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A new method for nonlinear state estimation problem

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A B S T R A C T

In this paper, a new filtering technique to solve a nonlinear state estimation problem has been developed with the help of the Gaussian integral. It is well known that for a nonlinear system, the prior and the posterior probability density functions (pdfs) are non-Gaussian in nature. However, in this work, they are assumed to be Gaussian; subsequently, the mean and the covariance are calculated. In the proposed method, nonlinear functions of process dynamics and measurements are expressed in a polynomial form with the help of the Taylor series expansion. In order to calculate the prior and the posterior mean and covariance, the functions are integrated over the Gaussian pdf with the Gaussian integral. The performance of the proposed method is tested on three nonlinear state estimation problems. The simulation results show that the proposed filter provides more accurate results than other existing deterministic sample point filters such as the cubature Kalman filter, the unscented Kalman filter, and the Gauss-Hermite filter.

1. Introduction

In the modern era, almost all systems are enabled with sensors. This summons the need for estimators to estimate hidden or unknown variables or states using a set of noisy measurements acquired by the sensors. The most challenging task is to estimate the hidden variables using noise corrupted measurements. In 1960, Rudolf E. Kalman proposed the Kalman filter [1] providing a recursive solution for the linear filtering problems.

However, in real life, most of the systems are nonlinear in nature, and the process and measurement model contain uncertainties that are generally represented with additive white Gaussian noise. For such systems, to estimate the states, several sub-optimal filtering algorithms are introduced. Those filtering algorithms find applications in various fields like aircraft and unmanned air vehicle (UAV) surveillance [2,3], torpedo tracking [4,5], autonomous underwater vehicle (AUV) navigation [6], GPS navigation [7], power system [8,9], state of charge estimation [10], biomedical [11], agriculture [12], meteorology [13], econometrics [14] and many more.

To estimate the states of a system, the most acknowledged approach is based on the Bayesian framework. In this process of solving nonlinear filtering problems, we come across a few integrals that are intractable. This issue was managed by using a method popularly known as the extended Kalman filter (EKF) [15,16] and its variants [17–19], where the nonlinear process and measurement model are linearized around the previous estimate using first-order Taylor series approximation. However, this method suffers from several limitations like the necessity of function's smoothness, restrictions due to the consideration of noise to be Gaussian and frequently the estimator loses track for highly nonlinear systems when the linearization error is large [16,20].

Due to the limitations of the EKF, several other nonlinear filtering techniques have been developed. There are mainly two approaches; in one of them, a set of points in state space (also called particles) and their corresponding weights are used to reconstruct the prior and the posterior pdfs. This approach is popularly known as the particle filter (PF) [21]. The locations and the weights of the particles are updated in each iteration. Although the accuracy of the PF is generally more, it has a very high computational burden, and to some extent, it suffers from the 'curse of dimensionality' problem. The ensemble Kalman filter (EnKF) was hence proposed [22] to estimate states of a high dimensional system. In the EnKF, a set of ensembles is generated using Monte Carlo runs, and then the algorithm of the KF works on the ensembles. Later a stochastic integrated filter (SIF) was proposed [23,24] which was based on stochastic integrated rules (SIRs).

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In another approach, the prior and the posterior pdfs are approximated as Gaussian [25–27], and represented with the mean and the covariance. Traditionally, they are approximated with the help of a few deterministic sample points and their associated weights. Some of the popular filtering techniques which use the above approach are the Gauss-Hermite filter (GHF) [25,26], the unscented Kalman filter (UKF) [27,28], the cubature Kalman filter (CKF) [29,30], and their variants [31,32]. Generally, these filters require less computational budget while computing the conditional moments.

As the above mentioned filters are suboptimal, there is a scope for improvement. This motivates us to explore the Gaussian integral, where an integral of a polynomial function over the Gaussian pdf is evaluated analytically. Once we assume the pdfs of states are Gaussian, the mean and covariance are calculated by evaluating an integral in the form $\int nonlinear function \times Gaussian\ pdf$. Now, if the nonlinear function is a polynomial, the Gaussian integral method provides an exact solution of the above integral which results the proposed estimation solution to be a near optimal. However, if the nonlinear function is not in a polynomial form, we use the Taylor series approximation to make it polynomial and apply the proposed method. Although, in some cases, Taylor series expansion is required at some point to solve the estimation problem, the proposed method is entirely different from the EKF.

It is worthy to mention here that there exist a few papers which use polynomial expansion for state estimation [33–36]. A filtering technique is presented in [33] only for a single-dimensional system by exploiting the full Taylor series expansion of a polynomial function. The Fourier Hermite Kalman filter (FHKF) [34] is developed based on the finite truncation of the Fourier-Hermite series. In this method, the expected value of a nonlinear function needs to be expressed in a closed form, which is always not possible for any arbitrary nonlinear function [37, p. 80]. The filtering algorithm presented in [35,36] uses Chebyshev polynomial series expansion, and Carleman approximation of a nonlinear system. To the best of our knowledge, the approach for state estimation presented in this paper is new and it does not exist in previous literature.

The proposed method is applied to three nonlinear filtering problems. The accuracy is evaluated in terms of the root mean square error (RMSE) of position and velocity and percentage of fail count, and it is compared with the EKF (first and second order), GHF, UKF, and CKF. It has been observed that the proposed filter is more accurate, and it provides the lowest percentage of fail count compared to the existing quadrature filters. Although the execution time of the filter is higher compared to the UKF and CKF, it does not suffer from the 'curse of dimensionality' problem. We also checked the robustness of the proposed filter against initial error uncertainty and found it to be more than the existing deterministic sample point filters.

2. Bayesian approach of filtering

Let us consider a system defined with the following state space model:

$$x_k = \phi(x_{k-1}) + \eta_{k-1}, \tag{1}$$

$$y_k = \gamma(x_k) + \nu_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state of a system, and $y_k \in \mathbb{R}^{n_y}$ is the measurement, $\phi(x_{k-1})$ and $\gamma(x_k)$ are nonlinear functions. η_{k-1} and ν_k are the process and the measurement noises, respectively. They are assumed to be zero mean, white, Gaussian with the covariance Q_{k-1} and R_k respectively, and they are uncorrelated with each other as well as uncorrelated with the initial states.

The objective of the filtering is to determine the probability density function (pdf) of x_k from the measurement data and approximate knowledge of the system. The most popular way of doing this is the Bayesian approach, where at each time instant, k, the pdf of the state of x_k is estimated recursively in two steps [16,20]: (i) prediction step, (ii) update step. The prediction step is governed by the Chapman-Kolmogorov equation:

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1}. \tag{3}$$

In the update step, Bayes' rule is used to compute the posterior density function of the state,

$$p(x_k|y_{1:k}) \propto p(y_k|x_k)p(x_k|y_{1:k-1}). \tag{4}$$

For a linear Gaussian system, the Kalman filter [16,20] provides an optimal solution of Eqs. (3)-(4). But for a nonlinear system, no such optimal solution is available in general.

The prior mean, $\hat{x}_{k|k-1}$, and the covariance, $P_{k|k-1}$, can be computed as follows [23,29]:

$$\hat{x}_{k|k-1} = \int \phi(x_{k-1})\mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})dx_{k-1}, \tag{5}$$

$$\begin{aligned} P_{k|k-1} = &\int \phi(x_{k-1})\phi^T(x_{k-1})\mathcal{N}(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})dx_{k-1} \\ &- \hat{x}_{k|k-1}\hat{x}^T_{k|k-1} + Q_{k-1}. \end{aligned} \tag{6}$$

The mean and the covariance of the projected measurement can be calculated as [23,29]

$$\hat{y}_{k|k-1} = \int \gamma(x_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k, \tag{7}$$

$$P^{yy}_{k|k-1} = \int \gamma(x_k)\gamma^T(x_k)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})dx_k - \hat{y}_{k|k-1}\hat{y}^T_{k|k-1} + R_k. \tag{8}$$

The cross-covariance of the state and measurement are given as [23,29]

$$P^{xy}_{k|k-1} = \int x_k \gamma^T(x_k)\mathcal{N}(x_k;\, \hat{x}_{k|k-1}, P_{k|k-1})dx_k - \hat{x}_{k|k-1}\hat{y}^T_{k|k-1}. \tag{9}$$

Finally, we compute the value of the posterior mean and covariance as follows:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \tag{10}$$

$$P_{k|k} = P_{k|k-1} - K_k P^{yy}_{k|k-1} K^T_k, \tag{11}$$

where K_k is the Kalman gain,

$$K_k = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}. \tag{12}$$

The integrals mentioned in Eqs. (5)-(9) are intractable for arbitrary $\phi(\cdot)$ and $\gamma(\cdot)$.

In this paper, the authors propose to use the Gaussian integral [38] to evaluate the Eqs. (5)-(9). Initially, the function $\phi(\cdot)$ and $\gamma(\cdot)$ are assumed to be polynomial, and for such a case, the evaluation is exact. If they are not in polynomial form, then they are expressed in a power series [39–41] with the help of the Taylor series expansion. Each term of the power series is integrated over the Gaussian pdf with the help of the Gaussian integral [38,42]. In doing so, a new filtering technique which is expected to be more accurate than the existing deterministic sample point filters is developed.

3. The proposed method

This section provides a theoretical explanation of the solution of the intractable integrals using the Gaussian integral method. It starts with Lemma 1, where a simple one-dimensional Gaussian integral problem is explained. It is further extended to its multidimensional form in Corollary 1. Theorem 1 discusses the expectation of an arbitrary polynomial function whose weight function is Gaussian.

Lemma 1. *For any variable $y_1 \in \mathbb{R}$, $d_1 \in \mathbb{R}^+$ and m_1 is any non-negative integer, the integral*

$$\begin{aligned} I &= \int_{-\infty}^{\infty} y_1^{m_1} \exp(-\frac{{y_1}^2}{2d_1})dy_1 \\ &= \begin{cases} (2d_1)^{\frac{m_1+1}{2}}\Gamma(\frac{m_1+1}{2}) & \text{if } m_1 \text{ is even,} \\ 0 & \text{if } m_1 \text{ is odd.} \end{cases} \end{aligned} \tag{13}$$

Proof. Let us consider an arbitrary function $f(y_1) = y_1^{m_1}\exp(-\frac{{y_1}^2}{2d_1})$. If m_1 is odd *i.e.* $f(y_1) = -f(-y_1)$, then the integral [40, p. 84]

$$\int_{-\infty}^{\infty} f(y_1)dy_1 = 0.$$

If m_1 is even *i.e.* $f(y_1) = f(-y_1)$, then the integral

$$I = 2\int_0^{\infty} y_1^{m_1}\exp(-\frac{{y_1}^2}{2d_1})\,dy_1. \tag{14}$$

By substituting $y_1 = \sqrt{2td_1}$, the above integral (14) becomes

$$\begin{aligned} I &= (2d_1)^{\frac{m_1+1}{2}} \int_0^{\infty} t^{\frac{m_1-1}{2}} \exp(-t)\,dt \\ &= (2d_1)^{\frac{m_1+1}{2}}\Gamma(\frac{m_1+1}{2}). \end{aligned} \tag{15}$$

Corollary 1. *The above Lemma can be easily extended for a variable $y \in \mathbb{R}^n$, where $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$, $d_i\ (i = 1, 2, \cdots, n) \in \mathbb{R}^+$ and m_i, any non-negative integer. In such a case, we can write*

$$\begin{aligned} &\int_{\mathbb{R}^n} \prod_{i=1}^n y_i^{m_i} \exp\big(-\frac{1}{2}\sum_{i=1}^n \frac{y_i^2}{d_i}\big)dy \\ &= \begin{cases} \prod_{i=1}^n \big\{(2d_i)^{\frac{m_i+1}{2}}\Gamma(\frac{m_i+1}{2})\big\} & \text{when } m_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{16}$$

Theorem 1. *For any arbitrary polynomial function $f_1(x)=\prod_{i=1}^{n} x_i^{m_i}$, where m_i is any non-negative integer, x_i is the i-th element of the state vector x, the integral*

$$I_1=\int_{-\infty}^{+\infty} f_1(x)\mathcal{N}(x;\mu,\ P)dx$$
$$=\begin{cases} \frac{1}{\sqrt{\pi^n}}\Big[\sum_{U_{a_{i,j}}}\Big\{\prod_{i=1}^{n}\Big(C_i\mu_i^{a_{i,1}}\big(\prod_{l=1}^{n}S_{il}^{a_{i,l+1}}\big)(2d_i)^{(\frac{1}{2}\sum_{l=1}^{n}a_{l,i+1})} \\ \qquad \Gamma\big(\frac{\sum_{l=1}^{n}a_{l,i+1}+1}{2}\big)\Big)\Big\}\Big] \quad \text{when } \sum_{l=1}^{n}a_{l,i+1} \text{ is even}, \\ 0 \qquad \text{otherwise}, \end{cases} \tag{17}$$

where $a_{i,j}$ $(i=1,2,\cdots,n;\ j=1,2,\cdots,n+1)$ are non-negative integer, and $U_{a_{i,j}}$ are all possible combinations of $a_{i,j}$ which satisfy $\sum_{j=1}^{n+1}a_{i,j}=m_i$. d_i is the i-th eigenvalue of the covariance matrix P, S is the orthogonal matrix which satisfies $S^{-1}PS=diag(d_1,d_2,\cdots,d_n)$, S_{il} are the (i,l) element of S, and C_i is the multinomial coefficient which satisfies $C_i=\frac{m_i!}{a_{i,1}!a_{i,2}!\cdots a_{i,n+1}!}$.

Proof. Consider the integral

$$I_1=\frac{1}{\sqrt{(2\pi)^n|P|}}\int_{-\infty}^{\infty}\prod_{i=1}^{n}x_i^{m_i}\exp\big\{-\frac{1}{2}(x-\mu)^T P^{-1}(x-\mu)\big\}dx. \tag{18}$$

We substitute $x-\mu=Sy$, where S is an orthogonal matrix which satisfies

$$S^{-1}PS=\text{diag}(d_1,d_2,\cdots,d_n), \tag{19}$$

$$\text{or}\quad S^{-1}P^{-1}S=\text{diag}\big(\frac{1}{d_1},\frac{1}{d_2},\cdots,\frac{1}{d_n}\big). \tag{20}$$

With the above expression of (20) and $dx=|S|dy$, the integral (18) can be written as

$$\begin{aligned} I_1&=\frac{|S|}{\sqrt{(2\pi)^n|P|}}\int_{-\infty}^{\infty}\prod_{i=1}^{n}\big(\mu_i+\sum_{l=1}^{n}S_{il}y_l\big)^{m_i}\exp\big\{-\frac{1}{2}y^TS^TP^{-1}Sy\big\}dy \\ &=\frac{1}{\sqrt{(2\pi)^n|P|}}\int_{-\infty}^{\infty}\prod_{i=1}^{n}\big(\mu_i+\sum_{l=1}^{n}S_{il}y_l\big)^{m_i}\exp\big\{-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{d_i}\big\}dy. \end{aligned} \tag{21}$$

With the help of multinomial expansion, the integral (21) can be written as

$$\begin{aligned} I_1=\frac{1}{\sqrt{(2\pi)^n|P|}}\int_{-\infty}^{\infty}&\sum_{U_{a_{1,j}}}C_1\big(\mu_1^{a_{1,1}}\prod_{l=1}^{n}(S_{1l}y_l)^{a_{1,l+1}}\big)\sum_{U_{a_{2,j}}}C_2\big(\mu_2^{a_{2,1}}\prod_{l=1}^{n}(S_{2l}y_l)^{a_{2,l+1}}\big) \\ &\cdots\sum_{U_{a_{n,j}}}C_n\big(\mu_n^{a_{n,1}}\prod_{l=1}^{n}(S_{nl}y_l)^{a_{n,l+1}}\big)\exp\big(-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{d_i}\big)dy, \end{aligned} \tag{22}$$

where $C_i=\frac{m_i!}{a_{i,1}!a_{i,2}!\cdots a_{i,n+1}!}$ is the multinomial coefficient, $U_{a_{i,j}}$ $(i=1,2,\cdots,n;\ j=1,2,\cdots,n+1)$ consists of all possible non-negative integer values of $a_{i,j}$, which satisfies $\sum_{j=1}^{n+1}a_{i,j}=m_i$. Now, the above integral can be written as

$$\begin{aligned} I_1=&\frac{1}{\sqrt{(2\pi)^n|P|}}\int_{-\infty}^{\infty}\sum_{U_{a_{i,j}}}\prod_{i=1}^{n}\Big\{C_i\mu_i^{a_{i,1}}\big(\prod_{l=1}^{n}S_{il}^{a_{i,l+1}}\big)\Big\}\Big(\prod_{i=1}^{n}y_i^{(\sum_{l=1}^{n}a_{l,i+1})}\Big) \\ &\exp\big(-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{d_i}\big)dy. \end{aligned} \tag{23}$$

Substituting $|P|=d_1d_2\cdots d_n$ and using Eq. (16) of Corollary 1, the above integral becomes Eq. (17).

Note 1. Theorem 1 can be extended for any arbitrary multidimensional polynomial function $f(x)$, $f:\mathbb{R}^n\to\mathbb{R}^p$, where $f(x)=\big[f_1(x)\quad f_2(x)\quad\cdots\quad f_p(x)\big]^T$; $f_i(x)\in\mathbb{R}$ by calculating the integral separately, row-wise, *i.e.* $I=\big[I_1\quad I_2\quad\cdots\quad I_p\big]^T$.

Note 2. An alternative expression of the integral I_1 (when $\sum_{j=i}^{n} a'_{j,i+1}$ is even) is derived using the Cholesky decomposition is as follows:

$$I_1 = \frac{1}{\sqrt{\pi}^n} \sum_{U_{a'_{i,j}}} \Big\{ \prod_{i=1}^{n} \Big(C'_i \mu_i^{a'_{i,1}} \big(\prod_{j=1}^{i} L_{ij}^{a'_{i,j+1}}\big) 2^{(\frac{1}{2}\sum_{j=i}^{n} a'_{j,i+1})} \Gamma\big(\frac{\sum_{j=i}^{n} a'_{j,i+1}+1}{2}\big)\Big)\Big\}, \tag{24}$$

where $a'_{i,j}(i=1,2,\cdots,n;\ j=1,2,\cdots,i+1)$ are non-negative integers, $U_{a'_{i,j}}$ are all possible combinations of $a'_{i,j}$ which satisfy $\sum_{j=1}^{i+1} a'_{i,j} = m_i$, and the multinomial coefficients are $C'_i = \frac{m_i!}{\prod_{j=1}^{i+1} a'_{i,j}!}$. L is the square root of the matrix P, *i.e.* $P = LL^T$, which can be calculated using Cholesky decomposition and $L_{i,j}$ are the (i,j)-th element of L.

Illustration. Let us consider a two dimensional polynomial function $f(x) = \begin{bmatrix} x_1^2 x_2 & x_1 x_2 \end{bmatrix}^T$. So, $f_1(x) = x_1^2 x_2$, where $m_1 = 2$ and $m_2 = 1$, and it satisfies

$$\sum_{j=1}^{3} a_{1,j} = a_{1,1} + a_{1,2} + a_{1,3} = 2,$$

and

$$\sum_{j=1}^{3} a_{2,j} = a_{2,1} + a_{2,2} + a_{2,3} = 1.$$

The possible values of $\{a_{1,1}, a_{1,2}, a_{1,3}\}$ include {2,0,0},{0,2,0},{0,0,2},{1,1,0},{1,0,1}, {0,1,1}; and $\{a_{2,1}, a_{2,2}, a_{2,3}\}$ are {1,0,0},{0,1,0},{0,0,1}. Similarly for $f_2(x) = x_1 x_2$, m_3 and m_4 are 1 each. The possible values of $\{a_{1,1}, a_{1,2}, a_{1,3}\}$ =$\{a_{2,1}, a_{2,2}, a_{2,3}\}$ = {1,0,0},{0,1,0},{0,0,1}. With the help of Theorem 1 and Note 1, the integral

$$I = \int_{-\infty}^{\infty} \begin{bmatrix} x_1^2 x_2 & x_1 x_2 \end{bmatrix}^T \mathcal{N}(x; \mu, P) dx = \begin{bmatrix} I_1 & I_2 \end{bmatrix}^T, \tag{25}$$

where

$$I_1 = (\mu_1^2 \mu_2 + \mu_2 S_{11}^2 d_1 + \mu_2 S_{12}^2 d_2 + 2\mu_1 S_{11} S_{21} d_1 + 2\mu_1 S_{12} S_{22} d_2)$$

and

$$I_2 = (\mu_1 \mu_2 + \frac{1}{2} S_{11} S_{21} + \frac{1}{2} S_{12} S_{22}).$$

Please note that for a particular problem when P is known, numerical values of $S_{11}, S_{12}, S_{21}, S_{22}$ are known and if μ_1, μ_2 are known, the value of I can be found out.

3.1. Expressing a function in polynomial form

Earlier the integrals (5)-(9) are evaluated assuming the integrand is a polynomial function. However, the process and/or measurement functions may not always be polynomial, and in such cases, the Taylor series expansion [39,41] is used to write them in polynomial form. Let us consider the previous estimate $\bar{x}$ is the nominal point, and a multi-variable function $f(x) \in \mathbb{R}^p$ is analytic at $\bar{x}$. Then by using the Taylor series, $f(x)$ can be expanded around the nominal point $\bar{x}$ [39, p. 642] as follows:

$$\begin{aligned} f(x) = f(\bar{x}) &+ \sum_{i=1}^{n} (x_i - \bar{x}_i) \left.\frac{\partial f(x)}{\partial x_i}\right|_{\bar{x}} + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - \bar{x}_i)(x_j - \bar{x}_j) \left.\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right|_{\bar{x}} \\ &+ \frac{1}{3!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_i - \bar{x}_i)(x_j - \bar{x}_j)(x_k - \bar{x}_k) \left.\frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k}\right|_{\bar{x}} + \cdots . \end{aligned} \tag{26}$$

The above expression can be written in polynomial form as

$$f(x) = A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} A_{ijk} x_i x_j x_k + \cdots , \tag{27}$$

where A_0 is a constant, and the coefficient A_{ijk} can be calculated by consolidating all the multiplicative terms of the respective polynomial. Once the Taylor series expansion transforms the function into a polynomial form, the desired integral can be evaluated by using Theorem 1.

Remark 1. In the EKF, the Taylor series expansion is used to calculate the Jacobian matrices and subsequently to linearize the nonlinear equations at the nominal points. The linearized process and measurement equations are then used in the Kalman filter algorithm to calculate the estimate. However, in the proposed method, the Taylor series approximation is used to write a (non polynomial) function into polynomial form which is then integrated over the Gaussian density function (using Gaussian integral) to receive the estimate and the error covariance. So, the proposed method and the EKF are fundamentally different and have no similarity; although at some point both the algorithms use the Taylor series expansion.

The intractable integrals mentioned in Eqs. (5)–(9) are solved using the Cholesky and eigenvalue based Gaussian integral method (Theorem 1, Note 1, and Note 2). Once the integral is evaluated, it is similar to the Kalman filter and the detailed algorithm is provided in Algorithm 1.

Algorithm 1 Algorithm for the proposed filtering.

Initialize the filter with $\hat{x}_{0|0}$ and $P_{0|0}$.

`Step 1: Time update`

- Calculate the eigenvector matrix ($S_{k-1|k-1}$) and the eigenvalue matrix ($d_{k-1|k-1}$) of $P_{k-1|k-1}$.
- Compute the prior mean $\hat{x}_{k|k-1}$, and the covariance $P_{k|k-1}$ by evaluating Eqs. (5)-(6) using Theorem 1.

`Step 2: Measurement update`

- Again eigenvector matrix ($S_{k|k-1}$) and eigenvalue matrix ($d_{k|k-1}$) of $P_{k|k-1}$ are calculated.
- Calculate $\hat{y}_{k|k-1}$, $P^{yy}_{k|k-1}$ and $P^{xy}_{k|k-1}$ by evaluating Eqs. (7)-(9) with the help of Theorem 1.
- Calculate the Kalman gain $K = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}$.
- Compute the posterior state estimate $\hat{x}_{k|k}$, and the covariance $P_{k|k}$ by using Eqs. (10)-(11).

3.2. Computation complexity

We discuss here the computation complexity of the filtering algorithm in terms of floating-point operations (flops). A flop is defined as one of the basic arithmetic operations such as addition, subtraction, multiplication, or division of any two floating-point numbers [43,44]. At first, we calculate the flops counts required by the eigenvalue based proposed algorithm. To compute the eigenvalue decomposition of any matrix of order $n \times n$, n^3 number of flops operations are required. The flops counts for various operations of the proposed algorithm are presented in Table 1.

Adding the flop counts of all the operations mentioned in the Table 1, we receive the total number of flop counts of the proposed algorithm as

$$\begin{aligned} C(n, n_y, \mathtt{m}) &= 2n^3 + (n + n_y + n^2 + n_y^2 + nn_y)G + 3(n^2 + n_y^2) \\ &\quad + 2nn_y(n + 2n_y) + n_y(1 + 2n) + n_y^3, \end{aligned} \tag{28}$$

where $\mathtt{m}$ is the order of Taylor series approximation, n and n_y are the dimension of state and measurement, respectively. Here, the multiplier term G involves the integral complexity of the proposed filter, which is dependent on the order of Taylor series expansion used to approximate the function.

For first, second and third order approximation, G can be expressed as

$$G(\mathtt{m} = 1) = (1 + n) + 2n(n + 1), \tag{29}$$

$$\begin{aligned} G(\mathtt{m} = 2) &= [1 + n + n^2] + 2n(n + 1)[1 + 2n] + (n + 1)[2(5n + 1) \\ &\quad + (2n - 2)(9n + 2)], \end{aligned} \tag{30}$$

and

$$\begin{aligned} G(\mathtt{m} = 3) &= [1 + n + n^2 + n^3] + 2n(n + 1)[1 + 2n + 3n^2] \\ &\quad + (n + 1)[2(5n + 1) + (2n - 2)(9n + 2) + 6(5n + 1) + 3(n^2 - 2)(9n + 2)] \\ &\quad + (n + 1)[(5n + 1) + 3(n - 1)(9n + 2) + (n - 1)(n - 2)(16n + 3)]. \end{aligned} \tag{31}$$

From the above equations, we can see that incorporation of higher order terms in Taylor series approximation incurs more computation burden.

Now, we calculate the flops count of CKF [29], UKF [45] and GHF [25]. For any matrix of order $n \times n$, the computational cost of the Cholesky decomposition is $n^3/3 + 2n^2$ [43]. The total flops required to implement the CKF, UKF and GHF with m number of sample points are

$$\begin{aligned} C(n, n_y, m) &= (6n^2 + (2 + n_y)2n + 3n_y + 2n_y^2)m + \frac{2}{3}n^3 + (7 + 2n_y)n^2 \\ &\quad + (3 + 4n)n_y^2 + (1 + 2n)n_y + n_y^3, \end{aligned} \tag{32}$$

where $m = 2n$ for CKF, $m = 2n + 1$ for UKF, and t-point GHF require t^n number of sample points *i.e.* $m = t^n$.

Finally, we summarize the different filters' computation complexity in Table 2. It can be noted that although the execution time of the proposed filter is high, unlike the GHF, it is free from the 'curse of dimensionality' problem.

4. Simulation results

The proposed filter is applied to three problems along with the EKF (first and second order), CKF, UKF, and GHF. Filtering performance is compared in terms of root mean square error (RMSE), percentage of track loss and execution time. We evaluate the RMSE at the k-th time step as,

$$\mathrm{RMSE}_k = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(x_{i,k} - \hat{x}_{i,k})^2}, \tag{33}$$

Table 1
Computation complexity of proposed algorithm.

Operation	Flops count
$[S_{k-1\|k-1}, d_{k-1\|k-1}] = \mathrm{eig}(P_{k-1\|k-1})$	n^3
$\hat{x}_{k\|k-1}$	nG
$P_{k\|k-1}$	$n^2G + 3n^2$
$[S_{k\|k-1}, d_{k\|k-1}] = \mathrm{eig}(P_{k\|k-1})$	n^3
$\hat{y}_{k\|k-1}$	n_yG
$P^{yy}_{k\|k-1}$	$n_y^2G + 3n_y^2$
$P^{xy}_{k\|k-1}$	$nn_yG + 2nn_y$
K_k	$2nn_y^2 - nn_y + n_y^3$
$\hat{x}_{k\|k}$	$2nn_y + n_y$
$P_{k\|k}$	$2n^2n_y + 2nn_y^2 - nn_y$

Table 2
Computation complexity of different filters.

Filter	Order of comp. complexity
CKF	n^3
UKF	n^3
GHF	$t^n n^2$
Proposed algorithm	n^{m+3}

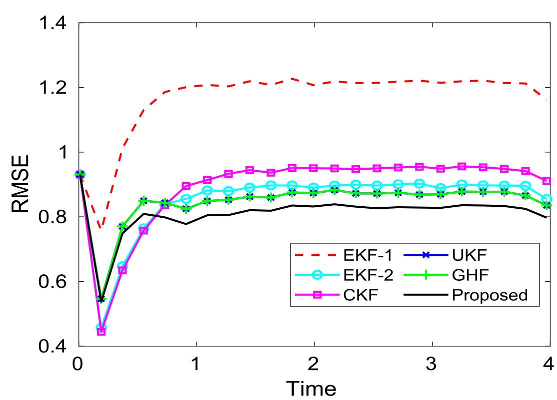


Fig. 1. RMSE of the state.

where M is the total number of Monte Carlo runs, $x_{i,k}$ represents the truth state at the k-th time step of i-th Monte Carlo runs, and $\hat{x}_{i,k}$ is the estimate of $x_{i,k}$.

Track is declared to be lost when the terminal estimation error goes beyond a specified limit e_{limit}. The percentage of track loss is evaluated as follows:

$$\mathrm{TL\%} = \frac{M - C}{M} \times 100\%, \tag{34}$$

where M is the total number of Monte Carlo runs and out of which the track loss occurs C times (terminal estimation error goes beyond e_{limit}).

Problem 1. A single dimensional problem [25] has been considered, with $\phi(x_{k-1}) = x_{k-1} + 5\delta t x_{k-1}(1 - x_{k-1}^2)$ and $\gamma(x_k) = \delta t(x_k - 0.05)^2$, where $\eta_{k-1} \sim \mathcal{N}(0, Q_{k-1})$ and $\nu_k \sim \mathcal{N}(0, R_k)$ are white Gaussian with $Q_{k-1} = b^2\delta t$ and $R_k = d^2\delta t$. The following values of the parameter are used for the simulation: $b = 0.5$, $d = 0.1$, $\delta t = 0.01$ s. The truth is initialized with $x_0 = -0.2$. The filter is initialized with $\hat{x}_{0|0} = 0.8$ and $P_{0|0} = 2$. Estimation is performed for a time span of 0 to 4 s. The system has two equilibrium points, among them it settles on one of the stable equilibria *i.e.* either at 1 or, -1. A moderate estimation error forces the estimate to settle in a wrong equilibrium point, and track loss situation occurs.

The problem has been solved by the EKF of first and second order, the CKF, the UKF, the three points GHF, and with the proposed filter. The RMSEs excluding fail count, obtained from 1,000 MC runs are plotted in Fig. 1. From the figure, it can be observed that the proposed filter performs better than the EKF-1, the EKF-2, the CKF, the UKF, and the GHF. We also provide the RMSE value averaged over the simulation time in Table 3. From the table we can see that the proposed filter attains the minimum average RMSE.

Filtering performance has also been compared in terms of percentage fail count which is defined above. Fail counts of different filters obtained from 10,000 MC runs are summarized in Table 3. From the Table 3, it can be observed that the proposed filter has the lowest fail count compared to the EKFs, CKF, UKF, and the GHF. The execution time of all the filters is compared and it has been seen that the run time of the proposed filter is higher (approximately five times) than other Gaussian filters.

Table 3
Average RMSE, percentage fail count and relative execution time of different filters.

Filter	Average RMSE	Fail count (%)	Execution time
EKF-1	1.16	37.59	0.18
EKF-2	0.84	16.60	0.19
CKF	0.88	17.55	0.185
UKF	0.83	14.03	0.19
GHF	0.83	14.03	0.19
Proposed	0.79	12.53	1

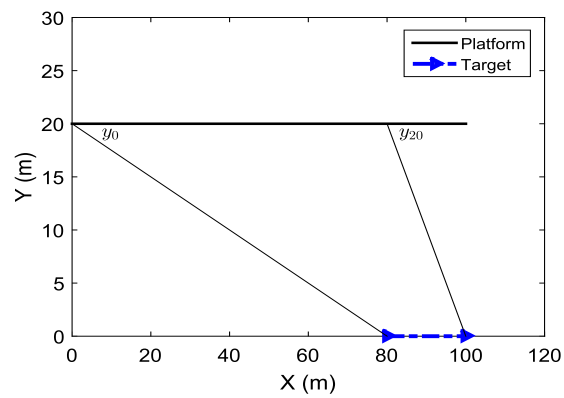


Fig. 2. Tracking platform kinematics.

Problem 2. A bearing only target tracking problem [2,46] has been considered where a moving target is tracked from an airborne platform. The engagement scenario is shown in Fig. 2.

Process model: The target dynamics is given by

$$x_k = \phi_k x_{k-1} + \beta_k \eta_{k-1}, \tag{35}$$

where $x_k = \begin{bmatrix} x_{1,k} & x_{2,k} \end{bmatrix}^T$, $\phi_k = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}$, $\beta_k = \begin{bmatrix} T_s^2/2 & T_s \end{bmatrix}^T$, $x_{1,k}$ and $x_{2,k}$ are position and velocity of the target respectively, and T_s is sampling time which is taken as 0.2 s, and total simulation time is 20 s. Process noise, η_k, is white Gaussian with mean zero and covariance intensity $q = 0.01$ m^2/s^4, and the initial truth of the state is taken as $x_0 = \begin{bmatrix} 80 & 1 \end{bmatrix}^T$.

Measurement model: The platform dynamics in discrete-time is represented as

$$x_{d,k} = \bar{x}_{d,k} + \Delta x_{d,k}, \tag{36}$$

$$y_{d,k} = \bar{y}_{d,k} + \Delta y_{d,k}, \tag{37}$$

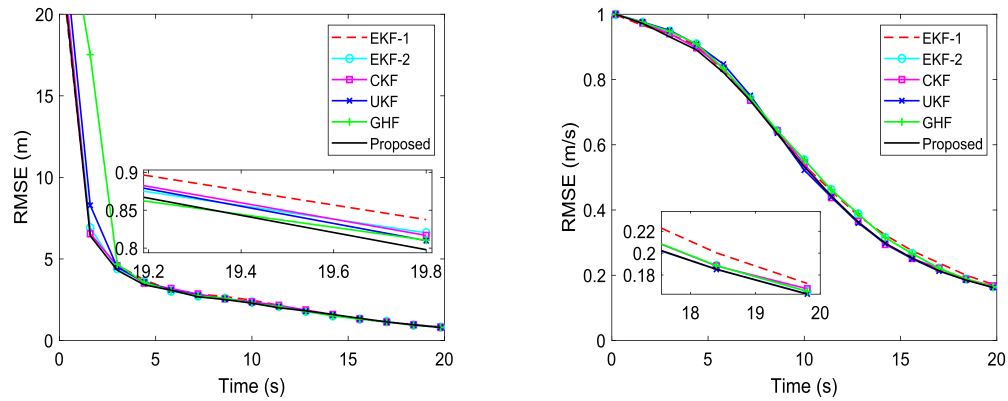
where $k = 1, 2, \cdots, n_{\text{step}}$, $\bar{x}_{d,k}$ and $\bar{y}_{d,k}$ are the average platform position in X and Y co-ordinates, respectively and $n_{\text{step}} = 100$. $\Delta x_{d,k} \sim \mathcal{N}(0, r_x)$ and $\Delta y_{d,k} \sim \mathcal{N}(0, r_y)$ are assumed to be white and mutually independent noises, where $r_x = 1\,\text{m}^2$ and $r_y = 1\,\text{m}^2$. The average platform positions are $\bar{x}_{d,k} = 4kT_s$ and $\bar{y}_{d,k} = 20$. The overall measurement model which includes platform noise [2] can be represented as

$$y_k = \tan^{-1}\left(\frac{\bar{y}_{d,k}}{x_{1,k} - \bar{x}_{d,k}}\right) + \nu_k, \tag{38}$$

where ν_k is the resultant measurement noise (including sensor noise and tracking platform uncertainty) with mean zero and covariance R_k, which is calculated as [2]

$$E[\nu_k^2] = R_k = \frac{\bar{y}_{d,k}^2 r_x + [x_{1,k} - \bar{x}_{d,k}]^2 r_y}{\{[x_{1,k} - \bar{x}_{d,k}]^2 + \bar{y}_{d,k}^2\}^2} + (3^\circ)^2. \tag{39}$$

Here, the measurement equation is not in polynomial form, so the Taylor series expansion is used. We implemented the proposed filter with second and third order Taylor series expansion along with the first, second order EKF [16], UKF, CKF and GHF-3 (3 points GHF). In this problem, we show the performance of proposed filter with second and third order Taylor series approximation because it has been observed that beyond third order approximation, no considerable improvement in performance has occurred. Moreover, in this problem, we stick to GHF-3 because it has been observed that the GHF beyond 3 points does not provide any improvement in estimation. Initialization of the filter is done as per [46]. The performance of the filter is shown in terms of RMSE obtained from 1000 MC runs. The RMSEs of position and velocity excluding track loss are shown in Fig. 3a and Fig. 3b and from the figures, it can be seen that the RMSE values for position and velocity of the EKFs, CKF, UKF, and the GHF are comparable to the proposed filter, that performs with a similar accuracy. We also provide the RMSE value averaged over 4 to 20 second obtained from 1000 MC runs in Table 4. From the table, we see that the at nominal condition of initialization all the filters perform with similar average RMSE.



(a) (b)

Fig. 3. RMSE against time plot for (a) Position (b) Velocity.

Table 4
Average RMSE in position and velocity.

	Position				Velocity			
Filter \ ξ	1	5	7.5	10	1	5	7.5	10
EKF-1	2.08	2.59	2.65	2.82	0.48	0.85	0.8	0.96
EKF-2	2.02	2.26	2.33	2.37	0.48	0.59	0.63	0.67
CKF	2.07	2.24	2.49	2.79	0.47	0.6	0.65	0.7
UKF	2.03	2.23	2.44	2.58	0.47	0.59	0.64	0.7
GHF	2.02	2.23	2.32	2.52	0.47	0.59	0.64	0.65
Proposed-2	2.01	2.26	2.33	2.38	0.47	0.58	0.63	0.67
Proposed-3	2	2.23	2.32	2.35	0.47	0.58	0.63	0.64

Table 5
Percentage track loss of different filters.

Filter \ ξ	1	5	7.5	10
EKF-1	0.22	1.48	1.50	1.72
EKF-2	0.02	0.03	0.05	0.08
CKF	0.01	0.04	0.15	0.23
UKF	0.02	0.03	0.08	0.09
GHF	0.01	0.03	0.06	0.12
Proposed-2	0	0.02	0.05	0.10
Proposed-3	0	0	0.01	0.01

To check the robustness of the estimators against a large initialization error, we vary the initial covariance $P_0 = \xi P_{0|0}$ (where ξ is a real number ≥ 1). The number of track loss, which is defined when the final position estimation error exceeds 15 m, is calculated from 100,000 MC runs and is tabulated in Table 5. From the table, we see that the UKF and GHF provide accuracy similar to the EKF-2 and the proposed filter has the lowest track loss compared to all other filters experimented with and is more robust against initial error uncertainty. Further, the proposed filter shows lower average RMSE compared to other filters for large initial uncertainty (see Table 4)

The relative execution times of the EKF-1, the EKF-2, the CKF, the UKF, and the GHF are 0.22, 0.28, 0.28, 0.30, and 0.32 respectively with respect to the Cholesky decomposition based proposed filter (which is considered to be unity). We also implemented the proposed filter using eigenvalue decomposition as mentioned in Theorem 1. The results remain the same however the execution time is approximately twice the Cholesky decomposition based method.

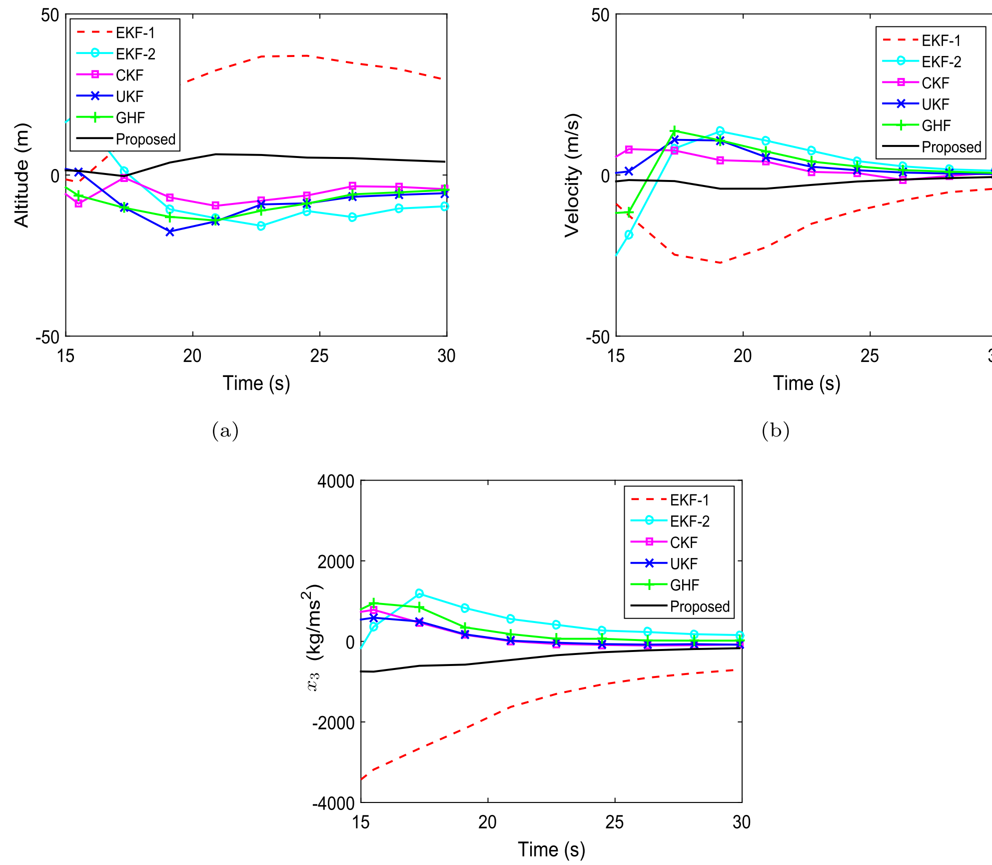
Problem 3. In this example, we have considered a problem on re-entry of ballistic target [47] in the atmosphere which is falling vertically under gravity and experiences a drag. The target kinematics in the continuous-time domain is presented by the following differential equations:

$$\dot{x}_1 = x_2, \tag{40}$$

$$\dot{x}_2 = g\Big(1 - \frac{\rho(x_1)x_2^2}{2x_3}\Big), \tag{41}$$

$$\dot{x}_3 = 0, \tag{42}$$

where x_1 is the altitude in meter, x_2 is velocity in m/s, $g = 9.81$ m/s^2 is the acceleration due to gravity, $\rho(x_1)$ is air density measured in kg/m^3, and x_3 is ballistic coefficient. Air density is the exponential function of altitude following $\rho(x_1) = a_1 \exp(-a_2 x_1)$ with $a_1 = 1.754$ and $a_2 = 1.49 \times 10^{-4}$. To implement the algorithm, the target dynamics are discretized by the Euler approximation with the small



(a) (b) (c)

Fig. 4. Mean error vs. time plot for (a) Position (b) Velocity (c) Ballistic coefficient.

integration step T_s as

$$x_k = \phi(x_{k-1}) + Gg + \eta_{k-1}, \tag{43}$$

where $x = [x_1 \quad x_2 \quad x_3]^T$, $G = [0 \quad T_s \quad 0]^T$, $\phi(x) = [x_1 - T_s x_2 \quad x_2 - T_s D(x) \quad x_3]^T$, $D(x)$ is the drag and is given by $D(x) = \dfrac{g\rho(x_1)x_2^2}{2x_3}$. The process noise, η_{k-1}, is taken as Gaussian with mean zero and covariance Q_{k-1} as described in Eq. (8) of [47].

The radar measures the altitude of the target and the measurement is corrupted by the noise which is assumed to be Gaussian. So the measurement equation becomes

$$y_k = [1 \quad 0 \quad 0]x_k + \nu_k, \tag{44}$$

where $\nu_k \sim \mathcal{N}(0, R_k)$.

The following parameters are used for simulation: $q_1 = q_2 = 5$, $T_s = 0.1$ s, and $R_k = (400\,\text{m})^2$. The truth of the filter is initialized with the initial altitude, x_1 = 60,960 m, velocity, x_2 = 3,048 m/s. The ballistic coefficient, x_3, is modeled with beta distribution [47] with the shape parameters $\lambda_1 = \lambda_2 = 1$, and lower and upper limits are 10,000 kg/ms^2, and 63,000 kg/ms^2 respectively. The initial estimate of the filter is $\hat{x}_{0|0} = [60960 \quad 3048 \quad \text{mean}(x_3)]^T$, and the initial error covariance is $P_{0|0}$ which is as described in Eq. (16) of [47].

The process equation has an exponential term, which is converted into a polynomial form by using the Taylor series expansion of order two and three. The states of the target are estimated by the EKF-1, EKF-2, CKF, UKF, GHF-3 (no improvement is observed beyond 3 points GHF), and the proposed filters. Similar to the previous problem, we show the performance of proposed filter with second and third order Taylor series approximation because it has been observed that beyond third order approximation, no improvement in performance has occurred. The performances of the filters are compared in terms of the RMSE and the mean of the estimation error, obtained from 1000 MC runs. The mean error and RMSE (excluding the diverged tracks) of altitude, velocity, and ballistic coefficient are plotted in Figs. 4 and 5, respectively. From Fig. 4, it has been observed that the mean error of the proposed filter has less ripples around zero compared to other filters. From Figs. 5a-5c, it can be seen that the proposed filter provides a better estimation result than the EKFs, CKF, UKF and the GHF. We also implemented the UKF with the eigenvalue decomposition method [48] but no decrease of the spike in RMSE plots is observed. We also tried to implement the proposed filter algorithm with Cholesky decomposition but it stops as the error covariance matrix becomes negative definite. The problem is well known in literature and it happens due to accumulated round-off error associated with the processing software [29,49]. The average RMSEs (after 15 s) obtained from 1000 MC runs of all the filters are listed in Table 6. From the table we observe that the proposed filter with third order Taylor series approximation achieves the lowest average RMSE for all the states.

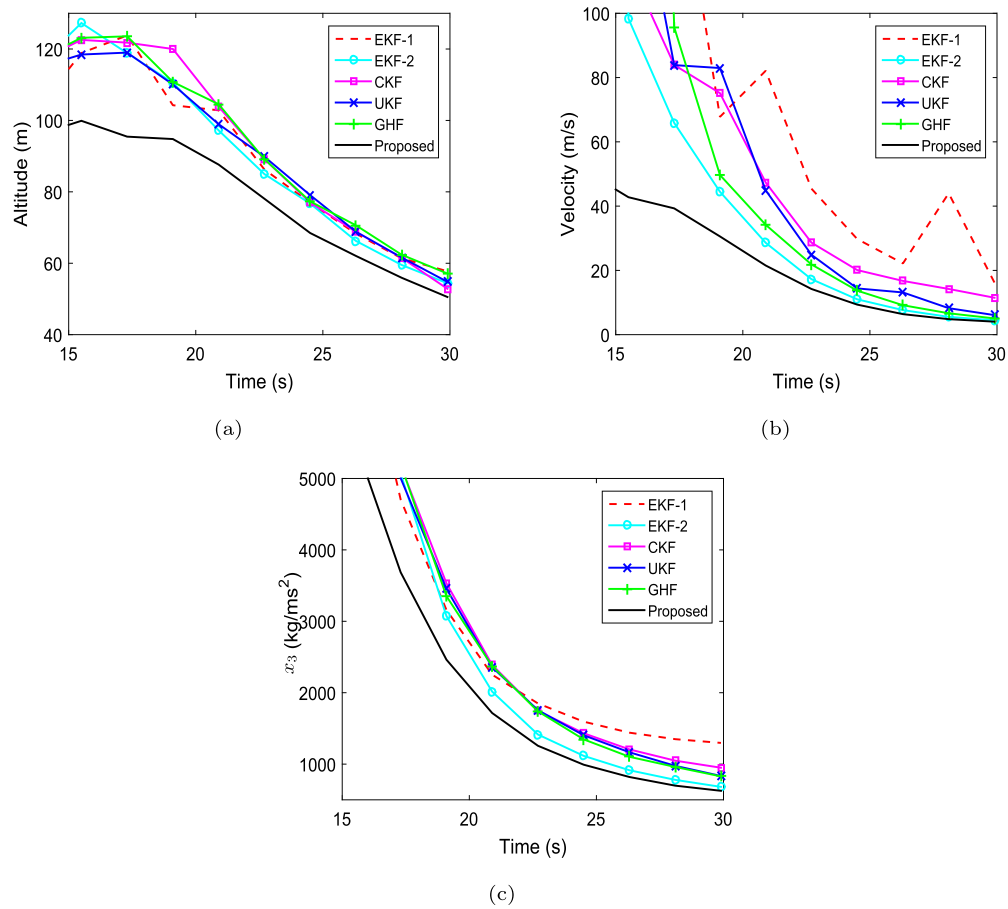


Fig. 5. RMSE vs. time plot for (a) Position (b) Velocity (c) Ballistic coefficient.

Table 6
Average RMSE of different filters.

Filter	Altitude (m)	Velocity (m/s)	Ballistic coef. (10^3 kg/ms^2)
EKF-1	89.62	66.83	2.76
EKF-2	89.14	52.90	2.50
CKF	88.05	53.56	2.51
UKF	87.68	52.62	2.48
GHF	87.56	52.47	2.45
Proposed-2	87.59	51.35	2.42
Proposed-3	77.47	19.19	1.92

Table 7
Percentage track loss and relative execution time of different filters.

Filter	Track loss (%)	Execution time
EKF-1	18.21	0.05
EKF-2	3.17	0.08
CKF	2.32	0.10
UKF	2.20	0.11
GHF	2.20	0.20
Proposed-2	2.21	0.32
Proposed-3	1.00	1

The performance of the filters is also compared in terms of percentage of track loss and relative execution time which is provided in Table 7. The filter is considered to be losing its track, when the terminal position error exceeds the predefined value $e_{\text{limit}} = 150$ m. From the table it has been observed that the proposed-3 attains the lowest track loss whereas the EKF-1 has the highest and other filters have comparable track loss. The relative execution time is highest for the proposed filter which is almost 10 times of the CKF.

5. Discussion and conclusion

Here, we propose a new filtering technique, where the prior and the posterior pdfs are assumed as Gaussian, and mean and covariance are calculated using the Gaussian integral. If the process and the measurement functions are polynomial, they directly fit into the proposed framework, if not the Taylor series expansion is used to convert them to a polynomial form. For a polynomial function, the proposed filter provides a near optimal solution under the Gaussian assumption and outperforms the available Gaussian filters. However, for a non-polynomial function (when we use the Taylor series approximation to make it a polynomial), the improvement may not always be prominent although more time is spent during the computation. In the simulation Problems 1 and 3, the proposed filtering algorithm consistently provides a more accurate estimation than the other filters, such as the EKF, CKF, UKF, and the GHF. In Problem 2, the proposed method provides comparable similar results in terms of RMSE but it shows more robustness compared to the existing filters against a large initial error uncertainty.

Although the execution time of the proposed filter is higher than the other quadrature filters, unlike the GHF, it does not suffer from the 'curse of dimensionality' problem. Due to high estimation accuracy and affordable computational burden, the proposed algorithm has the potential to become an indispensable state estimation technique for the designers.

The performance of the proposed algorithm is required to compare for more real life problems. To express a function in polynomial form instead of the Taylor series, orthogonal polynomial function can be used [50]. Further, implementation of Gaussian sum filter with the proposed estimator also remains under the scope of the future work.

CRediT authorship contribution statement

Kundan Kumar: Conceptualization, Methodology, Software, Validation, Writing-Original draft preparation. **Shreya Das**: Data curation, Writing-Reviewing and Editing. **Shovan Bhaumik**: Supervision, Visualization, Investigation, Writing-Reviewing and Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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