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IFAC PapersOnLine 58-15 (2024) 199–204

# Identification of Low Order Systems in a Loewner Framework [1]

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**Abstract:** This paper considers the problem of non-parametric identification of low-order models from time-domain experimental data using a combination of Caratheodory Fejer and Loewner-based interpolation, followed by a Loewner matrix Balanced Reduction (LBR) step. As we show in the paper, the Loewner matrix is an estimator for the trace norm of a system, playing a role similar to the one played by the Hankel matrix. However, utilizing Zolotarev numbers to establish decay rate bounds for singular values reveals that the decay of singular values in the Loewner matrix is considerably faster than that in the Hankel matrix. Thus, Loewner-based methods yield lower order systems, with the same error bound, than comparable ones based on Hankel matrices. The effectiveness of our method is demonstrated through a numerical example.



*Keywords:* Low-rank Approximation, Loewner Matrix, Hankel Matrix, Balanced Reduction, Subspace Methods, Linear Systems

## 1. INTRODUCTION

The primary objective of system identification is to find an appropriate system consistent with both data and prior information. Prior information is derived from our existing understanding of the system, such as its stability margins. The consistency set is defined as the set of all models that are consistent with the prior information and capable of generating the data. Numerous plants within the consistency set meet these criteria. To select a plant within a consistency set, the minimization of a suitable objective function is employed. This paper considers the order of the identified system as the objective function.

Tackling the challenge of identifying a low-order system requires a Rank Minimization Problem (RMP), which is generically NP-hard. To circumvent this issue, the minimization of the nuclear norm (recognized as the convex envelope of the rank) of the Hankel matrix has been proposed, as detailed in Fazel (2002). The endeavor to approximate a full-rank Hankel matrix with one that is rank-deficient has been the subject of extensive study. For a comprehensive understanding, the reader is directed to Benner et al. (2021); Antoulas et al. (2002); Fazel (2002).

In the frequency domain, model reduction techniques based on Loewner matrices are extensively discussed in (Perev and Shafai (1994); Perev (2012); Stykel (2004)). In this case, both the original system and the reduced order one are given in the Descriptor State Space (DSS) form.

Motivated by these results in this paper we consider the problem of non-parametric identification of low-order models from time-domain experimental data using a combination of Caratheodory Fejer and Loewner-based interpolation, followed by a Loewner matrix Balanced Reduction (LBR) step. The first contribution of this paper is an elucidation of the advantages of Loewner matrices over Hankel matrices in the context of identifying low-order plants. This comparison requires a theorem explaining the relationship between Zolotarev numbers and the decay rate of singular values in structured matrices (Beckermann, 2004; Beckermann and Townsend, 2019; Klippenstein, 2022).

The second contribution of the paper is a novel algorithm for the identification of low order systems using time-domain data. An early result in this direction is a two-step identification algorithm from Sznaier et al. (2014). The first phase of this algorithm formulates the problem as a Caratheodory Fejer Interpolation problem (CF interpolation )(Ball et al., 1990). The second phase employs Hankel rank minimization to approximate a system within the consistency set that closely aligns with a low order system. More recently, Singh and Sznaier (2020) proposed

[1] A. Honarpisheh and M. Sznaier were partially supported by NSF grants IIS–1814631 and CNS-2038493, AFOSR grant FA9550-19-1-0005, ONR grant N00014-21-1-2431, and the Sentry DHS Center of Excellence under Award 22STESE00001-03-03


10.1016/j.ifacol.2024.08.528

an algorithm that exploits the rank-revealing properties of the Loewner matrix to construct low order interpolants. It broadens the scope of time-domain identification by integrating frequency responses of the system, thereby facilitating the construction of the Loewner matrix. The algorithm then proceeds to minimize the rank of this matrix under the constraints imposed by both time and frequency conditions requiring solving the minimization problem subject to generalized interpolation constraints. Our proposed algorithm is a combination of these two algorithms. The initial phase of this combined algorithm parallels the first method, resulting in the parametrization of all possible interpolants. The second phase aligns with the second method, wherein the minimization of the Loewner matrix's rank is central. A notable feature of our approach is the incorporation of the Discrete Fourier Transform (DFT) to reframe the problem in the frequency domain. This technique allows for the utilization of CF interpolation in place of a generalized interpolation problem.

## 2. NOTATION

We denote the transpose, conjugate, and adjoint of the matrix $A$ by $A^\top$, $\bar{A}$, and $A^*$ respectively. $\sigma_{\mathbf{j}}(A)$ denotes the $j^{th}$ singular value of the matrix $A$ such that the largest one is $\sigma_{\mathbf{1}}$. The nuclear norm of a matrix $A$ is denoted by $\|A\|_*$ and is defined as the sum of the singular values of $A$. The Condition Number of A is defined as $\mathrm{CN}(A) = \frac{\sigma_{\mathbf{Max}}(A)}{\sigma_{\mathbf{Min}}(A)}$.

Let $\mathcal{D}(\rho)$ denote the open disk of radius $\rho > 1$ in $\mathbb{C}$. Then by $\mathcal{H}_\infty(\rho, M)$ we denote the space of functions holomorphic on $\mathcal{D}(\rho)$ equipped with the norm $\|f\|_{\rho,\infty} = \sup\{|f(z)|; z \in \mathcal{D}(\rho)\} \leq M$. We denote the space of sequences equipped with the norm $\|f\|_{\rho,\infty} = \sup\{|f(k)|\rho^k; k \in \mathbb{N}\} \leq M$ by $\ell_\infty(\rho, M)$. In our notation, we drop $\rho$ when $\rho = 1$, and both $M$ and $\rho$ when $M = \rho = 1$.

In this study, we focus on discrete-time, linear, time-invariant systems that are exponentially stable with a gain margin of $\rho - 1$. These systems are uniquely characterized by their impulse response $g(k) \in \ell_\infty(\rho, M)$. The transfer function of such a system is defined as $G = \sum_{k=0}^{\infty} g(k) z^k$, and for systems of this category, it holds that $G(z) \in \mathcal{H}_\infty(\rho, M)$. In the sequel, we will refer to $g_i$ as the Markov parameters of the system.

Given a finite sequence $\{x\}_{i=0}^{N}$, we define its corresponding (truncated) Toeplitz operator by

$$T_x = \begin{bmatrix} x_0 & 0 & 0 & \cdots & 0 \\ x_1 & x_0 & 0 & \cdots & 0 \\ x_2 & x_1 & x_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & x_{N-3} & \cdots & x_0 \end{bmatrix} \in \mathbb{R}^{N \times N}.$$

Similarly, $H_n$ denotes the Hankel matrix associated with a sequence $\{g\}_{i=0}^{2n-1}$, omitting $g_0$:

$$H_n = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \\ g_2 & g_3 & \cdots & g_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ g_n & g_{n+1} & \cdots & g_{2n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

$\theta$ is the grid of uniformly distributed frequency points on the upper unit semicircle:

$$\theta = \begin{bmatrix} \frac{\pi}{n+1} & \cdots & \pi - \frac{\pi}{n+1} \end{bmatrix}^\top \in \mathbb{R}^n \quad N = 2n$$

$$z^a = \exp i\theta \quad z^b = \exp -i\theta, \; z = \begin{bmatrix} z^a & z^b \end{bmatrix}^T \in \mathbb{C}^N.$$

FTM is the shifted or generalized discrete Fourier transform defined at the points $z$ on the unit circle:

$$\mathrm{FTM} \in \mathbb{C}^{N \times N} \quad \mathrm{FTM}_{i,j+1} = z_i^{-j}$$
$$i = 1, \cdots, N \quad j = 0, \cdots, N-1.$$

FTM approximates the frequency response of the system:

$$w = \begin{bmatrix} w^a \\ w^b \end{bmatrix} = \mathrm{FTM}(g) \approx G(e^{i\theta}).$$

$L_n$ and $L_s$ denote the Loewner and shifted Loewner matrices:

$$L_n = \begin{bmatrix} \frac{w_1^b - w_1^a}{z_1^b - z_1^a} & \frac{w_1^b - w_2^a}{z_1^b - z_2^a} & \cdots & \frac{w_1^b - w_n^a}{z_1^b - z_n^a} \\ \frac{w_2^b - w_1^a}{z_2^b - z_1^a} & \frac{w_2^b - w_2^a}{z_2^b - z_2^a} & \cdots & \frac{w_2^b - w_n^a}{z_2^b - z_n^a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n^b - w_1^a}{z_n^b - z_1^a} & \frac{w_n^b - w_2^a}{z_n^b - z_2^a} & \cdots & \frac{w_n^b - w_n^a}{z_n^b - z_n^a} \end{bmatrix} \in \mathbb{C}^{n \times n},$$

$$L_s = \begin{bmatrix} \frac{z_1^b w_1^b - w_1^a z_1^a}{z_1^b - z_1^a} & \frac{z_1^b w_1^b - w_2^a z_2^a}{z_1^b - z_2^a} & \cdots & \frac{z_1^b w_1^b - w_n^a z_n^a}{z_1^b - z_n^a} \\ \frac{z_2^b w_2^b - w_1^a z_1^a}{z_2^b - z_1^a} & \frac{z_2^b w_2^b - w_2^a z_2^a}{z_2^b - z_2^a} & \cdots & \frac{z_2^b w_2^b - w_n^a z_n^a}{z_2^b - z_n^a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{z_n^b w_n^b - w_1^a z_1^a}{z_n^b - z_1^a} & \frac{z_n^b w_n^b - w_2^a z_2^a}{z_n^b - z_2^a} & \cdots & \frac{z_n^b w_n^b - w_n^a z_n^a}{z_n^b - z_n^a} \end{bmatrix} \in \mathbb{C}^{n \times n}.$$

Finally, given $\rho$, we define $R \doteq \mathrm{diag}\begin{bmatrix} 1 & \rho & \ldots & \rho^{N-1} \end{bmatrix}$.

## 3. FORMULATION OF THE PROBLEM

Suppose the following state-space model represents the underlying dynamics

$$\begin{aligned} \dot{x}(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) + v(k) \end{aligned} \tag{1}$$

in which $\nu$ denotes the process noise. Given the time-domain experimental data $y$ and $u$:

$$y = [y_0 \; y_1 \; \ldots \; y_{N_t-1}]^T,$$
$$u = [u_0 \; u_1 \; \ldots \; u_{N_t-1}]^T,$$

we define prior information as the set of proper, rational transfer functions $G(z)$, which belong to $\mathcal{H}_\infty(\rho, M)$. The posterior information is characterized by the process noise, which is bounded by $\epsilon_t$ such that the noise vector $v$ belongs to the space $\ell_\infty(\epsilon_t)$. The objective is to identify a *low-order* system consistent with both the prior and posterior information.

## 4. TRACE NORM APPROXIMATION

The trace norm of a system is defined as $\mathrm{Tr}(\Sigma)$, where $\Sigma$ represents the controllability or observability Gramian of the system in its balanced form (see Lemma 6). It is a well-established fact that the nuclear norm of the truncated Hankel matrix $H_n$ approximates $\mathrm{Tr}(\Sigma)$. We will demonstrate that the Loewner matrix $L_n$ approximates $\mathrm{Tr}(\Sigma)$ in a similar fashion. Consequently, it can be considered analogous to the Hankel matrix in the frequency domain.

*Theorem 1.* Let $(A, b, c, d)$ be a balanced state space realization. Therefore, the controllability Grammian $\Sigma = W_c$ is diagonal and satisfies the Lyapunov equation $\Sigma - A\Sigma A^* = bb^*$. The following bounds hold for the approximation of $\mathrm{Tr}(\Sigma)$ with $\|H\|_*$ or $\frac{\|L\|_*}{n}$:

$$|\mathrm{Tr}(\Sigma) - \|H_n\|_*| \leq \frac{\|G_{u\to x}\|^2_{\rho,\infty}\rho^{-2n}}{\rho^2 - 1}, \tag{2}$$

$$|\mathrm{Tr}(\Sigma) - \|L_n\|_*/n| \leq \left(\frac{\pi^2\|G_{u\to x}\|^2_{\rho,\infty}(\rho+1)}{24(\rho-1)^2}\right)\frac{1}{n^2}. \tag{3}$$

**Proof.** Write the Hankel matrix as $H_n = K_{ho}{}^* K_{hc}$, where

$$K_{hc} = \left[\, b \;\, Ab \;\, \dots \;\, A^{n-1}b \,\right],$$
$$K_{ho} = \left[\, c \;\, cA \;\, \dots \;\, cA^{n-1} \,\right]^*$$

are the controllability and observability matrices respectively. Utilizing Lemma 6, we represent $\mathrm{Tr}(\Sigma)$ as a sum

$$\mathrm{Tr}(\Sigma) = \sum_{k=0}^{\infty} \|A^k b\|_2^2 \approx \mathrm{Tr}(K^*_{hc}K_{hc}). \tag{4}$$

Summing the tail of the RHS of (4) results in

$$|\mathrm{Tr}(\Sigma) - \mathrm{Tr}(K^*_{hc}K_{hc})| \leq \sum_{k=n+1}^{\infty} \|A^{k-1}b\|_2^2 \tag{5}$$

As $\|H_n\|_* = \mathrm{Tr}(K^*_{hc}K_{hc})$, we have

$$|\mathrm{Tr}(\Sigma) - \|H_n\|_*| \leq \frac{\|G_{u\to x}\|^2_{\rho,\infty}\rho^{-2n}}{\rho^2 - 1} \tag{6}$$

which proves (2).

Similarly, $L_n = -K^*_{lo}K_{lc}$ (Mayo and Antoulas, 2007), where

$$K_{lc} = \left[\, (z_1^a I - A)^{-1} b \;\, (z_2^a I - A)^{-1} b \;\, \dots \;\, (z_n^a I - A)^{-1} b \,\right],$$
$$K_{lo} = \left[\, c\left(z_1^b I - A\right)^{-1} \;\, c\left(z_2^b I - A\right)^{-1} \;\, \dots \;\, c\left(z_n^b I - A\right)^{-1} \,\right]^*$$

are the generalized controllability and observability matrices respectively. Utilizing Lemma 6, we represent $\mathrm{Tr}(\Sigma)$ as an integral

$$\mathrm{Tr}(\Sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|(e^{i\omega}I - A)^{-1}b\|_2^2 \, d\omega \approx \frac{2\,\mathrm{Tr}(K^*_{lc}K_{lc})}{2n}. \tag{7}$$

If we approximate the integral as a finite sum, the approximation error can be bounded using Lemma 7 in the Appendix. To apply this, it is necessary to bound the second derivative of the integrand in (7). Let $G_{u\to x}$ denote the transfer function from the inputs to the states, then by the definition of $\mathcal{H}_\infty$ norm we have:

$$\|(zI - A)^{-1}b\|_2^2 \leq \|G_{u\to x}\|^2_{\infty} \leq \|G_{u\to x}\|^2_{\rho,\infty}. \tag{8}$$

Cauchy's inequality for holomorphic functions yields:

$$\left|\frac{1}{2}\frac{d^2\,\|(zI - A)^{-1}b\|_2^2}{d^2\,z}\right| \leq \frac{\|G_{u\to x}\|^2_{\rho,\infty}}{(\rho-1)^2}, \tag{9}$$

$$\left|\frac{d\,\|(zI - A)^{-1}b\|_2^2}{d\,z}\right| \leq \frac{\|G_{u\to x}\|^2_{\rho,\infty}}{\rho - 1}. \tag{10}$$

Additionally, we have

$$\begin{aligned}\frac{d\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d\,\omega} &= \frac{d\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d\,e^{i\omega}}\frac{d\,e^{i\omega}}{d\,\omega} \\ &= \frac{d\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d\,e^{i\omega}} ie^{i\omega},\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{d^2\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d^2\,\omega} &= \frac{d^2\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d^2\,e^{i\omega}}\left(ie^{i\omega}\right)^2 \\ &\quad - \frac{d\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d\,e^{i\omega}} e^{i\omega}.\end{aligned} \tag{12}$$

Thus, the second derivative of the argument of the integral in (7) is bounded above as follows

$$\begin{aligned}\left|\frac{d^2\,\|(e^{i\omega}I - A)^{-1}b\|_2^2}{d^2\,\omega}\right| &\leq 2\frac{\|G_{u\to x}\|^2_{\rho,\infty}}{(\rho-1)^2} + \frac{\|G_{u\to x}\|^2_{\rho,\infty}}{\rho - 1} \\ &= \|G_{u\to x}\|^2_{\rho,\infty}\left(\frac{\rho+1}{(\rho-1)^2}\right).\end{aligned} \tag{13}$$

If we apply Lemma 7 from the Appendix to (7) and use (13) as a bound for the second derivative, we conclude

$$\left|\mathrm{Tr}(\Sigma) - \frac{\mathrm{Tr}(K^*_{lc}K_{lc})}{n}\right| \leq \frac{\pi^2\|G_{u\to x}\|^2_{\rho,\infty}(\rho+1)}{24(\rho-1)^2 n^2}. \tag{14}$$

As $\|L_n\|_* = \mathrm{Tr}(K^*_{lc}K_{lc})$, this completes the proof of (3).

## 5. DECAY RATE OF SINGULAR VALUES

Sylvester equations are linear matrix equalities of the matrix $X$ of the form $AX - XB = C$. Many structured matrices like Loewner and Hankel matrices can be formulated as the solution of a Sylvester equation in which $C$ is a low-rank matrix. The following theorem from Beckermann and Townsend (2019) can be used to explain the decay rate of the singular values of structured matrices.

*Theorem 2.* For an integer $k$, let $\mathcal{R}_{k,k}$ denote the set of irreducible rational functions of the form $r(x) = p(x)/q(x)$, where $p$ and $q$ are polynomials of degree at most $k$. Given two closed disjoint sets $E, F \subset \mathbb{C}$, the corresponding Zolotarev number, $Z_k(E, F)$, is defined by

$$Z_k(E, F) := \inf_{r\in\mathcal{R}_{k,k}} \frac{\sup_{z\in E}|r(z)|}{\inf_{z\in F}|r(z)|},$$

where the infimum is attained for some extremal rational function. Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be normal matrices with $m \geq n$, and let $E$ and $F$ be complex sets such that $\sigma(A) \subseteq E$ and $\sigma(B) \subseteq F$. Suppose that the matrix $X \in \mathbb{C}^{m\times n}$ satisfies

$$AX - XB = MN^*, \quad M \in \mathbb{C}^{m\times\nu}, \quad N \in \mathbb{C}^{n\times\nu}$$

where $1 \leq \nu \leq n$ is an integer. Then, for $j \geq 1$, the singular values of $X$ satisfy

$$\sigma_{j+\nu k}(X) \leq Z_k(E, F)\sigma_j(X), \quad 1 \leq j + \nu k \leq n$$

Therefore, the Zolotarev number $Z_k(\sigma(A), \sigma(B))$ indicates the singular values decay rate of the solution of the Sylvester equation $AX - XB = C$ assuming $C$ has low rank. In the case of Sylvester equations, an interpretation of the Zolotarev number can be given as follows:

$$\begin{aligned}Z_k(\sigma(A), \sigma(B)) &= \inf_{r\in\mathcal{R}_{k,k}} \frac{\sup_{z\in\sigma(A)}|r(z)|}{\inf_{z\in\sigma(B)}|r(z)|} \\ &= \inf_{r\in\mathcal{R}_{k,k}} \|r(A)\|_2\|r(B)^{-1}\|_2\end{aligned} \tag{15}$$

where the second equality comes from Lemma 5. Therefore, for every rational function $r(z)$ the decay of the singular values is bounded above by

$$\frac{\sigma_{\mathbf{j}+\nu\mathbf{k}}(X)}{\sigma_{\mathbf{j}}(X)} \leq \|r(A)\|_2\|r(B)^{-1}\|_2 = \frac{\sigma_{\mathbf{Max}}(r(A))}{\sigma_{\mathbf{Min}}(r(B))} \tag{16}$$

The Zolotarev number is the best upper bound among all rational functions of degree at most $k$. We will also use the extension of Theorem 2 mentioned in Beckermann and Townsend (2019).

*Theorem 3.* Suppose that the assumptions of Theorem 2 hold, except that the matrices $A$ and $B$ are not necessarily normal. Also suppose that $E$ and $F$ are $K$-spectral sets for $A$ and $B$ for fixed constants $K_A, K_B > 0$, respectively. Then we have $\sigma_{j+\nu k}(X) \leq K_A K_B Z_k(E,F)\sigma_j(X)$.

### 5.1 Zolotarev numbers of Hankel Matrices

Consider a Hankel matrix $H_n$ constructed from the impulse response of the system described in (1), assuming there is no noise. The Lyapunov equation for this system is

$$\begin{aligned} \Sigma - A\Sigma A^* &= bb^* \\ A^{-1}\Sigma - \Sigma A^* &= A^{-1}bb^*. \end{aligned} \tag{17}$$

Take $E = \{z; |z| \leq \frac{1}{\rho}\}$ and $F = \{z; |z| \geq \rho\}$. According to the Von Neumann inequalities for closed disks on the Riemann sphere, as discussed in Badea and Beckermann (2013) and Neumann (1950), these configurations ensure that the Sylvester equation meets the requirements of Theorem 3. This condition is satisfied because $E$ is a $\rho\|A^*\|_2$-spectral set for $A^*$, and $F$ is a $\rho^{-1}\|A^{-1}\|_2$-spectral set for $A^{-1}$. Consequently, it becomes necessary to compute the Zolotarev number for the sets $E$ and $F$:

$$Z_k\left(\{z; |z| \leq \frac{1}{\rho}\}, \{z; |z| \geq \rho\}\right) = \left(\frac{1/\rho}{\rho}\right)^k = \rho^{-2k}$$

Which is given in Lemma 4. Utilizing Theorem 3, we have

$$\begin{aligned} \sigma_{\mathbf{j+1}}(H) &\leq \rho\|A^*\|_2\rho^{-1}\|A^{-1}\|_2\rho^{-2k}\sigma_{\mathbf{j}}(H) \\ &= \|A\|_2\|A^{-1}\|_2\rho^{-2k}\sigma_{\mathbf{j}}(H) \\ &= \mathrm{CN}(A)\rho^{-2k}\sigma_{\mathbf{j}}(H) \end{aligned} \tag{18}$$

### 5.2 Zolotarev numbers of Loewner Matrices

Loewner matrices solve the following Sylvester equation:

$$\begin{bmatrix} z_1^b & & \\ & \ddots & \\ & & z_n^b \end{bmatrix} L - L \begin{bmatrix} z_1^a & & \\ & \ddots & \\ & & z_n^a \end{bmatrix} = \begin{bmatrix} w_1^b & -1 \\ \vdots & \vdots \\ w_n^b & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ w_1^a & \cdots & w_n^a \end{bmatrix}. \tag{19}$$

To compute the Zolotarev number of (19), define $F_+ = \{e^{i\theta}; \delta \leqslant \theta \leqslant \pi - \delta\}$ and $F_- = \{e^{i\theta}; -\pi + \delta \leqslant \theta \leqslant -\delta\}$. Since $(z^a) \subset F_+$ and $(z^b) \subset F_-$, computing $Z_k(F_+, F_-)$ provides a bound on the singular values decay of the Loewner matrix as per Theorem 2. Given that the Zolotarev number is invariant under the Möbius transformation (Klippenstein, 2022), we apply the transformation $\mathbf{M} = \frac{1}{i}\frac{z-1}{z+1}$ to $F_+$ and $F_-$ in order to obtain:

$$\mathbf{M}(F_+) = [l, 1/l],\ \mathbf{M}(F_-) = [-1/l, -l]$$
$$l = \tan(\frac{\delta}{2}), \quad \delta = \frac{\pi}{n+1}.$$

The Zolotarev number for symmetric intervals of the real line has a well-known upper bound. Using this upper bound mentioned in Lemma 4 in Appendix, we have

$$\begin{aligned} Z_k(F_+, F_-) &= Z_k(\mathbf{M}(F_+), \mathbf{M}(F_-)) \leq 4e^{\frac{-k\pi^2}{\ln(16\gamma)}} \\ \gamma &= \frac{1}{\sin^2(\delta)}. \end{aligned} \tag{20}$$

Utilizing Theorem 2, we have

$$\sigma_{\mathbf{j+2k}}(L) \leq 4e^{\frac{-k\pi^2}{2\ln(4/\sin(\delta))}}\sigma_{\mathbf{j}}(L). \tag{21}$$

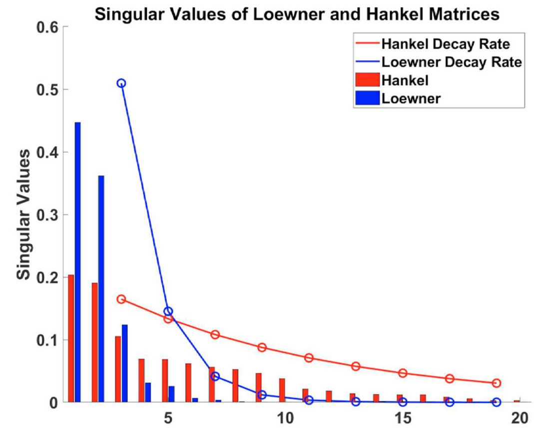


Fig. 1. Comparison between the decay rates of singular values of Loewner and Hankel matrices for a system generated by the 'drss(1000)' command in MATLAB. The red and blue curves correspond to the bounds derived in (18) and (21), respectively. To facilitate a fair comparison, the singular values are normalized by dividing them by the sum of the singular values.

Comparing the bounds derived in (18) and (21) reveals that Loewner matrices are inherently more effective at eliminating less important singular values and distribute $\mathrm{Tr}(\Sigma)$ over a smaller number of significant ones. This comparison is illustrated in Figure 1.

## 6. IDENTIFICATION ALGORITHM

In Section 4, we demonstrated that both $\frac{\|L_n\|_*}{n}$ and $\|H_n\|_*$ act as estimators for $\mathrm{Tr}(\Sigma)$. However, as shown in Section 5, Loewner matrices are approximated by low-rank matrices with less error compared to Hankel matrices. This results in a decrease in error when applying balanced reduction to the Loewner matrices rather than the Hankel matrices. This motivates us to modify a two-stage identification process proposed in Sznaier et al. (2014) to minimize the rank of the Loewner matrix in the second stage. Although the experimental data is given in the time domain, we need frequency responses to form the Loewner matrix. To overcome this challenge, we use FTM to transfer the time-domain data into the frequency-domain.

Algorithm 1 illustrates our identification process. It begins with a CF interpolation problem to identify a plant - $g$ - in the consistency set. In the second step, having $\rho, M, \epsilon_t$, and $g$, we construct $F$: the constant component of the Linear Fractional Transformation (LFT), which parameterizes all interpolants in the consistency set. A comprehensive description of $F$ can be found in Parrilo et al. (1998). In the third step, the algorithm employs the FTM to transform $g$ into the frequency domain, thereby facilitating the formation of the Loewner matrix without the introduction of additional variables. The minimization of the rank of the Loewner matrix is an NP-hard problem. To tackle this challenge, the Reweighted Nuclear Norm Minimization (RNNM) approach, as detailed in Mohan and Fazel (2010), is utilized. The final step involves applying balanced reduction to Loewner matrices, using the algorithm from Stykel (2004) which is discussed in the next section. Note that in Algorithm 1, we need to solve two optimization problems in the first and third steps. All the conditions of these optimization problems reduce to linear or semidefinite constraints.

**Algorithm 1** Loewner rank Minimization and Balanced Reduction (LBR)

**Input:** $\rho > 1, \quad M < \infty, \quad \epsilon_t > 0$

1: **Find** $g$ **such that** $\|y - T_U g\|_\infty \leq \epsilon_t$ **and** $\|RT_g R^{-1}\|_2 \leq M$

2: **Find** $F$ **such that** $G = \text{LFT}(F, Q)$ **parameterizes all interpolants as** $Q \in \mathcal{H}_\infty(\rho)$

3: **Minimize** $\text{Rank}(L(z, w))$ **such that** $g, p, v, w$ **satisfy**
$w = \text{FTM}(g)$
$\|y - T_U g\|_\infty \leq \epsilon_t$
$w = F_{11}(z) + F_{12}(z) \cdot p$
$v = F_{21}(z) + F_{22}(z) \cdot p$ **and** $p^* p \leq \frac{v^* v}{\rho^2}$

4: **Identify** the reduced order system from $L(z, w)$ illustrated in algorithm 2

## 7. MODEL ORDER REDUCTION

We denote the system's true, identified, and reduced-order transfer functions by $\hat{G}$, $G$, and $\bar{G}$ respectively. Within the Loewner framework, a realization of the high-order system obtained in the third step of Algorithm 1 can be expressed as a descriptor state space model (Mayo and Antoulas (2007)):

$$\begin{aligned} -Lx(k+1) &= -L_s x(k) + w_b u(k) \\ y(k) &= w_b^* x(k). \end{aligned} \tag{22}$$

The transfer function for (22) is $G = w_b^* \left(L_s - zL\right)^{-1} w_b$. Given that $z^a = (z^b)^*$, both $L$ and $L_s$ are Hermitian matrices. Consequently, if $z_0$ is a zero of $G$, then $z_0^*$ is also a zero of $G$. A similar argument applies to $\det(L_s - zL)$, indicating that if $z_0$ is a pole of $G$, then $z_0^*$ is a pole of $G$ as well. Since all poles and zeros of $G$ are either real or complex conjugate pairs, it follows that $G$ is a rational function with real coefficients. The balanced reduction technique for descriptor systems, as described in Perev (2012), can be applied to reduce the order of (22).

**Algorithm 2** Model Order Reduction from Perev (2012)

**Input:** $z^b = \overline{z^a}$ **and** $w^b = \overline{w^a}$

1: **Configure** $L(z, w)$ **and** $L_s(z, w)$

2: **reduced SVD** of $-L = USU^*$

3: $\text{T} = \begin{bmatrix} S^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$

4:
$$A = TU^*(-L_s), \; B = TU^* w_b, \; C = w_b^* U$$

5:
$$\begin{aligned} \bar{A} &= A_{11} - A_{12} {A_{22}}^\dagger A_{21}, \; \bar{B} = B_1 - A_{12} {A_{22}}^\dagger B_2 \\ \bar{C} &= C_1 - C_2 {A_{22}}^\dagger A_{21}, \; \bar{D} = -C_2 {A_{22}}^\dagger B_2 \end{aligned}$$

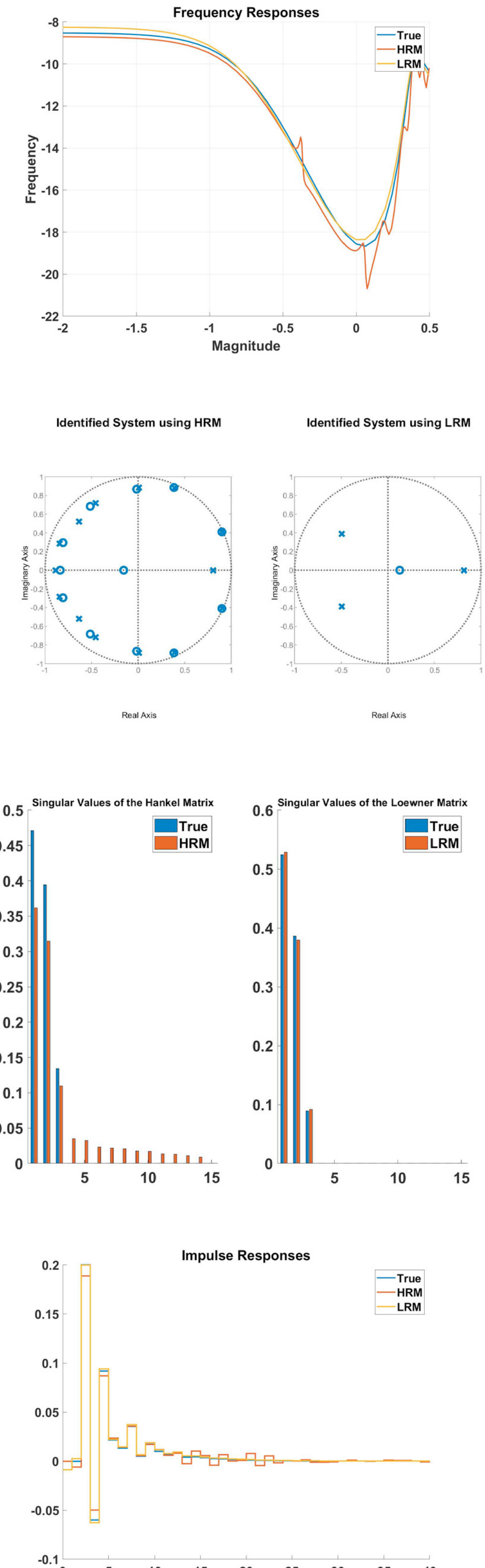


Fig. 2. Identification of a third order system. From top to bottom: The frequency responses; The pole-zero maps; The singular values; The impulse responses

## 8. NUMERICAL EXPERIMENTS

We conducted an experiment on the system $\hat{G} = 0.2\frac{z-0.1}{(z-0.8)(z^2+z+0.4)}$ with the following parameters: $N_t = 40, K = 1, \rho = 1/0.85$ and Signal to Noise Ratio (SNR) = 10. The results of this experiment are illustrated in Figure 2. It was observed that the singular values of the Loewner matrix decay more rapidly than those of the Hankel matrix, leading to the identification of a lower-order system. The orders of the identified systems for HBR and LBR are respectively 14 and 3. The identification error, defined as $\|\hat{\hat{G}} - \bar{G}\|_\infty$, for both methods is approximately 0.05. Regarding identification error, there is no guarantee that the LBR outperforms HBR or vice versa.

## 9. CONCLUSION

This paper compared the properties of Hankel and Loewner based model reduction, in the context of non-parametric identification of low order models. Our main result shows that, while both provide approximations of the trace norm of the system, the singular values of the Loewner matrix decay at a notably faster rate than those of the Hankel matrix. This characteristic of the Loewner matrix enables it to distribute the system's trace norm across fewer states. Thus, Loewner based methods yield, generically, substantially lower order systems, with the same error bound, than comparable ones based on Hankel matrices. These insights pave the way for more efficient system identification processes in scenarios where lower-order models are desirable and offer an avenue to analyze benign overfitting in systems identification.

## Appendix A. SUPPORTING LEMMAS

*Lemma 4.* The Zolotarev number for symmetric, real intervals is bounded as follows

$$Z_k([a,b],[-b,-a]) \leq 4\left[\exp\left(\frac{\pi^2}{2\ln(16\gamma)}\right)\right]^{-2k}$$
$$\gamma = \frac{\left(1+\frac{a}{b}\right)^2}{4\frac{a}{b}}. \quad \text{(A.1)}$$

The Zolotarev number for disjoint concentric circles are:

$$Z_k\left(\{z;|z-c|\leq r_1\},\{z;|z-c|\geq r_2\}\right) = \left(\frac{r_1}{r_2}\right)^k. \quad \text{(A.2)}$$

**Proof.** Refer to Beckermann and Townsend (2019); Klippenstein (2022).

*Lemma 5.* For the normal matrix $A$ and rational function $r(z)$; If $\sigma$ is a singular value of $A$ then $r(\sigma)$ is a singular value of $r(A)$.

*Lemma 6.* Let $(A,b,c,d)$ be a state space realization in the balanced form; Therefore, $\Sigma = W_c = W_o$ satisfies the Lyapunov equation $\Sigma - A\Sigma A^* = bb^*$. Then we have

$$\text{Tr}(\Sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\|(e^{i\omega}I - A)^{-1}b\|_2^2\, d\omega = \sum_{k=0}^{\infty}\|A^k b\|_2^2. \quad \text{(A.3)}$$

**Proof.** Follows from combining the explicit solution to Lyapunov's equation with Parseval's Theorem.

*Lemma 7.* suppose $f(t)$ is a twice continuously differentiable function over the interval $[a,b]$. Partition $[a,b]$ into $n$ equally spaced intervals and show the middle point of intervals by $z_1,\ldots,z_n$. If $|f''(t)| \leq M$ then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{n}\sum_{k=1}^{n} f(z_k)\right| \leq \frac{M(b-a)^2}{24n^2}. \quad \text{(A.4)}$$

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