ . For the tangent lines  $y = x \pm 2$ ,  $m=1$ ,  $c=\pm 2$ .

1950

1951 **So**, substituting into the parabola equation, we can find the points of contact.

1952

1953 **Alternatively**, using the formula for the point of contact, for the parabola  $y^2 = 4ax$ , the point of contact for a tangent  $y = mx + c$  is  $(a/m^2, 2a/m)$ .

1954

1955 **So**, for our case,  $a=2$ ,  $m=1$ , **so** the point is  $(2/1, 4/1) = (2,4)$ .

1956

1957 Similarly, for  $m=-1$ , the point is  $(2/1, -4/1) = (2,-4)$ .

1958

1959 Thus, the points of tangency on the parabola are  $(2,4)$ ,  $(2,-4)$ ,  $(-2,4)$ ,  $(-2,-4)$ . **Wait, but** according to the formula, it's  $(a/m^2, 2a/m)$ . **So**, if  $m=1$ , it's  $(2,4)$ . If  $m=-1$ , it's  $(2, -4)$ . Similarly, for  $m=1$ , it's  $(2,4)$ , and for  $m=-1$ , it's  $(2,-4)$ . **So**, the four points are  $(2,4)$ ,  $(2,-4)$ ,  $(-2,4)$ ,  $(-2,-4)$ .

1960

1961 **Wait, but** in the Asymptote code, the points are  $(-1,1)$ ,  $(2,4)$ ,  $(-1,-1)$ ,  $(2,-4)$ . **So, perhaps** the points on the circle are  $(-1,1)$ ,  $(1,-1)$ ,  $(1,1)$ ,  $(-1,-1)$ , and on the parabola,  $(2,4)$ ,  $(2,-4)$ ,  $(-2,4)$ ,  $(-2,-4)$ . **So**, the quadrilateral is formed by connecting these points in order:  $(-1,1)$ ,  $(2,4)$ ,  $(-1,-1)$ ,  $(2,-4)$ , and back to  $(-1,1)$ . That would form a kite-shaped quadrilateral.

1962

1963 **Wait, but** the user mentioned four points of tangency, **so perhaps** each tangent touches the circle and the parabola, resulting in four distinct points. **So**, the points are  $(-1,1)$ ,  $(1,-1)$ ,  $(1,1)$ ,  $(-1,-1)$  on the circle, and  $(2,4)$ ,  $(2,-4)$ ,  $(-2,4)$ ,  $(-2,-4)$  on the parabola.

1964

1965 **So**, the quadrilateral is formed by connecting these points. Let me list them in order:  $(-1,1)$ ,  $(2,4)$ ,  $(-1,-1)$ ,  $(2,-4)$ , and back to  $(-1,1)$ . This should form a convex quadrilateral.

1966

1967 Now, to find the area of this quadrilateral, I can use the shoelace formula.

1968

1969 First, list the coordinates in order:

1970

1971 1.  $(-1,1)$   
2.  $(2,4)$   
3.  $(-1,-1)$   
4.  $(2,-4)$   
5.  $(-1,1)$  # Closing the polygon

1972

1973 Apply the shoelace formula:

1974

1975  $\text{Area} = 1/2 \sum \text{over } i (x_i y_{i+1} - x_{i+1} y_i)$

1976

1977 Compute each term:

1978

1979 Between  $(-1,1)$  and  $(2,4)$ :

1980

1981  $x_i y_{i+1} = (-1)(4) = -4$

1982

1983  $x_{i+1} y_i = (2)(1) = 2$

1984

1985

1986

1987

1988

1989

1990

1991

1992

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1994

1995

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1998

1999

2000

2001

2002

Term:  $-4 - 2 = -6$ 

2003

Between (2,4) and (-1,-1):

2004

Between (-1,-1) and (2,-4):

2005

Between (2,-4) and (-1,1):

2006

Between (-1,1) and (2,4):

2007

Term:  $-2 - (-4) = 2$ 

2008

Between (2,-4) and (-1,1):

2009

Term:  $4 - (-2) = 6$ 

2010

Between (-1,-1) and (2,-4):

2011

Term:  $2 - 4 = -2$ 

2012

Sum all terms:  $-6 + 2 + 6 - 2 = 0$ 

2013

Wait, that can't be right because the area can't be zero. I must have made a mistake in the order of

2014

the points or in the calculation.

2015

Wait, perhaps the order of the points is incorrect. Let me try a different order or check the points.

2016

Alternatively, maybe the quadrilateral is actually a rectangle or another shape, but the shoelace

2017

formula requires the points to be ordered either clockwise or counterclockwise without crossing.

2018

Let me list the points again:

2019

(-1,1), (2,4), (-1,-1), (2,-4)

2020

Compute the terms:

2021

1. (-1,1) to (2,4):

2022

 $x_i y_{i+1} = (-1)(4) = -4$ 

2023

 $x_{i+1} y_i = (2)(1) = 2$ 

2024

Term:  $-4 - 2 = -6$ 

2025

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2052  
 2053  
 2054  
 2055  
 2056 2. (2,4) to (2,-4):  
 $x_i y_{\{i+1\}} = (2)(-4) = -8$   
 2057  $x_{\{i+1\}} y_i = (2)(4) = 8$   
 2058 Term:  $-8 - 8 = -16$   
 2059  
 2060 3. (2,-4) to (-1,-1):  
 $x_i y_{\{i+1\}} = (2)(-1) = -2$   
 2061  $x_{\{i+1\}} y_i = (-1)(-4) = 4$   
 2062 Term:  $-2 - 4 = -6$   
 2063  
 2064 4. (-1,-1) to (-1,1):  
 $x_i y_{\{i+1\}} = (-1)(1) = -1$   
 2065  $x_{\{i+1\}} y_i = (-1)(-1) = 1$   
 2066 Term:  $-1 - 1 = -2$   
 2067  
 2068 Sum all terms:  $-6 - 16 - 6 - 2 = -30$   
 2069  
 2070 Area =  $1/2 | -30 | = 15$   
 2071  
 2072 **So**, the area is 15.  
 2073  
 2074 **Wait**, that seems large, **but** given the points, it might be correct. Let me verify.  
 2075  
 2076 **Alternatively**, maybe using vectors or another method would give the same result.  
 2077  
 2078 **Alternatively**, since the quadrilateral is symmetric, I can compute the area by finding the area of  
 2079 two triangles and adding them.  
 2080  
 2081 **But** the shoelace formula gave me 15, which seems plausible.  
 2082  
 2083 **So**, the area of the quadrilateral is 15.  
 2084  
 2085  
 2086  
 2087  
 2088  
 2089  
 2090  
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 2101  
 2102  
 2103 Figure 5: Detail visualization of model outputs on both easy and hard problems.  
 2104  
 2105