Recent Developments of an Optimal Control approach to Nonrigid Image Registration

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Abstract. The Variational Principle (VP) forms diffeomorphisms (non-folding grids) with prescribed Jacobian determinant (JD) and curl under an optimal control set-up, which satisfies the properties of a Lie group. To take advantage of that, it is meaningful to regularize the resulting deformations of the image registration problem into the solution pool of VP. In this research note, (1) we provide an optimal control formulation of the image registration problem under a similar optimal control set-up as is VP; (2) numerical examples demonstrate the confirmation of diffeomorphic solutions as expected.

Keywords: Diffeomorphic Image Registration · Computational Diffeomorphism · Jacobian Determinant · Curl · $Green$'s identities

1 Our approach to Image Registration

This work connects the resulting registration deformations to the solution pool of VP in [1], which achieves a recent progression in describing non-folding grids in a diffeomorphism group. Hence, to restrict the image registration method built in [3] satisfying the constraint of VP, it is reformulated and proposed as follows: let $I_{\boldsymbol{m}}$ be a ***moving*** image is to be registered to a ***fixed*** image $I_{\boldsymbol{f}}$ on the fixed and bounded domain ($\boldsymbol{\omega} = <x, y, z> \in)\Omega \subset \mathbb{R}^3$, the energy function $Loss$ is minimized over the form $\boldsymbol{\phi} = \boldsymbol{id} + \boldsymbol{u}$ on Ω with $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$,

$$Loss(\boldsymbol{\phi}) = \frac{1}{2}\int_\Omega [I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}}]^2 d\boldsymbol{\omega} \quad \text{subjects to } \Delta\boldsymbol{\phi} = \boldsymbol{F}(f, \boldsymbol{g}) \text{ in } \Omega, \quad (1)$$

where the scalar-valued f and the vector-valued $\boldsymbol{g}$ are the control functions in the sense of VP that mimic the prescribed JD and curl, respectively.

1.1 Gradient with respect to control $\boldsymbol{F}$
The variational gradient of (1) with respect to $\delta\Delta\boldsymbol{\phi} = \delta\Delta\boldsymbol{u} = \delta\boldsymbol{F}$ is derived. For all $\delta\boldsymbol{F}$ vanishing on $\partial\Omega$ and by $Green$'s identities with fixed boundary condition,

$$\delta Loss(\boldsymbol{\phi}) = \delta(\frac{1}{2}\int_\Omega [I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}}]^2 d\boldsymbol{\omega}) = \int_\Omega [(I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}})\nabla I_{\boldsymbol{m}}(\boldsymbol{\phi}) \cdot \delta\boldsymbol{\phi}] d\boldsymbol{\omega}$$

$$= \int_\Omega [\Delta\boldsymbol{b} \cdot \delta\boldsymbol{\phi}] d\boldsymbol{\omega} = \int_\Omega [\boldsymbol{b} \cdot \delta\Delta\boldsymbol{\phi}] d\boldsymbol{\omega} = \int_\Omega [\boldsymbol{b} \cdot \delta\boldsymbol{F}] d\boldsymbol{\omega} \quad \Rightarrow \frac{\partial Loss}{\partial\boldsymbol{F}} = \boldsymbol{b}, \quad (2)$$

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where $\Delta \boldsymbol{b} = (I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}})\nabla I_{\boldsymbol{m}}(\boldsymbol{\phi})$, so, a gradient-based algorithm can be formed.

1.2 Hessian Matrix with respect to control function $\boldsymbol{F}$

In case of a Newton optimizing scheme is applicable, from (2), one can derive the Hessian matrix $\boldsymbol{H}$ of (1) with respect to $\boldsymbol{F}$ as follows,

$$\delta^2 Loss(\boldsymbol{\phi}) := \delta(\delta Loss(\boldsymbol{\phi})) = \delta\left(\int_\Omega [(I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}})\nabla I_{\boldsymbol{m}}(\boldsymbol{\phi}) \cdot \delta\boldsymbol{\phi}]d\boldsymbol{\omega}\right) = \int_\Omega [\delta\boldsymbol{\phi}^\top \boldsymbol{K}\delta\boldsymbol{\phi}]d\boldsymbol{\omega},$$

$$\text{where } \Delta^2 \boldsymbol{H} = \boldsymbol{K} = \nabla I_{\boldsymbol{m}}(\boldsymbol{\phi})[\nabla I_{\boldsymbol{m}}(\boldsymbol{\phi})]^\top + (I_{\boldsymbol{m}}(\boldsymbol{\phi}) - I_{\boldsymbol{f}})\nabla^2 I_{\boldsymbol{m}}(\boldsymbol{\phi}),$$

$$\text{and } \nabla^2 I_{\boldsymbol{m}}(\boldsymbol{\phi}) = \begin{pmatrix} I_{\boldsymbol{m}}(\boldsymbol{\phi})_{xx} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{xy} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{xz} \\ I_{\boldsymbol{m}}(\boldsymbol{\phi})_{yx} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{yy} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{yz} \\ I_{\boldsymbol{m}}(\boldsymbol{\phi})_{zx} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{zy} & I_{\boldsymbol{m}}(\boldsymbol{\phi})_{zz} \end{pmatrix},$$

$$\text{so, } \delta^2 Loss(\boldsymbol{\phi}) = \int_\Omega [\delta\boldsymbol{\phi}^\top \Delta^2 \boldsymbol{H}\delta\boldsymbol{\phi}]d\boldsymbol{\omega} = \int_\Omega [\delta\Delta\boldsymbol{\phi}^\top \boldsymbol{H}\delta\Delta\boldsymbol{\phi}]d\boldsymbol{\omega} \Rightarrow \frac{\partial^2 Loss}{(\partial \boldsymbol{F})^2} = \boldsymbol{H}. \quad (3)$$

A necessary condition that ensures a Newton scheme works is to show such Hessian $\boldsymbol{H}$ must be of Semi-Positive Definite matrix. This is left for future study.

1.3 Partial Gradients with respect to control functions $\hat{f}$ and $\boldsymbol{g}$

To ensure (1) producing diffeomorphic solutions that is controlled by $J_{min} \in (0,1)$, instead of optimizing along $\boldsymbol{F}$ by (2), it can be set that $f := J_{min} + \hat{f}^2$ in (1). Since it is known $\delta\Delta\boldsymbol{u} = \delta\boldsymbol{F} = \delta(\nabla f - \nabla \times \boldsymbol{g})$, then, it carries to,

$$\delta Loss(\boldsymbol{\phi}) = \int_\Omega [\boldsymbol{b} \cdot \delta\Delta\boldsymbol{\phi}]d\boldsymbol{\omega} = \int_\Omega [\boldsymbol{b} \cdot \delta\boldsymbol{F}]d\boldsymbol{\omega} = \int_\Omega [\boldsymbol{b} \cdot \delta(\nabla f - \nabla \times \boldsymbol{g})]d\boldsymbol{\omega}$$

$$= \int_\Omega [\boldsymbol{b} \cdot (\nabla\delta(J_{min} + \hat{f}^2)]d\boldsymbol{\omega} + \int_\Omega [-\boldsymbol{b} \cdot \nabla \times \delta\boldsymbol{g}]d\boldsymbol{\omega}$$

$$= \int_\Omega [\boldsymbol{b} \cdot (2\hat{f}\nabla\delta\hat{f})]d\boldsymbol{\omega} + \int_\Omega [-\boldsymbol{b} \cdot \nabla \times \delta\boldsymbol{g}]d\boldsymbol{\omega} = \int_\Omega [-2\hat{f}\nabla \cdot \boldsymbol{b}\delta\hat{f}]d\boldsymbol{\omega} + \int_\Omega [-\nabla \times \boldsymbol{b} \cdot \delta\boldsymbol{g}]d\boldsymbol{\omega}$$

$$\Rightarrow \quad \frac{\partial Loss}{\partial \hat{f}} = -2\hat{f}\nabla \cdot \boldsymbol{b} \quad \text{and} \quad \frac{\partial Loss}{\partial \boldsymbol{g}} = -\nabla \times \boldsymbol{b}. \quad (4)$$

2 Numerical Examples

In our algorithms, $J_{min} = 0.5$ is artificially set. It is desirable to design a mechanism that yields optimal values of J_{min}. The gradient-based algorithms can be structured with (1) the coarse-to-fine **multiresolution** technique, which fits better in large deformation problems over binary images, as it did in [2]; and (2) the function composition **regriding** technique, which divides the problem difficulty and prevent non-diffeomorphic solutions on medical image registrations. These observations are demonstrated by the next example.

2.1   A Large Deformation Test and A MRI Registration Test

The J-to-V part of this example is done with **multiresolution** and the Brain Morph part is done with **regriding**. In Fig.1(c, j), $\boldsymbol{\phi}$ is the diffeomorphic solution found by the proposed method; Fig.1(d, k), $I_{\boldsymbol{m}}(\boldsymbol{\phi})$ is the registered image that is close to $I_{\boldsymbol{f}}$, Fig.1(b, i). Next, $\boldsymbol{\phi}_{vp}^{-1}$ is the inverse of $\boldsymbol{\phi}$ that constructed by VP. In Fig.1(f,m,t), $\boldsymbol{\phi}$ is composed by $\boldsymbol{\phi}^{-1}$, in Red grid, and superposed on Black grid $\boldsymbol{id}$ but the Black grid barely shows. This shows the composition $\boldsymbol{T} = \boldsymbol{\phi}_{vp}^{-1} \circ \boldsymbol{\phi}$ is very close to $\boldsymbol{id}$. Therefore, $\boldsymbol{\phi}_{vp}^{-1}$ can be treated as the inverse to $\boldsymbol{\phi}$ and they are of the same diffeomorphism group which VP focuses.

Table 1: Evaluation of the Proposed Image Registration

e.g.	Ω	$ratio = Loss(\boldsymbol{\phi})/Loss(\boldsymbol{id})$	$\min(\det\nabla\boldsymbol{\phi})$	JSC	DICE
J-to-V	$[1,128]^2$	0.0034	0.2191	0.9337	0.9657
Brain Morph	$[1,128]^2$	0.0605	0.2540	0.9849	0.9924

(a) $I_{\boldsymbol{m}}$ (b) $I_{\boldsymbol{f}}$ (c) $\boldsymbol{\phi}$ (d) $I_{\boldsymbol{m}}(\boldsymbol{\phi})$ (e) $\boldsymbol{\phi}_{vp}^{-1}$ (f) $\boldsymbol{T}$ vs $\boldsymbol{id}$ (g) $I_{\boldsymbol{f}}(\boldsymbol{\phi}_{vp}^{-1})$

(h) $I_{\boldsymbol{m}}$ (i) $I_{\boldsymbol{f}}$ (j) $\boldsymbol{\phi}$ (k) $I_{\boldsymbol{m}}(\boldsymbol{\phi})$ (l) $\boldsymbol{\phi}_{vp}^{-1}$ (m) $\boldsymbol{T}$ vs $\boldsymbol{id}$ (n) $I_{\boldsymbol{f}}(\boldsymbol{\phi}_{vp}^{-1})$
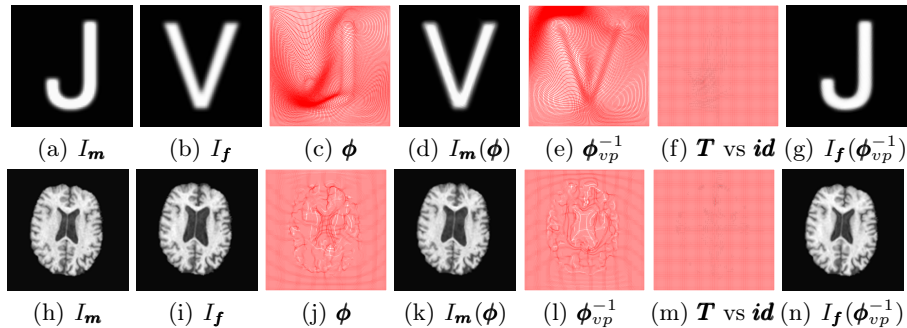
Fig. 1: Resulting Registration Deformations and their Inverses by VP

The question is whether $\boldsymbol{\phi}_{vp}^{-1}$ is also a valid inverse registration deformation that moves $I_{\boldsymbol{f}}$ back to $I_{\boldsymbol{m}}$. The answer is YES, at least in our tested examples. $I_{\boldsymbol{f}}(\boldsymbol{\phi}_{vp}^{-1})$ is indeed close to $I_{\boldsymbol{m}}$. That means $\boldsymbol{\phi}_{vp}^{-1}$ can be treated as a valid registration deformation from $I_{\boldsymbol{f}}$ to $I_{\boldsymbol{m}}$, as it is confirmed by the Table. 2 records.

Table 2: Evaluation of $\boldsymbol{\phi}_{vp}^{-1}$ by VP in the sense of Image Registration

e.g.	$ratio$ (of $Loss$ from $I_{\boldsymbol{f}}(\boldsymbol{\phi}_{vp}^{-1})$ to $I_{\boldsymbol{m}}$)	$\min(\det\nabla\boldsymbol{\phi}_{vp}^{-1})$	JSC	DICE
J-to-V	0.0029	0.1520	0.9195	0.9581
Brain morph	0.0657	0.3212	0.9832	0.9915

3   Discussion

This note is merely the analytic description and a short demonstration of the proposed method. A full and detailed paper will be available in a soon future.

References

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