Consensus over matrix-weighted networks with time-delays

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Abstract

This paper studies consensus conditions for leaderless and leader-follower matrix-weighted consensus networks under the presence of constant time-delays. Several delayed consensus algorithms for networks of single- and double-integrators using only the relative positions are considered. Conditions for the networks to asymptotically converge to a consensus or clustering configuration are derived based on direct eigenvalue evaluation or the Lyapunov-Krasovkii theorem. Furthermore, an application of these algorithms in bearing-based network localization is also considered. The theoretical results are supported by numerical simulations.

1 Introduction

Recently, matrix-weighted consensus, a multi-dimensional extension of the well-known scalar-weighted consensus algorithm [20], has received a considerable amount of research attention. A matrix-weighted consensus system models diffusion dynamics in a multi-layer system with intra- and cross-layer interactions between multiple subsystems (or agents). Several applications of matrix-weighted consensus systems include multi-dimensional opinion dynamics models in [1, 33], bearing-based formation control [7,37], distributed localization of wireless sensor networks [3,4], and network synchronization [31].

A matrix-weighted consensus network can be described by a graph with both positive definite and positive semidefinite matrix weights. Associated with the graph, a corresponding Laplacian matrix, whose the kernel (aka the nullspace) may contain further subspaces in addition to the consensus space [2, 12, 31], can be defined. Necessary and sufficient conditions for a matrix-weighted consensus network to asymptotically achieve consensus or clustering were given in [29, 30]. Discrete-time and randomized matrix-weighted consensus were studied in [14, 16, 28]. The authors in [22] investigated the continuous-time consensus protocol with switching matrix-weighted graphs. A consensus is asymptotically achieved if the weighted integral network over some fixed time period always contains a positive spanning tree, or equivalently, the kernel of the Laplacian matrix of the integrated network contains only the consensus space [22]. The works [15, 16] examined the consensus problems over matrix-weighted networks for double-integrator agents. Controllability of the matrix-weighted consensus network was discussed in [21]. Recent studies on bipartite and multi-partite matrix-weighted consensus have been proposed in [10, 17, 32].

It practice, time delays are unavoidable if agents communicate their state variables via a wireless network, especially when the agents are separated by significant distances. When restricted to linear systems, time delay yields phase lags and alters both the transient and steady-state responses of the system. If the magnitude of the time delay is sufficiently large, the whole system could be destabilized. For this reason, it is essential to examine the stability conditions of matrix-weighted consensus networks under different assumptions on the time delays. It is noteworthy that even with delayed linear differential equations, the exact analysis via characteristic equations will lead to transcendental equations, of which solutions are often complicated [5, 23]. An alternative approach for analysing the stability of time-delayed systems is based on Lyapunov-Krasovskii or Lyapunov-Razumikhin theorems [11, 13]. In the literature, a sufficient condition for reaching a consensus in a scalar consensus network with a uniform time delay was given in [20]. Lyapunov-Razumikhin type functional was used for finding sufficient conditions for consensus networks with heterogeneous edge delays and switching interaction topology in [27]. The authors in [25] studied the consensus problem with uniform delay communication and provided consensus conditions by considering some Lyapunov–Razumikhin functionals. The author in [24] considered a consensus problem with heterogeneous communication time delays and introduced a delayed weighted Laplacian for the analysis. The consensus of double-integrator agents with time delay was studied in [34] based on an approximated characteristic equation under the assumption that the delays are sufficiently small. An exact analytic method for a second-order delayed scalar-consensus protocol was proposed in [6]. Stabilization control laws for double- and chain of integrators using delays were proposed in [19], and in the consensus problem over a scalar-weighted graph [26].

In this paper, we derive stability conditions of several delayed matrix-weighted consensus models having either a leaderless or a leader-follower topology. A leader-follower network contains several leader agents acting as stationary references during the dynamic process. First, we consider a matrix-weighted consensus network where all the edges have the same constant time delay. For this network, a necessary and sufficient stability condition related to the magnitude of the time delay and the maximum eigenvalue of the matrix-weighted Laplacian is established. Second, we study the matrix-weighted consensus with multiple heterogeneous constant time delays. A stability condition is given in terms of the feasibility of a linear matrix inequality (LMI). Third, we consider a matrix-weighted consensus network of double integrators, and show that the network can asymptotically reach the kernel of the matrix-weighted Laplacian by using only the delayed relative positions. As it is assumed that the kernel of the matrix-weighted Laplacian is not restricted to the consensus space, the applicability of the considered models is beyond a consensus problem. In particular, an application of the theoretical results in bearing-based network localization [36] is also discussed.

The rest of the paper is organized as follows. In Section 2, the theoretical background is provided and three delayed matrix-weighted consensus models studied in this paper are presented. Sections 3–5 give stability conditions and detailed analysis of each consensus model. An application in bearing-based network localization is discussed in Section 6, and simulation results are provided in Appendix A.5 to support the analysis. Lastly, Section 7 concludes the paper.

Notations: In this paper, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^d$, $\mathbb{R}^{m\times n}$ respectively denote the sets of real numbers, positive real numbers, d-dimensional vectors with real entries and $m \times n$ matrix with real entries. Let $\mathbf{0}_d$ and $\mathbf{\Theta}_d$ respectively denote the zero vector of dimension d and the zero matrix of dimension $d \times d$. For a real $m \times n$ matrix $\mathbf{A}$, we use $\mathbf{A}^\top$, rank$(\mathbf{A})$, det$(\mathbf{A})$, ker$(\mathbf{A})$, and im$(\mathbf{A})$ to denote the transposition, rank, determinant, kernel space and image space of $\mathbf{A}$, respectively. If $\mathbf{A}$ is symmetric positive definite (positive semidefinite), we write $\mathbf{A} > 0$ (resp., $\mathbf{A} \geq 0$). Given a vector $\mathbf{x} \in \mathbb{R}^d$, the Euclidean norm of $\mathbf{x}$ is denoted by $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^d x_i^2}$.

2 Preliminaries

2.1 Matrix-weighted networks

Consider an undirected, matrix-weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the vertex set $\mathcal{V} = \{1, \ldots, n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of $m = |\mathcal{E}|$ edges, and the set of nonnegative definite matrix weights $\mathcal{A} = \{\mathbf{A}_{ij} \in \mathbb{R}^{d\times d}\}_{i,j\in\mathcal{V}}$ with $\mathbf{A}_{ij} = \mathbf{A}_{ij}^\top \geq 0, \forall i, j$ and $d \geq 2$. Each edge $(i, j) \in \mathcal{E}$ captures the interactions between two agents i and j, and the existence of (i, j) implies the existence of (j, i) since the graph is undirected. If $(i, j) \in \mathcal{E}$, then $\mathbf{A}_{ij} \neq 0$; and if $(i, j) \notin \mathcal{E}$ or $i = j$, then $\mathbf{A}_{ij} = \mathbf{\Theta}_d$. The neighbor set of a vertex $i \in \mathcal{V}$ is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} | \ (i, j) \in \mathcal{E}\}$. Then, the degree matrix of a vertex i is defined as $\mathbf{D}_i = \sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}$.

Now, we can define matrix-weighted- adjacency and degree matrices $\mathbf{A} = [\mathbf{A}_{ij}] \in \mathbb{R}^{dn\times dn}$ and $\mathbf{D} = \text{blkdiag}(\mathbf{D}_1, \ldots, \mathbf{D}_n) \in \mathbb{R}^{dn\times dn}$. A matrix weighted Laplacian $\mathbf{L} = [\mathbf{L}_{ij}] \in \mathbb{R}^{dn\times dn}$ has

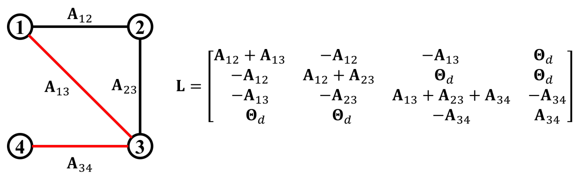


Figure 1: A matrix-weighted graph of four vertices and four edges and its matrix-weighted Laplacian. Each red edge corresponds to a positive definite matrix weight and each black edge corresponds to a positive semi-definite matrix weight.

block entries

$$\mathbf{L}_{ij} = \begin{cases} -\mathbf{A}_{ij}, & \text{if } i \neq j, \\ \sum_{j=1}^{n} \mathbf{A}_{ij}, & \text{if } i = j. \end{cases} \tag{1}$$

Note that $\mathbf{L}$ is symmetric, positive semi-definite, and $\text{ker}(\mathbf{L}) \supseteq \text{im}(\mathbf{1}_n \otimes \mathbf{I}_d)$. A matrix-weighted graph and its corresponding matrix-weighted Laplacian is depicted in Fig. 1 as an example.

We order the edges in $\mathcal{E}$ such that $\mathcal{E} = \{e_1, \ldots, e_m\}$, and adopt the notation $\mathbf{A}_{ij} \equiv \mathbf{A}_k, \forall e_k = (i, j,\ k = 1, \ldots, m)$. For each edge (i, j), we specify a vertex to be the starting vertex and the other vertex as the end vertex. The incidence matrix $\mathbf{H} = [h_{ki}] \in \mathbb{R}^{m \times n}$ is defined as follows

$$h_{ki} = \begin{cases} -1, & \text{if } i \text{ is the starting vertex of } e_k, \\ +1, & \text{if } i \text{ is the end vertex of } e_k, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then, $\mathbf{L} = \bar{\mathbf{H}}^\top \text{blkdiag}(\mathbf{A}_k)\bar{\mathbf{H}}$, where $\bar{\mathbf{H}} = \mathbf{H} \otimes \mathbf{I}_d$, and '$\otimes$' denotes the Kronecker product.

Suppose that the matrix-weighted Laplacian $\mathbf{L}$ has $l \geq d$ eigenvalues 0 with l linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_l$. This assumption allows the possibilities of achieving a consensus or and clustering when the following consensus algorithm is performed on a matrix-weighted network of single integrators

$$\dot{\mathbf{x}}_i(t) = \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),\ i = 1, \ldots, n. \tag{3}$$

where $\mathbf{x}_i \in \mathbb{R}^d$ is the state vector of agent $i \in \mathcal{V}$. Let $\mathbf{x} = [\mathbf{x}_1^\top, \ldots, \mathbf{x}_n^\top]^\top \in \mathbb{R}^{dn}$, the matrix-weighted consensus algorithm (3) can be rewritten in matrix form as

$$\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t), \tag{4}$$

and it has been shown that $\mathbf{x}(t) \to \bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i(0) \in \text{ker}(\mathbf{L})$, as $t \to +\infty$ [18, 29]. Throughout the paper, the shorthand $\mathbf{x}_{ij}(t) = \mathbf{x}_j(t) - \mathbf{x}_i(t)$ will be used.

From the assumption that zero is a semi-simple eigenvalue of multiplicity $l \geq d$, and $\mathbf{L}$ is symmetric, positive semi-definite, there exists an orthonormal matrix $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_{dn}] = [\mathbf{R}, \mathbf{Q}] \in \mathbb{R}^{dn \times dn}$ such that $\mathbf{R} = [\mathbf{p}_1, \ldots, \mathbf{p}_l] \in \mathbb{R}^{dn \times l}$, $\mathbf{Q} = [\mathbf{p}_{l+1}, \ldots, \mathbf{p}_{dn}] \in \mathbb{R}^{dn \times (dn-l)}$,

$$\mathbf{p}_i^\top \mathbf{p}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

and $\|\mathbf{p}_i\| = 1, \forall i, j = 1, \ldots, dn$ so that the matrix-weighted Laplacian is diagonalizable as $\mathbf{P}^\top \mathbf{L}\mathbf{P} = \mathbf{\Lambda}$, where $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Theta}_l & \mathbf{0}_{l \times (dn-l)} \\ \mathbf{0}_{(dn-l) \times l} & \bar{\mathbf{\Lambda}} \end{bmatrix} = \text{diag}(\lambda_1, \ldots, \lambda_{dn})$ (and $\bar{\mathbf{\Lambda}} = \text{diag}(\lambda_{l+1}, \ldots, \lambda_{dn})$, respectively) the diagonal matrix containing all eigenvalues (all positive eigenvalues) of $\mathbf{L}$. Note that $\mathbf{R} \supseteq \text{im}(\mathbf{1}_n \otimes \mathbf{I}_d)$ since the kernel of a matrix-weighted Laplacian always contains the consensus space. Also, $\mathbf{Q}^\top \mathbf{R} = \mathbf{0}_{(dn-l) \times l},\ \mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_{dn-l}$.

Consider a partition of the vertex set into two disjoint subsets $\mathcal{V}_a$ and $\mathcal{V}_b$ such that $\mathcal{V}_a \cup \mathcal{V}_b = \mathcal{V}$, $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$, $|\mathcal{V}_a| = n_a$, $|\mathcal{V}_b| = n_b$, $n_a + n_b = n$. The agents associated with the vertices in

$\mathcal{V}_a$ and $\mathcal{V}_b$ are referred to as leaders and followers, respectively. By labeling the vertices such that $\mathcal{V}_a = \{1, \ldots, n_a\}$, $\mathcal{V}_b = \{n_a + 1, \ldots, n\}$, the matrix-weighted Laplacian is partitioned as

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_a & \mathbf{L}_{ab}^\top \\ \mathbf{L}_{ab} & \mathbf{L}_b \end{bmatrix}, \tag{5}$$

where $\mathbf{L}_a = \mathbf{L}_a^\top \in \mathbb{R}^{dn_a \times dn_a}$, $\mathbf{L}_{ab}^\top \in \mathbb{R}^{dn_a \times dn_b}$, and $\mathbf{L}_b = \mathbf{L}_b^\top \in \mathbb{R}^{dn_b \times dn_b}$. Let $\mathbf{L}'$ denote the matrix-weighted Laplacian corresponding to the subgraph induced by the vertices in $\mathcal{V}_b$ and edges in $\mathcal{G}$. If $n_a = 0$, we have a leaderless network while for $n_a \geq 1$, we have a leader-follower network. We prove the following lemma on the matrix-weighted Laplacian (5).

Lemma 2.1. *Let rank*$(\mathbf{L}) = dn - l$, *rank*$(\mathbf{L}') = dn_b - l$, $n_a \geq 1$ *and* $l \geq d + 1$. *If* $\forall \boldsymbol{\xi} \in$ *ker*$(\mathbf{L}')$, $[\mathbf{0}_{dn_a}^\top, \boldsymbol{\xi}^\top]^\top \notin$ *ker*$(\mathbf{L})$, *then the matrix* $\mathbf{L}_b$ *is symmetric positive definite.*

Proof. Let $\mathbf{B} = \text{blkdiag}(\mathbf{L}_{ab}(\mathbf{1}_{n_a} \otimes \mathbf{I}_d)) = \text{blkdiag}(\mathbf{B}_1, \ldots, \mathbf{B}_{n_b}) \in \mathbb{R}^{dn_b \times dn_b}$, we have $\mathbf{L}_b = \mathbf{L}' - \mathbf{B}$. Suppose that $\mathbf{L}_b$ is not positive definite, then there exists $\boldsymbol{\xi} = [\boldsymbol{\xi}_1^\top, \ldots, \boldsymbol{\xi}_{n_b}^\top]^\top \in \mathbb{R}^{dn_b}$ such that $\boldsymbol{\xi}^\top \mathbf{L}_b \boldsymbol{\xi} = \boldsymbol{\xi}^\top (\mathbf{L}' - \mathbf{B}) \boldsymbol{\xi} = \mathbf{0}_{dn_b}$. From the assumption on $\mathbf{L}'$, it follows that $\boldsymbol{\xi} \in \ker(\mathbf{L}')$. Furthermore, $\boldsymbol{\xi}^\top \mathbf{B} \boldsymbol{\xi} = \sum_{k=1}^{n_b} \boldsymbol{\xi}_k^\top \mathbf{B}_k \boldsymbol{\xi}_k = 0$. Since each matrix weight in $\mathbf{B}_k = \sum_{j=1}^{n_a} [\mathbf{L}_{ab}]_{kj}$ is negative semidefinite, it follows that $\boldsymbol{\xi}_k \in \ker([\mathbf{L}_{ab}]_{kj}),\ \forall j = 1, \ldots, n_a$, or equivalently $\mathbf{L}_{ab}^\top \boldsymbol{\xi} = \mathbf{0}_{dn_a}$. Then, we have $\mathbf{L} \begin{bmatrix} \mathbf{0}_{dn_a} \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{ab}^\top \boldsymbol{\xi} \\ \mathbf{L}_b \boldsymbol{\xi} \end{bmatrix} = \mathbf{0}_{dn}$, which shows that $[\mathbf{0}_{dn_a}^\top, \boldsymbol{\xi}^\top]^\top \in \ker(\mathbf{L})$. This contradiction implies that $\mathbf{L}_b$ must be positive definite. □

2.2 Problem formulation

This paper aims to give some conditions for stability and/or reaching a consensus when time delays are present in (4) and its expanded versions. Particularly, the following matrix-weighted consensus models with time delays will be studied.

Model 1 Matrix-weighted consensus of single-integrators with a uniform constant time-delay $\tau > 0$:

$$\dot{\mathbf{x}}_i(t) = \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij} \mathbf{x}_{ij}(t - \tau), \tag{6}$$

$\forall i \in \mathcal{V}_b$, and $\dot{\mathbf{x}}_i(t) = \mathbf{0}_d, \forall i \in \mathcal{V}_a$.

Model 2 Matrix-weighted consensus of single-integrators with heterogeneous constant time-delays

$$\dot{\mathbf{x}}_i(t) = \sum_{j=1}^{n} \mathbf{x}_{ij}(t - \tau_{ij}), \tag{7}$$

where $i \in \mathcal{V}_b$, $\tau_{ij} \geq 0$ is the time-delay associated with an edge $(i, j) \in \mathcal{E}$, and $\dot{\mathbf{x}}_i(t) = \mathbf{0}_d, \forall i \in \mathcal{V}_a$.

Model 3 Matrix-weighted consensus of double-integrators with two constant time-delays:

$$\dot{\mathbf{x}}_i^1(t) = \mathbf{x}_i^2(t), \tag{8a}$$

$$\dot{\mathbf{x}}_i^2(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1(t - \tau_1) - \mathbf{x}_j^1(t - \tau_1)) - \alpha \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1(t - \tau_2) - \mathbf{x}_j^1(t - \tau_2)), \tag{8b}$$

where $\mathbf{x}_i^k = [x_{1i}^k, \ldots, x_{di}^k]^\top \in \mathbb{R}^d$, $i \in \mathcal{V}_b$, and $\dot{\mathbf{x}}_i^k(t) = \mathbf{0}_d$, $\forall i \in \mathcal{V}_a$, $k = 1, 2$. Here, $\mathbf{x}_i = [(\mathbf{x}_i^1)^\top, (\mathbf{x}_i^2)^\top]^\top$ and $\mathbf{x}_1^1$, $\mathbf{x}_i^2$ are referred to as the position and the velocity of agent i, and $\alpha > 0$ is a control gain.

For each model, the initial condition is given as $\mathbf{x}(\theta) = \mathbf{x}(0), \forall \theta \in [-\tau_k, 0]$.

3 Matrix-weighted consensus of single-integrators with a uniform time-delay

In this section, we give condition on the time-delay to ensure the model (6) to asymptotically achieve a consensus for leaderless and leader-follower matrix-weighted networks.

3.1 Leaderless network

The following theorem provides necessary and sufficient consensus condition for a leaderless matrix-weighted consensus network.

Theorem 3.1. *Consider a leaderless n-agent network with $\mathcal{V}_a = \emptyset$, and rank$(\mathbf{L}) = dn - l$, $l \geq d$. Under the consensus algorithm* (6), $\mathbf{x}(t)$ *asymptotically converges to* $\mathbf{x}^* = \mathbf{R}\mathbf{R}^\top \mathbf{x}(0) \in ker(\mathbf{L})$ *if and only if* $\tau < \frac{\pi}{2\lambda_{dn}}$, *where* λ_{dn} *is the largest eigenvalue of* $\mathbf{L}$.

Proof. The proof of this theorem is given in Appendix A.2. □

Remark 3.2. Observe that if rank$(\mathbf{L}) = dn - d$ and the stability condition $\tau < \frac{\pi}{2\lambda_{dn}}$ holds, then $l = d$, $\sum_{p=1}^{d} \mathbf{p}_k \mathbf{p}_k^\top = \frac{1}{n}(\mathbf{1}_n \mathbf{1}_n^\top \otimes \mathbf{I}_d)$ and the system asymptotically achieves a consensus. A similar consensus condition was given in [20] for scalar-weighted consensus networks but the proof is different from that of Theorem 3.1.

3.2 Leader-follower network

Next, we consider the leader-follower network under the consensus law (6). Let $\mathbf{x}_a = [\mathbf{x}_1^\top, \ldots, \mathbf{x}_{n_a}^\top]^\top$ and $\mathbf{x}_b = [\mathbf{x}_{n_a+1}^\top, \ldots, \mathbf{x}_n^\top]^\top$ respectively denote the stacked vectors of the leader and the follower agents. The behaviors of the network is given in the following theorem.

Theorem 3.3. *Consider a leader-follower n-agent network with $n_a \geq 1$, rank$(\mathbf{L}) = dn - l$, $l \geq d$, and $\mathbf{L}_b$ is positive definite. Under the consensus algorithm* (6), $\mathbf{x}_b$ *asymptotically converges to* $\mathbf{x}_b^* = \mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}_a$ *if and only if* $\tau < \frac{\pi}{2\lambda_{b\max}}$, *where* $\lambda_{b\max}$ *is the largest eigenvalue of* $\mathbf{L}_b$.

Proof. We can write the n-agent network in matrix form as follows

$$\begin{bmatrix} \dot{\mathbf{x}}_a(t) \\ \dot{\mathbf{x}}_b(t) \end{bmatrix} = -\begin{bmatrix} \mathbf{\Theta}_{dn_a} & \mathbf{0}_{dn_a \times dn_b} \\ \mathbf{L}_{ab} & \mathbf{L}_b \end{bmatrix} \begin{bmatrix} \mathbf{x}_a(t-\tau) \\ \mathbf{x}_b(t-\tau) \end{bmatrix}. \tag{9}$$

As $\mathbf{x}_a(t) = \mathbf{x}_a(0),\ \forall t \geq -\tau$, we consider the variable transformation $\boldsymbol{\delta}_b(t) = \mathbf{x}_b(t) + \mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}_a$, and derive the equation

$$\dot{\boldsymbol{\delta}}_b(t) = -\mathbf{L}_b \boldsymbol{\delta}_b(t-\tau). \tag{10}$$

The proof that the delayed system (10) is asymptotically stable if and only if $\tau < \frac{\pi}{2\lambda_{b\max}}$ is similar to the proof of Thm. 3.1 and will be omitted. □

Remark 3.4. It is remarked that if a consensus algorithm is performed in a leader-follower scalar-weighted graph with non-collocated leaders, the followers will asymptotically converge to fixed points inside the convex hull of the leaders' position. In contrast, as shown in Thm. 3.3, for a matrix-weighted consensus, the convergence points of follower agents may lie outside the convex hull of the leaders' positions. This property finds application in the bearing-based network localization problem discussed in Section 6.

4 Matrix-weighted consensus of single integrators with heterogeneous delays

In this section, we study the matrix-weighted consensus algorithms with heterogeneous time delays (7). We first study the problem for a leaderless matrix-weighted network and then consider the problem for a leader-follower network.

4.1 Leaderless network

Due to symmetry, we have $\tau_{ij} = \tau_{ji}, \forall (i,j) \in \mathcal{E}$. We can rewrite the dynamics (7) in the matrix form as follows:

$$\dot{\mathbf{x}}(t) = -\sum_{k=1}^{r} \mathbf{L}_k \mathbf{x}(t-\tau_k), \tag{11}$$

where $r \leq |\mathcal{E}|$, $\tau_k = \tau_{ij}$ if $e_k = (i, j)$, for $k = 1, \ldots, r$, and $\mathbf{L}_k = [\mathbf{L}_{kij}] \in \mathbb{R}^{dn \times dn}$ is a matrix whose $d \times d$ blocks are defined by

$$\mathbf{L}_{kij} = \begin{cases} -\mathbf{A}_{ij}, & j \neq i,\ \tau_k = \tau_{ij}, \\ \mathbf{\Theta}_d, & j \neq i,\ \tau_k \neq \tau_{ij}, \\ -\sum_{j=1, j\neq i}^{n} \mathbf{L}_{kij}, & j = i. \end{cases}$$

It is observed that $\mathbf{L}_k$ is a part of the Laplacian matrix corresponding to an update with time delay τ_k, and $\mathbf{L} = \sum_{k=1}^{r} \mathbf{L}_k$. As in the previous section, $\mathbf{R}^\top \mathbf{L}_k = \mathbf{0}_{l \times nd}$, for $k = 1, \ldots, r$. It follows that $\mathbf{x}^* = \mathbf{R}\mathbf{R}^\top \mathbf{x}(t)$ is time-invariant.

Moreover, we have $\mathbf{L}_k = \mathbf{P}\mathbf{\Lambda}_k\mathbf{P}^\top$, where $\mathbf{\Lambda}_k = \begin{bmatrix} \mathbf{\Theta}_l & \mathbf{0}_{l \times (dn-l)} \\ \mathbf{0}_{(dn-l)\times l} & \bar{\mathbf{\Lambda}}_k \end{bmatrix}$ and $\bar{\mathbf{\Lambda}}_k = \mathbf{Q}^\top \mathbf{L}_k \mathbf{Q} \in \mathbb{R}^{(dn-l)\times(dn-l)}$. Define $\boldsymbol{\delta}(t) = \mathbf{Q}^\top \mathbf{x}(t) \in \mathbb{R}^{dn-l}$, then the equation (11) can be rewritten in the following form [13]:

$$\begin{aligned} \dot{\boldsymbol{\delta}}(t) &= -\sum_{k=1}^{r} \mathbf{Q}^\top \mathbf{L}_k \mathbf{x}(t - \tau_k) = -\sum_{k=1}^{r} \bar{\mathbf{\Lambda}}_k \boldsymbol{\delta}(t - \tau_k) \\ &= -\bar{\mathbf{\Lambda}}\boldsymbol{\delta}(t) + \sum_{k=1}^{r} \bar{\mathbf{\Lambda}}_k (\boldsymbol{\delta}(t) - \boldsymbol{\delta}(t - \tau_k)) \\ &= -\bar{\mathbf{\Lambda}}\boldsymbol{\delta}(t) + \sum_{k=1}^{r} \bar{\mathbf{\Lambda}}_k \int_{t-\tau_k}^{t} \dot{\boldsymbol{\delta}}(s) ds. \end{aligned} \tag{12}$$

The stability of the system (12) is stated in the following theorem, whose proof can be found in Appendix A.3.

Theorem 4.1. *Consider the leaderless matrix-weighted consensus network with time delays* (12)*, where rank*$(\mathbf{L}) = dn - l$, $n_a = 0$ *and* $l \geq d$*. Suppose that the time delays* τ_k *are sufficient small such that the LMI* (13) *holds, where* $\tau = \sum_{i=1}^{r} \tau_i$.[1] *Then, the origin is a globally uniformly asymptotically equilibrium of* (12) *and* $\mathbf{x}(t) \to \mathbf{x}^* \in ker(\mathbf{L})$ *as* $t \to +\infty$.

$$\mathbf{M} = \begin{bmatrix} -2\bar{\mathbf{\Lambda}} & \bar{\mathbf{\Lambda}}_1 & \bar{\mathbf{\Lambda}}_2 & \cdots & \bar{\mathbf{\Lambda}}_r \\ * & -\tau_1^{-1}\mathbf{I}_{dn-l} & \mathbf{\Theta}_{dn-l} & \cdots & \mathbf{\Theta}_{dn-l} \\ * & * & \ddots & \ddots & \vdots \\ * & * & * & -\tau_{r-1}^{-1}\mathbf{I}_{dn-l} & \mathbf{\Theta}_{dn-l} \\ * & * & * & * & -\tau_r^{-1}\mathbf{I}_{dn-l} \end{bmatrix} + \tau \begin{bmatrix} -\bar{\mathbf{\Lambda}} \\ \bar{\mathbf{\Lambda}}_1 \\ \vdots \\ \bar{\mathbf{\Lambda}}_{r-1} \\ \bar{\mathbf{\Lambda}}_r \end{bmatrix} \begin{bmatrix} -\bar{\mathbf{\Lambda}} \\ \bar{\mathbf{\Lambda}}_1 \\ \vdots \\ \bar{\mathbf{\Lambda}}_{r-1} \\ \bar{\mathbf{\Lambda}}_r \end{bmatrix}^\top < 0. \tag{13}$$

4.2 Leader-follower network

Next, we consider the leader-follower network under the consensus algorithm (7). Similar to the previous section, we can define $\boldsymbol{\delta}_b(t) = \mathbf{x}_b(t) + \mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}_a$, where $\mathbf{L}_{ab} = \sum_{k=1}^{r} \mathbf{L}_{abk}$ and $\mathbf{L}_b = \sum_{k=1}^{r} \mathbf{L}_{bbk}$. That is, each matrix $\mathbf{L}_k$ contributes a part to the matrices $\mathbf{L}_{ab}$ and $\mathbf{L}_b$. Then,

$$\begin{aligned} \dot{\boldsymbol{\delta}}_b(t) &= -\sum_{k=1}^{r} \mathbf{L}_{bk}\mathbf{x}_b(t - \tau) - \sum_{k=1}^{r} \mathbf{L}_{abk}\mathbf{x}_a = -\sum_{k=1}^{r} \mathbf{L}_{bk}\boldsymbol{\delta}_b(t - \tau) \\ &= -\mathbf{L}_b\boldsymbol{\delta}_b(t) + \sum_{k=1}^{r} \mathbf{L}_{bk} \int_{t-\tau_k}^{t} \dot{\boldsymbol{\delta}}_b(s) ds. \end{aligned} \tag{14}$$

We can now state a theorem on the delayed-system (14), whose proof is similar to the proof of Theorem 4.1 and will be omitted.

Theorem 4.2. *Suppose that the* n*-agent network has a leader follower structure,* $n_a \geq 1$, *rank*$(\mathbf{L}) = dn - l$, $l \geq d$*, and* $\mathbf{L}_b$ *is positive definite. If the time delays* τ_k *are chosen such that the LMI* (15) *holds and* $\tau = \sum_{i=1}^{r} \tau_i$*, then* $\boldsymbol{\delta}_b = \mathbf{0}_{dn_b}$ *is globally unniformly asymptotically stable, and* $\mathbf{x}(t) \to \mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}_a$ *as* $t \to +\infty$.

[1] In each LMI, the asterisk '*' indicates that the matrix is symmetric, so it is no need to specify the block matrices below the diagonal.

$$\mathbf{N}=\begin{bmatrix}-2\mathbf{L}_b & \mathbf{L}_{b1} & \mathbf{L}_{b2} & \dots & \mathbf{L}_{br}\\ * & -\tau_1^{-1}\mathbf{I}_{dn_b} & \mathbf{\Theta}_{dn_b} & \dots & \mathbf{\Theta}_{dn_b}\\ * & * & \ddots & \ddots & \vdots\\ * & * & * & -\tau_{r-1}^{-1}\mathbf{I}_{dn_b} & \mathbf{\Theta}_{dn_b}\\ * & * & * & * & -\tau_r^{-1}\mathbf{I}_{dn_b}\end{bmatrix}+\tau\begin{bmatrix}-\mathbf{L}_b\\ \mathbf{L}_{b1}\\ \vdots\\ \mathbf{L}_{br-1}\\ \mathbf{L}_{br}\end{bmatrix}\begin{bmatrix}-\mathbf{L}_b\\ \mathbf{L}_{b1}\\ \vdots\\ \mathbf{L}_{br-1}\\ \mathbf{L}_{br}\end{bmatrix}^\top<0, \quad (15)$$

5 Matrix-weighted consensus of double-integrators without relative velocity measurements using two time delays

5.1 Leaderless network

Consider a leaderless matrix-weighted network. We express the network (8) in the matrix form as follows

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}\dot{\mathbf{x}}^1(t)\\ \dot{\mathbf{x}}^2(t)\end{bmatrix}=\begin{bmatrix}\mathbf{\Theta}_{dn} & \mathbf{I}_{dn}\\ \mathbf{\Theta}_{dn} & \mathbf{\Theta}_{dn}\end{bmatrix}\begin{bmatrix}\mathbf{x}^1(t)\\ \mathbf{x}^2(t)\end{bmatrix}+\begin{bmatrix}\mathbf{\Theta}_{dn} & \mathbf{\Theta}_{dn}\\ -\mathbf{L} & \alpha\mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{x}^1(t-\tau_1)\\ \mathbf{x}^1(t-\tau_2)\end{bmatrix}$$

First, observe that $(\mathbf{1}_n^\top\otimes\mathbf{I}_d)\dot{\mathbf{x}}^2(t)=-(\mathbf{1}_n^\top\otimes\mathbf{I}_d)\mathbf{L}\mathbf{x}^1(t-\tau_1)+\alpha(\mathbf{1}_n^\top\otimes\mathbf{I}_d)\mathbf{L}\mathbf{x}^1(t-\tau_2)=\mathbf{0}_{dn}$. Hence, $(\mathbf{1}_n^\top\otimes\mathbf{I}_d)\mathbf{x}^2(t)=(\mathbf{1}_n^\top\otimes\mathbf{I}_d)\mathbf{x}^2(0)=\mathbf{0}_{dn}$. This property will be used in proving the main theorem of this subsection.

Second, since $\mathbf{x}^1(t-\tau_1)=\mathbf{x}^1(t)\underbrace{-\int_{t-\tau_1}^{t}\mathbf{x}^2(s)ds}_{:=\mathbf{r}_1(t)}$ and,

$$\begin{aligned}\mathbf{x}^1(t-\tau_2)&=\mathbf{x}^1(t)-\tau_2\mathbf{x}^2(t)+\left(\tau_2\mathbf{x}^2(t)-(\mathbf{x}^1(t)-\mathbf{x}^1(t-\tau_2))\right)\\ &=\mathbf{x}^1(t)-\tau_2\mathbf{x}^2(t)+\underbrace{\left(\tau_2\mathbf{x}^2(t)-\int_{t-\tau_2}^{t}\mathbf{x}^2(s)ds\right)}_{:=\mathbf{r}_2(t)},\end{aligned}$$

we can rewrite the system as

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}\mathbf{\Theta}_{dn} & \mathbf{I}_{dn}\\ -(1-\alpha)\mathbf{L} & -\alpha\tau_2\mathbf{L}\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}\mathbf{0}_{dn}\\ -\mathbf{L}(\mathbf{r}_1(t)-\alpha\mathbf{r}_2(t))\end{bmatrix}.$$

Let $\mathbf{z}^1=\mathbf{Q}^\top\mathbf{x}^1$, $\mathbf{z}^2=\mathbf{Q}^\top\mathbf{x}^2$, and $\mathbf{z}=[(\mathbf{z}^1)^\top,(\mathbf{z}^2)^\top]^\top$. The differential equation governing the $\mathbf{z}$-system is

$$\begin{aligned}\dot{\mathbf{z}}&=\begin{bmatrix}\mathbf{\Theta}_{dn-l} & \mathbf{I}_{dn-l}\\ -(1-k)\bar{\mathbf{\Lambda}} & -\alpha\tau_2\bar{\mathbf{\Lambda}}\end{bmatrix}\mathbf{z}+\begin{bmatrix}\mathbf{0}_{dn-l}\\ -\bar{\mathbf{\Lambda}}\mathbf{Q}^\top(\mathbf{r}_1(t)-\alpha\mathbf{r}_2(t))\end{bmatrix}\\ &=\mathbf{F}(\tau_2)\mathbf{z}+\begin{bmatrix}\mathbf{0}_{dn-l}\\ \bar{\mathbf{\Lambda}}\int_{t-\tau_1}^{t}\mathbf{z}^2(s)ds\end{bmatrix}+\begin{bmatrix}\mathbf{0}_{dn-l}\\ \alpha\bar{\mathbf{\Lambda}}\left(\tau_2\mathbf{z}^2(t)-\int_{t-\tau_2}^{t}\mathbf{z}^2(s)ds\right)\end{bmatrix}.\end{aligned}$$

The eigenvalues of $\mathbf{F}(\tau_2)\in\mathbb{R}^{2(dn-l)\times 2(dn-l)}$ satisfy the characteristic equation

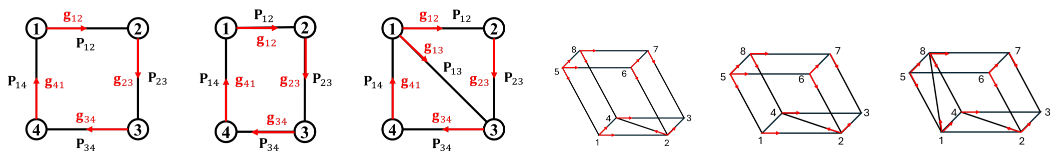
$$\det(s^2\mathbf{I}_{dn-l}+\alpha\tau_2\bar{\mathbf{\Lambda}}s+(1-\alpha)\bar{\mathbf{\Lambda}})=0\iff\prod_{i=dn-l+1}^{dn}(s^2+\alpha\tau_2\lambda_i s+(1-\alpha)\lambda_i)=0,$$

where $\lambda_i>0,\ i=l+1,\dots,dn$, are the positive eigenvalues of the matrix-weighted Laplacian matrix $\mathbf{L}$. Thus, for $\alpha<1$ and $\tau_2>0$, $\mathbf{F}(\tau_2)$ is Hurwitz, and we can find a symmetric positive definite matrix $\mathbf{\Pi}\in\mathbb{R}^{2(dn-l)\times 2(dn-l)}$ satisfying the Lyapunov equation

$$\mathbf{\Pi}\mathbf{F}(\tau_2)+\mathbf{F}(\tau_2)^\top\mathbf{\Pi}=-\tau\mathbf{I}_{2(dn-l)}, \quad (16)$$

where $\tau=\tau_2-\tau_1$.

Finally, we can state the following theorem whose proof can be found in Appendix A.4.



(a) $(\mathcal{G}_1, \mathbf{x})$ (b) $(\mathcal{G}_1, \mathbf{y})$ (c) $(\mathcal{G}_2, \mathbf{z})$ (d) $(\mathcal{G}_3, \mathbf{x}')$ (e) $(\mathcal{G}_3, \mathbf{y}')$ (f) $(\mathcal{G}_4, \mathbf{z}')$

Figure 2: Consider three networks (a), (b), (c) in the two-dimensional space. Two networks $(\mathcal{G}_1, \mathbf{x})$ and $(\mathcal{G}_1, \mathbf{y})$ have $\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|} = \frac{\mathbf{y}_i - \mathbf{y}_j}{\|\mathbf{y}_i - \mathbf{y}_j\|}, \forall (i,j) \in \mathcal{E}$ but are not related by a combination of translations and scaling. Their corresponding matrix-weighted Laplacian has $\text{rank}(\mathbf{L}) = 4 < 2n - 3$. In contrast, the network $(\mathcal{G}_2, \mathbf{z})$ (having one more edge $(1,3)$ satisfies $\text{rank}(\mathbf{L}) = 5 = 2n - 3$; Three networks (d), (e), (f) are considered in the three dimensional space, the matrix-weighted Laplacian of networks $(\mathcal{G}_3, \mathbf{x}')$ and $(\mathcal{G}_3, \mathbf{y}')$ has $\text{rank}(\mathbf{L}) = 19 < 3n - 4$, while network $(\mathcal{G}_4, \mathbf{z}')$ (have an additional edge $(1, 8)$) has $\text{rank}(\mathbf{L}) = 20 = 3n - 4$.

Theorem 5.1. *Consider the leaderless delayed second-order consensus model* (8)*, where* $\text{rank}(\mathbf{L}) = dn - l$*,* $n_a = 0$*,* $\alpha < 1$*,* $\mathbf{x}_i^2(0) = \mathbf{0}_d, \forall i = 1, \ldots, n$*, and* $\tau_1 > 0$*. Suppose that there exist positive definite matrices* $\mathbf{W} \in \mathbb{R}^{(dn-l)\times(dn-l)}$*,* $\mathbf{Z} \in \mathbb{R}^{(dn-l)\times(dn-l)}$ *and* $\mathbf{\Pi} \in \mathbb{R}^{2(dn-l)\times 2(dn-l)}$ *such that the matrix*

$$\mathbf{\Xi}(\tau_2) = \begin{bmatrix} \mathbf{X} & \mathbf{Y} & \mathbf{Y} & \tau_2^2 \mathbf{F}(\tau_2)^\top \begin{bmatrix} \mathbf{\Theta}_{dn-l} & \bar{\mathbf{\Lambda}} \end{bmatrix}^\top \mathbf{W} \\ * & -\mathbf{Z} & \mathbf{\Theta}_{dn-l} & -\tau_2^2 \bar{\mathbf{\Lambda}}^2 \mathbf{W} \\ * & * & -\frac{\pi^2}{4}\mathbf{W} & -k\tau_2^2 \bar{\mathbf{\Lambda}}^2 \mathbf{W} \\ * & * & * & -\mathbf{W} \end{bmatrix} \tag{17}$$

is negative definite, where

$$\mathbf{X} = \mathbf{\Pi}\mathbf{F}(\tau_2) + \mathbf{F}(\tau_2)^\top \mathbf{\Pi} + \begin{bmatrix} \mathbf{\Theta}_{dn-l} & \mathbf{\Theta}_{dn-l} \\ \mathbf{\Theta}_{dn-l} & \tau_1^2 \bar{\mathbf{\Lambda}}\mathbf{Z}\bar{\mathbf{\Lambda}} \end{bmatrix}, \ \mathbf{Y} = \mathbf{\Pi} \begin{bmatrix} \mathbf{\Theta}_{dn-l} \\ \bar{\mathbf{\Lambda}} \end{bmatrix}. \tag{18}$$

Then, $\mathbf{x}^1(t) \to \ker(\mathbf{L})$*,* $\mathbf{x}^2(t) \to \mathbf{0}_{dn}$ *as* $t \to +\infty$*.*

Remark 5.2. The condition $\alpha < 1$ is only sufficient for our analysis, which is based on (16) to held. Indeed, for certain choices of τ_1 and τ_2, $\alpha = 1$ may still make the system achieve asymptotic consensus.

5.2 Leader-follower network

We now consider the consensus algorithm (8) when the matrix-weighted graph has a leader-follower structure. The leaders' positions are time-invariant, thus $\mathbf{x}_a^1(t) = \mathbf{x}_a^1(0) := \mathbf{x}_a^1$, $\mathbf{x}_a^2(t) = \mathbf{0}_{dn_a}, \forall t \geq -\tau$. The equations governs followers' dynamics are given as follows

$$\dot{\mathbf{x}}_b^1(t) = \mathbf{x}_b^2(t), \tag{19a}$$

$$\dot{\mathbf{x}}_b^2(t) = -\mathbf{L}_b \mathbf{x}_b^1(t - \tau_1) - \mathbf{L}_{ab}\mathbf{x}_a^1 + \alpha \mathbf{L}_b \mathbf{x}_b^1(t - \tau_2) + \alpha \mathbf{L}_{ab}\mathbf{x}_a^1. \tag{19b}$$

Using the variable transformation $\boldsymbol{\delta}_b^1(t) = \mathbf{x}_b^1(t) + \mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}^1$ and $\boldsymbol{\delta}_b^2(t) = \mathbf{x}_b^2(t)$, we have the equations with the transformed variables

$$\dot{\boldsymbol{\delta}}_b^1(t) = \boldsymbol{\delta}_b^2(t), \tag{20a}$$

$$\dot{\boldsymbol{\delta}}_b^2(t) = -\mathbf{L}_b \boldsymbol{\delta}_b^1(t - \tau_1) + \alpha \mathbf{L}_b \boldsymbol{\delta}_b^1(t - \tau_2). \tag{20b}$$

Defining $\mathbf{E}(\tau_2) = \begin{bmatrix} \mathbf{\Theta}_{dn_b} & \mathbf{I}_{dn_b} \\ -(1-\alpha)\mathbf{L}_b & -\alpha\tau_2 \mathbf{L}_{dn_b} \end{bmatrix}$, then $\mathbf{E}(\tau_2)$ is Hurwitz for $\alpha < 1$ and $\tau_2 > 0$. Thus, there exists a symmetric positive definite matrix $\mathbf{\Pi}_b$ satisfying the following equation

$$\mathbf{\Pi}_b \mathbf{E}(\tau_2) + \mathbf{E}(\tau_2)^\top \mathbf{\Pi}_b = -\tau \mathbf{I}_{2dn_b}, \tag{21}$$

where $\tau = \tau_2 - \tau_1$. Similar to the proof of Theorem 5.1, the following theorem can be proved.

Theorem 5.3. *Consider the delayed second-order consensus model* (8) *in a leader-follower network with rank*$(\mathbf{L}) = dn - l$*,* $n_a \geq 1$*,* $\mathbf{L}_b > 0$*,* $\alpha < 1$ *and* $\tau_1 > 0$*. Suppose that there exist positive definite matrices* $\mathbf{W}_b \in \mathbb{R}^{(dn-l)\times(dn_b)}$*,* $\mathbf{Z}_b \in \mathbb{R}^{dn_b \times dn_b}$*, and* $\mathbf{\Pi}_b \in \mathbb{R}^{2dn_b \times 2dn_b}$ *such that the matrix*

$$\mathbf{\Xi}_b(\tau_2) = \begin{bmatrix} \mathbf{X}_b & \mathbf{Y}_b & \mathbf{Y}_b & \tau_2^2 \mathbf{E}(\tau_2)^\top \begin{bmatrix} \mathbf{\Theta}_{dn_b} & \mathbf{L}_b \end{bmatrix}^\top \mathbf{W}_b \\ * & -\mathbf{Z}_b & \mathbf{\Theta}_{dn_b} & -\tau_2^2 \mathbf{L}_b^2 \mathbf{W}_b \\ * & * & -\frac{\pi^2}{4}\mathbf{W}_b & -k\tau_2^2 \mathbf{L}_b^2 \mathbf{W}_b \\ * & * & * & -\mathbf{W}_b \end{bmatrix}$$

is negative definite, where

$$\mathbf{X}_b = \mathbf{\Pi}_b \mathbf{E}(\tau_2) + \mathbf{E}(\tau_2)^\top \mathbf{\Pi}_b + \begin{bmatrix} \mathbf{\Theta}_{dn_b} & \mathbf{\Theta}_{dn_b} \\ \mathbf{\Theta}_{dn_b} & \tau_1^2 \mathbf{L}_b \mathbf{Z}_b \mathbf{L}_b \end{bmatrix},\ \mathbf{Y}_b = \mathbf{\Pi}_b \begin{bmatrix} \mathbf{\Theta}_{dn_b} \\ \mathbf{L}_b \end{bmatrix}.$$

Then, $\mathbf{x}_b^1(t) \to -\mathbf{L}_b^{-1}\mathbf{L}_{ab}\mathbf{x}_a$*, and* $\mathbf{x}_b^2(t) \to \mathbf{0}_{dn_b}$*.*

6 Bearing-based network localization under time delays

We consider a wireless sensor network of n nodes in the $d \geq 2$ dimensional space. Consider a global coordinate system ${}^g\Sigma$, and let the position of the i-th sensor in the network referred in ${}^g\Sigma$ be denoted as $\mathbf{x}_i \in \mathbb{R}^d$.

The network is characterized by $(\mathcal{G}, \mathbf{x})$, where $\mathcal{G}$ is the interaction graph and $\mathbf{x} = [\mathbf{x}_1^\top, \ldots, \mathbf{x}_n^\top]^\top \in \mathbb{R}^{dn}$, the stacked vector of the global positions of n nodes, is referred to as a realization. Each node (or agent), located at $\mathbf{x}_i \in \mathbb{R}^d$, can measure the bearing vector $\mathbf{g}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|}$, which contains the directional information from node i to a neighboring node $j \in \mathcal{N}_i$. The global position $\mathbf{x}_i$ is unknown to each agent i, so it needs to update an estimate $\hat{\mathbf{x}}_i(t) \in \mathbb{R}^d$ of $\mathbf{x}_i$ and exchange this information with its neighbors. The process of determining the positions of the network's nodes is called *network localization*. We assume that the information about the origin of the global coordinate system is unavailable to each agent and each agent maintains a local coordinate systems ${}^i\Sigma$, whose axes are aligned with ${}^g\Sigma$. This assumption is feasible since we can firstly conduct an orientation alignment algorithm before performing the network localization process.

For each bearing vector $\mathbf{g}_{ij}$, there is a corresponding symmetric positive semidefinite matrix $\mathbf{P}_{\mathbf{g}_{ij}} = \mathbf{I}_d - \mathbf{g}_{ij}\mathbf{g}_{ij}^\top \in \mathbb{R}^{d \times d}$ satisfying ker$(\mathbf{P}_{\mathbf{g}_{ij}}) = \text{im}(\mathbf{g}_{ij})$ and $\mathbf{P}_{\mathbf{g}_{ij}} = \mathbf{P}_{\mathbf{g}_{ij}}^\top = \mathbf{P}_{\mathbf{g}_{ij}}^2$. Observe that $\mathbf{P}_{\mathbf{g}_{ij}}$ is an orthogonal projection onto ker$(\mathbf{g}_{ij})$. The bearing-based network localization algorithm [35, 36]

$$\dot{\hat{\mathbf{x}}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{P}_{\mathbf{g}_{ij}}(\hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_j(t)),\ i = 1, \ldots, n, \tag{22}$$

can be considered as a matrix-weighted consensus algorithm (3). The network localization algorithm (22) induces the bearing Laplacian $\mathbf{L}$ with the ij-th off-diagonal block matrix $-\mathbf{P}_{\mathbf{g}_{ij}}$. It has been shown that the necessary and sufficient condition for the network under the update law (22) to be determined up to a translation and a scaling is rank$(\mathbf{L}) = dn - d - 1$ [37]. Thus, the bearing Laplacian corresponds to $l = d + 1$, and all theoretical results in Sections 3–5 are applicable for the bearing-based network localization problem with time delays.

7 Conclusions

In this paper, three leaderless and leader-follower matrix-weighted consensus models with constant time-delays were studied. The stability of the considered models was analysed and several conditions for the system to asymptotically converge to a point in the kernel of the matrix-weighted Laplacian were provided. An application in bearing-based network localization with time-delays was also given. Since the current work only focuses on constant time delay, for further studies, it will be interesting to consider time-varying time-delays or adaptive algorithms for stabilizing the matrix-weighted consensus network with time-delays.

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A Appendix / supplemental material

A.1 Time-delay systems and the Lyapunov-Krasovskii theorem

Consider the functional differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t), t \geq t_0, \tag{23a}$$

$$\mathbf{x}_{t_0}(\theta) = \boldsymbol{\varphi}(\theta),\ \forall \theta \in [-\tau, 0], \tag{23b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, and the notation $\mathbf{x}_t = \mathbf{x}(t+\theta), \forall \theta \in [-\tau, 0]$ is adopted. The function $\mathbf{f} : \mathbb{R} \times \mathcal{C}_{n,\tau} \to \mathbb{R}^n$ is continuous in both arguments and is locally Lipschitz in the second argument.[2] Furthermore, it is assumed that $\mathbf{f}(t, \mathbf{0}_n) = \mathbf{0}_n, \forall t \in \mathbb{R}$ so that $\mathbf{x} \equiv \mathbf{0}_n$ is a solution of the system.

[2] $\mathcal{C}_{n,\tau} = \mathcal{C}[-\tau, 0]$ denotes the Banach space of absolutely continuous vector functions $\boldsymbol{\varphi} : [-\tau, 0] \to \mathbb{R}^n$ with $\dot{\boldsymbol{\varphi}} \in L_2(-\tau, 0)$ (the space of square-integrable functions) equipped with the norm $\|\boldsymbol{\varphi}\|_{\mathcal{C}} = \max_{\theta \in [-\tau, 0]} \|\boldsymbol{\varphi}(\theta)\| + \left(\int_{-\tau}^{0} \|\dot{\boldsymbol{\varphi}}(s)\|^2 ds\right)^{\frac{1}{2}}$.

Lemma A.1 (Lyapunov-Krasovskii Theorem). *[8] Suppose that* $\mathbf{f}$ *maps* $\mathbb{R}\times$ *(bounded sets of* $\mathcal{C}_{n,\tau}$*) into bounded sets of* $\mathbb{R}^n$*, and there exist functions* $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ *which are continuous, nondecreasing functions,* $u(s) > 0$*,* $v(s) > 0$*,* $w(s) > 0$*,* $\forall s > 0$*,* $u(0) = v(0) = 0$*. If there exists a continuous function* $V : \mathbb{R} \times \mathcal{C}_n \times L_2(-h, 0) \to \mathbb{R}^+$*, such that*

(i) $u(\|\mathbf{x}\|) \leq V(t, \mathbf{x}_t, \dot{\mathbf{x}}_t) \leq v(\|\mathbf{x}_t\|_{\mathcal{C}})$*,*

(ii) $\dot{V}(t, \mathbf{x}_t, \dot{\mathbf{x}}_t) \leq -w(\|\mathbf{x}\|)$*,*

then, the solution $\mathbf{x}(t) \equiv \mathbf{0}_n$ *is uniformly asymptotically stable. If in addition,*

(iii) $\lim_{s\to+\infty} u(s) = +\infty$*,*

then the solution $\mathbf{x}(t) \equiv \mathbf{0}_n$ *is globally uniformly asymptotically stable.*

The following lemmas are useful for analysing the stability of time-delay systems. A short introduction to time-delay systems and the Lyapunov-Krasovskii method are given in Appendix A.1 for quick reference, while we refer the reader to [8] for a tutorial on the topic.

Lemma A.2 (Jensen's inequality). *Denote*

$$\mathbf{G} = \int_a^b f(s)\mathbf{x}(s)ds,$$

where $a \leq b$*,* $f : [a, b] \to [0, \infty)$*,* $x(s) \in \mathbb{R}^n$*. Then, for any positive definite matrix* $\mathbf{K} \in \mathbb{R}^{n\times n}$*, there holds*

$$\mathbf{G}^\top \mathbf{K}\mathbf{G} \leq \int_a^b f(\theta)d\theta \int_a^b f(s)\mathbf{x}^\top(s)\mathbf{K}\mathbf{x}(s)ds.$$

Lemma A.3 (Wirtinger's Inequality). *Let* $\mathbf{z}(t) : (a, b) \to \mathbb{R}^n$ *be absolutely continuous with* $\dot{\mathbf{z}} \in L_2(a, b)$ *and* $\mathbf{z}(a) = \mathbf{0}_n$ *or* $\mathbf{z}(b) = \mathbf{0}_n$*. Then, for any positive definite matrix* $\mathbf{W} \in \mathbb{R}^{n\times n}$*, there holds*

$$\int_a^b \mathbf{z}(\xi)^\top \mathbf{W}\mathbf{z}(\xi)d\xi \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{\mathbf{z}}(\xi)^\top \mathbf{W}\dot{\mathbf{z}}(\xi)d\xi.$$

A.2 Proof of Theorem 3.1

We rewrite the consensus system (6) in the following matrix form

$$\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t-\tau). \tag{24}$$

Consider the variable transformation $\boldsymbol{\delta}(t) = \mathbf{Q}^\top \mathbf{x}(t)$. By expressing $\mathbf{L} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^\top$, we have

$$\dot{\boldsymbol{\delta}}(t) = -\bar{\boldsymbol{\Lambda}}\boldsymbol{\delta}(t-\tau). \tag{25}$$

As $\mathbf{R}^\top \dot{\mathbf{x}}(t) = \mathbf{0}_l, \mathbf{R}^\top \mathbf{x}(t)$, which shows that $\mathbf{R}^\top \mathbf{x}(t) = \mathbf{R}^\top \mathbf{x}(0) = \sum_{i=1}^l \mathbf{p}_i^\top \mathbf{x}(0)$ is time invariant. The n-agent system (25) asymptotically converges to a point in ker($\mathbf{L}$) if and only if $\boldsymbol{\delta}(t) \to \mathbf{0}_{dn-l}$, as $t \to +\infty$, or all roots of the characteristic equation

$$\det(s\mathbf{I}_{dn} + \bar{\boldsymbol{\Lambda}}e^{-\tau s}) = 0 \tag{26}$$

must have negative real parts. Equation (26) is equivalent to $s + \lambda_k e^{-\tau s} = 0, \forall k = l+1, \ldots, dn$. Let $s = \sigma + \jmath\omega$, where $\sigma, \omega \in \mathbb{R}$, we have

$$\begin{aligned}\sigma + \jmath\omega + \lambda_k e^{-\tau(\sigma+\jmath\omega)} &= \sigma + \jmath\omega + \lambda_k e^{-\tau\sigma}(\cos(\omega\tau) - \jmath\sin(\omega\tau)) \\ &= \sigma + \lambda_k e^{-\tau\sigma}\cos(\omega\tau) + \jmath(\omega - \lambda_k e^{-\tau\sigma}\sin(\omega\tau)).\end{aligned}$$

Thus, the roots of (26) satisfy $\sigma = -\lambda_k e^{-\tau\sigma}\cos(\omega\tau),\ \omega = \lambda_k e^{-\tau\sigma}\sin(\omega\tau)$.

(Necessity) If $\sigma < 0,\ \forall\omega$, it follows that $\cos(\omega\tau) = \cos(\lambda_k\omega e^{-\tau\sigma}\tau\sin(\tau\omega)) > 0,\ \forall\omega$. This implies that $|\lambda_k\tau e^{-\tau\sigma}\sin(\tau\omega)| \leq \lambda_k\tau e^{-\tau\sigma} < \frac{\pi}{2}, \forall k = l+1, \ldots, dn$, or $\tau < \frac{\pi e^{\tau\sigma}}{2\lambda_{dn}} \leq \frac{\pi}{2\lambda_{dn}}$.

(Sufficiency) if $\tau < \frac{\pi}{2\lambda_{dn}}$, because $\sigma^2 + \omega^2 = \lambda_k^2 e^{-2\tau\sigma}$, it follows that $|\omega| \leq \lambda_k e^{-\tau\sigma}$, and $\tau|\omega| \leq \frac{\pi e^{-\tau\sigma}}{2}$. If $\sigma \geq 0$, then $e^{-\tau\sigma} \leq 1$. It follows that $\cos(\tau\omega) \geq \cos(\frac{\pi}{2}) \geq 0$, and $\sigma = -\lambda_k e^{-\tau\sigma}\cos(\frac{\pi}{2}) \leq 0$. This contradiction implies that $\sigma < 0$.

Therefore, we conclude that $\sigma < 0$ if and only if $\tau < \frac{\pi}{2\lambda_{dn}}$.

Next, let the condition $\tau < \frac{\pi}{2\lambda_{dn}}$ be satisfied, and $\mathbf{x}(t) = \Phi(t), \forall t \in [-\tau, 0]$, and $\Phi(t) = \mathbf{x}(0)$, the Laplace transform of (25) gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = -e^{-s\tau}\mathbf{L}\mathbf{X}(s) - \mathbf{L}\int_{-\tau}^{0}\mathbf{x}(\xi)e^{-s(\xi+\tau)}d\xi$$

$$\mathbf{X}(s) = -(s\mathbf{I}_{dn} + e^{-s\tau}\mathbf{L})^{-1}\left(\mathbf{x}(0) + \mathbf{L}\int_{-\tau}^{0}\mathbf{x}(\xi)e^{-s(\xi+\tau)}d\xi\right)$$

Using the final value theorem [9], we have

$$\begin{aligned}\lim_{t\to+\infty}\mathbf{x}(t) &= \lim_{s\to 0} s(s\mathbf{I}_{dn} + e^{-s\tau}\mathbf{L})^{-1}\left(\mathbf{x}(0) - \mathbf{L}\int_{-\tau}^{0}\mathbf{x}(\xi)e^{-s(\xi+\tau)}d\xi\right)\\ &= \lim_{s\to 0}\mathbf{P}\mathrm{diag}\left(\frac{s}{s+\lambda_k e^{-s\tau}}\right)\mathbf{P}^\top\left(\mathbf{x}(0) + \mathbf{L}\int_{-\tau}^{0}\mathbf{x}(\xi)d\xi\right)\\ &= \mathbf{R}\mathbf{R}^\top\left(\mathbf{x}(0) + \mathbf{L}\int_{-\tau}^{0}\mathbf{x}(\xi)d\xi\right) = \mathbf{R}\mathbf{R}^\top\mathbf{x}(0), \end{aligned} \tag{27}$$

which completes the proof.

A.3 Proof of Theorem 4.1

Consider the functional $V(t, \boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}_t) = V_1(\boldsymbol{\delta}(t)) + V_2(\dot{\boldsymbol{\delta}}_t)$, where $V_1 = \boldsymbol{\delta}(t)^\top\boldsymbol{\delta}(t)$ and $V_2 = \sum_{k=1}^{r}\int_0^{\tau_k} ds \int_{t-s}^{t}\dot{\boldsymbol{\delta}}(h)^\top\dot{\boldsymbol{\delta}}(h)dh$. The derivatives of V_1 and V_2 along a trajectory of (12) are given by

$$\begin{aligned}\dot{V}_1 =& 2\boldsymbol{\delta}(t)^\top\left(-\bar{\boldsymbol{\Lambda}}\boldsymbol{\delta}(t) + \sum_{k=1}^{r}\bar{\boldsymbol{\Lambda}}_k\int_{t-\tau_k}^{t}\dot{\boldsymbol{\delta}}(s)ds\right)\\ =& -2\boldsymbol{\delta}(t)^\top\bar{\boldsymbol{\Lambda}}\boldsymbol{\delta}(t) + 2\boldsymbol{\delta}(t)^\top\sum_{k=1}^{r}\bar{\boldsymbol{\Lambda}}_k\int_{t-\tau_k}^{t}\dot{\boldsymbol{\delta}}(s)ds,\end{aligned} \tag{28}$$

and

$$\begin{aligned}\dot{V}_2 &= \tau\dot{\boldsymbol{\delta}}^\top(t)\dot{\boldsymbol{\delta}}(t) - \sum_{k=1}^{r}\int_{t-\tau_k}^{t}\dot{\boldsymbol{\delta}}^\top(s)\dot{\boldsymbol{\delta}}(s)ds\\ &\leq \tau\dot{\boldsymbol{\delta}}^\top(t)\dot{\boldsymbol{\delta}}(t) - \sum_{k=1}^{r}\tau_k^{-1}\left(\int_{t-\tau_k}^{t}\dot{\boldsymbol{\delta}}(s)ds\right)^\top\left(\int_{t-\tau_k}^{t}\dot{\boldsymbol{\delta}}(s)ds\right)\end{aligned} \tag{29}$$

where $\tau = \sum_{i=1}^{r}\tau_i$, and in (29) we have used the Jensen's inequality in Lemma A.2. Define the $(r+1)(dn-l)$ vector

$$\mathbf{y}(t) \triangleq \left[\boldsymbol{\delta}^\top(t), \int_{t-\tau_1}^{t}\dot{\boldsymbol{\delta}}^\top(s)ds, \ldots, \int_{t-\tau_r}^{t}\dot{\boldsymbol{\delta}}^\top(s)ds\right]^\top,$$

from Eqs. (28) and (29), one gets

$$\dot{V}(\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}_t) \leq (\mathbf{y}(t))^\top\mathbf{M}\mathbf{y}(t), \tag{30}$$

where $\mathbf{M}$ is given in (13). From the assumption that $\mathbf{M} < 0$, there exists $\gamma > 0$ such that

$$\dot{V}(\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}_t) \leq -\gamma\|\boldsymbol{\delta}(t)\|^2. \tag{31}$$

or the origin is a globally uniformly asymptotically stable equilibrium of the system (12) (Appendix A.1). Thus, $\mathbf{x}(t) \to \mathbf{x}^* \in \ker(\mathbf{L})$, as $t \to +\infty$.

The matrix $\mathbf{M}$ is a summation of two matrices, the first one is positive definite when τ_k are small, and the second one can be made arbitrarily small by choosing τ_k small. This implies that the LMI (13) is feasible if τ_k, $k = 1, \ldots, r$, are sufficiently small.

A.4 Proof of Theorem 5.1

Consider the following functionals

$$\begin{aligned} V_1(\mathbf{z}(t)) &= \mathbf{z}^\top \mathbf{\Pi} \mathbf{z}, \\ V_2(\mathbf{z}_t) &= \tau_1 \int_{t-\tau_1}^{t} (s - t + \tau_1)(\mathbf{z}^2(s))^\top \bar{\mathbf{\Lambda}} \mathbf{Z} \bar{\mathbf{\Lambda}} \mathbf{z}^2(s) ds, \\ V_3(\dot{\mathbf{z}}_t) &= \alpha^2 \tau_2^3 \int_{t-\tau_2}^{t} (s - t + \tau_2)(\dot{\mathbf{z}}^2(s))^\top \bar{\mathbf{\Lambda}} \mathbf{W} \bar{\mathbf{\Lambda}} \dot{\mathbf{z}}^2(s) ds, \end{aligned}$$

where $\mathbf{Z}$, $\mathbf{W} \in \mathbb{R}^{(dn-l)\times(dn-l)}$ are positive definite matrices. Denoting $\boldsymbol{\beta}^1(t) = \int_{t-\tau_1}^{t} \mathbf{z}^2(s)ds$, and $\boldsymbol{\beta}^2(t) = \tau_2 \mathbf{z}^2(t) - (\mathbf{z}^1(t) - \mathbf{z}^1(t - \tau_2))$, and taking the time derivatives of $V_j, j = 1, 2, 3$, we have

$$\dot{V}_1 = \mathbf{z}^\top \mathbf{\Pi} \left(\mathbf{F}(\tau_2)\mathbf{z} + \begin{bmatrix} \mathbf{\Theta}_{dn-l} \\ \bar{\mathbf{\Lambda}} \end{bmatrix} \boldsymbol{\beta}^1(t) + \begin{bmatrix} \mathbf{\Theta}_{dn-l} \\ k\bar{\mathbf{\Lambda}} \end{bmatrix} \boldsymbol{\beta}^2(t) \right) \tag{32a}$$

$$\dot{V}_2 = \tau_1^2 (\mathbf{z}^2(t))^\top \hat{\mathbf{Z}} \mathbf{z}^2(t) - \tau_1 \int_{t-\tau_1}^{t} (\mathbf{z}^2(s))^\top \hat{\mathbf{Z}} \mathbf{z}^2(s) ds, \ \hat{\mathbf{Z}} = \bar{\mathbf{\Lambda}} \mathbf{Z} \bar{\mathbf{\Lambda}} \tag{32b}$$

$$\dot{V}_3 = \alpha^2 \tau_2^4 (\dot{\mathbf{z}}^2(t))^\top \hat{\mathbf{W}} \dot{\mathbf{z}}^2(t) - \alpha^2 \tau_2^3 \int_{t-\tau_2}^{t} (\dot{\mathbf{z}}^2(s))^\top \hat{\mathbf{W}} \dot{\mathbf{z}}^2(s) ds, \ \hat{\mathbf{W}} = \bar{\mathbf{\Lambda}} \mathbf{W} \bar{\mathbf{\Lambda}}. \tag{32c}$$

Based on Jensen's inequality, the second term in $\dot{V}_2$ can be evaluated as follows

$$\begin{aligned} \tau_1 \int_{t-\tau_1}^{t} (\mathbf{z}^2(s))^\top \hat{\mathbf{Z}} \mathbf{z}^2(s) ds &= \int_{t-\tau_1}^{t} d\theta \int_{t-\tau_1}^{t} (\mathbf{z}^2(s))^\top \hat{\mathbf{Z}} \mathbf{z}^2(s) ds \\ &\geq \left(\int_{t-\tau_1}^{t} (\mathbf{z}^2(s))^\top ds \right) \hat{\mathbf{Z}} \left(\int_{t-\tau_1}^{t} \mathbf{z}^2(s) ds \right) \\ &= (\boldsymbol{\beta}^1(t))^\top \hat{\mathbf{Z}} \boldsymbol{\beta}^1(t). \end{aligned} \tag{33}$$

Thus,

$$\dot{V}_2 \leq \tau_1^2 (\mathbf{z}^2(t))^\top \hat{\mathbf{Z}} \mathbf{z}^2(t) - (\boldsymbol{\beta}_1(t))^\top \hat{\mathbf{Z}} \boldsymbol{\beta}_1(t). \tag{34}$$

Next, based on Wirtinger's and Jensen's inequalities, we have

$$\begin{aligned} \frac{4\tau_2^2}{\pi^2} \int_{t-\tau_2}^{t} (\dot{\mathbf{z}}^2(s))^\top \hat{\mathbf{W}} \dot{\mathbf{z}}^2(s) ds &\geq \int_{t-\tau_2}^{t} (\mathbf{z}^2(t) - \mathbf{z}^2(s))^\top \hat{\mathbf{W}} (\mathbf{z}^2(t) - \mathbf{z}^2(s)) ds \\ &\geq \frac{1}{\tau_2} \left(\int_{t-\tau_2}^{t} (\mathbf{z}^2(t) - \mathbf{z}^2(s)) ds \right)^\top \hat{\mathbf{W}} \left(\int_{t-\tau_2}^{t} (\mathbf{z}^2(t) - \mathbf{z}^2(s)) ds \right) \\ &= \frac{1}{\tau_2} \left(\tau_2 \mathbf{z}^2(t) - \int_{t-\tau_2}^{t} \mathbf{z}^2(s) ds \right)^\top \hat{\mathbf{W}} \left(\tau_2 \mathbf{z}^2(t) - \int_{t-\tau_2}^{t} \mathbf{z}^2(s) ds \right) \\ &= \frac{1}{\tau_2} (\boldsymbol{\beta}^2(t))^\top \hat{\mathbf{W}} \boldsymbol{\beta}^2(t). \end{aligned} \tag{35}$$

Thus, $\dot{V}_3 \leq \tau_2^4 (\dot{\mathbf{z}}^2(t))^\top \hat{\mathbf{W}} \dot{\mathbf{z}}^2(t) - \frac{\pi^2}{4} (\boldsymbol{\beta}^2(t))^\top \hat{\mathbf{W}} \boldsymbol{\beta}^2(t)$. Choosing the Lyapunov functional $V(\mathbf{z}(t), \dot{\mathbf{z}}_t) = V_1(\mathbf{z}(t)) + V_2(\mathbf{z}_t) + V_3(\dot{\mathbf{z}}_t)$, and let $\boldsymbol{\eta} = [\mathbf{z}(t)^\top, (\bar{\mathbf{\Lambda}} \boldsymbol{\beta}^1(t))^\top, (\bar{\mathbf{\Lambda}} \boldsymbol{\beta}^2(t))^\top]^\top$, we can compute

$$\dot{V} \leq \boldsymbol{\eta}(t)^\top \underbrace{\begin{bmatrix} \mathbf{X} & \mathbf{Y} & \mathbf{Y} \\ * & -\mathbf{Z} & \mathbf{\Theta}_{dn-l} \\ * & * & -\frac{\pi^2}{4}\mathbf{W} \end{bmatrix}}_{:=\mathbf{\Xi}_1(\tau_2)} \boldsymbol{\eta}(t) + \tau_2^4 (\dot{\mathbf{z}}^2(t))^\top \bar{\mathbf{\Lambda}} \mathbf{W} \bar{\mathbf{\Lambda}} \dot{\mathbf{z}}^2(t), \tag{36}$$

where $\mathbf{X}$, $\mathbf{Y}$ are defined as in (18). Since

$$\begin{aligned} \bar{\mathbf{\Lambda}} \dot{\mathbf{z}}^2(t) &= \begin{bmatrix} \mathbf{\Theta}_{dn-l} & \bar{\mathbf{\Lambda}} \end{bmatrix} \mathbf{F}(\tau_2)\mathbf{z}(t) - \bar{\mathbf{\Lambda}}^2 \boldsymbol{\beta}^1(t) - \alpha \bar{\mathbf{\Lambda}}^2 \boldsymbol{\beta}^2(t) \\ &= \begin{bmatrix} \begin{bmatrix} \mathbf{\Theta}_{dn-l} & \bar{\mathbf{\Lambda}} \end{bmatrix} \mathbf{F}(\tau_2) & -\bar{\mathbf{\Lambda}}^2 & -\alpha \bar{\mathbf{\Lambda}}^2 \end{bmatrix} \boldsymbol{\eta}, \end{aligned} \tag{37}$$

we have the following equation $(\dot{\mathbf{z}}^2(t))^\top \bar{\mathbf{\Lambda}}\mathbf{W}\bar{\mathbf{\Lambda}}\dot{\mathbf{z}}^2(t) =$

$$\boldsymbol{\eta}^\top \underbrace{\begin{bmatrix} \mathbf{F}(\tau_2)^\top \begin{bmatrix} \mathbf{\Theta}_{dn-l} & \mathbf{\Theta}_{dn-l} \\ \mathbf{\Theta}_{dn-l} & \bar{\mathbf{\Lambda}}\mathbf{W}\bar{\mathbf{\Lambda}} \end{bmatrix} \mathbf{F}(\tau_2) & -\mathbf{F}(\tau_2)^\top \begin{bmatrix} \mathbf{\Theta}_{dn-l} \\ \bar{\mathbf{\Lambda}}\mathbf{W}\bar{\mathbf{\Lambda}}^2 \end{bmatrix} & -\alpha \mathbf{F}(\tau_2)^\top \begin{bmatrix} \mathbf{\Theta}_{dn-l} \\ \bar{\mathbf{\Lambda}}\mathbf{W}\bar{\mathbf{\Lambda}}^2 \end{bmatrix} \\ * & \bar{\mathbf{\Lambda}}^2\mathbf{W}\bar{\mathbf{\Lambda}}^2 & \alpha\bar{\mathbf{\Lambda}}^2\mathbf{W}\bar{\mathbf{\Lambda}}^2 \\ * & * & \alpha^2\bar{\mathbf{\Lambda}}^2\mathbf{W}\bar{\mathbf{\Lambda}}^2 \end{bmatrix}}_{:=\mathbf{\Xi}_2(\tau_2)} \boldsymbol{\eta} \quad (38)$$

Thus, if the LMI $\mathbf{\Xi}_1 + \tau_2^4 \mathbf{\Xi}_2 < 0$ is feasible, the $\mathbf{z}$-system is globally uniformly asymptotically stable. By Schur's complement, this condition is equivalent to

$$\mathbf{\Xi}(\tau_2) < 0. \quad (39)$$

Thus, $\dot{V}(\mathbf{z}_t, \dot{\mathbf{z}}_t) \leq -c\|\mathbf{z}\|^2$ for some $c > 0$, or equivalently, $\mathbf{z} = \mathbf{0}$ is globally uniformly asymptotically stable (Appendix A.1) and $\mathbf{x}^k(t) \to$ker$(\mathbf{L})$, $k = 1, 2$, if the LMI (39) is satisfied. Since $\mathbf{x}_i^2(0) = \mathbf{0}_d,\ \forall i = 1, \ldots, n$, due to the observation at the beginning of the Subsection 5.1, we conclude that $\mathbf{x}_i^2(t) \to \mathbf{0}_d,\ \forall i = 1, \ldots, n$.

Finally, we consider the feasibility of the LMI (39). As $\mathbf{F}(\tau_2)$ is affinely dependent on τ_2, let $\mathbf{\Pi}$ be a solution of the Lyapunov equation (16), then $\mathbf{\Pi}$ does not have any term that is affine dependent on τ_2, i.e., $\mathbf{\Pi} = \mathcal{O}(1)$. Let τ_1 be selected such that $\tau_1 = \mathcal{O}(\tau_2^2)$,

$$\mathbf{F}(\tau_2)^\top \mathbf{\Pi} + \mathbf{\Pi}\mathbf{F}(\tau_2) = -\tau \mathbf{I}_{2dn-2l} + \mathcal{O}(\tau_2^2).$$

Choose $\mathbf{R} = \tau_1^{-1}\mathbf{I}_{dn}$, $\mathbf{W} = \tau_2^{-2}\mathbf{I}_{dn}$, by Schur complement, the LMI $\mathbf{\Xi}(\tau_2) < 0$ gives the approximated evaluation

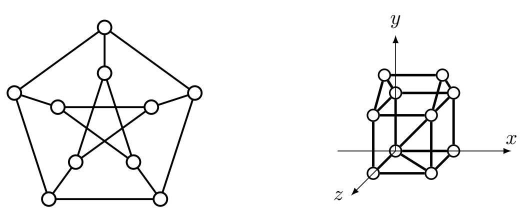
$$\mathbf{\Pi}\mathbf{F}(\tau_2) + \mathbf{F}(\tau_2)^\top \mathbf{\Pi} + \mathcal{O}(\tau^2) < 0$$

which is satisfied for small positive τ_2.

A.5 Simulation results

A.5.1 Matrix-weighted consensus models with time delays

In this subsection, we consider a matrix-weighted network of 10 agents in $\mathbb{R}^3$ with the interaction graph as depicted in Fig. 4(a). The edge weights are selected so that rank$(\mathbf{L}) = 3n - 3 = 27$. We will below simulate the network of 10 agents under different assumptions of the time-delays.

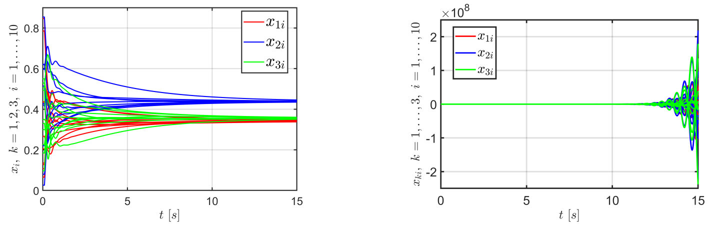


(a) (b)

Figure 3: (a) The topological graph $\mathcal{G}$ of the 10-agent matrix-weighted consensus network; (b) the graph $\mathcal{G}$ and the true positions $\mathbf{x}_i$ of 10-sensor network in Subsection 6.2.

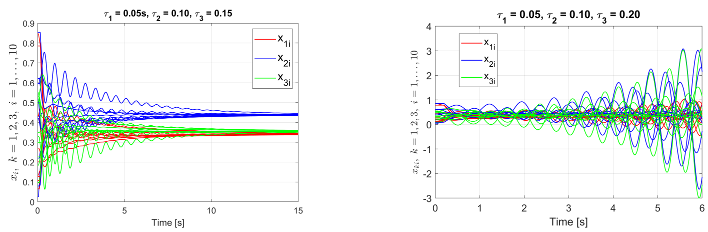
The network has a uniform time-delay: We consider the consensus network with a uniform constant delay. The maximum eigenvalue of the matrix-weighted Laplacian is calculated to be 10.9235, and thus, the upper bound of the delay is $\tau_{\max} \approx 0.1438$ (seconds). For $\tau = 0.1 < \tau_{\max}$, Fig. 4(b) shows that the n-agent system asymptotically consents on a common vector. However, for $\tau = 0.25 > \tau_{\max}$, simulation result in Fig. 4(c) shows that the consensus system becomes unstable.

The network has heterogeneous time delays: Next, let the matrix-weighted network has heterogeneous edge time delays as given in Table 1. In Simulation 1, the time delays are $\tau_1 = 0.05$, $\tau_2 = 0.10$, $\tau_3 = 0.15$. The system asymptotically achieves a consensus. As shown in Figs. 5(a), heterogeneous delays cause significant fluctuations on the process of reaching an agreement.



(a) $\tau = 0.1$s (b) $\tau = 0.25$s

Figure 4: The simulations results with (a) $\tau_1 = 0.1$ and (b) $\tau_2 = 0.25$ are given.



(a) (b)

Figure 5: The simulation results of the matrix-weighted consensus model (7) with multiple delays. The system asymptotically achieves consensus for $\tau_1 = 0.05$, $\tau_2 = 0.1$ and $\tau_3 = 0.15$ but being unstable for $\tau_1 = 0.05$, $\tau_2 = 0.1$ and $\tau_3 = 0.2$.

Table 1: Simulation parameters of the matrix-weighted consensus model (7).

	$e_1, \ldots, e_3$	$e_4, \ldots, e_9$	$e_{10}, \ldots, e_{15}$
Simulation 1	$\tau_1 = 0.05$	$\tau_2 = 0.10$	$\tau_3 = 0.15$
Simulation 2	$\tau_1 = 0.05$	$\tau_2 = 0.10$	$\tau_3 = 0.20$

For Simulation 2, the time delays are changed to $\tau_1 = 0.05$, $\tau_2 = 0.10$, $\tau_3 = 0.20$. In this case, the system becomes unstable as shown in Fig. 5(c).

Consensus of double integrators without velocity measurements: We consider the same matrix weighted graphs and conduct simulations for different values of the time delays τ_1, τ_2 and the control gain k to demonstrate the continuous dependencies of the MWC algorithm (8) with regard to the design parameters.

We first fix the time delays $\tau_1 = 0.05$, $\tau_2 = 0.25$ and vary the control gain k from 1.1 to 0.2. It can be seen that if $k = 1.1$ (exceeding 1) and $k = 0.2$ (being too small so that the LMI does not hold), the system becomes unstable (see Figs. 6(a)– (f)). For $k = 0.3,\ 0.5,\ 0.85,\ 1$, the agents asymptotically achieve a consensus. It can be observed from Figs. 6(b)–6(e) that when k is smaller, the interaction between agents becomes weaker and thus, more fluctuations are exhibited during the process of reaching a consensus.

Second, we fix $k = 0.85$, $\tau_1 = 0.05$ (sec), and vary τ_2. Simulation results corresponding to $\tau_2 = 0.25, 0.6,$ and 0.66 are shown in Figs. 6(c), (g), (h), respectively. Clearly, after τ_2 exceeds the limit (about 0.658 (sec)), the network becomes unstable.

Third, we fix $k = 0.85$, $\tau_2 = 0.25$ (sec), and vary τ_1. Simulation results are depicted in Figs. 6 (a), (c), (j)–(l), corresponding to $\tau_1 = 0,\ 0.05,\ 0.1,\ 0.2,\ 0.22$, respectively. As τ_1 gradually reaches to τ_2, the network tends to be less stable, and when $\tau_2 = 0.22$ (sec), the network becomes unstable. Thus, simulation results are consistent with the analysis.

A.5.2 Bearing-based network localization with time delays

Below, we give simulations of the bearing-based network localization laws with time delays to reinforce our analysis. Specifically, in all simulations in this subsection, a 10-agent network will be

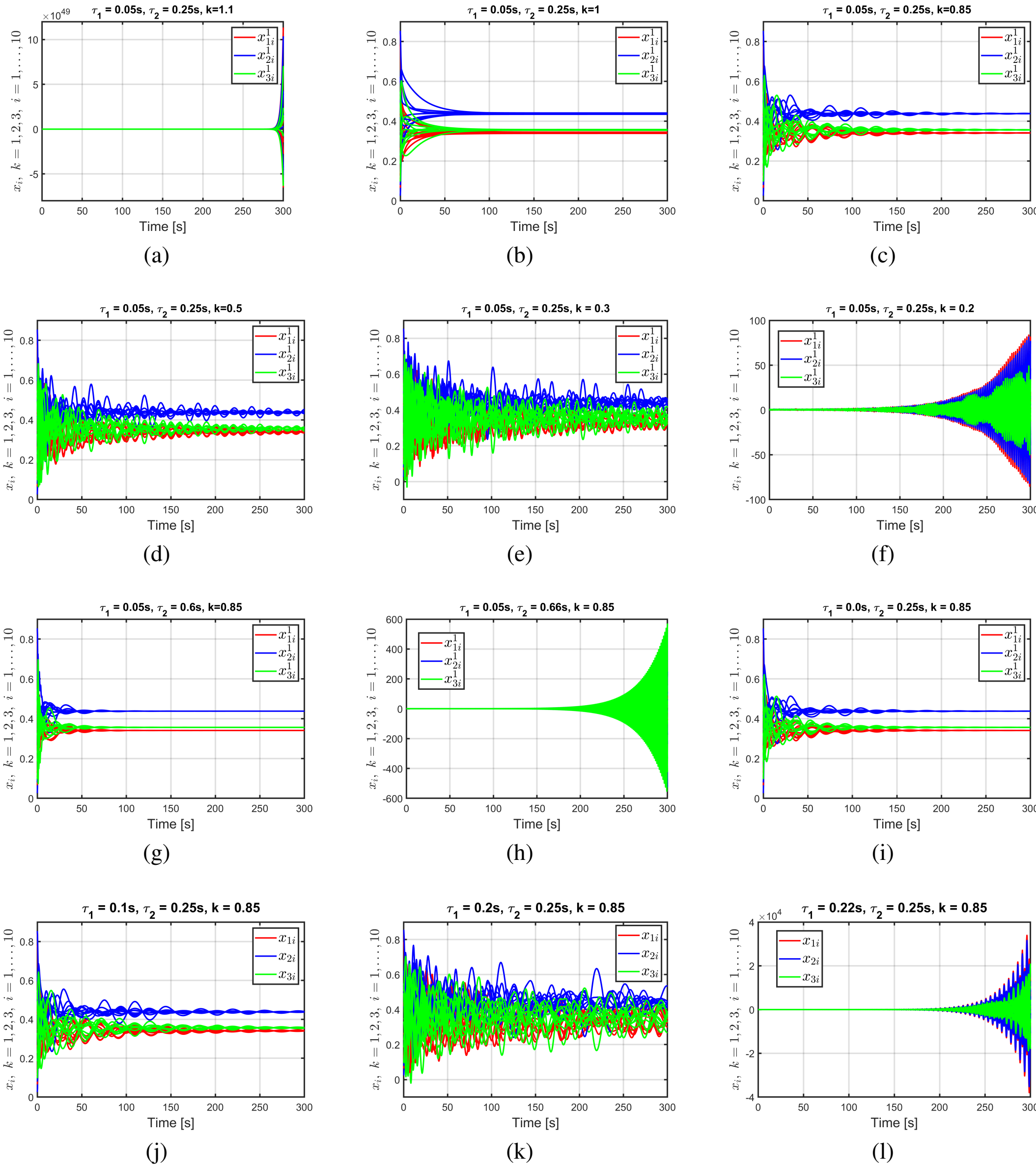
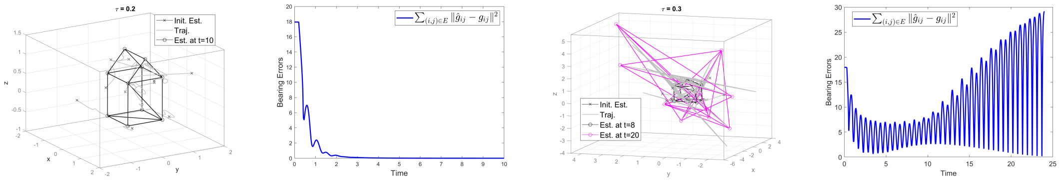


Figure 6: The simulation results of the matrix-weighted consensus model (8) with different values of τ_1, τ_2 and k.

considered. The graph $\mathcal{G}$ and the true position of the nodes are given as follows. It can be checked that the bearing Laplacian satisfies $\text{rank}(\mathbf{L}) = 26$.

Bearing-based network localization with uniform constant time delays: Consider the bearing-based network localization (6) with a constant time delay. The simulation results are depicted in Figs. 7(a)–(b) for $\tau = 0.1$, and Figs. 7(c)–(d). For $\tau = 0.1$, the estimate $\hat{\mathbf{x}}$ asymptotically converges to an $\mathbf{x}^*$, which differs from the correct position $\mathbf{x}$ by a translation and a scaling. For $\tau = 0.2$, after 20 seconds of simulation, it can be observed that $\hat{\mathbf{x}}$ tends to grow unbounded (instability).

Bearing-based network localization with heterogeneous time-delays: Next, we simulate the network localization algorithm (7) with parameters given in Table 2. For $\tau_3 = 0.2$ (sec), it is observed from Figures 8(a)–(b) that $\hat{\mathbf{x}}$ converges to a configuration $\mathbf{x}^*$, and the sum of squared bearing errors $\sum_{(i,j)\in\mathcal{E}} \|\mathbf{g}_{ij} - \mathbf{g}^*_{ij}\|^2$ asymptotically converges to zero. Thus, $\mathbf{x}^*$ is a configuration



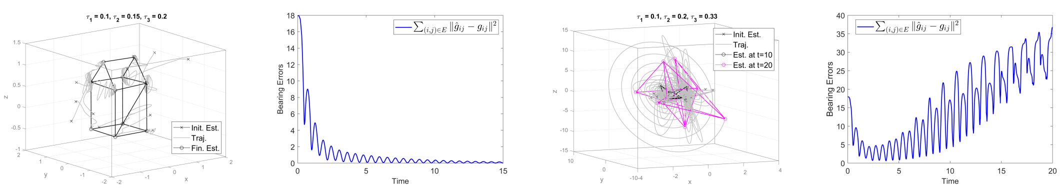
(a) (b) (c) (d)

Figure 7: The simulation results (trajectories of $\hat{\mathbf{x}}_i$ and bearing error) of the network localization update law (6) with $\tau = 0.1$ (sec) ((a)–(b)), and $\tau = 0.2$ (sec) ((c)–(d)).

satisfying all the sensed bearing vectors. As τ_3 changes from 0.2 (sec) to 0.3 (sec), the network becomes unstable, as shown in Figs. 8(c)–(d).

Table 2: Simulation parameters of the network localization algorithm (7).

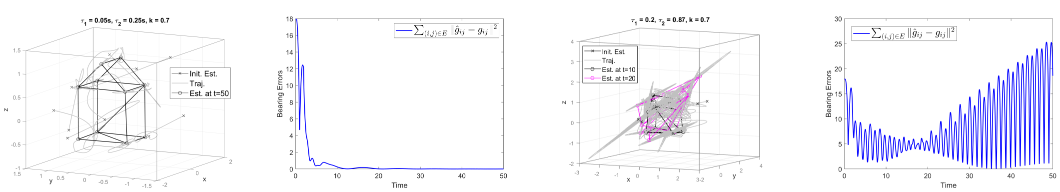
	$e_1, \ldots, e_3$	$e_4, \ldots, e_9$	$e_{10}, \ldots, e_{15}$
Simulation 1	$\tau_1 = 0.1$	$\tau_2 = 0.2$	$\tau_3 = 0.30$
Simulation 2	$\tau_1 = 0.1$	$\tau_2 = 0.2$	$\tau_3 = 0.35$



(a) (b) (c) (d)

Figure 8: The simulation results (trajectories of $\hat{\mathbf{x}}_i$ and bearing error) of the network localization update law (8) with (a) & (b) $\tau_1 = 0.1$, $\tau_2 = 0.2$, $\tau_3 = 0.3$ (sec) and (c) & (d) $\tau_1 = 0.1$, $\tau_2 = 0.2$, $\tau_3 = 0.35$ (sec).

Bearing-based network localization of double-integrators with two constant time delays: Finally, we conduct simulations of the network localization algorithms for double-integrator agents with two time-delays. The results are depicted as in Fig. 9. We can observe that



(a) (b) (c) (d)

Figure 9: The simulation results (trajectories of $\hat{\mathbf{x}}_i$ and bearing error) of the network localization update law (7) with (a) & (b) $\tau_1 = 0.05$, $\tau_2 = 0.25$, $\tau_3 = 0.7$ (sec) and (c) & (d) $\tau_1 = 0.05,\ \tau_2 = 0.87,\ \tau_3 = 0.7$ (sec).

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   Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

   Answer: [Yes]

   Justification: The paper is the first one studying effects of time-delay in matrix-weighted consensus networks. Our analytical tool is control theory for linear systems and the Lyapunov-Krasovskii theorem. Application of the considered consensus algorithms in network localization is also discussed and supported by simulations.

2. **Limitations**

   Question: Does the paper discuss the limitations of the work performed by the authors?

   Answer: [Yes] .

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Justification: The paper does provided descriptions of all theoretical results and numerical simulations of the paper (in the main body of the paper and in the appendix/supplementary files).

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