Pseudo Clifford Bandpass Prolates

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***Abstract*—We introduce operators which generalise the classical modulation and translation operators, now acting on functions defined on $\mathbb{R}^m$ and taking values in the associated Clifford algebra $\mathbb{C}_m$. The modulation operators are used to map orthonormal bases for Paley-Wiener spaces associated with balls in $\mathbb{R}^m$ to incomplete orthonormal sets in Paley-Wiener spaces PW_A associated with annuli A in $\mathbb{R}^m$. The complementary spaces are characterised and an orthonormal basis for them is given. These bases are used to construct an orthonormal basis for PW_A composed of pseudo bandpass prolates.**

I. Introduction

Given $c > 0$, the Paley-Wiener space $PW_c(\mathbb{R})$ is the collection of functions $f \in L^2(\mathbb{R})$ whose Fourier transforms $\mathcal{F}f$ are supported on the interval $[-c, c]$. Let Q_c be the *time-limiting* projection operator $Q_c f(t) = \mathbf{1}_{[-c,c]}(t)f(t)$ where $\mathbf{1}_{[-c,c]}$ is the characteristic function of the interval $[-c, c]$. The *band-limiting* operator P_c is given by $P_c = \mathcal{F}Q_c\mathcal{F}$. The (one-dimensional) prolate spheroidal wavefunctions (PSWFs) are eigenfunctions of the self-adjoint Hilbert-Schmidt integral operator P_cQ_1. They are most efficiently computed by observing that they are also eigenfunctions of a second order differential operator $\mathcal{L}_c$ which commutes with P_cQ_1 [9], [6].

For $0 < c < c' < \infty$, bandpass prolates (BPPs) are eigenfunctions of $(P_{c'} - P_c)Q_1$. BPPs are members of $PW_{c',c}(\mathbb{R})$ – the space of square-integrable functions on the line whose Fourier transforms are supported on $[-c', -c] \cup [c, c']$. It was shown [8] that bandpass prolates are not eigenfunctions of a differential operator. Nevertheless [7] the construction of BPPs can be achieved by solving an appropriate matrix eigenvalue problem. The key observation in the construction is that if $\{\varphi_n\}_{n=0}^\infty$ is an orthonormal basis for PW_c and $t > c$, then $\{e^{-2\pi itx}\varphi_n(x)\}_{n=0}^\infty \cup \{e^{2\pi itx}\varphi_n(x)\}_{n=0}^\infty$ is an orthonormal basis for $PW_{t-c,t+c}$.

In this paper we investigate the multidimensional analogue of this construction in which intervals are replaced by balls and the union of intervals is replaced by spherically symmetric annuli in m-dimensional euclidean space $\mathbb{R}^m$ ($m \geq 3$ odd). This requires the application of techniques from *Clifford analysis* [3].

Let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis for m-dimensional euclidean space $\mathbb{R}^m$. The associative *Clifford algebra* $\mathbb{R}_m$ is the 2^m-dimensional algebra spanned by the collection

$$\bigcup_{j=1}^m \{e_A : A = \{i_1, i_2, \dots, i_j\} \text{ with } 1 \leq i_1 < i_2 < \cdots < i_j \leq m\}$$

with algebraic properties $e_\emptyset = 1$ (the identity), $e_j^2 = -1$, and if $j < k$ then $e_{\{j,k\}} = e_je_k = -e_ke_j$. Here $\emptyset$ is the null set and we often abuse notation and write $e_\emptyset = e_0 = 1$. Notice that for convenience we write $e_{\{j\}} = e_j$. In particular we have $\mathbb{R}_m = \{\sum_A x_Ae_A;\ x_A \in \mathbb{R}\}$. Similarly, we have the complexified Clifford algebra $\mathbb{C}_m = \{\sum_A z_Ae_A;\ z_A \in \mathbb{C}\}$.

The canonical mapping of $\mathbb{R}^m$ into $\mathbb{R}_m$ maps the vector $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ to $\sum_{j=1}^m x_je_j \in \mathbb{R}_m$. For this reason, elements of $\mathbb{R}_m$ of the form $\sum_{j=1}^m x_je_j$ are also known as *vectors*. Notice that $\mathbb{R}_m$ decomposes as $\mathbb{R}_m = \Lambda_0 \oplus \Lambda_1 \oplus \dots \Lambda_m$, where $\Lambda_j = \{\sum_{|A|=j} x_Ae_A\}$. A similar decomposition applies to $\mathbb{C}_m$. In particular, Λ_0 is the collection of scalars while Λ_1 is the collection of vectors. Given $x \in \mathbb{R}_m$ of the form $x = \sum_A x_Ae_A$ and $0 \leq p \leq m$ we write $[x]_p$ to mean the "Λ_p-part" of x, i.e, $[x]_p = \sum_{|A|=p} x_Ae_A$.

If $x, y \in \mathbb{R}_m$ are vectors, then

$$x^2 = -|x|^2 \quad \text{and} \quad xy = -\langle x, y\rangle + x \wedge y \in \Lambda_0 \oplus \Lambda_2. \tag{1}$$

Here $\langle x, y\rangle$ is the usual dot product of x and y while $x \wedge y$ is their *wedge product*. The linear involution $\overline{u}$ of $u \in \mathbb{R}_m$ is determined by the rules $\overline{x} = -x$ if $x \in \Lambda_1$ while

$$\overline{uv} = \overline{v}\,\overline{u} \tag{2}$$

for all $u, v \in \mathbb{R}_m$. On $\mathbb{C}_m$, (2) still applies, but we also have $\overline{\lambda e_j} = \lambda^*\overline{e_j}$ where λ^* is the complex conjugate of the complex number λ. Further details are available in [3].

Given an open domain Ω in $\mathbb{R}^m$, the Dirac operator acts on functions in $C^1(\Omega, \mathbb{C}_m)$ by

$$Df(x) = \sum_{j=1}^m e_j \frac{\partial f}{\partial x_j}.$$

Functions in the kernel of the Dirac operator are said to be *monogenic*. A *spherical monogenic* of degree $k \geq 0$ is a polynomial Y defined on $\mathbb{R}^m$ of homogeneous degree k (i.e., $Y(\lambda x) = \lambda^kY(x)$ for all $\lambda > 0$) with coefficients in $\mathbb{C}_m$, which is also monogenic.

We consider the right Clifford module $\mathcal{H} = L^2(\mathbb{R}^m, \mathbb{C}_m)$ of measurable $\mathbb{C}_m$-valued functions f defined on $\mathbb{R}^m$ for which $\int_{\mathbb{R}^m} |f(x)|^2\,dx < \infty$. The $\mathbb{C}_m$-valued inner product on $\mathcal{H}$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^m} \overline{f(x)}g(x)\,dx.$$

The $\mathbb{C}_m$-valued inner product on the sequence space $\ell^2(\mathbb{N},\mathbb{C}_m)$ is given by

$$\langle \mathbf{a},\mathbf{b}\rangle_{\ell^2}=\sum_{n=1}^{\infty}\overline{a_n}b_n.$$

The Fourier transform $\mathcal{F}$ is defined on $L^1(\mathbb{R}^m,\mathbb{C}_m)$ by $\mathcal{F}f(\xi)=\int_{\mathbb{R}^m}f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx$ and extends to a unitary operator on $\mathcal{H}$. The Paley-Wiener space PW_c^m associated with the ball $B(c)$ (centred at the origin and of radius $c>0$) is the class of those functions $f\in\mathcal{H}$ for which $\mathcal{F}f(\xi)=0$ for $|\xi|>c$. Similarly, if $0\le c<c'<\infty$, the Paley-Wiener space $PW_{c',c}^m$ associated with the annulus $A(c',c)$ is the class of those functions $f\in\mathcal{H}$ for which $\mathcal{F}f(\xi)=0$ for $|\xi|<c$ or $|\xi|>c'$.

A. *Clifford-Legendre polynomials and Clifford Prolate Spheroidal Wavefunctions*

In [1] and [2], the Clifford-Legendre polynomials (C-L polys) $\bar{C}_n^0(Y_k^\ell)$ are studied. Here $\{Y_k^\ell\}_{\ell=1}^{d_k^m}$ is an orthonormal basis for the space $\mathcal{H}_k$ of spherical monogenics of degree k, and d_k^m is the dimension of $\mathcal{H}_k$. Each C-L poly is an $\mathbb{R}_m$-valued function on $\mathbb{R}^m$, normalised so that $\int_{B(1)}|\bar{C}_n^0(Y_k^\ell)(x)|^2\,dx=1$, and takes one of two forms, depending on whether n is even or odd:

$$\begin{aligned}\bar{C}_{2N}^0(Y_k^\ell)(x)&=P_N(|x|^2)Y_k^\ell(x)\\ \bar{C}_{2N+1}^0(Y_k^\ell)(x)&=xQ_N(|x|^2)Y_k^\ell(x)\end{aligned}$$

where P_N and Q_N are real-valued polynomials of degree N on $[0,1]$. The C-L polys satisfy the *Bonnet formula* [2]:

$$\begin{aligned}x\bar{C}_{2N}^0(Y_k^\ell)(x)&=\alpha_N\bar{C}_{2N+1}^0(Y_k^\ell)(x)+\beta_N\bar{C}_{2N-1}^0(Y_k^\ell)(x)\\ x\bar{C}_{2N+1}^0(Y_k^\ell)(x)&=\alpha_N'\bar{C}_{2N+2}^0(Y_k^\ell)(x)+\beta_N'\bar{C}_{2N}^0(Y_k^\ell)(x)\end{aligned}\tag{3}$$

where α_N, β_N, α_N', β_N' are explicit real constants which are bounded independent of N.

Given $0\le c<c'<\infty$ and $r>0$, we define orthogonal projections P_c, $P_{c',c}$ and Q_r on $\mathcal{H}$ as follows: $Q_rf(x)=\mathbf{1}_{B(r)}(x)f(x)$ and

$$P_cf(x)=\mathcal{F}^{-1}Q_c\mathcal{F};\quad P_{c',c}=\mathcal{F}^{-1}(Q_{c'}-Q_c)\mathcal{F}.$$

Here $\mathbf{1}_{B(r)}$ is the characteristic function of $B(r)$, Q_r is the *space-limiting operator* which truncates functions outside $B(r)$, P_c and $P_{c',c}$ are *frequency-limiting operators* which truncate the Fourier transforms of functions outside $B(c)$ and the annulus $A(c',c)$ respectively. By Q we mean the operator Q_1. Clifford-Prolate Spheroidal Wavefunctions (CPSWFs) are eigenfunctions of P_cQ. In [1] and [2], orthonormal bases $\{\psi_{n,k}^\ell:(n,k,\ell)\in\Lambda\}$ for PW_c^m consisting of CPSWFs are constructed and their properties developed. Here

$$\Lambda=\{(n,k,\ell):\,n,k\ge 0,\,1\le\ell\le d_k^m\}.$$

We have $P_cQ\psi_{n,k}^\ell=\lambda_n^k\psi_{n,k}^\ell$ with $\lambda_n^k\ge 0$ and $\lambda_n^k\downarrow 0$ as $n\to\infty$. The functions $\phi_{n,k}^\ell=(\lambda_n^k)^{-1/2}Q\psi_n^k$ $((n,k,\ell)\in\Lambda)$ form an orthonormal basis for $L^2(B(1),\mathbb{C}_m)$. The construction of CPSWFs given in [1] and [2] involves the numerical computation of real constants $d_{N,i}^{k,\ell}$ and $b_{N,i}^{k,\ell}$ for which

$$\begin{aligned}\phi_{2N,k}^\ell&=\sum_{i=0}^{\infty}\bar{C}_{2i}^0(Y_k^\ell)d_{N,i}^{k,\ell}\\ \phi_{2N+1,k}^\ell&=\sum_{i=0}^{\infty}\bar{C}_{2i+1}^0(Y_k^\ell)b_{N,i}^{k,\ell}.\end{aligned}\tag{4}$$

It is shown in [2] that the CPSWFs have the functional form

$$\begin{aligned}\phi_{2N,k}^\ell(x)&=p_N(|x|)Y_k^\ell(x)\\ \phi_{2N+1,k}^\ell(x)&=xq_N(|x|)Y_k^\ell(x)\end{aligned}\tag{5}$$

with p_N, q_N radial functions whose expansions in Jacobi polynomials are explicitly computed.

B. *Clifford translations and modulations*

As a consequence of (1), we have for each $t\in\mathbb{R}$ and $x\in\mathbb{R}^m$,

$$e^{tx}=\sum_{n=0}^{\infty}\frac{t^nx^n}{n!}=\cos(t|x|)+\frac{x}{|x|}\sin(t|x|).\tag{6}$$

Since $(\frac{x}{|x|})^2=-1$, we may view equation (6) as a generalisation of Euler's formula for complex exponentials. Given $\xi,y\in\mathbb{R}^m$, the classical modulation operator M_ξ and classical translation operator T_y act on $\mathcal{H}$ by $M_\xi f(x)=e^{-2\pi i\langle x,\xi\rangle}f(x)$ and $T_yf(x)=f(x-y)$ and are intertwined by the Fourier transform: $\mathcal{F}T_y=M_y\mathcal{F}$. Analogously, given $t\in\mathbb{R}$, we define the *Clifford modulation operator* m_t on $\mathcal{H}$ by

$$m_tf(x)=e^{tx}f(x)$$

and the *Clifford translation operator* τ_t by $\tau_t=\mathcal{F}^{-1}m_t\mathcal{F}$. The multiplication operator S given by $Sf(x)=xf(x)$ is intertwined with the Dirac operator by the Fourier transform: $\mathcal{F}D=2\pi iS\mathcal{F}$. Since $D^2=-\Delta$ (the space Laplacian), we see that $u(\xi,t)=e^{itD_\xi}f(\xi)$ satisfies the following initial value problem for the wave equation in $\mathbb{R}^m$:

$$\begin{aligned}\frac{\partial^2u(\xi,t)}{\partial t^2}&=\Delta_xu(\xi,t)\quad(\xi\in\mathbb{R}^m,\ t>0)\\ u(\xi,0)&=f(\xi)\qquad(\xi\in\mathbb{R}^m)\\ \left.\frac{\partial u(\xi,t)}{\partial t}\right|_{t=0}&=iDf(\xi)\qquad(\xi\in\mathbb{R}^m).\end{aligned}$$

When the ambient dimension m is odd, the solution of this problem is obtained by taking mean-values of the initial data over spheres in $\mathbb{R}^m$: if $\mu=\frac{m-3}{2}$ (an integer) then

$$\begin{aligned}u(\xi,t)=\frac{1}{\gamma_m|S^{m-1}|}\Bigg[\partial_t\left(\frac{1}{t}\partial_t\right)^{\mu}\left(\frac{1}{t}\int_{\partial B_t(\xi)}f(y)\,d\sigma(y)\right)\\ +\left(\frac{1}{t}\partial_t\right)^{\mu}\left(\frac{1}{t}\int_{\partial B_t(\xi)}iDf(y)\,d\sigma(y)\right)\Bigg]\end{aligned}\tag{7}$$

where $\gamma_m=1.3.5\ldots(m-2)$, S^{m-1} is the unit sphere in $\mathbb{R}^m$ and $|S^{m-1}|$ is its surface measure. From (7) we see that if f is supported on $B(c)$ and $t>c$, then τ_tf is supported on the annulus $A(t-c,t+c)$. Hence if $f\in PW_c$ and $t>c$, then $e^{2\pi tx}f\in PW^m_{t-c,t+c}$.

C. Clifford translates of C-L polys

We now investigate the action of Clifford modulations on the C-L polys. Since $\bar{C}^0_n(Y^\ell_k)$ is supported on $B(1)$, if $t>1$ the Clifford translates $\tau_{\varepsilon t}\bar{C}^0_n(Y^\ell_k)$ ($\varepsilon\in\{\pm1\}$) are supported on $A(t-1,t+1)$. From the unitarity of $\tau_{\varepsilon t}$ and the orthonormality of the C-L polys on $B(1)$, we see that if $t>1$, the Clifford translates

$$\tilde{\mathcal{B}}_t=\{\tau_{\varepsilon t}\bar{C}^0_n(Y^\ell_k):(n,k,\ell)\in\Lambda,\,\varepsilon\in\{\pm1\}\}\tag{8}$$

form an orthonormal collection in $L^2(A(t-1,t+1),\mathbb{C}_m)$, or equivalently, the Clifford modulates

$$\mathcal{B}_t=\{m_{\varepsilon t}\mathcal{F}\bar{C}^0_n(Y^\ell_k):\,(n,k,\ell)\in\Lambda,\,\varepsilon\in\{\pm1\}\}$$

form an orthonormal collection in $PW^m_{t-1,t+1}$.

For $c>0$, let D_c be the unitary isotropic dilation on $\mathcal{H}$, i.e., $D_cf(x)=c^{-m/2}f(\frac{x}{c})$. Since $\tau_tD_c=D_c\tau_{t/c}$, for $t>c$ the collection

$$\mathcal{B}_{t,c}=\{m_{\varepsilon t}\mathcal{F}D_c\bar{C}^0_n(Y^\ell_k):\,(n,k,\ell)\in\Lambda,\,\varepsilon\in\{\pm1\}\}$$

forms an orthonormal collection in $PW^m_{t-c,t+c}$. As we shall see, this collection does not span $PW^m_{t-c,t+c}$.

We have the following explicit description of the Clifford translates of the C-L polys. In what follows, let m be odd and $\nu=k+\frac{m}{2}$.

Theorem 1. *Let $C^{(\nu)}_n$ be the Gegenbauer polynomial of degree n on the real line. Then for each pair of integers $N,k\geq0$, there are real constants a^k_N, b^k_N, c^k_N, d^k_N for which*

$$\tau_t\bar{C}^0_{2N}(Y_k)(\xi)=$$
$$a^k_NY_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{3}{2}}\frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{3}{2}}C^{(\nu-1)}_{2N+1}(s-t)]$$
$$+\,b^k_N\xi Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{1}{2}}\frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{1}{2}}C^{(\nu)}_{2N}(s-t)],$$
$$\tau_t\bar{C}^0_{2N+1}(Y_k)=$$
$$c^k_N\xi Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{1}{2}}\frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{1}{2}}C^{(\nu)}_{2N+1}(s-t)]$$
$$+\,d^k_NY_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{3}{2}}\frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{3}{2}}C^{(\nu-1)}_{2N+2}(s-t)]$$

where $s=|\xi|$.

Let $t>c$ and

$$\tilde{X}_{t,c}=\overline{\mathrm{sp}}\{\tau_{\varepsilon t}D_c\bar{C}^0_n(Y^\ell_k):\,(n,k,\ell)\in\Lambda,\,\varepsilon\in\{\pm1\}\}.$$

$\tilde{X}_{t,c}$ is a closed submodule of $L^2(A(t-c,t+c),\mathbb{C}_m)$. Let $\tilde{X}^\perp_{t,c}$ be the collection of those $g\in L^2(A(t-c,t+c),\mathbb{C}_m)$ which are orthogonal to all functions in $\tilde{X}_{t,c}$. Then $X_{t,c}:=\mathcal{F}(\tilde{X}_{t,c})$ is a closed submodule of $PW^m_{t-c,t+c}$.

D. An orthonormal basis for the complementary space $\tilde{X}^\perp_t$

As a consequence of Theorem 1, we have the following description of $\tilde{X}^\perp_{t,c}$:

Corollary 2. *Let $t>c$. Then*

$$\tilde{X}^\perp_{t,c}=\left\{\sum_{k=0}^{\infty}\sum_{\ell=1}^{d^m_k}[p^{(\ell)}_{2\nu-2}(|x|)+\omega q^{(\ell)}_{2\nu-3}(|x|)]\frac{Y^\ell_k(\omega)}{|x|^{2\nu-1}}\right\}\tag{9}$$

where $\omega=\dfrac{x}{|x|}$, $p_{2\nu-2}$ is the restriction of an odd polynomial of degree less than or equal to $2\nu-2$ to the interval $[t-c,t+c]$ and $q_{2\nu-3}$ is the restriction of an even polynomial of degree less than or equal to $2\nu-3$ to the interval $[t-c,t+c]$.

For $t>c$ fixed, let I_t be the interval $I_t=[t-c,t+c]$. For each non-negative integer k, consider the Hilbert space $L^2_k(I_t)$ consisting of measurable functions $f:I_t\to\mathbb{R}$ for which

$$\int_{t-c}^{t+c}|f(s)|^2\frac{ds}{s^{2\nu-1}}<\infty.$$

The inner product in $L^2_k(I_t)$ is $\langle f,g\rangle_k=\int_{t-c}^{t+c}f(s)g(s)\,\frac{ds}{s^{2\nu-1}}$. Within $L^2_k(I_t)$, we identify the closed subspaces $\pi_n(I_t)$ of restrictions to I_t of polynomials of degree less than or equal to n, $\pi^e_n(I_t)$ of restrictions to I_t of even polynomials of degree less than or equal to n and $\pi^o_n(I_t)$ of restrictions of odd polynomials of degree less than or equal to n.

For a fixed positive integer k, we apply Gram-Schmidt orthogonalisation within $L^2_k(I_t)$ to the even monomials $\{1,s^2,\ldots,s^{2\nu-3}\}$ to produce an orthonormal basis $\{P^k_{2j}\}_{j=0}^{\nu-\frac{3}{2}}$ for $\pi^e_{2\nu-3}(I_t)$. Similarly, we apply Gram-Schmidt orthogonalisation within $L^2_k(I_t)$ to the odd monomials $\{s,s^3,\ldots,s^{2\nu-2}\}$ to produce an orthonormal basis $\{P^k_{2j+1}\}_{j=0}^{\nu-\frac{3}{2}}$ for $\pi^o_{2\nu-2}(I_t)$. Note that the collection $\{P^k_{2j}\}_{j=0}^{\nu-\frac{3}{2}}$ need not be orthogonal to the collection $\{P^k_{2j+1}\}_{j=0}^{\nu-\frac{3}{2}}$.

Define functions $\chi^\varepsilon_{j,k,l}$ ($k\geq0$, $1\leq\ell\leq d^m_k$, $0\leq j\leq\nu-\frac{3}{2}$, $\varepsilon\in\{\pm1\}$) by

$$\chi^\varepsilon_{j,k,l}(x)=[P^k_{2j+1}(|x|)+\varepsilon\omega P^k_{2j}(|x|)]\frac{Y^\ell_k(\omega)}{\sqrt{2}|x|^{k+m-1}}.$$

Theorem 3. *The collection*

$$\mathcal{B}'_{t,c}=\left\{\mathcal{F}\chi^\varepsilon_{j,k,\ell}:\,k\geq0,\,1\leq\ell\leq d^m_k,\,0\leq j\leq\nu-\frac{3}{2},\,\varepsilon\in\{\pm1\}\right\}$$

is an orthonormal basis for $X^\perp_{t,c}$.

II. Pseudo Clifford Bandpass Prolates (PCBPs)

From Theorem 3 we see that the collection $\mathcal{B}_{t,c}\cup\mathcal{B}'_{t,c}$ is an orthonormal basis for $PW^m_{t-c,t+c}$ provided $t>c$. By a *pseudo prolate* we mean an eigenfunction of either $P_{X_{t,c}}Q$ or $P_{X^\perp_{t,c}}Q$.

A. Matrix formulation of the eigenvalue problem for PBCPs

Let $\{\psi^\ell_{n,k} : (n,k,\ell) \in \Lambda\}$ be the orthonormal basis for PW^m_c consisting of the CPSWFs of section IA. Since $\mathcal{V}_{k,\ell} := \overline{\mathrm{sp}}\{e^{2\pi t x}\psi^\ell_{n,k}\}_{n=0}^\infty \oplus \overline{\mathrm{sp}}\{e^{-2\pi t x}\psi^\ell_{n,k}\}_{n=0}^\infty$ is invariant under $P_{X_{t,c}}Q$ for each $k \geq 0$ and $1 \leq \ell \leq d^m_k$, we may seek eigenfunctions of $P_{X_{t,c}}Q$ within each $\mathcal{V}_{k,\ell}$. Suppose

$$f = \sum_{n=0}^\infty e^{2\pi t x}\psi^\ell_{n,k}a^+_n + \sum_{n=0}^\infty e^{-2\pi t x}\psi^\ell_{n,k}a^-_n \in \mathcal{V}_{k,\ell}$$

is an eigenfunction of $P_{X_{t,c}}Q$. Then

$$\begin{aligned}
P_{X_{t,c}}Qf &= \sum_{m=0}^\infty e^{2\pi t x}\psi^\ell_{m,k}\langle e^{2\pi t x}\psi^\ell_{m,k}, Qf\rangle \\
&\quad + \sum_{m=0}^\infty e^{-2\pi t x}\psi^\ell_{m,k}\langle e^{-2\pi t x}\psi^\ell_{m,k}, Qf\rangle \\
&= \sum_{m,n} e^{2\pi t x}\psi^\ell_{m,k}[\langle Q\psi^\ell_{m,k}, \psi^\ell_{n,k}\rangle a^+_n + \langle Q\psi^\ell_{m,k}, e^{-4\pi t x}\psi^\ell_{n,k}\rangle a^-_n] \\
&+ \sum_{m,n} e^{-2\pi t x}\psi^\ell_{m,k}[\langle Q\psi^\ell_{m,k}, e^{4\pi t x}\psi^\ell_{n,k}\rangle a^+_n + \langle Q\psi^\ell_{m,k}, \psi^\ell_{n,k}\rangle a^-_n] \\
&= \sum_m \psi^\ell_{m,k}[e^{2\pi t x}[D\mathbf{a}^+ + G\mathbf{a}^-]_m + e^{-2\pi t x}[G^*\mathbf{a}^+ + D\mathbf{a}^-]_m]
\end{aligned}$$

where D is the (doubly-infinite) diagonal matrix with diagonal entries $D_{n,n} = \lambda^k_n$ and G is the (doubly-infinite) matrix with entries $G_{n,m} = \langle Q\psi^\ell_{m,k}, e^{4\pi t x}\psi^\ell_{n,k}\rangle$. We conclude that $P_{X_{t,c}}Qf = f\lambda$ for some Clifford constant λ if and only if the vector $\mathbf{a} = \begin{pmatrix}\mathbf{a}^+ & \mathbf{a}^-\end{pmatrix}^T$ satisfies the matrix equation

$$C\mathbf{a} = \begin{pmatrix} D & G \\ G^* & D\end{pmatrix}\begin{pmatrix}\mathbf{a}^+ \\ \mathbf{a}^-\end{pmatrix} = \begin{pmatrix}\mathbf{a}^+ \\ \mathbf{a}^-\end{pmatrix}\lambda. \qquad (10)$$

It can be shown that the entries of G are real. The matrix C on the left hand side of (10) is self-adjoint and by the orthonormality of $\{\psi^\ell_{n,k}\}_{n=0}^\infty$ on $B(1)$ we have

$$\begin{aligned}
\sum_{m,n}|C_{m,n}|^2 &= 2\sum_n(\lambda^k_n)^2 + 2\sum_{m,n}|G_{m,n}|^2 \\
&= 2\sum_n(\lambda^k_n)^2 + 2\sum_{m,n}|\langle\phi^\ell_{m,k}, e^{4\pi t x}Q\phi^\ell_{n,k}\rangle|^2 \\
&\leq 2\sum_n(\lambda^k_n)^2 + 2\left(\sum_n \lambda^k_n\right)^2 < \infty.
\end{aligned}$$

We conclude that C has a complete system of eigenvectors $\mathbf{a}_n = \begin{pmatrix}\mathbf{a}^+_n & \mathbf{a}^-_n\end{pmatrix}^T$ with real eigenvalues θ_n. Let $\mathbf{a}_n$ be such an eigenvector and

$$\Psi^\ell_{n,k} = \sum_m e^{2\pi t x}\psi^\ell_{m,k}(\mathbf{a}^+_n)_m + \sum_m e^{-2\pi t x}\psi^\ell_{m,k}(\mathbf{a}^-_n)_m.$$

Then $\Psi^\ell_{n,k}$ is an eigenfunction of $P_{X_{t,c}}Q$ with eigenvalue θ_n and

$$\langle\Psi^\ell_{n,k}, \Psi^\ell_{m,k}\rangle = \langle\mathbf{a}^+_n, \mathbf{a}^+_m\rangle + \langle\mathbf{a}^-_n, \mathbf{a}^-_m\rangle = \langle\mathbf{a}_n, \mathbf{a}_m\rangle = \delta_{n,m}.$$

We have:

Theorem 4. *The collection $\mathcal{B}_\Psi = \{\Psi^\ell_{n,k} : (n,k,\ell) \in \Lambda\}$ is an orthonormal basis for $X_{t,c}$ consisting of eigenfunctions of $P_{X_{t,c}}Q$ (i.e., consisting of pseudo Clifford bandpass prolates).*

B. Matrix formulation of the eigenvalue problem for the complementary space

The elements of the orthonormal basis $\mathcal{B}'_{t,c}$ for $X^\perp_{t,c}$ may be computed as follows:

$$\begin{aligned}
\mathcal{F}\chi^{\ell,\varepsilon}_{j,k}(x) &= \int_{A_{t-c,t+c}} \frac{P^k_{2j+1}(|\xi|) + \varepsilon\omega P^k_{2j}(|\xi|)}{\sqrt{2}|\xi|^{k+m-1}}Y^\ell_k(\omega)e^{-2\pi i\langle x,\xi\rangle}\,d\xi \\
&= A^\ell_{j,k}(x) + \varepsilon B^\ell_{j,k}(x)
\end{aligned}$$

where

$$A^\ell_{j,k}(x) = \frac{2\pi}{\sqrt{2}}(-i)^k\frac{Y^\ell_k(\eta)}{|x|^{\nu-1}}\int_{t-c}^{t+c}\frac{P^k_{2j+1}(r)}{r^{\nu-1}}J_{\nu-1}(2\pi r|x|)\,dr$$

$$B^\ell_{j,k}(x) = \frac{2\pi}{\sqrt{2}}(-i)^{k+1}\eta\frac{Y^\ell_k(\eta)}{|x|^{\nu-1}}\int_{t-c}^{t+c}\frac{P^k_{2j}(r)}{r^\nu}J_\nu(2\pi r|x|)\,dr,$$

$\eta = \frac{x}{|x|}$ and J_μ is a Bessel function of the first kind. To obtain these expressions for $A^\ell_{j,k}$ and $B^\ell_{j,k}$, we have used the Funk-Hecke theorem [5]. Closed forms for $A^\ell_{j,k}(x)$ and $B^\ell_{j,k}(x)$ may be written in terms of the values of P^k_{2j} and P^k_{2j+1} and their derivatives evaluated at $t \pm c$ and the values of various Bessel functions J_μ at $2\pi(t \pm c)|x|$. Details will appear elsewhere.

If $f = \sum_j \mathcal{F}\chi^{\ell,+}_{j,k}b^+_j + \sum_j \mathcal{F}\chi^{\ell,-}_{j,k}b^-_j \in X^\perp_{t,c}$ is an eigenfunction of $P_{X^\perp_{t,c}}Q$, then the coefficients b^+_j and b^-_j satisfy the matrix equation

$$\begin{pmatrix} H^{++} & H^{+-} \\ H^{-+} & H^{--}\end{pmatrix}\begin{pmatrix}\mathbf{b}^+ \\ \mathbf{b}^-\end{pmatrix} = \begin{pmatrix}\mathbf{b}^+ \\ \mathbf{b}^-\end{pmatrix}\lambda \qquad (11)$$

where H^{++}, H^{+-}, H^{-+}, H^{--} are (doubly-infinite) matrices given by $H^{\varepsilon_1,\varepsilon_2}_{j',j} = \langle\mathcal{F}\chi^{\ell,\varepsilon_1}_{j,k}, Q\mathcal{F}\chi^{\ell,\varepsilon_2}_{j',k}\rangle$. The matrix on the left hand side of (11) is self-adjoint. Before invoking the spectral theory, we must show that this matrix is Hilbert-Schmidt. Let

$$\begin{aligned}
K_k(r,s) &:= \int_0^1 uJ_{k+\nu-1}(2\pi r u)J_{\nu-1}(2\pi s u)\,du \\
&= \frac{2\pi s J_{\nu-1}(2\pi s)J_{\nu-1}(2\pi r) - 2\pi r J_{\nu-2}(2\pi r)J_{\nu-1}(2\pi s)}{r^2 - s^2}.
\end{aligned}$$

The second equality is obtained from 6.521 (1) in [4]. We find that

$$\begin{aligned}
&\frac{\langle A^\ell_{j,k}, QA^\ell_{j',k}\rangle}{2\pi^2} \\
&= \int_{t-c}^{t+c}\frac{P^k_{2j+1}(r)}{r^{\nu-1}}\int_{t-c}^{t+c}\frac{P^k_{2j'+1}(s)}{s^{\nu-1}}K_k(r,s)\,ds\,dr \\
&= \mathcal{G}^k_1\mathcal{G}^k_2(K_k)(j,j')
\end{aligned}$$

where $\mathcal{G}^k_1\mathcal{G}^k_2F(j,j')$ is the generalised (j,j')-th Fourier coefficient of $F(r,s)$ $(r,s \in [t-c,t+c])$ relative to the orthonormal collection $\{P^k_{2j}\}_{j=0}^{\nu-\frac{3}{2}}$. By Bessel's inequality we have

$$\sum_{j,j'}|\langle A^\ell_{j,k}, QA^\ell_{j',k}\rangle|^2 \leq \int_{t-c}^{t+c}\int_{t-c}^{t+c}|K_k(r,s)|^2\frac{dr\,ds}{(sr)^{\nu-1}} < \infty.$$

A similar estimate may be made for $\sum_{j,j'}|\langle B^\ell_{j,k},QB^\ell_{j',k}\rangle|^2$. Since $\langle A^\ell_{j,k},B^\ell_{j',k}\rangle=0$, we have

$$\begin{aligned}\sum_{j,j'}|H^{\varepsilon_1,\varepsilon_2}_{j,j'}|^2&=\sum_{j,j'}|\langle A^\ell_{j,k}+\varepsilon_1B^\ell_{j,k},Q(A^\ell_{j',k}+\varepsilon_2B^\ell_{j',k}\rangle|^2\\&=\sum_j[|\langle A^\ell_{j,k},QA^\ell_{j,k}\rangle|^2+|\langle B^\ell_{j,k},QB^\ell_{j,k}\rangle|^2\\&\quad+2\varepsilon_1\varepsilon_2\Re(\langle A^\ell_{j,k},QA^\ell_{j,k}\rangle\overline{\langle B^\ell_{j,k},QB^\ell_{j,k}\rangle})\end{aligned}$$

Summing over $\varepsilon_1,\varepsilon_2\in\{\pm1\}$ gives

$$\begin{aligned}&\sum_{\varepsilon_1,\varepsilon_2}\sum_{j,j'}|H^{\varepsilon_1,\varepsilon_2}_{j,j'}|^2\\&=4\sum_j[|\langle A^\ell_{j,k},QA^\ell_{j,k}\rangle|^2+|\langle B^\ell_{j,k},QB^\ell_{j,k}\rangle|^2]\\&\le16\pi^4\int_{t-1}^{t+1}\int_{t-c}^{t+c}[|K_k(r,s)|^2+|K_{k+1}(r,s)|^2]\,\frac{dr\,ds}{(rs)^{\nu-1}}<\infty.\end{aligned}$$

C. Computation of G

The construction of bandpass prolates within $X_{t,c}$ still requires the computation of the matrix G in (10). One scheme involves another eigenfunction property of the CPSWFs ([1], [2]):

$$\int_B\psi^{k,\ell}_n(x)e^{2\pi ic\langle x,y\rangle}\,dx=\mu^k_m\psi^\ell_{n,k}(x)\tag{12}$$

where $\mu^k_n=\pm i^{k+n}\sqrt{\frac{\lambda^k_k}{c^m}}$. From (12) we have the following:

Theorem 5. *Let* $\alpha_{n,n';k}=\frac{\mu^k_n\overline{\mu^k_{n'}}(-1)^{n+k+1}}{2\pi ic((\mu^k_n)^2+(\overline{\mu^k_{n'}})^2)}$. *Then*

$$\int_B\overline{\psi^{\ell'}_{n',k'}(x)}x\psi^\ell_{n,k}(x)\,dx=\delta_{k,k'}\delta_{\ell,\ell'}\sqrt{\lambda^k_n\lambda^k_{n'}}\gamma_{n,n,;k}$$

where

$$\gamma_{n,n';k}=\alpha_{n,n';k}\begin{cases}-\overline{p^k_{\frac{n'}{2}}(1)}q^k_{\frac{n-1}{2}}(1)&\text{if } n' \text{ even and } n \text{ odd}\\\overline{q^k_{\frac{n'-1}{2}}(1)}p^k_{\frac{n}{2}}(1)&\text{if } n' \text{ odd and } n \text{ even}\\0&\text{if } n-n' \text{ even}\end{cases}\tag{13}$$

with p_N and q_N as in (5).

The orthonormality of the CPSWFs on $B(1)$ now allows for the following description of the matrix G:

Theorem 6. *Let $\Gamma^{(k)}$ be the doubly-infinite matrix with entries $\Gamma^{(k)}_{n,n'}=\gamma_{n,n';k}$ with $\gamma_{n,n';k}$ as in (13). Then $G=\exp(4\pi t\Gamma^{(k)})$.*

Finally, we provide another method of computation of the matrix G of (10). We recall the expansions (4) of $L^2(B)$-normalised CPSWFs as linear combinations of C-L polys. By iterating the Bonnet formulae (3) we find negative definite, tri-diagonal, symmetric, doubly-infinite matrices M^e and M^o for which

$$\begin{aligned}x^2\sum_{i=0}^\infty\bar C^0_{2i}(Y^\ell_k)a_i&=\sum_{i=0}^\infty\bar C^0_{2i}(Y^\ell_k)(M^e\mathbf{a})_i\\x^2\sum_{i=0}^\infty\bar C^0_{2i+1}(Y^\ell_k)b_i&=\sum_{i=0}^\infty\bar C^0_{2i+1}(Y^\ell_k)(M^o\mathbf{b})_i\end{aligned}$$

As a consequence, we have the following result:

Theorem 7. *The entries $G_{2N,2N'}$ of the matrix G are given by*

$$\begin{aligned}&\int_B\overline{\psi^\ell_{2N,k}(x)}e^{4\pi tx}\psi^{\ell'}_{2N',k'}(x)\,dx\\&=\sqrt{\lambda^k_{2N}\lambda^k_{2N'}}\delta_{k,k'}\delta_{\ell,\ell'}\langle\mathbf{d}^{k,\ell}_N,\cos(4\pi t\sqrt{-M^e})\mathbf{d}^{k,\ell}_{N'}\rangle_{\ell^2}\end{aligned}$$

where the sequence $\mathbf{d}^{k,\ell}_N$ with i-th entry $d^{k,\ell}_{N,i}$ is as in (4). Similar formulae hold for other entries of G.

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