Hierarchical Retrieval: The Geometry and a Pretrain-Finetune Recipe

Chong You **Rajesh Jayaram** **Ananda Theertha Suresh** **Robin Nittka**
Felix Yu **Sanjiv Kumar**
Google
{cyou,rkjayaram,theertha,rnittka,felixyu,sanjivk}@google.com

Abstract

Dual encoder (DE) models, where a pair of matching query and document are embedded into similar vector representations, are widely used in information retrieval due to their simplicity and scalability. However, the Euclidean geometry of the embedding space limits the expressive power of DEs, which may compromise their quality. This paper investigates such limitations in the context of *hierarchical retrieval (HR)*, where the document set has a hierarchical structure and the matching documents for a query are all of its ancestors. We first prove that DEs are feasible for HR as long as the embedding dimension is linear in the depth of the hierarchy and logarithmic in the number of documents. Then we study the problem of learning such embeddings in a standard retrieval setup where DEs are trained on samples of matching query and document pairs. Our experiments reveal a *lost-in-the-long-distance* phenomenon, where retrieval accuracy degrades for documents further away in the hierarchy. To address this, we introduce a *pretrain-finetune* recipe that significantly improves long-distance retrieval without sacrificing performance on closer documents. We experiment on a realistic hierarchy from WordNet for retrieving documents at various levels of abstraction, and show that *pretrain-finetune* boosts the recall on long-distance pairs from 19% to 76%. Finally, we demonstrate that our method improves retrieval of relevant products on a shopping queries dataset.

1 Introduction

Information retrieval [23] is the task of finding the most relevant documents within a large database in response to a user query. **Dual Encoder (DE)** is one of the most popular modeling architectures for information retrieval due to their simplicity and scalability. It functions by encoding each query and each document by a vector representation, e.g., by using a deep neural network [16, 9, 36]. Then, similarity between a query and a document is calculated using their Euclidean distance or inner product, enabling scalable retrieval through approximate nearest neighbor search [19, 17, 10].

This paper considers **Hierarchical Retrieval (HR)**, a particular case of information retrieval where the set of documents is organized into a (hidden) hierarchical structure. To motivate our retrieval task on a hierarchy, consider the keyword targeting problem in online advertising where ad platforms aim to display ads based on the relevance of the associated keywords to a user's query. While many notions of relevance exist, a particularly important case, known as *Phrase Match* [2, 1, 4], defines relevant keywords as those that are semantically more general than the user's query. This definition motivates the modeling of advertiser keywords as a hierarchy, where higher level keywords are semantically more general than the lower level keywords (see Figure 1 Left). Then, Phrase Match may be expressed as the problem of retrieving not only the keyword that matches exactly the meaning of a user query, but also all those at a higher level in the hierarchy. Motivated by Phrase Match, we

39th Conference on Neural Information Processing Systems (NeurIPS 2025).

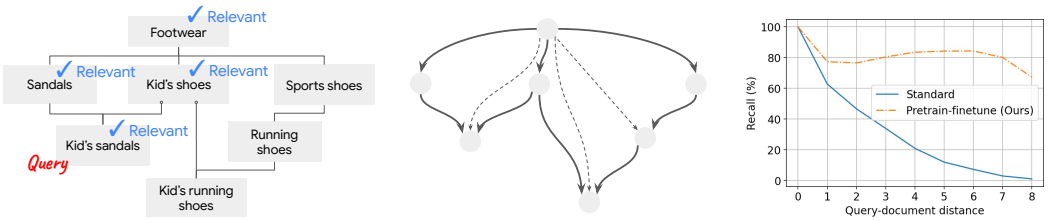


Figure 1: (Left) A illustrative example of a document set that forms a hierarchy. Given a query, e.g. "Kid's sandals", the goal is to retrieve all its ancestors in the hierarchy. (Middle) An abstraction of this hierarchy using a DAG where the edges are represented as solid arrows. For each node u of the graph, the goal of **Hierarchical Retrieval (HR)** is to retrieve all v such that v is reachable from u. (Right) The *lost-in-the-long-distance* phenomenon of a Dual Encoders (DEs) trained for HR on the WordNet DAG. Documents further away from the query in the hierarchy are more difficult to retrieve. We introduce a **pretrain-finetune** recipe to drastically improve long-distance retrieval.

define HR as the retrieval problem on a hierarchy that can be described by a direct acyclic graph (DAG, see Figure 1 Middle). Then, the goal is to retrieve, for each node u, all nodes v reachable from u with a directed path. In particular, we assume that the DAG is unobserved, which is the typical case in information retrieval where the model is learned on a data set of matching query-document pairs.

Why is Hierarchical Retrieval hard? DEs solve information retrieval tasks by finding embeddings such that a relevant document is closer in distance to the query than an irrelevant one. For HR, this distance measure needs to be *asymmetric*. For example, if "Kid's sandals" is the query then "Sandals" is considered a relevant keyword, but the inverse is not true: if "Sandals" is the query then "Kid's sandals" is not a relevant keyword. This makes *asymmetric* DE a natural choice, where the same node is embedded differently by a query encoder $Q()$ and a document encoder $D()$.

However, asymmetric DEs can still be limited for HR due to the properties of the Euclidean geometry [21]. To illustrate this, consider the query "Kid's sandals" for which a DE needs to place the two document embeddings $D(\text{"Sandals"})$ and $D(\text{"Kid's shoes"})$ in close proximity to each other, since they need to be both close to $Q(\text{"Kid's sandals"})$. On the other hand, for the query "Kid's running shoes", a DE needs to place the aforementioned two document embeddings far apart since only one of them is close to $Q(\text{"Kid's running shoes"})$. This apparent inconsistency leads to the critical question:

*Q1: Does there **exist** Dual Encoders that solve Hierarchical Retrieval?*

We will provide an affirmative answer to Q1, establishing the feasibility of DEs for HR.

Nonetheless, the existence of such DEs does not mean that they can be learned from data. In practice, the following question is of great importance:

*Q2: Can we **learn** (from train data) Dual Encoders that solve Hierarchical Retrieval?*

We show through experimental evidence that if the embedding dimension of DE is sufficiently high, then the answer to Q2 is positive as well. This result justifies DE as a feasible architecture for HR.

Lost in the long distance? While the answer to Q2 is positive with a high embedding dimension, practical retrieval systems have memory and latency requirement hence a low embedding dimension is desirable and critical. Towards improving the practice of DEs for HR, we examine cases where learned embeddings fail due to an insufficient embedding dimension, and discover an intriguing *lost-in-the-long-distance* phenomenon. This phenomenon states that documents further away from the query in the underlying hierarchy are more difficult to retrieve, hence compromises the quality of the retrieval (see Figure 1 Right). To mitigate this, we introduce a *pretrain-finetune* recipe, where a pretrained DE is finetuned on a dataset focusing solely on long-distance pairs. Such a recipe enhances the practicality of DEs for HR by improving long-distance retrieval capabilities.

We summarize the contribution of this paper as follows.

- **Dual Encoders are feasible for Hierarchical Retrieval.** We formally establish that asymmetric DEs are feasible for solving HR. Specifically, with a constructive algorithm that maps an arbitrary DAG to a set of asymmetric embeddings, we prove that such embeddings solve the associated HR task with a high probability. This holds as long as the dimension of the embedding space is larger than a threshold determined by the underlying hierarchy (Section 3).

- **Dual Encoders can be learned from training data to solve Hierarchical Retrieval.** The constructive algorithm above requires the DAG as input. On the other hand, information retrieval tasks often do not directly provide the DAG but require learning embeddings from a training dataset of matching query-document pairs. Next, we conduct an experimental study of this learning problem, starting from a synthetic tree-structured hierarchies for gaining insights. We verify that the learned DEs successfully solve HR with a sufficiently high embedding dimension (Section 4).
- **A Pretrain-Finetune recipe improves the training of Dual Encoders for Hierarchical Retrieval.** We reveal the *lost-in-the-long-distance* phenomenon under a toy setup with a synthetic tree-structured hierarchy. Critically, we show that the standard approach based on rebalancing the sampling of short vs. long distance pairs in the training dataset fails to solve the problem. This highlights the importance of our *pretrain-finetune* recipe, where a pretrained DE further finetuned on long-distance pairs gains enhanced long-distance retrieval capabilities without compromising the quality on short-distance documents (Section 5). Finally, the effectiveness of this recipe extends beyond the toy setup to real datasets, including 1) WordNet, a large lexical database of English, and 2) ESCI, a shopping queries dataset (Section 6).

1.1 Related Work

Graph embedding. Graph embedding broadly refers to methods that learn a set of node embeddings that preserves certain properties of the graph [35]. While Euclidean embedding is a natural choice, the symmetric nature of the distance metric makes it problematic for handling graphs with directed edges, such as DAGs. To address the issue, numerous works have explored ideas going beyond Euclidean embeddings, e.g., by representing each node with a geometric region [31, 32, 11, 29] or a probability distribution [33, 6]. These ideas may be coupled with non-Euclidean metrics [24, 30, 14] to better model the hierarchical structures. However, such embedding models are not suited for retrieval due to a lack of an efficient nearest neighbor search that hinders their applicability to large scale data. Another work more related to ours is [25], where asymmetric embeddings are used to capture the asymmetry and transitivity of the DAG. However, the focus of [25] is on computing embeddings on a given graph. In contrast, retrieval applications often do not have access to the underlying graph, and embeddings are learned from a dataset of matching query-document pairs.

Pretrain-finetune. The paradigm of pretraining and finetuning, where a model is first trained on a large, general-purpose dataset then adapted to downstream tasks with a finetuning procedure, is a cornerstone of modern machine learning [13, 27, 7]. For information retrieval, this paradigm demonstrates effectiveness for improved DE quality, particularly for sparse or noisy downstream data [8]. Our pretraining-finetuning recipe adapts this paradigm by treating the retrieval of long-distance matches as a downstream task, which we demonstrate to be effective in addressing the *loss-in-the-long-distance* challenge in HR.

2 Problem Setup

Hierarchical Retrieval (HR). Let $\mathcal{Q} = \{q_i\}_{i=1}^n$ be a collection of n queries and $\mathcal{D} = \{x_j\}_{j=1}^m$ be a collection of m documents, respectively. For each $i \in \{1, \ldots, n\}$, let $S(q_i) \subseteq \mathcal{D}$ be the set that contains all documents that are the most relevant to the query q_i. The goal of information retrieval is to return a ranked list of the documents in $\mathcal{D}$ for each q_i, with the top ones being those in $S(q_i)$.

This paper considers HR, a particular case of information retrieval where the document set is associated with a hierarchy. Let us assume that each query has an *exact match* document, e.g., for a query "Sandals for kids", the document "Kid's sandals" from the document set illustrated in Figure 1 is considered an exact match. Then, the relevant document set $S(q_i)$ contains both its *exact match* and all its descendants in the hierarchy. Formally, HR is defined as follows.

Definition 2.1 (Hierarchical Retrieval (HR))**.** Assume that there is a directed acyclic graph (DAG), denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, associated with the document set $\mathcal{D}$, i.e., with $\mathcal{V} = \mathcal{D}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Also, assume that there is a $E : \mathcal{Q} \to \mathcal{D}$, where $E(q_i)$ is referred to as the *exact match* to q_i. Then, HR refers to information retrieval with the relevant documents given by $S(q_i) = \{x \in \mathcal{D} : x \text{ is reachable from } E(q_i)\}$ for each $i \in \{1, \ldots, n\}$.

Dual Encoders (DEs). DEs are embedding models that map the query and document to the same embedding space, where the inner product may be used to measure relevance. A DE is composed of a query encoder, denoted as $f_q(\cdot, \boldsymbol{\theta}_q) : \mathcal{Q} \to \mathbb{R}^d$, and a document encoder, denoted as $f_x(\cdot, \boldsymbol{\theta}_x) : \mathcal{D} \to \mathbb{R}^d$, where $\boldsymbol{\theta}_q$ and $\boldsymbol{\theta}_x$ are parameters to be learned from data.

Following standard practice in information retrieval, we assume that the HR training dataset is composed of matching query-document pairs. That is, there is a collection of pairs $\{(q^{(k)}, x^{(k)})\}_{k=1}^N \subseteq \mathcal{Q} \times \mathcal{D}$ that satisfies $x^{(k)} \in S(q^{(k)})$ for each $k = 1, \ldots, N$. The parameters $\boldsymbol{\theta}_q$ and $\boldsymbol{\theta}_x$ of a DE may be learned from minimizing the following softmax loss on the training dataset:

$$L(\boldsymbol{\theta}_q, \boldsymbol{\theta}_x) = \frac{1}{N} \sum_{k=1}^{N} \text{CE}\left(\sigma\left(\frac{\boldsymbol{D}(\boldsymbol{\theta}_x)^\top \cdot f_q(q^{(k)}, \boldsymbol{\theta}_q)}{T} \right), \mathbf{1}_k \right), \tag{1}$$

where

$$\boldsymbol{D}(\boldsymbol{\theta}_x) = \left[f_x(x^{(1)}, \boldsymbol{\theta}_x), \ldots, f_x(x^{(N)}, \boldsymbol{\theta}_x) \right] \in \mathbb{R}^{d \times N} \tag{2}$$

is a matrix containing all document embeddings as columns. In above, $\mathbf{1}_k \in \mathbb{R}^N$ is a vector with the k-th entry being 1 and all other entries being 0, and T is a hyper-parameter that is fixed to be 20 in all our experiments. CE means the cross-entropy loss and $\sigma()$ represents the softmax function.

Our ultimate objective is to understand whether a DE from optimizing Equation (1) solves the HR problem. The answer will necessarily depend on multiple factors including the specificity of $\mathcal{G}$, the architecture of $f_q(\cdot, \boldsymbol{\theta}_q)$ and $f_x(\cdot, \boldsymbol{\theta}_x)$, and the optimization algorithm, etc, signficantly complicating the study. In the following, we start with a study of the geometry of HR which is agnostic to the choice of model architecture and optimization procedure.

3 The Geometry of Dual Encoders for Hierarchical Retrieval

This section studies the following problem: Is there a collection of query and document embeddings that solves the HR? An affirmative answer to this question asserts the *existence* of embeddings that solve HR, which is a necessary condition for the minimizer of Equation (1) to solve HR.

To answer this question, we start with considering the special case when the embedding dimension d is as large as the size m of the document set. Here, one may simply take $\boldsymbol{x}_j = \mathbf{1}_j$ as the embedding for each x_j. Subsequently, we set the query embedding for each q_j as $\boldsymbol{q}_i = \sum_{j \in S(q_i)} \boldsymbol{x}_j$. It can be verified that $\langle \boldsymbol{q}_i, \boldsymbol{x}_j \rangle$ take a value 1 if $j \in S(q_i)$, and 0 otherwise, i.e., these embeddings solve HR.

Algorithm 1 A constructive algorithm for Hierarchical Retrieval

1: **Input:** A query set $\mathcal{Q} = \{q_i\}_{i=1}^n$, a document set $\mathcal{D} = \{x_j\}_{j=1}^m$, and relevant document sets $\{S(q_i) \subseteq \mathcal{D}\}_{i=1}^n$.
2: Sample $\{\widehat{\boldsymbol{x}}_j\}_{j=1}^m \subseteq \mathbb{R}^d$ *i.i.d.* from the standard Gaussian distribution.
3: For each j, take $\boldsymbol{x}_j = \frac{\widehat{\boldsymbol{x}}_j}{\|\widehat{\boldsymbol{x}}_j\|_2}$.
4: For each i, take $\boldsymbol{q}_i = \frac{\widehat{\boldsymbol{q}}_i}{\|\widehat{\boldsymbol{q}}_i\|_2}$, where $\widehat{\boldsymbol{q}}_i = \sum_{j \in S(q_i)} \widehat{\boldsymbol{x}}_j$.
5: **Output:** Query embeddings $\{\boldsymbol{q}_i\}_{i=1}^n$ and document embeddings $\{\boldsymbol{x}_j\}_{j=1}^m$.

The construction above is feasible only when the embedding dimension d is allowed to be very large. Towards deriving a tighter bound, we consider another construction where the document embeddings are drawn from a Gaussian distribution, in lieu of the one-hot embeddings, see Algorithm 1. We will use that random embeddings are with a high probability *sufficiently uncorrelated* to each other to show that this construction provides a solution to HR with a much relaxed requirement on d. This is stated formally below.

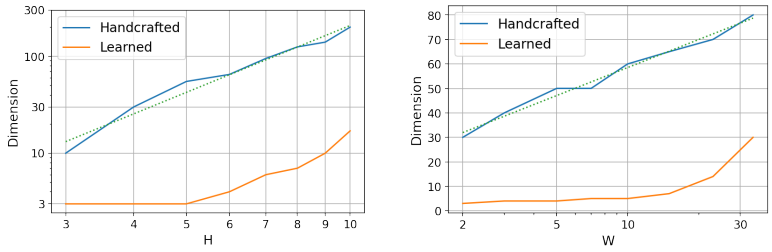
Theorem 3.1. *Consider the HR problem in Definition 2.1, and fix any $\epsilon \in (0, 1/2)$. Assume that the hierarchy $\mathcal{G}$ satisfies $|S(q_i)| \leq s, \forall i \in [n]$ for some integer s. Then there exists a dimension d with*

$$d = O(\max\{s \log m, 1/\epsilon^2 \log m\}), \tag{3}$$

a threshold[1] *r, and a collection of embeddings $\{\boldsymbol{q}_i\}_{i=1}^n, \{\boldsymbol{x}_j\}_{j=1}^m \subset \mathbb{R}^d$, such that for all $i \in [n]$ and $j \in [m]$ the following holds:*

- ***Case 1:** If $d_j \in S(q_i)$, then $\langle \boldsymbol{q}_i, \boldsymbol{x}_j \rangle \geq r + \epsilon$.*
- ***Case 2:** If $d_j \notin S(q_i)$, then $\langle \boldsymbol{q}_i, \boldsymbol{x}_j \rangle \leq r - \epsilon$.*

[1] This is a *universal* threshold r that separates the matching documents from no-matching ones for *all* queries. In practice, it is often sufficient to have a query-dependent threshold.



(a) Varying H for $W = 2$ (b) Varying W for $H = 4$

Figure 2: Comparing *learned* (*i.e.*, from optimizing Equation (1)) vs *handcrafted* (*i.e.*, from running Algorithm 1) DEs for a HR task defined on a W, H-tree. For each (H, W) pair, we experiment with an increasing sequence of embedding dimensions d until the retrieval is successful, and report that dimension on the y-axis. Successful retrieval means that recall is above 95% on the evaluation set. Dotted lines are least squares fittings of the *Handcrafted*.

Moreover, the vectors $\{\boldsymbol{q}_i\}_{i=1}^n, \{\boldsymbol{x}_j\}_{j=1}^m$ *can be constructed to satisfy these properties in time* $\tilde{O}(md + nsd)$ *with high probability.*

Theorem 3.1 asserts that the required embedding dimension is on the order specified in Equation (3). This order is logarithmic in m, implying that a small dimension can be sufficient for handling a large document set. For a fixed ϵ and m, this order is linear in s, the maximum number of relevant documents per query, which in practice is usually much smaller than m. For instance, when $\mathcal{G}$ is a perfect tree with edges pointing from each child node to its parent node, s becomes the number of levels of the tree and is on the order of $\log(m)$; plugging this into Equation (3) we obtain $M = O((\log m)^2)$, which again scales benignly with m. Finally, Equation (3) is independent of n, the number of queries. This is because the theorem only requires us to handle a subset of queries $\{\boldsymbol{q}_i\}_{i\in T}, T \subseteq \{1, \ldots, n\}$ for which their relevant document sets are distinct from each other, i.e., $S(\boldsymbol{q}_i) \neq S(\boldsymbol{q}_j), \forall \{i, j\} \subseteq T$ and $i \neq j$. For HR, the size of such a T is upper bounded by m according to Definition 2.1.

The construction procedure in Algorithm 1 requires knowing the sets $\{S(q_i)\}_{i=1,\ldots,m}$. In practice, these sets are often not directly observed. Instead, it is often the case that the training dataset is composed of a set of matching query-document pairs, see Section 2. Hence, while Theorem 3.1 establishes the correctness of such embeddings, Algorithm 1 may not be applicable in practice. Instead, the embeddings are often learned by optimizing a proper training loss such as Equation (1). Understanding if such learned embeddings solve the HR problem is the subject of the next section.

Comparing Theorem 3.1 with Prior Work [15]. Our Theorem 3.1 relates to [15] which also establishes logarithmic bounds on embedding dimensions. However, [15] addresses a different problem. The focus of [15] is multi-label classification, for which their Theorem 2.1 establishes a bound for representing the multi-label set. In contrast, our Theorem 3.1 specifically addresses HR, proving the existence of asymmetric query and document embeddings in Euclidean space that satisfy the specific ancestor-retrieval property. That being said, [15]'s result may be applied to HR by associating their multi-label set with a hierarchy. With this specification, their result states the following (informally):

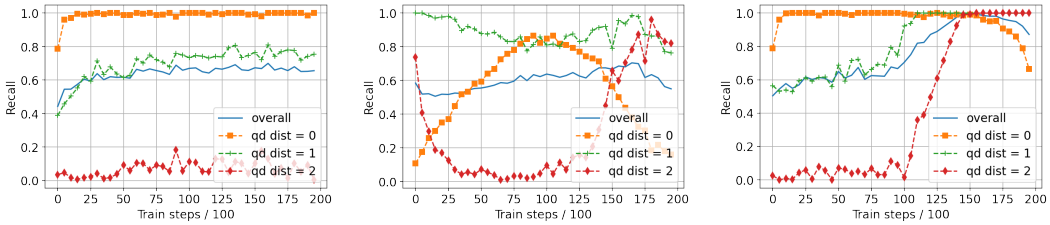
> There is a dimension $d = O(s \log m)$ such that for $d_j \in S(q_i)$, it has $\langle \boldsymbol{q}_i, \boldsymbol{x}_j \rangle > 2/3$, and for $d_j \notin S(q_i)$, it has $\langle \boldsymbol{q}_i, \boldsymbol{x}_j \rangle < 1/3$.

Comparing to this result which has a fixed gap of $1/3$ between matching and no-matching pairs, ours in Theorem 3.1 holds for an arbitrary gap of 2ϵ, hence is more general.

4 Towards Learning Dual Encoders for Hierarchical Retrieval

This section studies whether learned DEs from optimizing Equation (1) can solve HR. To understand the effect of depth and size of the hierarchy, this section focuses on synthetic, tree-structured hierarchies. Experiments on real hierarchies from practical data are provided in Section 6.

A toy setup. We consider *perfect trees*, where each non-leaf node has the same number child nodes and all tree leaf nodes are at the same level. A perfect tree is described by two parameters, namely



(a) Regular sampling of train data (b) Re-balanced train data (c) Pretraining-finetuning

Figure 3: Recall on query-document pairs at varying distances $\in \{0, 1, 2\}$ (denoted as *qd dist* in the legend). We train DEs with $d = 3$ dimensional embeddings via optimizing Equation (1) on a tree with $H = 4$ and $W = 5$. (a) Regular training data. The recall for (q, x) pairs with distances 1 and 2 are low. (b) Re-balanced training data, where (q, x) pairs with a distance of 1 or 2 are up-sampled. The recall for such pairs are significantly improved at the cost of a drastic decrease in recall for pairs with distance 0. (c) Pretrain on regular data for 10k steps then finetune on long distance pairs for another 10k steps. Recalls at all distances are close to 100% near 15k steps. In particular, the overall recall (i.e., averaged over the 3 distances) improves to 97%, compared to 66% in (a) and 70% in (b).

the number of child nodes for each non-leaf node (i.e., the *width* W), and the number of levels (i.e., the *height* H). We use H, W*-tree* to refer to a perfect tree with width W and height H.

Given an H, W-tree, we consider a HR problem where both the query set $\mathcal{Q}$ and the document set $\mathcal{D}$ have a one-to-one correspondence to the set of all nodes of the tree[2]. Then, the hierarchy associated with $\mathcal{D}$ is naturally described by a DAG with edges pointing from each child node to its parent node. Hence, the relevant document set $S(q)$ for a query $q \in \mathcal{Q}$ contains the document that corresponds to the same node in the tree as q, as well as all nodes reachable from it. We consider solving this HR problem by a lookup-table DE, a standard choice for studying embedding models [12, 26, 5]. In such a DE, the encoders f_q and f_x are lookup tables with one embedding associated with each $q \in \mathcal{Q}$ and $x \in \mathcal{D}$, respectively.

We sample training data using the following procedure. First, a query q is sampled by drawing a node with equal probabilities from all nodes of the tree. Then, we obtain a matching document to q by sampling a node with equal probabilities from the set of all of its matching documents. Training is conducted by optimizing Equation (1) with gradient descent. Evaluation is conducted on data sampled using the same procedure described above. We use the standard *recall* metric, which is the percentage of (q, x) pairs in the evaluation set for which x is one of the k documents that have the largest inner product score with q, with $k = |S(q)|$ being the total number of relevant documents for q. We are interested in the smallest embedding dimension (i.e., d) sufficient for a successful retrieval. To obtain this dimension, for each (H, W), we experiment with an increasing sequence of d and terminate the process when the evaluation recall metric is >95%.

Results. In Figure 2, we report the dimension d needed for a successful retrieval as a function of (H, W). Towards that, we vary H for a fixed $W = 2$ on the left, and vary W for a fixed $H = 4$ on the right. Both cases show that a reasonably large d is sufficient even for H up to 10 and W up to 30.

We further compare such *learned* embeddings with *handcrafted* ones from Algorithm 1. From Theorem 3.1, the handcrafted embeddings solve HR with $d = O(s \log m)$. For H, W-trees, we have $s = H$, $m = O(W^{H-1})$, which gives $d = O(H^2 \log W)$. This aligns well with our simulation results which we report in Figure 2. For example, in Figure 2a, we perform a line fitting in the log-log space and obtain a slope of 2.29, whereas the slope derived from $d = O(H^2 \log W)$ is 2. In Figure 2b, we perform a line fitting in the space of $\log(W)$ and obtains a slope of 16.5, whereas the slope derived from $d = O(H^2 \log W)$ is 16. Finally, Figure 2 shows that that learned embeddings achieve successful retrieval with a much smaller d compared to our handcrafted embeddings.

5 Improving Dual Encoders for Hierarchical Retrieval

With the establishment that standard training of DEs solves HR, this section takes one step further and asks the following practical question: Can we improve our training algorithm to minimize the dimension required for solving HR? We approach this by examining the failure cases of learned

[2]Excluding the root node since if it were included then all queries should trivially retrieve it.

embeddings from standard training *when the dimension is insufficient*. This leads us to discover a common failure case called the *lost-in-the-long-distance* phenomenon. In addressing this issue, we present a pretrain-finetune recipe that leads to an improved retrieval quality.

5.1 *Lost-in-the-Long-Distance*

We again consider HR on a H, W-tree as described in Section 4, but focus on a particular case where standard training of DE fails to retrieve relevant documents. In particular, we consider the case of $H = 4$, $W = 5$, and $d = 3$. Towards understanding this failure case, we introduce the notion of distance between a matching pair (q, x), defined as the difference between the level of the tree nodes corresponding to q and x. For example, a distance of 0 means that q and x correspond to the same node, and a distance of 1 means that x corresponds to the parent node of q.

For the tree with $H = 4$, $W = 5$, any query that corresponds to a leaf node has 3 matching documents with distance 0, 1, and 2, respectively (recall that the root node does not correspond to any query / document). We evaluate recall for query-document pairs at these three distances separately, and report results in Figure 3a. It can be seen that the learned embeddings achieve almost perfect retrieval for pairs with a distance 0, but do not work well for pairs with distance 1 and 2. We refer to the phenomenon that matching documents at longer distances to the query tend to be lost in retrieval as *lost-in-the-long-distance*.

Failure of re-balanced sampling. A tempting approach to alleviate *lost-in-the-long-distance* is to re-balance the training set, so that more pairs of longer distances are included. To test this, we consider two sampling distributions:

- *Regular* sampling which refers to the sampling procedure described in Section 4. For $H = 4$, $W = 5$, distances 0, 1, and 2 pairs are sampled with probabilities 38%, 35%, and 27%, respectively.
- *Heavy-Tail* sampling, where pairs with distances 0, 1, and 2 are sampled with probabilities 0%, 50%, and 50%, respectively.

By mixing regular and heavy-tail sampling with a ratio of $p : 1 - p$, we may create training datasets with a controllable ratio between short and long distance pairs. In Figure 3b we report the result with $p = 0.03$. It can be seen that the recall for pairs with distance 1 and 2 are significantly improved and reaches a level of beyond 80% towards the end of training. However, this comes at the cost of a significant recall degradation on distance 0 pairs. Finally, this tradeoff cannot be fixed by tuning p, as illustrated in Figure 6 (see Appendix) which contains further results with varying p in $\{0.01, 0.1, 0.3\}$.

5.2 Main Algorithm: A Pretrain-Finetune Recipe

We introduce a pretrain-finetune recipe to address the challenge of *lost-in-the-long-distance*. This approach simply means that the DE is first pretrained on a standard training set, then finetuned on a long-distance dataset. Notably, the finetuning stage requires long-distance data *only* and does *not* require tuning the ratio of short vs long distance pairs as a hyperparameter.

We conduct an experiment with pretraining and finetuning using data from regular and heavy-tail sampling, respectively, and report the results in Figure 3c. We observe that in the finetuning stage, the retrieval quality for pairs with distance 1 and 2 quickly improves and reaches nearly 100% at 15,000 train step. Notably, at this point the recall for distance 0 pairs remains close to 100%, and the overall recall (i.e., averaged over pairs of all distances) is 97%, far exceeding the regular data sampling (which has 66% recall) or re-balanced data sampling (which has 70% near 17000 steps). Finally, after 15,000 steps the quality of distance 0 pairs starts to decline. This is expected since the finetuning stage does not have any training data with distance 0. However, this quality degradation does not compromise the practicality of our approach since one can apply early stopping during the finetuning stage by monitoring the model quality on a validation set.

Discussion on data requirement. In applying the pretrain-finetune recipe, a practical question is how to construct the long-distance dataset for finetuning when the underlying hierarchy, and thus the query-document distances, is unobserved as is typical in many retrieval applications. The key point is that our recipe does not require precise path lengths or knowledge of the full DAG. Instead, it only requires a practical proxy for distance that can be used to partition the training data into short-distance and long-distance subsets. This proxy is often readily available from the data or the problem definition itself. For instance, in our shopping dataset experiment (see Section 6), we treat Exact query-product matches as the short-distance set for pretraining and Substitute matches as

Table 1: Quality of DE for HR on WordNet. *Regular sampling* refers to first sampling a query then a document uniformly at random from the set of all matching documents. *Rebalanced* means a mixture of data from regular sampling with a proportion p and a heavy-tail data of proportion $1-p$ where long-distance pairs are upsampled proportionally to their distance. *Pretrain-finetune* (ours) refers to first pretraining on regular sampling data then finetuning on heavy-tail data. Quality is measured by averaged recall on a test set. *Our method (i.e., pretrain-finetune) enables good retrieval quality for query-document pairs at all distances.*

	Query-document distance										
Method	0	1	2	3	4	5	6	7	8	Min	Overall
Embedding dimension = 16:											
Regular sampling	**100.0**	**62.8**	**46.6**	33.9	20.9	11.9	7.2	2.9	1.0	1.0	43.0
Pretrain-finetune (Ours)	**100.0**	57.1	46.4	**47.9**	**50.2**	**53.6**	**53.1**	**47.3**	**32.0**	**32.0**	**60.1**
Embedding dimension = 32:											
Regular sampling	**100.0**	**90.8**	**79.2**	62.3	46.4	31.8	20.1	14.8	8.4	8.4	61.8
Pretrain-finetune (Ours)	**100.0**	77.3	76.5	**80.4**	**83.5**	**84.2**	**84.3**	**80.1**	**67.3**	**67.3**	**87.3**
Embedding dimension = 64:											
Regular sampling	**100.0**	**93.9**	86.9	76.8	60.2	46.8	36.1	28.8	19.4	19.4	71.4
Rebalanced (p=0.01)	0.6	46.9	69.7	67.2	56.3	44.7	39.4	34.9	36.6	0.6	41.8
Rebalanced (p=0.03)	2.8	49.6	69.4	64.2	52.1	43.2	37.7	31.8	32.7	2.8	40.7
Pretrain-finetune (Ours)	**100.0**	90.8	**91.6**	**92.7**	**92.6**	**91.8**	**90.9**	**87.3**	**75.7**	**75.7**	**92.3**

the long-distance set for finetuning. In other scenarios, this partition could be based on whether a document is a direct parent versus a more remote ancestor in a known but partial hierarchy. Finally, human annotation can be another viable path towards obtaining such a dataset, which is a significantly easier task than annotating the full DAG. This flexibility allows our pretrain-finetune recipe to be applied in a wide range of practical settings where the full hierarchy is not explicitly given.

Finally, our recipe implicitly assumes that there is sufficient short-distance data to learn a meaningful initial representation during pretraining. If this data is extremely sparse, pretraining may be ineffective, and a mixed training approach might indeed perform better.

6 Experiments on Real Data

In this section, we experiment with the pretrain-finetune recipe on two real datasets, namely WordNet and ESCI. On WordNet, which is a large lexical database of English, our method improves the retrieval of hypernyms that are several levels more general than the query. On ESCI, which is a shopping queries dataset where each query has both exact matching products and substitute products, our method enables a single DE to retrieve both categories at a higher recall.

6.1 WordNet Experiments

WordNet [22] is a large lexical database of English where the nouns, verbs, adjectives, and adverbs are grouped into *synsets* that represent synonyms. The set of synsets is equipped with a binary *hypernym* relation, e.g., "chair" is the hypernym of "armchair". This relation may be described by a DAG with nodes corresponding to synsets and edges pointing from a synset to its hypernym synset.

In our experiments, we use the 82,115 noun synsets as our document set $\mathcal{D}$. We take the query set $\mathcal{Q}$ to be the same as $\mathcal{D}$. For each query $q \in \mathcal{Q}$, the matching documents $S(q)$ include itself, its hypernyms, and hypernyms of all hypernyms, etc. For example, matching documents for the query "cat" include "cat", "feline", "carnivore", "placental", etc. In practice, we make a slight modification to this definition by restricting to (q, d) pairs with a distance of at most 8. Here, the distance between two synsets is defined as the length of the shortest path that connects them in the hypernym DAG.

Unless specified otherwise, we use the following *regular sampling* procedure to generate training and evaluation data. First, a query q is sampled uniformly at random among all 82,115 synsets. Then, a document is sampled uniformly at random from the set of all matching documents to q.

Lost-in-the-long-distance. We train a lookup-table DE by optimizing Equation (1) using SGD for 50k iterations on 10M matching pairs from regular sampling. We use learning rate 0.5, momentum 0.9, and batch size 4096. To evaluate the learned DE, we use the recall metric defined as the percentage of (q, x) pairs for which x is one of the k documents that have the largest inner product score with q, with $k = |S(q)|$. We use a validation set of size 10k to pick the best checkpoint. Then,

Table 2: Spearman correlation score ρ on Hyperlex for DE trained on WordNet. We used 5-dimensional embeddings to be consistent with prior work. *Our method (i.e., pretrain-finetune) obtains the best correlation score.*

Method	OrderEmb [34]	WN-Basic [34]	WN-Euclidean [24]	Regular sampling	**Pretrain-finetune (ours)**
ρ	0.195	0.240	0.389	0.350	**0.415**

we report in Table 1 the recall computed on a test set of size 10k, including an overall recall that is averaged over all pairs in the test set, and recall on slices with different query-document distances (i.e., 0, 1, ..., 8). For a varying dimension of the embedding space in $\{16, 32, 64\}$, we observe that quality degrades rapidly as a function of the distance between the query and document. A similar qualitative behavior is also observed in [18].

Rebalanced data sampling is insufficient. The *lost-in-the-long-distance* phenomenon may be attributed to the distribution of data from regular sampling, which is biased towards pairs with short distances (see Figure 4). A natural choice is to use a heavy-tail sampling of the training dataset, which works as follows. First, a query q is sampled uniformly at random from all synsets. Then, the matching document for q is sampled with a probability *proportional to the distance between the document and* q.

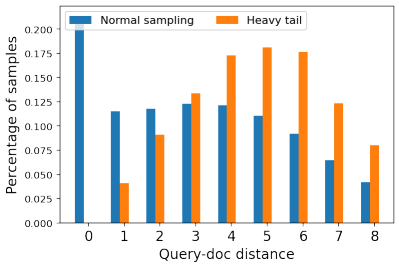


Figure 4: Distribution of regular sampling and heavy-tail data over varying query-document distances on WordNet.

We create a *rebalanced* dataset where each batch has $p \times 4096$ pairs from regular sampling and $(1-p) \times 4096$ from heavy-tailed data. Results with $p = 0.01$ or $p = 0.03$ for embedding dimension 64 are reported in Table 1. It can be seen that rebalanced data improves the retrieval quality on long distance pairs but at the cost of compromising quality on short distance pairs, aligning with the observation in Section 5.

Our pretrain-finetune recipe offers a solution. We pretrain the DE on data from normal sampling, then finetune on the heavy-tail dataset. During finetuning, we reduce the learning rate to 1,000 times smaller and increase the temperature in Equation (1) from 20 to 500; an ablation study on these two hyper-parameters is provided in Appendix E. In both of the two stages, we pick the best checkpoint on the validation set. The results in Table 1 demonstrate a significant retrieval quality improvement for long distance document, leading to much higher overall recall. We further provide an example of the retrieved documents for selected queries in Table 3. These examples show that regular sampling tends to miss the long distance pairs and the pretrain-finetune recipe fixes many such errors.

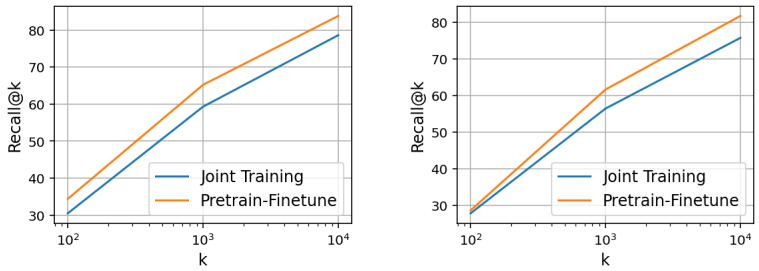
Hypernymy evaluation. We supplement our evaluation by using HyperLex [34], a dataset for evaluating how well a model captures the hyponymy-hypernymy relation between concept pairs. Here, we evaluate our DE models learned on WordNet using regular sampling as well as the pretrain-finetune recipe. We also compare with results from previous papers and report the results in Table 2. This result confirms the effectiveness of the pretrain-finetune recipe.

6.2 Experiment on ESCI Shopping Dataset

ESCI [28] is a public Amazon search dataset, containing 2.6 million manually labeled query-product relevance judgements in four categories, namely, *Exact, Substitute, Complement*, and *Irrelevant*. For our experiment, we focus on *Exact*, where the product is relevant for the query and satisfies all query specifications, and *Substitute* where the product is somewhat relevant and fails to fulfill some aspects of the query. We consider the task of retrieving, given a user query, both Exact and Substitute documents by a DE[3].

Training and evaluation data. ESCI comes with a train vs test data splitting. We take the Exact and Substitute pairs from the train split as our training sets, denoted as E_{train} and S_{train}, respectively.

[3]This task may not fit exactly the HR problem definition. However, the similarity to HR is that a Substitute match may be considered as having a longer distance to the query than an Exact match, leading to the same *lost-in-long-distance* challenge as HR.



(a) Results on *Exact* (b) Results on *Substitute*

Figure 5: Quality of *Exact* (left) and *Substitute* (right) retrieval on the ESCI dataset using a single DE. By first pretraining on Exact then finetuning on Substitute matches, *our pretrain-finetune recipe performs better than naively joint training on Exact and Substitute matches.*

E_{train} and S_{train} contain 1.3 million and 0.4 million matches, respectively. We sample 5k Exact and 2k Substitute pairs from the test split for evaluating our model. These two sets are denoted as E_{test} and S_{test}, respectively. To evaluate our model, we use

$$\text{Recall@}k = \frac{\#\{(q,d) \in T \mid d \text{ is among the top-}k \text{ matches for } q\}}{|T|}, \tag{4}$$

where T is either E_{test} or S_{test}. In words, it is the percentage of (q, x) pairs in the set T with the property that the inner product score between q and x is among the k largest ones across all $x \in \mathcal{D}$. Here, $\mathcal{D}$ is all products provided as part of ESCI and has a size of approximately 1 million.

We use the SentencePiece tokenizer, Transformers for the encoder models in DE, and the Lazy Adam optimizer [3]. Details are provided in Appendix B.

Methods and Results. A naive approach for this task is *Joint Training*, where DE is trained on the union of E_{train} and S_{train}. We compare this with our pretrain-finetune recipe, where a DE is pretrained on E_{train} then finetuned on S_{train}. The results are presented in Figure 5 on Exact matches (Left) and Substitute matches (Right). It shows that the pretrain-finetune recipe performs better than joint training in terms of recall@k for varying values of $k \in \{100, 1000, 10000\}$.

7 Conclusion

This paper studies the theory and practice of dual encoders (DE) for hierarchical retrieval (HR), the task where the document set is organized into a hierarchy. Through a geometric analysis, we first validated rigorously that DEs are capable of solving the HR problem despite the constraints from the Euclidean geometry. We then demonstrated through experiments that such DEs can be found in practice via standard DE training. Towards improving the practical performance of DE, we introduced a pretrain-finetune recipe which addresses the challenge associated with long-distance pairs. Finally, the effectiveness of this recipe is verified on real datasets including WordNet and ESCI shopping queries.

References

[1] Amazon Ads. `https://advertising.amazon.com/help/GHTRFDZRJPW6764R`. Accessed: 2024-12-15.

[2] Google Ads. `https://support.google.com/google-ads/answer/11586965?hl=en#:~:text=A%20keyword%20match%20type%20that,specific%20form%20of%20the%20meaning`. Accessed: 2024-12-15.

[3] Lazy Adam. `https://www.tensorflow.org/addons/api_docs/python/tfa/optimizers/LazyAdam`. Accessed: 2025-1-24.

[4] Microsoft Ads. `https://help.ads.microsoft.com/apex/index/3/en-us/50822#!` Accessed: 2024-12-15.

[5] Kaikai An, Ce Zheng, Bofei Gao, Haozhe Zhao, and Baobao Chang. Coarse-to-fine dual encoders are better frame identification learners. In *Findings of the Association for Computational Linguistics: EMNLP 2023*, pages 13455–13466, 2023.

[6] Ben Athiwaratkun and Andrew Gordon Wilson. Hierarchical density order embeddings. In *International Conference on Learning Representations*, 2018.

[7] Tom Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared D Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, et al. Language models are few-shot learners. *Advances in neural information processing systems*, 33:1877–1901, 2020.

[8] Wei-Cheng Chang, X Yu Felix, Yin-Wen Chang, Yiming Yang, and Sanjiv Kumar. Pre-training tasks for embedding-based large-scale retrieval. In *International Conference on Learning Representations*, 2020.

[9] Wei-Cheng Chang, Yu Felix X, Yin-Wen Chang, Yiming Yang, and Sanjiv Kumar. Pre-training tasks for embedding-based large-scale retrieval. In *International Conference on Learning Representations*, 2020.

[10] Felix Chern, Blake Hechtman, Andy Davis, Ruiqi Guo, David Majnemer, and Sanjiv Kumar. Tpu-knn: K nearest neighbor search at peak flop/s. *Advances in Neural Information Processing Systems*, 35:15489–15501, 2022.

[11] Tejas Chheda, Purujit Goyal, Trang Tran, Dhruvesh Patel, Michael Boratko, Shib Sankar Dasgupta, and Andrew Mccallum. Box embeddings: An open-source library for representation learning using geometric structures. In *Proceedings of the 2021 Conference on Empirical Methods in Natural Language Processing: System Demonstrations*, pages 203–211, 2021.

[12] Bhuwan Dhingra, Christopher Shallue, Mohammad Norouzi, Andrew Dai, and George Dahl. Embedding text in hyperbolic spaces. In *Proceedings of the Twelfth Workshop on Graph-Based Methods for Natural Language Processing (TextGraphs-12)*, pages 59–69, 2018.

[13] Jeff Donahue, Yangqing Jia, Oriol Vinyals, Judy Hoffman, Ning Zhang, Eric Tzeng, and Trevor Darrell. Decaf: A deep convolutional activation feature for generic visual recognition. In *International conference on machine learning*, pages 647–655. PMLR, 2014.

[14] Octavian Ganea, Gary Bécigneul, and Thomas Hofmann. Hyperbolic entailment cones for learning hierarchical embeddings. In *International conference on machine learning*, pages 1646–1655. PMLR, 2018.

[15] Chuan Guo, Ali Mousavi, Xiang Wu, Daniel N Holtmann-Rice, Satyen Kale, Sashank Reddi, and Sanjiv Kumar. Breaking the glass ceiling for embedding-based classifiers for large output spaces. *Advances in Neural Information Processing Systems*, 32, 2019.

[16] Jiafeng Guo, Yixing Fan, Qingyao Ai, and W Bruce Croft. A deep relevance matching model for ad-hoc retrieval. In *Proceedings of the 25th ACM international on conference on information and knowledge management*, pages 55–64, 2016.

[17] Ruiqi Guo, Philip Sun, Erik Lindgren, Quan Geng, David Simcha, Felix Chern, and Sanjiv Kumar. Accelerating large-scale inference with anisotropic vector quantization. In *International Conference on Machine Learning*, pages 3887–3896. PMLR, 2020.

[18] Yuan He, Zhangdie Yuan, Jiaoyan Chen, and Ian Horrocks. Language models as hierarchy encoders. *arXiv preprint arXiv:2401.11374*, 2024.

[19] Jeff Johnson, Matthijs Douze, and Hervé Jégou. Billion-scale similarity search with gpus. *IEEE Transactions on Big Data*, 7(3):535–547, 2019.

[20] Beatrice Laurent and Pascal Massart. Adaptive estimation of a quadratic functional by model selection. *Annals of statistics*, pages 1302–1338, 2000.

[21] Aditya Menon, Sadeep Jayasumana, Ankit Singh Rawat, Seungyeon Kim, Sashank Reddi, and Sanjiv Kumar. In defense of dual-encoders for neural ranking. In *International Conference on Machine Learning*, pages 15376–15400. PMLR, 2022.

[22] George A Miller. Wordnet: a lexical database for english. *Communications of the ACM*, 38(11):39–41, 1995.

[23] Bhaskar Mitra, Nick Craswell, et al. An introduction to neural information retrieval. *Foundations and Trends® in Information Retrieval*, 13(1):1–126, 2018.

[24] Maximillian Nickel and Douwe Kiela. Poincaré embeddings for learning hierarchical representations. *Advances in neural information processing systems*, 30, 2017.

[25] Mingdong Ou, Peng Cui, Jian Pei, Ziwei Zhang, and Wenwu Zhu. Asymmetric transitivity preserving graph embedding. In *Proceedings of the 22nd ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 1105–1114, 2016.

[26] Maulik Parmar and Apurva Narayan. Hyperbox: A supervised approach for hypernym discovery using box embeddings. In *Proceedings of the Thirteenth Language Resources and Evaluation Conference*, pages 6069–6076, 2022.

[27] Colin Raffel, Noam Shazeer, Adam Roberts, Katherine Lee, Sharan Narang, Michael Matena, Yanqi Zhou, Wei Li, and Peter J Liu. Exploring the limits of transfer learning with a unified text-to-text transformer. *Journal of machine learning research*, 21(140):1–67, 2020.

[28] Chandan K Reddy, Lluís Màrquez, Fran Valero, Nikhil Rao, Hugo Zaragoza, Sambaran Bandyopadhyay, Arnab Biswas, Anlu Xing, and Karthik Subbian. Shopping queries dataset: A large-scale esci benchmark for improving product search. *arXiv preprint arXiv:2206.06588*, 2022.

[29] Benjamin Rozonoyer, Michael Boratko, Dhruvesh Patel, Wenlong Zhao, Shib Sankar Dasgupta, Hung Le, and Andrew McCallum. Learning representations for hierarchies with minimal support. In *The Thirty-eighth Annual Conference on Neural Information Processing Systems*, 2024.

[30] Alexandru Tifrea, Gary Becigneul, and Octavian-Eugen Ganea. Poincare glove: Hyperbolic word embeddings. In *International Conference on Learning Representations*, 2018.

[31] Ivan Vendrov, Ryan Kiros, Sanja Fidler, and Raquel Urtasun. Order-embeddings of images and language. *arXiv preprint arXiv:1511.06361*, 2015.

[32] Luke Vilnis, Xiang Li, Shikhar Murty, and Andrew Mccallum. Probabilistic embedding of knowledge graphs with box lattice measures. In *Proceedings of the 56th Annual Meeting of the Association for Computational Linguistics (Volume 1: Long Papers)*, pages 263–272, 2018.

[33] Luke Vilnis and Andrew McCallum. Word representations via gaussian embedding. 2015.

[34] Ivan Vulić, Daniela Gerz, Douwe Kiela, Felix Hill, and Anna Korhonen. Hyperlex: A large-scale evaluation of graded lexical entailment. *Computational Linguistics*, 43(4):781–835, 2017.

[35] Mengjia Xu. Understanding graph embedding methods and their applications. *SIAM Review*, 63(4):825–853, 2021.

[36] Wayne Xin Zhao, Jing Liu, Ruiyang Ren, and Ji-Rong Wen. Dense text retrieval based on pretrained language models: A survey. *ACM Transactions on Information Systems*, 42(4):1–60, 2024.

A Proof of Theorem 3.1

We now present the proof that our construction of embeddings solves the dual encoder embedding construction task. In what follows, we write $a = x \pm y$ to indicate the containment $a \in [x-y, x+y]$, and we assume $n = m$ without loss of generality.

Proof of Theorem 3.1. We begin by drawing standard Gaussian vectors $x_1, \ldots, x_n \sim \mathcal{N}(0, I_d) \in \mathbb{R}^d$ (we will later normalize them). Next, we set $q_i = \frac{1}{\sqrt{|S_i|}} \sum_{j \in S_i} x_j$ (here we abuse notation and think of $S_i \subset [m]$ to be the corresponding indices). Note that, by stability of Gaussian random variables and independence of the x_j's, it follows that each coordinate of q_i is distributed independently as a standard normal distributed (i.e. $\mathcal{N}(0,1)$). By standard χ^2 concentration (e.g. Lemma 1 [20]), for any single Gaussian vector g and value $\lambda > 0$, if $g \sim \mathcal{N}(0, I_d)$ is a vector of i.i.d. standard Gaussian variables, we have $|\|g\|_2^2 - d| \leq 2\sqrt{d\lambda} + 2\lambda$ with probability at least $1 - 2 \cdot 2^{-\lambda}$. Setting $\lambda = 4 \log n$ and taking $d = \Omega(\log n)$ with a sufficiently large constant, we have

$$\mathbf{Pr}\left[\left|\|g\|_2^2 - d\right| \leq 5\sqrt{d \log n}\right] \geq 1 - n^{-4}$$

We can thus condition on $\|x_i\|_2^2 = d \pm 5\sqrt{d \log n}$ and $\|q_i\|_2^2 = d \pm 5\sqrt{d \log n}$ occurring for all $i \in [n]$, which holds with probability at least $1 - 2n^{-3}$ by a union bound over the $2n$ vectors. Call this event $\mathcal{E}_1$.

Let E be the edge set of the HR problem, namely, $(i,j) \in E$ iff $j \in S_i$. To further analyze the construction, first define the event $\mathcal{E}_2$ that for all $(i,j) \notin E$, we have $|\langle q_i, x_j \rangle| \leq 100\sqrt{d \log n}$. Also define the event $\mathcal{E}_3$ that for all $(i,j) \in E$ we have $|\langle \sum_{t \in S_i \setminus j} x_t, x_j \rangle| < 100\sqrt{ds \log n}$.

We first analyze $\mathbf{Pr}[\mathcal{E}_2]$. If $(i,j) \notin E$, then q_i and x_j are independent Gaussian vectors, thus by Gaussian stability we have

$$|\langle q_i, x_j \rangle| \sim |g| \cdot \|x_j\|_2 \leq |g|\sqrt{d}\left(1 + 5\sqrt{\frac{\log(n)}{d}}\right)^{1/2} \leq |g|\sqrt{d}(1 + \tfrac{1}{100})$$

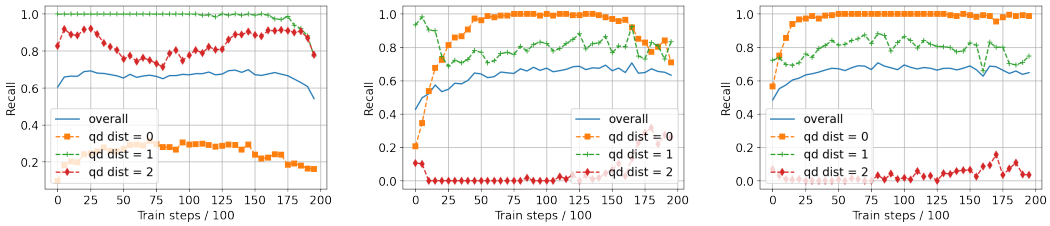
where $g \sim \mathcal{N}(0,1)$ and we took $d = \Omega(\log n)$. Via the density function of a Gaussian, we have $\mathbf{Pr}\left[|g| \cdot \|x_j\|_2 > 100\sqrt{d \log n}\right] < 1/n^4$. Thus, by a union bound over at most n^2 pairs, we have $\mathbf{Pr}[\mathcal{E}_2] > 1 - 1/n^2$. For $\mathcal{E}_3$, note that for any $i \in [n]$ with $j \in S_i$, the vector $\sum_{t \in S_i \setminus j} x_t$ is distributed like $\mathcal{N}(0, \sqrt{|S_i| - 1} \cdot I_d)$; namely each coordinate is i.i.d. Gaussian distributed with variance $|S_i| - 1$. Thus

$$\left|\left\langle \sum_{t \in S_i \setminus j} x_t, x_j \right\rangle\right| \sim \sqrt{|S_i| - 1} \cdot |g| \cdot \|x_j\|_2 < |g|\sqrt{sd}(1 + \tfrac{1}{100})$$

where again $g \sim \mathcal{N}(0,1)$. Following the same argument as above yields $\mathbf{Pr}[\mathcal{E}_3] > 1 - n^{-2}$.

In what follows, let $\gamma = 10 \cdot \max\{s, \frac{1}{\epsilon^2}\}$, and set the dimension $d = C\gamma \log n$ for a sufficiently large constant C. Conditioned on $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, we claim that the vectors $q_1/\|q_1\|_2, \ldots, q_n/\|q_n\|_2, x_1/\|x_1\|_2, \ldots, x_n/\|x_n\|_2$ satisfy the desired properties with threshold $r = \frac{1}{4\sqrt{\gamma}}$. For case one, if $j \in S_i$ we have

$$\begin{aligned}
\left\langle \frac{q_i}{\|q_i\|_2}, \frac{x_j}{\|x_j\|_2} \right\rangle &= \frac{1}{\|q_i\|_2 \|x_j\|_2 \sqrt{|S_i|}} \left(\|x_j\|_2^2 + \left\langle \sum_{t \in S_i \setminus j} x_t, x_j \right\rangle \right) \\
&\geq \frac{2}{3d\sqrt{\gamma}} \left(\frac{2}{3} d - 100\sqrt{ds \log n} \right) > \frac{4}{9\sqrt{\gamma}} - \frac{200}{3} \cdot \sqrt{\frac{\log n}{d}} \qquad (5) \\
&\geq \frac{1}{3\sqrt{\gamma}}
\end{aligned}$$



(a) $p = 0.01$ (b) $p = 0.1$ (c) $p = 0.3$

Figure 6: Effect of re-balanced training data with a varying ratio p between the regular and the heavy-tail data on retrieval quality.

Table 3: Retrieved synsets for selected queries with 64-dimensional embeddings. For each of the three queries considered here, we list the relevant documents in the row "Groundtruth" with an ascending order in their distance to the query. For "Regular sampling" and "Pretrain-finetune", we list top-k documents in an ascending order of k. Documents retrieved but not in the groundtruth are underscored.

Query = "cat"									
Groundtruth	cat	feline	carnivore	placental	mammal	vertebrate	chordate	animal	organism
Regular sampling	cat	feline	carnivore	placental	mammal	wildcat	domestic cat	vertebrate	canine
Pretrain-finetune	cat	feline	chordate	vertebrate	animal	placental	mammal	carnivore	wildcat
Query = "recliner"									
Groundtruth	recliner	armchair	chair	seat	furniture	furnishing	instrumentality	artifact	whole
Regular sampling	recliner	armchair	seat	chair	furnishing	furniture	article	ware	toy dog
Pretrain-finetune	recliner	armchair	seat	chair	furnishing	furniture	instrumentality	artifact	cleaning pad
Query = "motorist"									
Groundtruth	motorist	driver	operator	causal agent	physical entity				
Regular sampling	motorist	operator	driver	floridian	foe				
Pretrain-finetune	motorist	operator	driver	physical entity	causal agent				

Where we used the bounds on d. Next, for case two, when $j \notin S_i$, using the events $\mathcal{E}_1, \mathcal{E}_2$, we have

$$\left|\left\langle \frac{q_i}{\|q_i\|_2}, \frac{x_j}{\|x_j\|_2} \right\rangle\right| < \frac{1}{\|q_i\|_2 \|x_j\|_2} |\langle q_i, x_j \rangle| \leq \frac{3}{2d} 100\sqrt{d \log n} = \frac{150}{\sqrt{C\gamma}} \leq r/2 \tag{6}$$

Where we took $C > (2 \cdot 4 \cdot 150)^2$, which completes the proof as $r/2 < r - \epsilon$. Finally, note for runtime, one needs only generate $O(m)$ d-dimensional Gaussian vectors, and then compute each q_i which takes $\tilde{O}(sd)$ time each, thus the total time is $\tilde{O}(md + nsd)$ as desired.

□

B Implementation Details on ESCI Dataset

Here we provide additional details for experiments on ESCI. We use the SentencePiece model to tokenize the queries and products which are fed to standard 8-layer Transformers as the architecture for the encoder models in DE. For the Transformer, we use model dimension 512, 8 attention heads, two-layer MLP with GELU activation and a hidden dimension of 4096 as the feedforward network. The output embeddings from the Transformer are mean-pooled and projected to 128 dimensions, followed by a normalization to the unit ℓ_2 sphere as the final embedding. The model is trained with the Lazy Adam optimizer [3] with a warmup stage of 2000 steps to a learning rate of 1e-4, followed by a linear decay to 1e-6 at step 50000.

C Additional Experiments for the Toy Setup in Section 4

In Figure 6, we provide additional results complementing Figure 3b. These results reconfirm that rebalanced sampling cannot effectively solve the lost-in-the-long-distance issue.

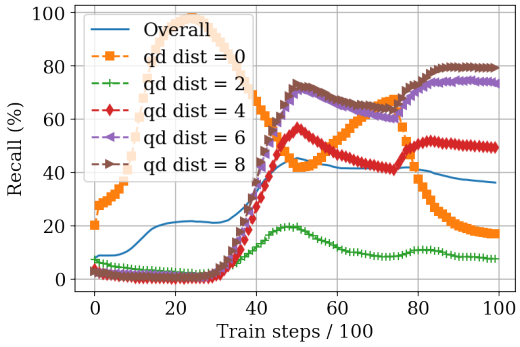


Figure 7: Retrieval quality at varying query-document distances on WordNet, **using hyperbolic embeddings of dimension** 16. The peak performance is a recall of 45.4% obtained at step 49. This recall is higher than that with the Euclidean embeddings of the same dimension *with standard training*, which is at 43.0% (see Table 1), demonstrating the superiority of hyperbolic spaces. Nonetheless, it is worse compared to our pretrain-finetune recipe, which has a recall of 60.1%.

D Comparison with Hyperbolic Embeddings on WordNet

For hierarchical relations, hyperbolic space is a popular choice for addressing the shortcomings of the Euclidean space [24]. Unfortunately, practical large-scale retrieval systems cannot widely adopt hyperbolic embeddings, due to a lack of an efficient approximate k-nearest neighbor search algorithms in hyperbolic spaces. Nonetheless, for the purpose of scientifically understanding the capability of hyperbolic geometry for solving the HR problem, here we implement hyperbolic embeddings and perform experiments on the WordNet dataset.

Specifically, we train 16-dimensional hyperbolic embeddings on the same 10M normal sampling data as in Section 6. Among many options for implementing hyperbolic embeddings, we use the reparameterization form in [12], using a learning rate of 0.01. All other training details are the same as those for Euclidean embeddings.

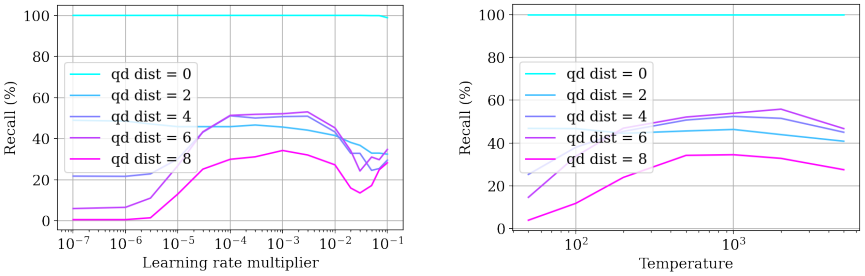
We evaluate the recall for query-document pairs at varying distances and report the results in Figure 7. We see that hyperbolic embeddings struggle to obtain a good balance between pairs with short vs long distances. Specifically, the model first learns to retrieve pairs with short distances, i.e. with distance 0 and 2. As it starts to retrieve longer distance pairs, the quality on the short distance pairs drops rapidly. The best overall recall (i.e., averaged over all distances) is 45.4% obtained at step 49. This recall is better than that of a DE trained on the same data, which is 43.0% (see Table 1), showing the superiority of embedding in hyperbolic space. However, it is still worse than our pretrain-finetune approach, which obtains a recall of 60.1%.

E Ablation Studies for Finetuning on WordNet

In this section, we study the effect of hyper-parameters in our pretrain-finetune recipe for the WordNet experiments presented in Section 6. In particular, the results in Table 1 are obtained with a finetuning learning rate that is 0.001 times the one used during pretraining, and a temperature that is increased from 20 during pretraining to 500 when finetuning. Here, we vary the choice of this learning rate multiplier and temperature during finetuning, and present the results in Figure 8.

For varying learning rate multiplier (see Figure 8a), we observe that the recall on long distance pairs improves as this multiplier is increased from a very small number of 1e-7 up to 1e-3. Crucially, we observe that the recall on short distance pairs are not significantly affected, despite the fact that such pairs are not included in the finetuning data. However, when this multiplier is further increased from 1e-3, the model performance starts to deteriorate on both short and long distance pairs.

In terms of temperature (see Figure 8b), we see that the best recall is obtained with a temperature of around 500. Both much smaller and much larger values of temperature lead to a quality loss.



(a) Effect of Learning Rate

(b) Effect of Temperature

Figure 8: Effect of learning rate and temperature during the finetuning stage of the pretrain-finetune recipe for HR on WordNet.

Finally, Figure 8 shows that the model quality in terms of recall is not sensitive to the choice of these two hyper-parameters, making our method practically easy to tune.

NeurIPS Paper Checklist

1. **Claims**

   Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

   Answer: [Yes]

   Justification: The claims in the abstract and introduction—(1) feasibility of dual encoders for hierarchical retrieval, (2) empirical validation via learned embeddings, and (3) a pretrain-finetune recipe that mitigates performance issues for long-distance pairs—are all rigorously supported by theory and experiments detailed in the main paper (see Sections 3–6).

   Guidelines:

   - The answer NA means that the abstract and introduction do not include the claims made in the paper.
   - The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
   - The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
   - It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

2. **Limitations**

   Question: Does the paper discuss the limitations of the work performed by the authors?

   Answer: [Yes]

   Justification: The paper discusses limitations such as the dependency on embedding dimension for success (Section 4), the failure mode of standard training for long-distance pairs (Section 5.1), and the risk of degrading performance on short-distance retrieval during fine-tuning. These are acknowledged and addressed with empirical findings and mitigation strategies.

   Guidelines:

   - The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
   - The authors are encouraged to create a separate "Limitations" section in their paper.
   - The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
   - The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
   - The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
   - The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
   - If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
   - While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

3. **Theory assumptions and proofs**

   Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

   Answer: [Yes]

   Justification: The main theoretical result (Theorem 3.1) includes all assumptions (e.g., bounded size of relevant document set per query) and provides a complete constructive proof in Appendix A. The theorem is carefully motivated and used to justify design decisions in experiments.

   Guidelines:

   - The answer NA means that the paper does not include theoretical results.
   - All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
   - All assumptions should be clearly stated or referenced in the statement of any theorems.
   - The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
   - Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
   - Theorems and Lemmas that the proof relies upon should be properly referenced.

4. **Experimental result reproducibility**

   Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

   Answer: [Yes]

   Justification: The paper includes sufficient information for reproducing its results: synthetic data generation is described (Section 4), datasets (WordNet and ESCI) are public (Section 6), and implementation details (e.g., model architecture, optimization settings) are provided in Appendix C. Evaluation metrics and sampling procedures are clearly explained.

   Guidelines:

   - The answer NA means that the paper does not include experiments.
   - If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
   - If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
   - Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general. releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
   - While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
     (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
     (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
     (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).

(d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

5. **Open access to data and code**

   Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

   Answer: [No]

   Justification: We do not provide open access to the code.

   Guidelines:

   - The answer NA means that paper does not include experiments requiring code.
   - Please see the NeurIPS code and data submission guidelines (`https://nips.cc/public/guides/CodeSubmissionPolicy`) for more details.
   - While we encourage the release of code and data, we understand that this might not be possible, so "No" is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
   - The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (`https://nips.cc/public/guides/CodeSubmissionPolicy`) for more details.
   - The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
   - The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
   - At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
   - Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

6. **Experimental setting/details**

   Question: Does the paper specify all the training and test details (e.g., data splits, hyperparameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

   Answer: [Yes]

   Justification: The paper provides sufficient experimental details. Section 4 outlines the synthetic setup including tree structures and sampling methods; Section 6 provides information on WordNet and ESCI datasets. Appendix C gives model architecture, optimization settings, tokenizer, training steps, and learning rates for ESCI experiments.

   Guidelines:

   - The answer NA means that the paper does not include experiments.
   - The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
   - The full details can be provided either with the code, in appendix, or as supplemental material.

7. **Experiment statistical significance**

   Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

   Answer: [No]

   Justification: The paper does not report confidence intervals, variance, or statistical significance tests for the experimental results. Trends and improvements (e.g., in recall) are clear and consistent across conditions.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).
- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).
- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

8. **Experiments compute resources**

   Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

   Answer: [Yes]

   Justification: The paper provides a reasonable description of compute used. Section 6 and Appendix C mention model size (8-layer Transformers with 512 hidden dimensions), batch size (4096), number of steps (up to 50k), and training procedures (e.g., learning rate schedules). While exact hardware details (e.g., GPU model, number of GPUs) are not specified, the scale of the experiments is modest and consistent with common academic setups.

   Guidelines:

   - The answer NA means that the paper does not include experiments.
   - The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
   - The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
   - The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

9. **Code of ethics**

   Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics `https://neurips.cc/public/EthicsGuidelines`?

   Answer: [NA]

   Justification: The research does not involve sensitive data, human subjects, biometric information, or real-world deployment that would invoke ethical concerns or require adherence to a formal code of ethics. The work focuses on synthetic and public benchmark datasets (WordNet and ESCI) for algorithmic research in retrieval.

   Guidelines:

   - The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
   - If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.

- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

10. **Broader impacts**

    Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

    Answer: [Yes]

    Justification: The paper discusses real-world applications in advertising (Phrase Match in online ads) and product search (Amazon ESCI) where hierarchical retrieval is directly applicable (Section 1, Section 6.2). While no dedicated "societal impact" section is included, the implications of better retrieval systems are clearly articulated in motivating examples, and no negative societal impacts are anticipated from this line of work.

    Guidelines:

    - The answer NA means that there is no societal impact of the work performed.
    - If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.
    - Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.
    - The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
    - The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
    - If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

11. **Safeguards**

    Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

    Answer: [NA]

    Justification: The paper does not propose or deploy models in high-risk domains such as healthcare, legal, financial, or safety-critical systems. The research is focused on information retrieval using public datasets (WordNet and ESCI) and does not require safeguards for fairness, robustness, privacy, or misuse prevention in its current scope.

    Guidelines:

    - The answer NA means that the paper poses no such risks.
    - Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
    - Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
    - We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

12. **Licenses for existing assets**

    Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

    Answer: [Yes]

    Justification: The datasets used in this work—WordNet and ESCI (Amazon Shopping Queries)—are both publicly available. WordNet is distributed under a permissive license (Princeton WordNet License), and ESCI is a publicly released benchmark dataset as noted in [28]. These datasets are used in accordance with their respective terms, and no unlicensed or proprietary assets are used.

    Guidelines:

    - The answer NA means that the paper does not use existing assets.
    - The authors should cite the original paper that produced the code package or dataset.
    - The authors should state which version of the asset is used and, if possible, include a URL.
    - The name of the license (e.g., CC-BY 4.0) should be included for each asset.
    - For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.
    - If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, `paperswithcode.com/datasets` has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
    - For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.
    - If this information is not available online, the authors are encouraged to reach out to the asset's creators.

13. **New assets**

    Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

    Answer: [No]

    Justification: This work does not introduce a new dataset, model, or benchmark. All assets used (WordNet, ESCI, synthetic hierarchies) are pre-existing and publicly available. The primary contributions are theoretical analysis, algorithmic design (pretrain-finetune recipe), and empirical evaluation.

    Guidelines:

    - The answer NA means that the paper does not release new assets.
    - Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
    - The paper should discuss whether and how consent was obtained from people whose asset is used.
    - At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

14. **Crowdsourcing and research with human subjects**

    Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

    Answer: [NA]

    Justification: The paper does not involve any human participants, crowdsourced annotations, interviews, surveys, or behavioral experiments.

    Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

15. **Institutional review board (IRB) approvals or equivalent for research with human subjects**

    Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

    Answer: [Yes]

    Justification: This study does not involve human subjects research and therefore does not require IRB approval.

    Guidelines:

    - The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
    - Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.
    - We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
    - For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.

16. **Declaration of LLM usage**

    Question: Does the paper describe the usage of LLMs if it is an important, original, or non-standard component of the core methods in this research? Note that if the LLM is used only for writing, editing, or formatting purposes and does not impact the core methodology, scientific rigorousness, or originality of the research, declaration is not required.

    Answer: [NA]

    Justification: the core method development in this research does not involve LLMs as any important, original, or non-standard components

    Guidelines:

    - The answer NA means that the core method development in this research does not involve LLMs as any important, original, or non-standard components.
    - Please refer to our LLM policy (`https://neurips.cc/Conferences/2025/LLM`) for what should or should not be described.