LLM-TOPLA: Efficient LLM Ensemble by Maximising Diversity

Anonymous ACL submission

Abstract

 Combining large language models during train- ing or at inference time has shown substan- tial performance gain over component LLMs. This paper presents LLM-TOPLA, a diversity- optimized LLM ensemble method with three unique properties: (i) We introduce the fo- cal diversity metric to capture the diversity- performance correlation among component LLMs of an ensemble. (ii) We develop a diversity-optimized ensemble pruning algo-**rithm to select the top-k sub-ensembles from** a pool of N base LLMs. Our pruning method recommends top-performing LLM subensem-014 bles of size S, often much smaller than N. (iii) We generate new output for each prompt **query by utilizing a learn-to-ensemble ap-** proach, which learns to detect and resolve 018 the output inconsistency among all compo- nent LLMs of an ensemble. Extensive eval- uation on four different benchmarks shows good performance gain over the best LLM ensemble methods: (i) In constrained solu- tion set problems, LLM-TOPLA outperforms the best-performing ensemble (Mixtral) by 3.5% in accuracy on MMLU and the best- performing LLM ensemble (MoreAgent) on **GSM8k** by 2.1%. (ii) In generative tasks, LLM-TOPLA outperforms the top-2 perform-029 ers (Llama70b/Mixtral) on SearchQA by $3.9x$ in F1, and on XSum by more than 38 in ROUGE-1. Our code and dataset, which con- tains outputs of 8 modern LLMs on 4 bench-**marks** is available at [https://anonymous.](https://anonymous.4open.science/r/llm_topla-891B) [4open.science/r/llm_topla-891B](https://anonymous.4open.science/r/llm_topla-891B)

035 1 Introduction

 Modern Large Language Models [\(Achiam et al.,](#page-8-0) [2023;](#page-8-0) [Jiang et al.,](#page-9-0) [2024;](#page-9-0) [Touvron et al.,](#page-9-1) [2023;](#page-9-1) [Team](#page-9-2) [et al.,](#page-9-2) [2024\)](#page-9-2) are characterized by architectures with billions of parameters, massive training datasets, and remarkable performance across many zero and one-shot tasks. Recently, there has been a myr-iad of open-sourced models, aiming for improving

generalizability in a subset of tasks e.g., question **043** answering, code generation, and summarization, **044** with smaller sizes (1b to 70b), and yet performing equally or better compared to larger sizes on **046** that particular task [\(Zhao et al.,](#page-9-3) [2023\)](#page-9-3). This en- **047** ables LLM consumers to access many open-source **048** LLMs of small-mid-large sizes and choose to run **049** them locally or via an API from an LLM infer- **050** ence service provider. A widely recognized chal- **051** lenge is how to select among the large collection **052** of open/close-sourced LLMs the best model com- **053** bination, and how to combine possibly conflicting **054** output answers from multiple LLMs to reach the **055** best generative output for the target learning task. **056**

We argue that a practical LLM ensemble method **057** should provide an efficient solution approach to an- **058** swer both of the above questions. To this end, first, **059** we introduce LLM-TOPLA, a diversity-optimized **060** LLM ensemble method with three unique proper- **061** ties: (i) a focal diversity metric to capture the error **062** diversity and the diversity-performance correlation **063** among component LLMs of an ensemble; (ii) a 064 diversity-optimized ensemble pruning algorithm to **065** identify and select the top-k sub-ensembles from a **066** pool of N base LLMs, which shows equal or bet- **067** ter performance compared to the ensemble of N **068** models; (iii) a learn-to-combine approach, which **069** learns to detect and resolve the output inconsistency **070** among all component LLMs of an ensemble, and **071** generate the LLM-TOPLA output for each prompt **072** query. **073**

Related Work. We broadly categorize the re- **074** lated work in achieving better generalization perfor- **075** mance of LLMs into two threads: ensemble with 076 unsupervised or supervised learning. For unsuper- **077** vised approaches, prompt engineering, represented **078** by the Chain of Thought (CoT) [\(Wang et al.,](#page-9-4) [2022\)](#page-9-4) **079** prompts the creation of multiple output passes in **080** the solution space to ensemble the final output with **081** majority voting. The downside of majority vot- **082** ing is the definition of equality between divergent **083**

 answers. Compared to math problems or multiple- choice problems, consensus-based approaches like weighted majority voting may do poorly for gen- erative queries. Recently, two threads of research to further improve CoT. One advocates integrating more agents (models) from different LLM produc- ers [\(Li et al.,](#page-9-5) [2024\)](#page-9-5) and utilizing the BLEU score as the heuristic to compare answers. Another is to enhance the BLEU score-based answer combi- [n](#page-9-6)ation method by either assigning weights [\(Yao](#page-9-6) [et al.,](#page-9-6) [2024\)](#page-9-6) or by creating a debate environment [\(Wan et al.,](#page-9-7) [2024\)](#page-9-7). One caveat in common for these unsupervised methods is that they require lengthy and complex prompt strategies. Several supervised LLM ensemble methods are proposed: LLM-Blender [\(Jiang et al.,](#page-9-8) [2023\)](#page-9-8) performing two steps of training; one for model selection and one for generation. Yet, the proposed ranking model requires pairwise comparison of models in the pool and the ensemble method has a limited context window with the high cost of training. Alterna- [t](#page-9-7)ively, a distillation strategy is proposed in [\(Wan](#page-9-7) [et al.,](#page-9-7) [2024\)](#page-9-7) by performing a token alignment on the probability distributions of the models. In ad- dition to the high computational cost, this paper only ensembles LLama-2 architectures. Regarding the model selection, [\(Chen et al.,](#page-8-1) [2023\)](#page-8-1) reduced 111 the cost of inference by performing prompt adap- tation, caching, and model tuning to choose the strongest model in the pool. We extensively evalu- ate on multiple-choice, open-ended, and generative question benchmarks to show that the proposed LLM-TOPLA outperforms the best LLM ensem- ble methods on MMLU, GSM8k, SearchQA and **118** XSum.

¹¹⁹ 2 Problem Definition

 Let x denote an input query for task T under an **LLM M** and y represent the desired output. We assume a dataset D to be the collection of samples 123 for task T, such that $(x, y) \in \mathcal{D}$. For a pool of **LLMs** with the size N, denoted as M_1, \ldots, M_N , 125 we utilize D to find the optimal ensemble func- tion. This function takes outputs of each LLM **and yields one final answer, denoted as** \tilde{y} **, given** 128 by $f(M_1(x), \ldots, M_N(x)) = \tilde{y}$, such that the dif- ference between desired output is minimized, mea-**sured by the loss function** $\mathcal{L}(\tilde{y}, y)$ **. However, based on whether the task T, the desired output y can** represent different solution spaces. Here, we define three different types of solution spaces.

In the first type, $y^{(1)} \in \{1, \ldots, m\}$ represents 134 the choices in a multiple-choice question (MCQ), **135** where m is typically a small integer, such as 4. 136 The second type of outputs represents the type of **137** open-ended question (OEQ), such as an answer **138** to a multi-step descriptive math problem where **139** the expected answer is a real number and denoted **140** as $y^{(2)} \in \mathbb{R}$ or the expected answer can be a **141** word representing the short answer to a trivia ques- **142** tion denoted as $y^{(2)} \in \{w_1, \dots, w_{|V|}\}\)$, where $|V|$ **143** is the size of the vocabulary. Lastly, the third **144** type represents the outputs of generative question **145** (GQ) tasks such as machine translation, summariza- **146** tion, and open question-answering. The solution **147** space consists of a sequence of words, given by **148** $y^{(3)} = \{w_1, \ldots, w_t\}$. The key difference between 149 all three types of solution sets is that the third solu- **150** tion type is a sequence of words whereas the sec- **151** ond solution set consists exact word or a number. **152** Next, we describe how our proposed methodology **153** addresses all the solution sets by introducing en- **154** semble learning functions for each type of problem. **155**

3 Ensemble Learning Functions **¹⁵⁶**

We propose two learn-to-ensemble methods. The 157 first method, TOPLA-WEIGHTED, is lightweight **158** and applicable only to the first two types of out- **159** puts $y^{(1)}$ and $y^{(2)}$. The second method, TOPLA- 160 SUMMARY, applies to all three types at a higher **161** cost of complexity. **162**

3.1 LLM-TOPLA-Weighted **163**

An autoregressive language model predicts the next 164 token, w_t , based on the probability mass condi tioned on the input query, x , and the formerly gen erated tokens, $w_{\leq t}$, i.e. it models:

$$
p(w_t|x, w_{< t}) = \frac{\exp(c_{t-1})}{\sum_{j=1}^{|V|} \exp(c_j)},\tag{1}
$$

, (1) **168**

where *c* represents the final linear layer output of 169
the language model. For an MC question, as pro- 170 the language model. For an MC question, as pro- **170** posed in [\(Hendrycks et al.,](#page-8-2) [2020\)](#page-8-2), the probabilities **171** assigned to choices are obtained by calculating the **172** probability of the choice's token using equation [1](#page-1-0). **173** For instance, $p(w_t = A|x, w_{\leq t})$ is calculated for **174** choice A. However, a more popular methodology **175** proposed by [\(Gao et al.,](#page-8-3) [2023\)](#page-8-3) is used by the Hug- **176** gingFace Leader Board [\(Beeching et al.,](#page-8-4) [2023\)](#page-8-4) and **177** also in our paper. We aggregate the probabilities **178** of the tokens creating the whole choice to compute **179** the probability of an answer. After repeating the **180**

Figure 1: We present the different types of tasks with their solution spaces.

 procedure for all the choices, we obtain the prob- ability distribution over the choices, denoted by **q** = $[q_1, \ldots, q_m]$, where q represents the probabil-ity of a choice and m is the number of choices.

 As shown by[\(Holtzman et al.,](#page-9-9) [2021\)](#page-9-9), the high- est probability answer may not lead to a correct decision, and the probabilities assigned to other choices carry equal significance. Furthermore, we defend that the probability distribution of a model defines its characteristics and multiple models can be leveraged to reach the correct answer. To this end, we aim for the most robust way to combine 193 N different probability distributions, denoted by $\mathcal{M}_i(x) = \mathbf{q}_i$ where $i = 1, \dots, N$ to generate the **ensemble output,** \tilde{y} , against the query x sampled from a dataset D. Our goal is to maximize the probability of the correct choice conditioned on the probabilities of base models:

199
$$
\max \sum_{(x,y)\in\mathcal{D}} p(y|\mathcal{M}_1(x), \dots, \mathcal{M}_N(x)).
$$
 (2)

 We approximate this likelihood using an ensemble learner parameterized by θ : $f(\mathbf{q}_1, ..., \mathbf{q}_N; \theta) = \tilde{y}$. This ensemble learner can be a machine learn- ing model such as decision trees, or a neural net- work. In this paper, we use a Multi-layer Percep- tron (MLP) containing multiple layers of fully con- nected weights with sigmoid activation functions. At the final layer, the model performs softmax to produce the output probability:

$$
\tilde{y} = \text{softmax}(\mathbf{W}_H(\dots \sigma(\mathbf{W}_1[\mathbf{q}_1, \dots, \mathbf{q}_N]) \dots)),
$$

209 (3)

 where H is the number of layers. The first layer takes the concatenation of the probabilities as the 212 input, i.e., $\mathbf{W}_1 \in \mathbb{R}^{(mN) \times d}$ where d is the input dimension of the second layer. We want to find the 214 best parameters $\theta = (\mathbf{W}_1, \dots, \mathbf{W}_H)$ to maximize the likelihood, which can be reduced to minimize the cross-entropy loss on a dataset which is the col- lection of probabilities for each component model. Thus, we split the dataset into train, validation, and test and use the training set to train the ensemble model, the validation set to stop the training, and finally, we use the test set to calculate the perfor- mance. In each iteration of training, the parameters are updated by minimizing the loss function:

$$
\theta_{\text{best}} = \underset{\theta}{\text{argmin}} \sum_{x, y \in \mathcal{D}^{\text{train}}} \mathcal{L}_{\text{vote}}(y, \tilde{y}),
$$

$$
\tilde{y} = f(\mathcal{M}_1(x), \dots, \mathcal{M}_N(x); \theta), \quad (4)
$$

$$
\mathcal{L}_{\text{vote}}(y, \tilde{y}) = -\sum_{i=1}^{m} y_i \log(\tilde{y}_i).
$$

We use SGD to perform updates on the parame-
ters for every iteration. The ensemble learner an-
226 ters for every iteration. The ensemble learner an- **226** alyzes the probabilities assigned by each model **227** and their confidence level. Thus, we train the **228** learn-to-ensemble model to learn how to efficiently **229** recognize the patterns among the predictions of **230** each component model. This allows the ensemble **231** learner to learn to make the correct choice even in **232** the absence of consensus, instead of blindly relying **233** on consensus voting algorithms, such as majority **234** or plurality voting. **235**

Generalizing the formulation for $y^{(2)}$ **: Con-** 236 sidering the size of the solution set for $y^{(2)}$ can be 237 large, concatenating probabilities for each token is **238** impractical, especially since an answer may com- **239** prise a long sequence of tokens. It is essential to **240** reduce the size of the solution set. Inspired by the **241** Chain-of-Thought (CoT) prompting [\(Wang et al.,](#page-9-4) **242** [2022\)](#page-9-4), we consider two scenarios: (i) If a model is **243** certain of its answer, multiple passes of the same **244** query would result in the same reasoning paths **245** with the same answers. (ii) When a model is uncer- 246 tain, the decision is dispersed into multiple paths **247** with different answers. Hence, we need a mech- **248** anism to find the correct output when the model **249** is uncertain. To address both problems, we iterate **250** the input query K times with CoT prompting and **251** count the occurrences of answers and divide by K **252** indicating the probability distribution of the model **253** for that query. The answers sampled from a model **254** create its solution set. For N number of models, **255** we can have at most $K \times N$ different answers. Let **256** $Y_j = {\hat{y}_1, \dots, \hat{y}_K}$ represent the solution set of 257 j^{th} model where \hat{y}_i is the i^{th} answer of the model. 258 We define a counting function to count the occur- **259** rence of an answer in the solution set denoted by **260** $g(\hat{y}_i, Y_j) = \sum_{y \in Y_j} \mathbb{1}(\hat{y}_i = y)$. However, each 261 model can have its own solution set that is different **262** than the others. By selecting the top- K answers 263 in all of the solution sets, we create one final so- **264**

1 lution set, denoted by Y^{final} . Next, we compute the probability distribution for the final solution set generated by each model. This is done by dividing the frequency of each answer in the solution set of the model by the total number of passes, given by:

$$
\mathbf{q}_{j} = [q_{1}, \dots, q_{K}], \ j = 1, \dots, N
$$

$$
q_{i} = \frac{g(\hat{y}_{i}, Y_{j})}{K}, \ \hat{y}_{i} \in Y^{\text{final}}, \tag{5}
$$

271 where \mathbf{q}_j is the probability distribution of the j^{th} model on the solution set. By obtaining the proba- bilities, we use the same ensemble learner in Equa- tio[n3](#page-2-0) to learn the correct answer, leveraging the confidences of models for the input query.

276 3.2 LLM-TOPLA-Summary

 We design the LLM-TOPLA learn-to-ensemble by summarization (LLM-TOPLA-Summary for short) with two objectives in mind. First, considering certain generative tasks, such as machine transla- tion, and question-answering, the LLM-TOPLA- Weighted is not applicable without relaxing the definition of equality between different solutions. Even if the definition is relaxed by using compari- son metrics such as BLEU score or distance metrics on the vector representation of the outputs, the use of TOPLA-weighted will select one of the answers generated by the best component model of the en- semble to create a TOPLA solution set. This may fail to produce the best generative output, even by utilizing heuristics on the relaxed definition. Sec- ond, our goal with TOPLA-Summary is to create an ensemble learner that applies to all types of tasks and generates its own output.

 LLM-TOPLA-summary performs learn to en- semble as follows. First, we employ another lan- guage model to generate a summary of the outputs produced by each model. Next, we use a sequence- to-sequence (seq2seq) model with encoder-decoder architecture [\(Jiang et al.,](#page-9-8) [2023\)](#page-9-8) by concatenating the outputs of the component models of a chosen ensemble of S base models with the input query and generating the final output of LLM-TOPLA. The fitness of the solution is limited by constraints such as context length, computation complexity, and training complexity. When the input length is short, it limits the number of models that can be fused and forces truncation on the outputs of component models. Also the self-attention mech- anisms in encoder-decoder models have quadratic complexity [\(Beltagy et al.,](#page-8-5) [2020\)](#page-8-5). In response to these limitations, we implement sparse attention and global attention such that we can increase the **313** context length up to 16396 tokens with 149 mil- **314** lion parameters, and utilize a small training dataset. **315 Recall,** $y^{(3)} = \{w_1, \ldots, w_T\}$, where T is the se- 316 quence length of the desired output. Each model in **317** the pool generates the predicted sequence denoted **318 by** $\mathcal{M}_i(x) = \{\hat{w}_1, \dots, \hat{w}_{T_i}\} = z_i$ and T_i is the 319 sequence length of the ith model output which can 320 be different than T. Let h be the seq2seq model **321** with ϕ parameters, and $\mathcal{Z} = \{z_1, \ldots, z_N\}$ be the 322 collection of candidates. Our goal is to approxi- **323** mate the desired sequence probability conditioned **324** on the input query and the model outputs, given by: **325**

$$
p(y|x, \mathcal{Z}) \approx h(x, \mathcal{Z}; \phi). \tag{6}
$$

326

334

(7) **335**

We give the input sequence, x_s , to the seq2seq 328 model in the format of $x_s = \text{concat}(x, z_1, \dots, z_N)$ 329 and use special tokens as separators to indicate the **330** beginning and end of the question or an answer. **331** Consider an ensemble from 3 base models, the **332** input below is sent to the TOPLA-summary model: **333**

$$
x_s = \langle \text{ boq } \rangle x \langle \text{ eoq } \rangle \langle \text{ bocl } \rangle z_1 \langle \text{ eoc1 } \rangle
$$

$$
\langle \text{ boc2 } \rangle z_2 \langle \text{ eoc2 } \rangle \langle \text{ boc3 } \rangle z_3 \langle \text{ eoc3 } \rangle. \tag{7}
$$

We use distinct tokens to indicate which model each **336** candidate belongs to. As the number of models **337** in an ensemble increases, the length of the input **338** sequence to the seq2seq model, ℓ , grows, resulting 339 in a high computational cost in self-attention: **340**

$$
\text{Attention}(Q, K, V) = \text{softmax}(\frac{QK^T}{\sqrt{d_k}})V \quad (8)
$$

where the operations are performed in each layer 342 [o](#page-9-10)f the Transformer model architecture [\(Vaswani](#page-9-10) **343** [et al.,](#page-9-10) [2017\)](#page-9-10) and Q , K , and V contain query, key, $\frac{344}{4}$ and value vectors for all the tokens. The result **345** of the softmax function produces scores for each **346** token. These scores are then multiplied by V , scor- 347 ing each token in the input sentence against every **348** other token. This process results in the complexity **349** of $O(\ell^2 \times d)$, where d is the embedding dimension. 350

To reduce the complexity and increase the con- **351** text length, we employ the *sliding window attention* **352** pattern by [\(Beltagy et al.,](#page-8-5) [2020\)](#page-8-5). A fixed-sized win- **353** dow slides through tokens in each layer increasing **354** the receptive field towards the top layers. For a win- **355** dow of size a, each token attends to the surrounding **356** tokens within a range of $a/2$. This reduces the com- 357 putation complexity to $O(\ell \times d \times a)$ which scales 358 linearly with the input sequence. **359**

Finally, the TOPLA ensemble learner evaluates **360** the relation between the question and the answer **361** given by each model to decide which answer suits the best. To stress the relation between the question and each candidate's answer, we employ *selective global attention* on the tokens of x of the input question. The global attention is the standard self- attention by scoring each token against every other token. With the sliding and global attention mech- anism, we increase the context window length, re- duce the computational complexity, and improve the performance.

 Overall, LLM-TOPLA-summary is optimized by finding the best model parameter ϕ that will max- imize the joint distribution over the target tokens $p(y|x, z_1, \ldots, z_N; \phi)$. It performs auto-regressive generation using the following cross-entropy loss for a target summary $y = \{w_1, \ldots, w_T\}$:

378
$$
\mathcal{L}_{sum} = -\sum_{t=1}^{T} \log p(w_t | w_{< t-1}, x, \mathcal{Z}; \phi) \tag{9}
$$

 We use SGD to perform updates on the parameters in each iteration. As the LLM-TOPLA-Summary model is trained, it learns to generate the correct token sequence by utilizing the information pro- vided by each candidate answer and the TOPLA-summary evaluation results.

³⁸⁵ 4 Focal Diversity and Ensemble Pruning

 Given a pool of base LLMs as an ensemble, LLM- TOPLA first performs the focal diversity-based en- semble pruning for two reasons: First, the diversity among base models improves the ensemble perfor- mance [\(Breiman,](#page-8-6) [1996;](#page-8-6) [Dietterich,](#page-8-7) [2000\)](#page-8-7). Second, as we add more models to the ensemble pool, it be- comes more expensive to prompt each model, and the input length of the ensemble model increases. Thus, the model selection for an ensemble set is essential. Consider a pool of N base models, the total number of possible ensemble teams with size $S (2 \le S \le N)$ is $2^N - N - 1$ [\(Wu et al.,](#page-9-11) [2021\)](#page-9-11). A key question is how to perform ensemble pruning efficiently. We argue that the smaller ensemble size and the higher ensemble diversity, the better the generation performance of the ensemble.

 Focal Negative Correlation & Focal Diversity. **The focal negative correlation metric,** ρ^{focal} is used to quantify the level of error diversity among the component models of an ensemble concerning each model within the ensemble. The focal diversity metric λ^{focal} is used to quantify the general error diversity of the ensemble by taking into account

all focal negative correlation scores of an ensem- **409** ble. Let \mathcal{E}^S denote an LLM ensemble composed 410 of S models: $\{M_1, \ldots, M_S\}$, we choose one of 411 the S base models each time as the focal model to **412** compute the focal negative correlation score of this **413** ensemble, denoted as $\rho^{focal}(\mathcal{E}^S; \mathcal{M}_i)$. We define 414 the focal diversity of this ensemble team by the **415** average of the s focal negative correlation scores. **416** The procedure of computing the focal negative cor- **417** relation score of ρ^{focal} is as follows: (i) select a 418 base model among the set of S base models as the **419** *focal* model, (ii) take all the validation episodes that **420** the focal model has failed and calculate the focal **421** negative correlation score, (iii) repeat the previous **422** steps until all S focal negative correlation scores **423** are obtained. $\{\rho_1^{focal}\}$ $j_1^{focal}, \ldots, \rho_S^{focal}$ $\begin{bmatrix} J^{local} \\ S \end{bmatrix}$, and (iv) compute the average over the scores to obtain the focal **425** diversity of ensemble \mathcal{E}^S , denoted by $\lambda^{focal}(\mathcal{E}^S)$): **426**

$$
\lambda^{focal}(\mathcal{E}^S) = \frac{1}{S} \sum_{\mathcal{E}^S \in \mathcal{M}_i} \rho^{focal}(\mathcal{E}^S; \mathcal{M}_i)
$$

$$
\rho^{focal}(\mathcal{E}^S; \mathcal{M}_i) = 1 - \frac{P(2)}{P(1)}
$$
(10) 427

$$
P(2) = \sum_{j=1}^{S} \frac{j(j-1)}{S(S-1)} p_j, \ P(1) = \sum_{j=1}^{S} \frac{j}{M} p_j
$$

Here p_i is the probability that i number of mod- 428 els fail together on a randomly chosen episode. We **429** calculate as $p_i = n_i/L^{val}$ where n_i is the total 430 number of episodes that i number of models failed 431 together on the validation set and L^{val} is the to- 432 tal number of validation episodes. The term $P(2)$ 433 represents the probability of two randomly cho- **434** sen models simultaneously failing on an episode, **435** while the denominator, $P(1)$, represents the proba- 436 bility of one randomly chosen model failing on an **437** episode. The terms beneath p_i values are the proba- 438 bility of the chosen model being one of the failures. **439** For example, when $S = 3$, there are three cases 440 of model failures; one, two, or three models can **441** fail simultaneously. If one model fails, the chance **442** of selecting the failed model is 1/3. Similarly, for **443** two models, it is $2/3$, and for three models, it is 1. 444 In the case of minimum diversity, the probability **445** of two randomly chosen models failing together **446** comes down to the probability of one of them fail- **447** ing, which makes the fraction term equal to 1 and **448** $\rho^{focal} = 0$. Similarly, in the case of maximum di- 449 versity, there are no simultaneous failures. Hence, **450** the nominator equals 0 and $\rho^{focal} = 1$. The defini- 451 tion of error changes according to the type of task **452** and its solution set y. For the MCQs and OEQs, **453**

Figure 2: An overview of TOPLA-Framework.

 the errors are inequality between the prediction of the model and the label, for the GQs, the errors are missed 1-grams between the prediction and the label. Thus, the focal diversity captures member models that are not correlated solely by their error diversity.

Figure 3: For each task, all candidate ensemble teams from the base model pools are plotted with their focal diversity scores and their performance metrics. The colors represent the size of each team, and the dotted line represents the best-performing individual model in the pool. We also plot the best-fit line with Pearson's Correlation Coefficient ρ to show the correlation between performance and the focal diversity.

 Ensemble Pruning Optimization. Figure [3](#page-5-0) shows the focal diversity scores for a given pool 462 of $N = 8$ base models with GSM8k [\(Cobbe et al.,](#page-8-8) [2021\)](#page-8-8) and XSum [\(Dunn et al.,](#page-8-9) [2017\)](#page-8-9) respectively $(N = 6$ and see Appendix-B for the base models trained on other datasets). For GSM8k, we get 247 466 candidate ensemble teams from the pool of $N = 8$ base models. For XSum, we get 57 candidate teams 468 from the pool of $N = 6$ base LLMs.

 We make three observations: (i) the focal- diversity metric is correlated with the model perfor- mance, (ii) there are multiple sub-ensemble teams of size 2-4 that outperform the largest ensemble of size 8, and (iii) a majority of the smaller en- semble teams also outperform the best-performing individual model in the base model pool.

476 To perform focal diversity-based ensemble prun-**477** ing, we need to compute the focal diversity scores

for all $2^N - N - 1$ sub-ensemble teams when given **478** a pool of N base models. The brute force (BF) ap- **479** proach requires computing the focal diversity for **480** each candidate ensemble of size S ($2 \le S \le N$). 481 For $N = 20$, we need to compute the focal diversity score for all 1, 048, 555 candidate ensemble **483** teams. To speed up this process, we leverage the **484** Genetic Algorithm (GA) [\(Mirjalili and Mirjalili,](#page-9-12) **485** [2019\)](#page-9-12), which takes significantly less time to reach **486** the best combination. Table [1](#page-5-1) shows a comparison. **487** For a pool of $N = 15$ base models, we complete 488 the focal diversity-based ensemble pruning in under **489** a minute, achieving 5 orders of magnitude speed **490** up (see Appendix C for further illustration and de- **491** tails). **492**

Table 1: Brute Force (BF) and GA pruning comparison.

5 LLM-TOPLA Framework **⁴⁹³**

The framework for LLM-TOPLA is shown in Fig- **494** ure [2.](#page-5-2) The user inputs the training data, which **495** includes queries with the desired outputs and a list **496** of N LLMs available in the pool. TOPLA will first **497** send $\alpha\%$ of the queries to each LLM and generate 498 N outputs for each query. Each question and multi- **499** ple answer pairs are stored to create \mathcal{D}^{train} . In the 500 second step, the focal diversity ensemble pruning 501 module selects the best ensemble set combination **502** to decrease pool size from N to S number of mod- **503** els, where $S < N$. It performs the Genetic Algo- 504 rithm boosted diversity pruning algorithm and out- **505** puts top-k ensemble sets. Among top-k candidates, **506** an ensemble set is selected randomly. Then, $\mathcal{D}^{\text{train}}$ 507 is updated based on the new S model selection. At **508** this point, the dataset can be populated with more **509** samples if $\alpha < 100\%$, yet our experiments show 510 that a small-sized dataset is enough to train a per- **511**

 formant ensemble learner. According to the task type, TOPLA-framework will match the generated train data with either TOPLA-Summary or TOPLA- Weighted ensemble learner. After the training, the framework outputs LLM-TOPLA model which can be directly used by the user during inference time.

⁵¹⁸ 6 Experiments

 We validate the effectiveness of LLM-TOPLA through extensive evaluations on MCQ, OEQ, and GQ benchmarks. We show that LLM-TOPLA out- performs the state-of-the-art LLM ensemble meth- ods. Due to the space constraint, we include the details on the datasets and experimental setups in Appendix D.

526 6.1 Performance of LLM-TOPLA

 Table [2](#page-7-0) shows experiments on MMLU and GSM8k datasets, where we compare scores of each base model in the pool with the ensemble learners TOPLA-Weighted and Summary. The Model IDs of TOPLA denote the models in the ensemble set which is selected by focal diversity pruning. The inference time is the average response latency for a sample by each model in the pool. TOPLA frame- work sends each response in a parallel process, thus, the bottleneck is the slowest model, Llama- 70b. In the MMLU dataset, TOPLA-Weighted reaches the best performance by surpassing the best-performing model Mixtral-8x7b by 2%. Since the HF leaderboard provides only the probabil- ity distribution of choices, we could only use the TOPLA-Weighted model. Thus, we used Together- AI API for both models. However, the performance improvement on the best base model is marginal \langle 1\%. We observe that the returned outputs do not change across multiple passes $(K > 1)$, thus preventing the ensemble model from considering alternative thoughts. In the GSM8k dataset, we 549 provide scores when $K = 1$ and $K = 10$, where the outputs have high variation. While the TOPLA- Weighted model can improve the best-performing model by up to $6 - 8\%$, TOPLA-Summary im-**proves** $4 - 5\%$. As K increases, the number of outputs leading to wrong thoughts rises, affecting the TOPLA-Summary model to reach the wrong conclusion, however, this effect is minimized by the frequency-based probability generation in the TOPLA-Weighted model. The full effect of K on the performance is shown in Figure [4.](#page-7-1)

560 Table [3](#page-7-2) shows experiments on SearchQA and **561** XSum datasets, where the TOPLA-Summary

model ensembles the base models selected by fo- 562 cal diversity pruning. In the SearchQA dataset, **563** TOPLA largely improves the best-performing **564** model by up to $> 30\%$ in the F1 score. When we 565 look at the outputs generated by the models and the **566** ensemble model as shown in Table [7](#page-12-0) in Appendix **567** F, we observe that the base models can gather re- **568** lated information about the question but the ex- **569** act term is missing or either model is wordy and **570** provides lots of unrelated information. TOPLA- **571** Summary successfully detects the asked informa- **572** tion gathered by each base model and generates **573** the correct output. Each model has its expertise **574** due to its training dataset's coverage and its learn- **575** ing capability. TOPLA can summarize and detect **576** the asked information by exploiting the wisdom **577** of models. Similarly, TOPLA-Summary surpasses **578** the best-performing base model, Gemma-7b, by up **579** $\text{to} > 30\%$ in ROGUE-L score. By looking at the 580 examples shown in Table [8](#page-13-0) in Appendix F, TOPLA- **581** Summary provides a dense answer covering all the **582** base model outputs and removing the redundancy. **583** Using multiple base models allows the ensemble **584** model to reach more-grained details on the sample 585 document. 586

Table [4](#page-7-3) shows the performance comparison be- **587** tween LLM-TOPLA and other ensemble methods **588** in the literature where More Agents is the previous **589** SOTA. Moreover, we also add the majority voting **590** method as a baseline to our approach. Secondly, **591** Table [5](#page-7-4) shows that LLM-TOPLA outperforms the **592** SOTA on XSum dataset in all measures. **593**

6.2 Ablation Studies **594**

To further observe the effect of the pruning and at- **595** tention mechanisms, we execute two ablation stud- **596** ies in Figure [4.](#page-7-1) First, we ensemble all the models **597** in the pool and compare their performances with **598** the ensemble model selected by the pruning mech- **599** anism. As shown in the first two figures, pruning **600** improves the TOPLA-Weighted and -Summary in **601** MMLU and GSM8k tasks and keeps the perfor- **602** mance in SearchQA and XSum tasks. Although 603 there is no improvement in the last two tasks, the **604** pruned ensemble set is reaching the equivalent per- **605** formance with fewer models. Second, we show **606** [t](#page-9-13)he effect of the Seq2seq model, BART[\(Lewis](#page-9-13) **607** [et al.,](#page-9-13) [2019\)](#page-9-13), sliding window attention, and selec- **608** tive global attention in the third figure by removing **609** them in order and observing the resulting model **610** performance in every dataset. In all of the tasks, all **611** three combinations show the best performance. **612**

Model Name	Model ID	Inf. Time (s)		MMLU [*] test split (Acc $\%$) ^{\uparrow}		GSM8k ^T (Acc %) \uparrow	
		MMLU	GSM8k	HuggingFace LB	Together-AI	$K=1$	$K=10$
$Phi-2$		$\overline{}$	1.29	$56.53_{0.91}$		51.09	65.93
Gemma-2b		0.72	0.82	$40.78_{0.57}$	$31.41_{0.56}$	9.92	19.56
Gemma-7b		1.44	0.87	$65.26_{0.35}$	$47.56_{0.53}$	53.50	70.63
Llama-7b	4	4.82	1.58	$42.62_{0.88}$	$25.05_{0.60}$	8.08	10.87
Mistral-7b		0.87	2.11	$58.70_{0.86}$	$40.04_{0.64}$	40.22	54.02
Llama-13b	6	12.46	2.80	$53.77_{0.53}$	$44.40_{0.48}$	13.73	19.02
Llama-70b		7.74	3.15	$69.39_{0.96}$	$51.60_{0.58}$	49.04	56.52
Mixtral-8x7b	8	1.25	1.55	$70.53_{0.95}$	$64.82_{0.54}$	60.83	71.16
LLM-TOPLA-Summary	378*1138 [†]	13.76	4.21		$65.44_{0.96}$	65.40	75.57
LLM-TOPLA-Weighted	378 138 1	12.46	4.05	$72.77_{1.18}$	$65.75_{0.93}$	66.82	79.01

Table 2: LLMTopla performance in MMLU and GSM8k dataset. We create the ensemble sets using focal-diversity on * MMLU and † GSM8k

Model Name	Model ID	Inf. Time (s) .		SearchOA			XSum		
		SearchOA	XSum	BLEU-11	EM(%)	$F1+$	$ROUGE-1+$	ROUGE-2 ⁺	ROUGE-L ⁺
Gemma-7h		0.41	0.83	10.60	4.43	12.39	26.43	7.43	20.13
Mistral-7b		0.36	.59	4.12	0.47	5.15	22.4	5.49	15.72
Llama-13b		0.39	. 97	8.77	0.63	$10.6\,$	22.99	6.25	15.85
Llama-70b		0.32	. .64	13.97	5.55	15.95	26.46	7.70	19.21
Mixtral-8x7b		0.38	1.21	13.13	2.20	16.04	19.29	5.47	14.28
LLM-TOPLA-Summary	378 13678	0.43	2.01	47.24	33.64	48.13	54.32	27.29	51.87

Table 3: LLMTopla performance in SearchQA and XSum dataset. We create the ensemble sets using focal-diversity on * SearchQA and † XSum

Figure 4: The effect of Focal-diversity Pruning is shown in the first two figures, and the effect of sliding window and selective global attention is shown in the third plot. Lastly, we show the effect of K on TOPLA-Summary, and Weighted models in the GSM8k dataset.

Method	Model ID	MMLU (Acc%)	GSM8k (Acc%)
More Agents	h	51.09	61.00
More Agents		60.05	77.00
LLM-Blender	12345678	44.01	40.41
Majority Voting	12345678	68.06	72.31
Mixtral-8x7b		69.26	71.16
LLM-TOPLA	378^* 1138 [†]	72.77	79.01

Table 4: We compare our approach with the other ensemble methods in the literature.

Table 5: We compare our approach with previous SOTA methods of XSum, [\(Zhao et al.,](#page-9-14) [2022\)](#page-9-14) and [\(Liu et al.,](#page-9-15) [2022\)](#page-9-15).

⁶¹³ 7 Conclusion

 In this paper, we tackled the problem of ensem- bling modern LLMs from a wide perspective. The problem was defined as a mapping from three types of solution sets into the correct solution, and we

introduced two different models. First, TOPLA- **618** Weighted, the model attends weights to each base **619** model output based on their confidence, and in **620** the second type, we introduce a Seq2seq model, **621** TOPLA-Summary, to perform summarization on **622** concatenated outputs and generate one final an- **623** swer. To stress the diversity, we created our en- **624** semble set with the most diverse selection within **625** seconds by Genetic Algorithm. The seq2seq model **626** is further improved by employing sliding window **627** attention to increase the context length and selec- **628** tive global attention to stress the relation between **629** questions and answers. Our evaluation on 4 dif- **630** ferent benchmarks and 8 different modern LLMs **631** shows that LLM-TOPLA framework outperforms **632** the compared models and reaches SOTA. **633**

Additionally, we provide a benchmark dataset **634** that includes answers to MMLU, GSM8k, **635** SearchQA, and Xsum, generated by the most pop- **636** ular large language models. This comprehensive **637** dataset serves as a valuable resource for evaluating **638** and comparing ensemble methods. **639**

Figure 5: The effect of training data size to the performance.

⁶⁴⁰ 8 Limitations

 The limitations of our study can be listed as the computational complexity and number of obser- vational examples \mathcal{D}^{train} . First, the main source of complexity is the cost of using multiple LLMs. In terms of user perspective, this burden is trans- ferred to servers by the available inference services. The user can access each LLM with an API re- quest. However, this aggregates the communica- tion latency to the whole system. Therefore, we implement our framework in parallel so that the bottleneck is the slowest model. Second, we target the complexity of the pruning algorithm by em- ploying the Genetic Algorithm, which allowed us to speed up the search by $> 100 \times$. Third, the com- plexity of the Seq2seq model is reduced by using a million-sized model, and we reduce the complex- ity coming from long input sequences by sliding window attention.

 On the other hand, we assume an observational data $\mathcal{D}^{\text{train}}$ which requires labeled samples. To in- vestigate the effect of the training data size, we plot the effect of training data against performance in Figure [5.](#page-8-10) The x-axis shows the percentage of train- ing data we used from our dataset, e.g. in a total of 40,000 XSum samples and we used 5% of them (8,000) to train and test it on the full portion of the test samples. The results demonstrate that even with a small ratio, the ensemble model enhances the performance of the best base model. However, as more data is used, the performance improves significantly. As a future direction, we will inves- tigate the usage of synthetic data to decrease the dependency on labeled samples.

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The Genetic algorithm requires (i) the representation of a candidate solution, α , and (ii) a fitness function, 813 r, to evaluate the solutions. We represent each solution as a binary vector, where each index represents **814** the presence of the base model in the ensemble set. For the fitness function, we create a focal pruning **815** score metric on the validation dataset, by taking the convex combination of the focal diversity and other **816** metrics such as the validation accuracy of each ensemble set (validation accuracy is applicable only for **817** MCQ and OEQ, thus we used only focal-diversity score in GQ) or cost of models. **818**

The pruning score calculation is given by, $r(\alpha_i) = w_1 \lambda_i + w_2 a_i$, where a_i is the validation accuracy, w_1 819 and w_2 are the significance of each metric for pruning score such that $w_1 + w_2 = 1$ and $w_1, w_2 \in [0, 1]$. The initial population contains randomly created candidate solutions. During selection, the most fitted **821** solutions survive to the next population. As the last step, we reproduce new solutions by performing **822** a cross-over among the best-fitted solutions. The procedure is repeated until we reach a plateau or a **823** predetermined fitness function value. **824**

Figure 6: For MMLU and SearchQA tasks, we show all ensemble teams with their focal diversity scores and their performance metrics. The colors represent the size of each team, and the dotted line represents the best-performing individual model in the pool. We also plot the best-fit line with Pearson's Correlation Coefficient ρ to show the correlation between performance and the focal diversity.

D Datasets and Solution Spaces. **⁸²⁵**

The experiments contains three different datasets targeting each type of solution spaces. For $y^{(1)}$, we use 826 MMLU [\(Hendrycks et al.,](#page-8-2) [2020\)](#page-8-2) which contains MCQs covering 57 subjects from STEM to social sciences **827** with varying difficulties and total 14,042 samples. Our experiments on this dataset coming from two **828** sources. The first is HuggingFace leader board [\(Beeching et al.,](#page-8-4) [2023\)](#page-8-4) providing probability distribution **829** of choices for each test sample. We also mimic a user who has only access to open-source LLMs through **830** an API such as Together-AI [\(TogetherAI,](#page-9-16) [2023\)](#page-9-16) or DeepInfra [\(DeepInfra,](#page-8-11) [2023\)](#page-8-11). However, currently, **831** these APIs do not support next token probability distribution. Therefore, we performed regular expression **832** parsing, plus, the highest BLUE-1 score between model output and the choices to find the output choice **833** of the model. Performing multiple passes $(K > 1)$ allows us to obtain probability distribution for each 834 sample, as we shown in equation [5.](#page-3-0) The dataset does not contain training samples; therefore, we perform 835 a train-test split with a 70% to 30% ratio, repeat the procedure 20 times, and report the mean test score **836** and standard deviation. **837**

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 For $y^{(2)}$ type of solution spaces, we use GSM8k [\(Cobbe et al.,](#page-8-8) [2021\)](#page-8-8). The GSM8k dataset contains 7,472 training samples and 1,318 test samples, each with open-ended mathematical questions and multi- step solutions. Following [\(Wang et al.,](#page-9-4) [2022\)](#page-9-4), we perform CoT prompting on the base models up to 841 $K = 10$.

842 Lastly, for $y^{(3)}$, we measure the performance on generative tasks by employing SearchQA [\(Dunn et al.,](#page-8-9) [2017\)](#page-8-9) and XSum [\(Narayan et al.,](#page-9-17) [2018\)](#page-9-17) datasets. The SearchQA dataset contains 172,908 train and 43,228 test samples where each sample is question-answer pairs with contexts. The questions are from*Jeopardy!* and answers are 1-4 words length. We remove the contexts and performed closed-book prompting (see Appendix for examples and prompts). We used only 20,000 samples from train dataset to train our models and all the test samples to measure performance. On the other hand, the XSum contains 204,045 train and 11,334 test samples. Each sample includes a news article and one sentence summary. We used only 40,000 samples from train dataset to train our models and all the test samples to measure performance.

Evaluation. We use accuracy to evaluate MMLU and GSM8k datasets. In SearchOA, we use BLUE-1, Exact Match (EM), and F1 scores, while in XSum, we use ROGUE-(1, 2, L) scores for evaluation of models.

Model Pool. In our model pool selection, we aim for three elements and their effect on performance: (i) size of the model, (ii) model variety, and (iii) being open-source.

 Fusion Model. LLM-TOPLA-Weighted model contains two fully-connected hidden layers with 100 neurons and sigmoid activations between the layers. The model weights starts from Xavier initialization and converges in 200 epochs optimized by Adam. To implement LLM-TOPLA-Summary, we employ Longformer-Encoder-Decoder [\(Beltagy et al.,](#page-8-5) [2020\)](#page-8-5) model which is initialized from BART weights [\(Lewis et al.,](#page-9-13) [2019\)](#page-9-13).

Ensemble Pruning. We selected $w_1 = 0.6$ and $w_2 = 0.4$ while scoring a candidate ensemble set to give more importance to the diversity in MMLU and GSM8k datasets. In SearchQA and XSum datasets, only focal diversity is used for pruning. The genetic algorithm stops when the fitness function does not change for 100 consecutive generations.

E Improvement by the Ensemble

Table 6: To analyze the improvement coming from the base models, we perform experiments on the Seq2Seq model, Longformer-Encoder-Decoder, by removing outputs of base LLM in the pool.

 Table [6](#page-11-0) shows the results of an experiment on the effect of base models on performance. In each task, we remove the outputs of each base LLM from the input sequence of the Seq2seq ensemble learner, Longformer-Encoder-Decoder model. and fine-tune it purely by using the questions and their answers or documents and their summaries. As shown in the results, LLM-TOPLA outperforms in every task including summarization. This indicates that the ensemble learner can exploit the information gathered by the base LLMs and provide correct results.

Prompt: "Answer very briefly by using at most 4 words."

Question: In devising the periodic table, Mendeleev laid out the elements in rows & columns to show the different relationships when read each way; he's said to have been inspired by a version of this card game.

Ground Truth: solitaire

Mixtral-8x7B: Solitaire, rearranging cards.

Gemma-7b: "Sure, here is the answer: The answer is "Uno".

Llama-2-70b: The periodic table was inspired by a version of the game Uno."

LLM-TOPLA: solitare

Table 7: A sample *Jeopardy!* question taken from the SearchQA dataset is shown with the ground truth, each base LLM answer according to the prompt, and the answer of LLM-TOPLA.

F Samples from SearchQA and XSUM **871**

Table [7](#page-12-0) shows an example output by each model and TOPLA-summary for a question taken from the **872** SearchQA dataset. While the majority of the models give incorrect answers, TOPLA agrees with the **873** minority and produces the correct output. 874 Table [8](#page-13-0) shows an example summarization task performed by each base model and the TOPLA-summary. **875**

While every model produces the expected information in the ground truth, LLM-TOPLA produces the **876** most dense and similar to the ground truth. **877** Prompt: "Try your best to summarize the main content of the given document. And generate a short summary in 1 sentence only. Summary:"

Document: "A judge had made the request because efforts to get them through "political channels" had failed, the lawyer said. France, the former colonial power, has not yet commented on the request. Mr Sankara's widow and supporters have repeatedly accused France of masterminding his 1987 killing because he was a Marxist revolutionary. He was killed by soldiers in a hail of bullets shortly before a meeting of his cabinet in Burkina Faso's capital, Ouagadougou. Mr Sankara's assassination led to his second-in-command, Blaise Compaore, seizing power in a coup. Mr Compaore steered Burkina Faso into a strong alliance with France, which retains close political, security and economic ties with many of its former colonies in Africa. Burkina Faso opened an investigation into the killing after he was ousted from power in 2014. Speaking at a press conference in Ouagadougou, the family lawyer, Benewende Stanislas Sankara, said an investigating judge had formally asked French authorities to declassify military documents to see whether France had played a role in the assassination. A request had also been made to interview French officials who were involved in Burkina Faso's affairs at the time, the lawyer added. Burkina Faso issued an arrest warrant for Mr Compaore in 2015, accusing him of involvement in Mr Sankara's killing. He has repeatedly denied the allegation, but has refused to return to Burkina Faso to stand trial. Mr Compaore is exiled in Ivory Coast, also a former French colony."

Ground Truth: "Burkina Faso has asked France to declassify documents documents related the assassination of former-President Thomas Sankara, a lawyer has the family has said. "

Mixtral-8x7b: " A lawyer for the family of Thomas Sankara, the former president of Burkina Faso who was assassinated in 1987, has asked French authorities to declassify military documents and interview French officials as part of an investigation into Sankara's death, accusing France of playing a role in the assassination."

Gemma-7b: **Summary: ** Burkina Faso has requested that France declassify military documents and interview officials regarding the 1987 assassination of its former president, Thomas Sankara. "

Llama-2-13b: Sure! Here is a summary of the given document in one sentence: A lawyer for the family of former Burkina Faso President Thomas Sankara has requested that French authorities declassify military documents to investigate whether France was involved in Sankara's 1987 assassination, which led to the coup by his successor Blaise Compaore, who has denied any involvement but is exiled in Ivory Coast." Burkina Faso has requested that France declassify military documents and interview French officials regarding the 1987 assassination of the country's former president, Thomas Sankara, amid allegations that France was involved in the killing. "

Llama-2-70b: Burkina Faso has requested that France declassify military documents and interview French officials regarding the 1987 assassination of the country's former president, Thomas Sankara, amid allegations that France was involved in the killing.

LLM-TOPLA-Summary: Burkina Faso has asked France to declassify military documents about the killing of ex-president Thomas Sankara, a lawyer for his family has said.

Table 8: A sample document taken from XSum dataset is shown with the ground truth, each base LLM summarization according to the prompt, and the answer of LLM-TOPLA.