Characterizing Flow Complexity in Transportation Networks Using Graph Homology

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***Abstract*—Series-parallel networks generally exhibit simplified dynamics, and lend themselves to computationally tractable optimization problems. We are interested in a systematic analysis of the flow complexity that emerges as a network deviates from a series-parallel topology. This letter introduces the notion of a robust p-path on a directed acyclic graph to localize and quantify this complexity. We develop a graph homology with robust p-paths as the bases of its p-chain spaces. We expect that this association between the collection of robust p-paths within a graph and an algebraic structure will provide a framework for the analysis of flow networks. To this end, we show that the simplicity of the series-parallel class corresponds to triviality of high-order chain spaces ($p > 2$). Consequently, the susceptibility of a flow network to the Braess Paradox is associated with the space of 3-chains. Moreover, the computational complexity of decision problems on a network can be related to the order of chains within the proposed homology.**

***Index Terms*—Network analysis, transportation networks, large-scale systems.**

I. Introduction

Directed graphs are widely used to model flows in many real-world transportation and logistic networks. A directed acyclic graph (DAG) is said to be a series-parallel graph if it can be constructed via sequential series and parallel combination of edges or smaller series-parallel graphs. The possession of a series-parallel topology is a global property of a DAG; it is not thoroughly characterized by localized subgraphs within the graph. Series-parallel networks exhibit simple behavior across many different contexts. For instance, combinatorics– a broad class of combinatorial problems can be solved in linear-time on series-parallel graphs [1]; electrical network analysis [2]; behavior in routing games [3], [4], [5]. Therefore, the deviation of a graph from a series-parallel topology can be considered an increase in its flow complexity.

Manuscript received 8 March 2024; revised 10 May 2024; accepted 30 May 2024. Date of publication 12 June 2024; date of current version 2 July 2024. This work was supported in part by NASA under Grant 80NSSC23M0220, and in part by the NASA University Leadership Initiative under Grant 80NSSC21M0071. Recommended by Senior Editor K. Savla. *(Corresponding author: Shashank A. Deshpande.)*

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Digital Object Identifier 10.1109/LCSYS.2024.3413597

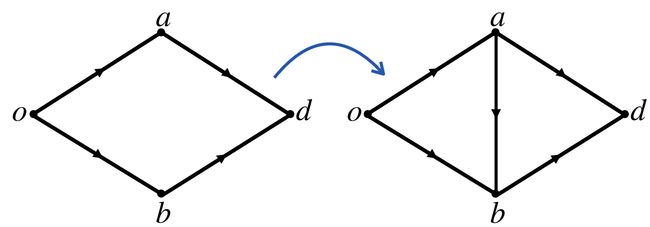

Fig. 1. Braess Paradox: Addition of the link a-b slows down the $o \to d$ flow.

This complexity can lead to inefficiencies in traffic behavior [6], [7].

The Braess paradox is a well-known example of such behavior, when the *addition* of a link to a traffic network *slows* the flow down. Fig. 1 shows the canonical example of the Braess paradox: Here, the addition of link a-b breaks the series-parallel topology of the network. As a result, the traffic that is initially split between the routes $o \to a \to d$ and $o \to b \to d$ concentrates along the faster route $o \to a \to b \to d$, leading to increased congestion and longer travel times than with the originally split flow [3]. The Braess paradox has been known to appear in transportation [8], power grids [9], [10] and ecological networks [11].

A popular approach to a systematic study of global features in complex networks is graph homology. Simplicial homology has been deployed as a generalized clustering mechanism that identifies interconnections within and among clustered communities on undirected graphs [12], [13]. Higher-order dynamics on networks have been studied using simplices [14]. Further, path homology [15] on directed graphs has been shown to identify topological characteristics that classify complex networks [16], [17]. While the intuition behind this classification remains largely intractable, a prior interpretation that path homology measures the consistency and robustness of directional flow in a graph [16], [17]. In this letter, we develop a graph homology of robust paths in line with this interpretation.

We introduce a notion of robust path of length k (or a robust k-path) on a DAG, where larger k is a reflection of larger flow complexity. For instance, we find that the presence of a robust 3-path is a necessary and sufficient condition for a network to deviate from a series-parallel topology, and that each robust 3-path is associated with a site susceptible to the Braess Paradox. Further, a robust k-path identifies the presence

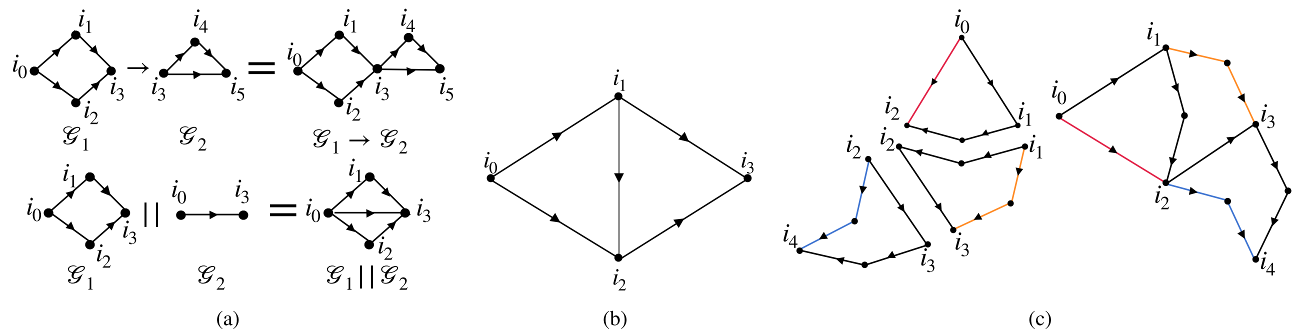

Fig. 2. (a) Series and Parallel Combination (b) The Braess Embedding (c) Robust 2-paths combine into a robust 4-path.

of $\binom{k+1}{4}$ distinct susceptible sites within the network. This motivates us to develop a systematic approach for the characterization of flow complexity in DAGs using robust paths as basic objects. For this purpose, we utilize the algebraic structure of graph homology. In particular, we associate the linear spans of robust k-paths with k-chains in a graph homological framework and prove that the association sets up a consistent chain complex. We demonstrate that the induced chain complex provides a representation of the underlying DAG where higher-order chains identify sites of high flow complexity within the graph. We illustrate the utility of this framework by showing that series-parallel topology of a DAG translates to triviality of 3-chains in the chain complex. This algebraic restatement of a known combinatorial result is validation of how the proposed homology can be used to systematically investigate flow complexity. We believe that our approach can be used for the systematic localization of flow complexity in networks, and to understand its implications.

The organization of this letter proceeds as follows. In the brief subsection that follows the notation subsection below, we introduce the concepts of series-parallel graphs and robust paths, and the role of the latter in reflecting the deviation of a flow from the series-parallel nature. In Section II, we develop a consistent algebraic structure for the formal study of these concepts. Subsequently in Section III, we formalise the notion of a series-parallel topology and use the developed structure to produce an algebraic characterization of the same, as well as to characterize deviations from this topology. We conclude with a brief discussion in Section IV.

A. Notation

(i) For a DAG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we denote a directed edge from $i \in \mathcal{V}$ to $j \in \mathcal{V}$ by e_{ij}.
(ii) $e_{i_0,\ldots i_p}$ is used to denote the tuple $(i_0, \ldots, i_p) \in \mathcal{V}^{p+1}$.
(iii) $i \in \mathcal{G}$ and $e_{ij} \in \mathcal{G}$ respectively mean $i \in \mathcal{V}$ or $e_{ij} \in \mathcal{E}$.
(iv) $[N]$ denotes the set $\{1, 2, \ldots, N\}$ for each $N \in \mathbb{N}$.
(v) Union and intersection on graphs are as usual, for $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $\cup_i \mathcal{G}_i = (\cup_i \mathcal{V}_i, \cup_i \mathcal{E}_i)$ and $\cap_i \mathcal{G}_i = (\cap_i \mathcal{V}_i, \cup_i \mathcal{E}_i)$.
(vi) For DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$, we say $\mathcal{G}_1 \cong \mathcal{G}_2$ if $\mathcal{G}_1 = \mathcal{G}_2$ up to relabelling of their vertices and edges.
(vii) K_{ij} denotes the edge graph $K_{ij} := (\{i, j\}, \{e_{ij}\}, i, j)$.
(viii) $\mathbb{K}$ denotes a field, the reader may specialise to $\mathbb{K} = \mathbb{R}$.

B. Series-Parallel Graphs and the Braess Embedding

We are interested in a class of DAGs called two-terminal graphs where directional flows emanate from an origin vertex (source) and are absorbed by a destination vertex (sink). Series-parallel graphs are two-terminal graphs obtained by serially or parallelly combining edges and/or smaller series-parallel graphs. See Fig. 2(a) for a depiction of series and parallel combination operations. The departure of a two-terminal graph from the series-parallel topology is known to follow from the appearance of the structure called the Braess embedding or a Braess site, shown in Fig. 2(b) as a graphical embedding within the network [3], [4]. The tuple of the vertices involved in the embedding (e.g., (i_0, i_1, i_2, i_3) in Fig. 2(b)) localizes the site within a network.

We wish to investigate the deviation of a graph from a series-parallel topology in a comprehensive manner. To this end, we introduce the notion of a robust k-path in a DAG. The basic object in our discussion is the robust 2-path, which we also call a *triangle*. We call $e_{i_0 i_1 i_2}$ a robust 2-path in $\mathcal{G}$ if i_0, i_1, i_2 are three vertices in $\mathcal{G}$, and there exists a *triangulating* pair of non-intersecting routes from i_0 to i_2, exactly one of which passes through i_1. If robust 2-paths occur as adjacent structures within the graph, they give rise to longer robust paths. Therefore, $e_{i_0,\ldots i_p}$ is a robust p-path if $e_{i_k i_{k+1} i_{k+2}}$ is a robust 2-path for each $k \in [p-2]$, and the triangulating route that evades i_{k+1} does not intersect the triangulating routes of the robust 2-path $e_{i_{k-1} i_k i_{k+1}}$. We will precisely define robust k-paths later. For illustration, see Fig. 2(c) where three adjacent robust 2-paths $(e_{i_0 i_1 i_2}, e_{i_1 i_2 i_3}, e_{i_2 i_3 i_4})$ are shown to merge and give rise to a robust 4-path $(e_{i_0 i_1 i_2 i_3 i_4})$.

We will show that the presence of a robust 3-path ensures the presence of a Braess-susceptible site or a Braess embedding within the network. More generally, long robust paths within a network contribute to the rising flow complexity as the network topology deviates from a series-parallel one.

II. Graph Homology of Robust Paths

In Section II-A below, we define routes, two-terminal graphs, and colored route simplices of two-terminal graphs. In Section II-B that follows, we formalise the notion of a robust path and embed the linear spaces spanned by the robust paths into the homological algebra.

A. Two-Terminal DAGs and Colored Route Simplices

A two-terminal DAG is induced by a union of linear graphs or routes, which we define as follows. We also introduce formal notation for segments of a route, which are shorter routes with different origin-destination pairs.

Definition 1: (i) A route $\mathcal{R}$ is a tuple $\mathcal{R} = (\mathcal{V}, \mathcal{E}, o, d, r)$ where $\mathcal{V}$ is a finite set of nodes or vertices, $r : \mathcal{V} \to \mathbb{N}$ is a strict order on $\mathcal{V}$, origin $o = arg\, min_{i \in \mathcal{V}} r(i)$, destination $d = arg\, max_{j \in \mathcal{V}} r(j)$, and, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ contains all edges $e_{ij} := (i, j)$ if and only if i and j are consecutive in the order r, that is, $r(i) < r(j)$ and $\nexists\, k \in \mathcal{V} : r(i) < r(k) < r(j)$.

(ii) Let $\mathcal{R} = (\mathcal{V}, \mathcal{E}, o, d, r)$ be a route and $i, j \in \mathcal{V}$ be two of its vertices. Define and denote another route from i to j as follows: $\mathcal{R}^{i \to j} := (\mathcal{V}^{i \to j}, \mathcal{E}^{i \to j}, i, j, r)$ where $\mathcal{V}^{i \to j} = \{k \in \mathcal{V} : r(i) \le r(k) \le r(j)\}$, $\mathcal{E}^{i \to j} = \{e_{ab} | a, b \in \mathcal{V}^{i \to j}\} \cap \mathcal{E}$. We regard $\mathcal{R}^{i \to i} = (\{i\}, \phi, i, i, r)$ as the vertex i.

Note: Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two arbitrary routes. If the intersection graph $\mathcal{R}_1 \cap \mathcal{R}_2$ is non-empty, then we take note of the fact that it is expressible in the following form:

$$\mathcal{R}_1 \cap \mathcal{R}_2 = \bigcup_{n=1}^{n_0} \mathcal{R}_1^{p_n \to q_n} = \bigcup_{n=1}^{n_0} \mathcal{R}_2^{p_n \to q_n} \quad (1)$$

where $n_0 \in \mathbb{N}$, $p_n, q_n \in \mathcal{V}_1 \cap \mathcal{V}_2$ for each n.

A union of routes that share the same origin-destination pair induces a two-terminal DAG as follows. Acyclicity of the induced DAG is ensured by requiring the routes to respect each other's order.

Definition 2: i) Let $\{\mathcal{R}_\alpha\}_{\alpha \in A} = \{(\mathcal{V}_\alpha, \mathcal{E}_\alpha, o_\alpha, d_\alpha, r_\alpha)\}_{\alpha \in A}$ be a finite collection of routes with the same origin o and destination d (i.e. $o_\alpha \equiv o, d_\alpha \equiv d$) that obey the partial order induced by $\{r_\alpha\}_{\alpha \in A}$:

$$\forall \delta, \beta \in A, \{i, j\} \in \mathcal{V}_\delta \cap \mathcal{V}_\beta \implies r_\delta(i) < r_\delta(j) \longleftrightarrow r_\beta(i) < r_\beta(j) \quad (2)$$

Then, the tuple $\mathcal{G} := (\mathcal{V} := \cup_{\alpha \in A} \mathcal{V}_\alpha, \mathcal{E} := \cup_{\alpha \in A} \mathcal{E}_\alpha, o, d, (r_\alpha)_{\alpha \in A})$ is called a two-terminal graph from origin o to destination d induced by the collection of routes $(\mathcal{R}_\alpha)_{\alpha \in A}$. We then write $\mathcal{G} = \bigcup_{\alpha \in A} \mathcal{R}_\alpha$ and say that $i < j$ if $i, j \in \mathcal{V}$ and $\exists \alpha \in A$ such that $r_\alpha(i) < r_\alpha(j)$.

Note: We call the collection $\{\mathcal{R}_i\}_{i \in [N]}$ a complete enumeration of routes in $\mathcal{G} = \bigcup_{i \in [N]} \mathcal{R}_i$ if it contains all o to d routes in $\mathcal{G}$. All collections in this letter are assumed to be complete enumerations. We also drop the underlying partial order $(r_i)_{i \in [N]}$ in our notation and use $\mathcal{G} = (\mathcal{V}, \mathcal{E}, o, d)$ to represent the two-terminal graph.

As we will see below, a route induces a two-terminal DAG, that we call a route-simplex. The route-simplex shares the vertex set of the underlying route $\mathcal{R}$ and contains an edge e_{ij} if j is reachable from i. A union of route simplices induced by the constituent routes of a two-terminal DAG is declared as the route simplex of the DAG. We attach a multi-coloring to each edge e_{ij} in a route-simplex to record the set of routes that reach j from i; this produces what we call a colored route simplex of a DAG. These notions are formalised by the definition below.

Definition 3: (i) The route-simplex of $\mathcal{R}_i = (\mathcal{V}_i, \mathcal{E}_i, o, d, r_i)$ denoted by $Sim(\mathcal{R}_i)$ is the two terminal graph $(\mathcal{V}_i, \mathsf{R}(\mathcal{E}_i), o, d)$ where $\mathsf{R}(\mathcal{E}) = \{e_{ij} : i, j \in \mathcal{V}, r(i) < r(j)\}$.

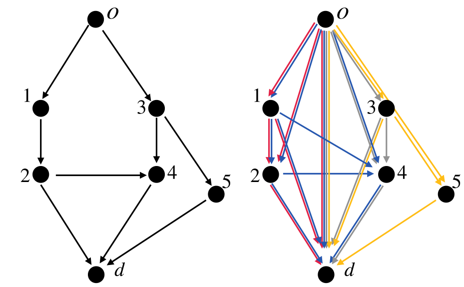

Fig. 3. $\mathcal{G} = \bigcup_\alpha \mathcal{R}_\alpha$; $\mathsf{R}(\mathcal{G})$: $\dim \Omega_3(\mathsf{R}(\mathcal{G})) \neq 0$.

(ii) The route-simplex of $\mathcal{G} = \bigcup_{i \in [N]} \mathcal{R}_i$ is defined to be the union $Sim(\mathcal{G}) := \bigcup_{i \in [N]} Sim(\mathcal{R}_i)$.

(iii) The colored route simplex of $\mathcal{G}$ is the tuple $\mathsf{R}(\mathcal{G}) := (\mathcal{V}, \mathsf{R}(\mathcal{E}), o, d, \mathcal{C})$ where the (multi)coloring $\mathcal{C} : \mathsf{R}(\mathcal{E}) \to 2^{[N]}$ obeys $\mathcal{C}(e_{pq}) = \{i \in [N] | e_{pq} \in Sim(\mathcal{R}_i)\}$.

Consider four routes $\{\mathcal{R}_\alpha\}_{\alpha \in [4]} = \{(\mathcal{V}_\alpha, \mathcal{E}_\alpha, o, d, r_\alpha)\}$ with $\mathcal{V}_1 = \{o, 1, 2, d\}, \mathcal{E}_1 = \{e_{o1}, e_{12}, e_{2d}\}, \mathcal{V}_2 = \{o, 1, 2, 4, d\}, \mathcal{E}_2 = \{e_{o1}, e_{12}, e_{24}, e_{4d}\}, \mathcal{V}_3 = \{o, 3, 4, d\}, \mathcal{E}_3 = \{e_{o3}, e_{34}, e_{4d}\}, \mathcal{V}_4 = \{o, 3, 5, d\}, \mathcal{E}_4 = \{e_{o3}, e_{35}, e_{5d}\}$ which constitute the two-terminal graph $\mathcal{G} = \cup_\alpha \mathcal{R}_\alpha$ shown in Fig. 3 below. $\mathsf{R}(\mathcal{G})$ is depicted alongside where Red, Blue, Gray and Yellow respectively depict the colors 1, 2, 3 and 4.

B. Development of the Robust Path Homology

We now develop our graph homology of robust paths. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, o, d) = \bigcup_{i=1}^{N} \mathcal{R}_i = \bigcup_{i=1}^{N} (\mathcal{V}_i, \mathcal{E}_i, o, d, r_i)$ be a two-terminal DAG and $\mathsf{R}(\mathcal{G}) = (\mathcal{V}, \mathsf{R}(\mathcal{E}), \mathcal{C})$ be the colored route simplex of $\mathcal{G}$. The space of vertex tuples $\mathcal{V}^{p+1}$ is refined to record graph topology in the refined subset of allowed paths.

Definition 4: (i) $e_{i_0, \ldots i_p}$ is an elementary allowed p-path in $\mathsf{R}(\mathcal{G})$ if $e_{i_{m-1} i_m} \in \mathsf{R}(\mathcal{E})$ for all $m \in [p]$. (Recall Section I-A(ii).)

(ii) We define the $\mathbb{K}$-linear span of all elementary allowed p-paths as the space of allowed p-paths:

$$\mathcal{A}_p(\mathsf{R}(\mathcal{G})) := \mathbb{K}\text{−span}\{e_{i_0, \ldots i_p} : e_{i_j i_{j+1}} \in \mathsf{R}(\mathcal{E}) \ \forall\, j \in [p-1]\}.$$

Next, we define a linear operator on the allowed path spaces.

Definition 5: The linear boundary operator $\partial_p : \mathcal{A}_p(\mathsf{R}(\mathcal{G})) \to \mathcal{A}_{p-1}(\mathsf{R}(\mathcal{G}))$ is a linear operator defined via its action on elementary paths: $\partial_p e_{i_0, \ldots i_p} = \sum_k (-1)^k e_{i_0, \ldots \widehat{i_k}, \ldots i_p}$ and extended over $\mathcal{A}_p(\mathsf{R}(\mathcal{G}))$ by linearity. Note that $\partial_p \equiv 0$.

Elementary allowed paths are further refined to exclude non-robust paths. The robust k-paths then become the basis set for the space of k-chains.

Definition 6: (i) An allowed $e_{i_0 i_1 i_2}$ is a robust 2-path or a triangle in $\mathcal{G}$ if there is a route from i_0 to i_2 that evades at least one route from i_0 to i_2 through i_1, i.e., $\exists (\alpha, \beta) \in \mathcal{C}(e_{i_0 i_2}) \times \mathcal{C}(e_{i_0 i_1}) \cap \mathcal{C}(e_{i_1 i_2})$ such that $\mathcal{V}_\alpha^{i_0 \to i_2} \cap \mathcal{V}_\beta = \{i_0, i_2\}$. We then call the tuple of routes $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$, a triangulating pair of the robust 2-path $e_{i_0 i_1 i_2}$. We denote the set of all triangles (robust 2-paths) by $\Delta_2(\mathsf{R}(\mathcal{G}))$.

(ii) An allowed $e_{i_0 i_1 i_2, \ldots i_p}$ is a robust p-path if there exists a collection of route tuples $\{(\mathcal{R}_{\alpha_k}, \mathcal{R}_{\beta_k})\}_{k=1}^{p-1}$ such that $(\mathcal{R}_{\alpha_k}, \mathcal{R}_{\beta_k})$ triangulates $e_{i_{k-1} i_k i_{k+1}}$ for each $k \in [p-1]$ and $\mathcal{R}_{\alpha_k}^{i_{k-1} \to i_{k+1}} \cap \mathcal{R}_{\alpha_{k+1}}^{i_k \to i_{k+2}} = \phi$ for each $k \in [p-2]$. We denote the set of all robust p-paths by $\Delta_p(\mathsf{R}(\mathcal{G}))$ and call the associated collection $\{(\mathcal{R}_{\alpha_a}, \mathcal{R}_{\beta_a})\}_{a=1}^{k-1}$ a k-triangulating pair of $e_{i_0 i_1 i_2,\ldots i_k}$.

Definition 7: (i) The sets of 0-chains and 1-chains are respectively defined as $\Omega_0(\mathsf{R}(\mathcal{G})) := \mathbb{K}\text{-span}\{\mathcal{V}\} = \mathcal{A}_0(\mathsf{R}(\mathcal{G}))$ and $\Omega_1(\mathsf{R}(\mathcal{G})) := \mathbb{K}\text{-span}\{\mathsf{R}(\mathcal{E})\} = \mathcal{A}_1(\mathsf{R}(\mathcal{G}))$.

(ii) The set of p-chains is defined as the $\mathbb{K}$-linear span of robust p-paths: $\Omega_p(\mathsf{R}(\mathcal{G})) = \mathbb{K}\text{-span}\{\Delta_p(\mathsf{R}(\mathcal{G}))\} \subseteq \mathcal{A}_p(\mathsf{R}(\mathcal{G}))$.

Note: $e_{i_0,\ldots i_p} \in \Delta_p(\mathsf{R}(\mathcal{G})) \longleftrightarrow e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}))$.

The following proposition sets up the desired homology of the introduced k-chain spaces $\{\Omega_k(\mathsf{R}(\mathcal{G}))\}_{k\in\mathbb{N}_0}$.

Proposition 1: For all $p \geq 1$, we have

(i) $\partial_{p-1} \circ \partial_p = 0$. (ii) $\partial\Omega_p(\mathsf{R}(\mathcal{G})) \subseteq \Omega_{p-1}(\mathsf{R}(\mathcal{G}))$.

Consequently, we obtain the following chain complex

$$\mathbb{K}\{0\} \xleftarrow{\partial_0} \Omega_0(\mathsf{R}(\mathcal{G})) \xleftarrow{\partial_1} \ldots \xleftarrow{\partial_n} \Omega_n(\mathsf{R}(\mathcal{G})) \xleftarrow{\partial_{n+1}} \ldots$$

Proof: (i) For an arbitrary $e_{i_0,\ldots i_p}$,

$$\partial_{p-1} \circ \partial_p e_{i_0,\ldots i_p} = \sum_{r=0}^{q-1} \sum_q (-1)^{q+r} e_{i_0,\ldots \widehat{i_r},\ldots \widehat{i_q},\ldots i_p} + \sum_{r=q+1}^{p} \sum_q (-1)^{q+r-1} e_{i_0,\ldots \widehat{i_q},\ldots \widehat{i_r},\ldots i_p} = 0.$$

Therefore Proposition 1(i) follows by linearity of ∂_p.

(ii) – For $p = 0, 1$, $\partial\Omega_0(\mathsf{R}(\mathcal{G})) = \mathbb{K}\{0\}$ and $\partial\Omega_1(\mathsf{R}(\mathcal{G})) \subset \mathbb{K}\{\mathcal{V}\} = \Omega_0(\mathsf{R}(\mathcal{G}))$ follow by definition as $\partial_0 = 0$, $\Omega_0(\mathsf{R}(\mathcal{G})) = \mathcal{A}_0(\mathsf{R}(\mathcal{G}))$ and $\Omega_1(\mathsf{R}(\mathcal{G})) = \mathcal{A}_1(\mathsf{R}(\mathcal{G}))$.

For $p \geq 2$, we show that $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G})) \implies e_{i_0,\ldots \widehat{i_k},\ldots i_p} \in \Omega_{p-1}(\mathsf{R}(\mathcal{G}))$ for all $k \in \{0, \ldots, p\}$ which implies $\partial_p\Omega_p(\mathsf{R}(\mathcal{G})) \subset \Omega_{p-1}(\mathsf{R}(\mathcal{G}))$ by linearity of ∂_p.

– For $p = 2$, notice that $e_{i_0 i_1 i_2} \in \Omega_2(\mathsf{R}(\mathcal{G})) \subset \mathcal{A}_2(\mathsf{R}(\mathcal{G})) \implies e_{i_0 i_1}, e_{i_1 i_2} \in \mathsf{R}(\mathcal{E})$ which implies the existence of routes

$$\mathcal{R}_a : e_{i_0 i_1} \in Sim(\mathcal{R}_a);\ \mathcal{R}_b : e_{i_1 i_2} \in Sim(\mathcal{R}_b).$$

It follows that $e_{i_0 i_2} \in Sim(\mathcal{R}_c)$ where $\mathcal{R}_c = \mathcal{R}_a^{o\to i_1} \cup \mathcal{R}_b^{i_1\to d}$. Thereby $\partial e_{i_0 i_1 i_2} \in \mathcal{A}_1(\mathsf{R}(\mathcal{G})) \equiv \Omega_1(\mathsf{R}(\mathcal{G}))$.

– For $p > 2$, let $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}))$. Then $e_{i_1,\ldots i_p}$ and $e_{i_0,\ldots i_{p-1}}$ both belong to $\Omega_{p-1}(\mathsf{R}(\mathcal{G}))$ as they are respectively $p-1$-triangulated by $\{(\mathcal{R}_{\alpha_k}, \mathcal{R}_{\beta_k})\}_{k=2}^{p-1}$ and $\{(\mathcal{R}_{\alpha_k}, \mathcal{R}_{\beta_k})\}_{k=1}^{p-2}$. Further, for $k \in [p-1]$, $e_{i_0,\ldots \widehat{i_k},\ldots i_p} \in \Omega_{p-1}(\mathsf{R}(\mathcal{G}))$ as it is $p-1$-triangulated by $\{(\mathcal{R}_{\alpha_a}, \mathcal{R}_{\beta_a})_{a=1}^{k-2}\} \cup \{(\mathcal{R}_{\alpha_{k-1}}^{o\to i_k} \cup \mathcal{R}_{\beta_k}^{i_k\to d}, \mathcal{R}_{\beta_{k-1}}^{o\to i_{k-1}} \cup \mathcal{R}_{\alpha_k}^{i_{k-1}\to d}), (\mathcal{R}_{\beta_{k-1}}^{o\to i_k} \cup \mathcal{R}_{\alpha_{k+1}}^{i_k\to d}, \mathcal{R}_{\alpha_k}^{i_{k-1}\to i_{k+1}} \cup \mathcal{R}_{\beta_{k+1}}^{i_{k+1}\to d})\} \cup \{(\mathcal{R}_{\alpha_a}, \mathcal{R}_{\beta_a})\}_{a=k+2}^{p-1}$. It follows that $\partial_p e_{i_0,\ldots i_p} \in \Omega_{p-1}(\mathsf{R}(\mathcal{G}))$. This completes the proof of this proposition. ■

III. Robust Paths in Series Parallel Graphs

We build a formal definition of a series-parallel graph in Section III-A and investigate chain complexes induced by them in Section III-B. We find that $\dim \Omega_p(\mathsf{R}(\mathcal{G})) = 0$ for all $p > 2$ if and only if $\mathcal{G}$ is a series-parallel graph which presents a notable correspondence between the developed homological algebra and emergent combinatorial complexity as $\mathcal{G}$ deviates from a series-parallel topology.

A. Series-Parallel Two-Terminal DAGs

We define parallel and series combinations below (Recall Fig. 2), following which an inductive definition for a series-parallel graph follows.

Definition 8: Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, o_1, d_1) = \bigcup_i \mathcal{R}_i^1$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, o_2, d_2) = \bigcup_j \mathcal{R}_j^2$ be two-terminal graphs.

(i) If $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy $d_1 = o_2, \mathcal{V}_1 \cap \mathcal{V}_2 = \{d_1\}$, then a series combination of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the two-terminal graph $\mathcal{G}_1 \to \mathcal{G}_2 := \mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, o_1, d_2) = \bigcup_{i,j}(\mathcal{R}_i^1 \to \mathcal{R}_j^2)$ where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2;\ \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy $o_1 = o_2 =: o$, $d_1 = d_2 =: d$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \{o, d\}$, then, a parallel combination $\mathcal{G}_1$ and $\mathcal{G}_2$ is the two-terminal graph $\mathcal{G}_1||\mathcal{G}_2 := \mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, o, d) = \bigcup_{j\in\{1,2\} i\in([N_j])} \mathcal{R}_i^j$ where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2;\ \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

(iii) A two-terminal graph $\mathcal{G}$ is a series-parallel graph if and only if 1) $\mathcal{G} \cong K_{12}$ or 2) $\mathcal{G} \cong \mathcal{G}_1 \to \mathcal{G}_2$ for series-parallel graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ or 3) $\mathcal{G} \cong \mathcal{G}_1||\mathcal{G}_2$ for series-parallel graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

A series-parallel graph $\mathcal{G}$ can hence be represented as a series and parallel combination of edges. For instance, the graph $\mathcal{G}_1$ in Fig. 2(a) expressible as follows in an 'edge-combinatorial' representation:

$$\mathcal{G}_1 \to \mathcal{G}_2 = \big((K_{i_0 i_1} \to K_{i_1 i_3})||(K_{i_0 i_2} \to K_{i_2 i_3})\big). \tag{3}$$

B. Path Complexes of Series-Parallel Graphs

Given $\mathcal{G}_1$ and $\mathcal{G}_2$ along with their respectively induced chain complexes $\{\Omega_k(\mathsf{R}(\mathcal{G}_1))\}_{k\in\mathbb{N}_0}$ and $\{\Omega_m(\mathsf{R}(\mathcal{G}_1))\}_{m\in\mathbb{N}_0}$, we state what can be inferred about the complex induced by their combinations in the two propositions that follow.

Proposition 2: Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, o, d) = \bigcup_{i\in[N_1]} \mathcal{R}_i^1$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, o, d) = \bigcup_{j\in[N_2]} \mathcal{R}_j^2$, and, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, o, d) = \mathcal{G}_1||\mathcal{G}_2 = \bigcup_{j\in\{1,2\} i\in([N_j])} \mathcal{R}_i^j \equiv \bigcup_{\alpha\in[N]} \mathcal{R}_\alpha$ be their parallel combination. Further, let $\mathsf{R}(\mathcal{G}_i) = (\mathcal{V}_i, \mathsf{R}(\mathcal{E}_i), \mathcal{C}_i)$ for each $i = 1, 2$ and $\mathsf{R}(\mathcal{G}) = (\mathcal{V}, \mathsf{R}(\mathcal{E}), \mathcal{C})$. Following relations then hold.

(i) $\dim \Omega_0(\mathsf{R}(\mathcal{G})) = \dim \Omega_0(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_0(\mathsf{R}(\mathcal{G}_2)) - 2$.

(ii) $\dim \Omega_1(\mathsf{R}(\mathcal{G})) = \dim \Omega_1(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_1(\mathsf{R}(\mathcal{G}_2)) - 1$.

(iii) $\Omega_2(\mathsf{R}(\mathcal{G})) \supseteq \Omega_2(\mathsf{R}(\mathcal{G}_1)) \cup \Omega_2(\mathsf{R}(\mathcal{G}_2))$.

(iv) $\dim \Omega_p(\mathsf{R}(\mathcal{G})) = \dim \Omega_p(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_p(\mathsf{R}(\mathcal{G}_2)), p > 2$.

Proof: (i) Follows since $\Omega_0(\mathsf{R}(\mathcal{G}))$ is a linear space spanned by all vertices of $\mathsf{R}(\mathcal{G})$: $\dim \Omega_0(\mathsf{R}(\mathcal{G})) = |\mathcal{V}_1 \cup \mathcal{V}_2| = |\mathcal{V}_1| + |\mathcal{V}_2| - 2 = |\dim \Omega_0(\mathsf{R}(\mathcal{G}_1))| + |\dim \Omega_0(\mathsf{R}(\mathcal{G}_2))| - 2$.

(ii) Follows since $\Omega_1(\mathsf{R}(\mathcal{G}))$ is a linear space spanned by all edges of $\mathsf{R}(\mathcal{G})$: $\dim \Omega_1(\mathsf{R}(\mathcal{G})) = |\mathsf{R}(\mathcal{E}_1) \cup \mathsf{R}(\mathcal{E}_2)| = |\mathsf{R}(\mathcal{E}_1)| + |\mathsf{R}(\mathcal{E}_2)| - |\mathsf{R}(\mathcal{E}_1) \cap \mathsf{R}(\mathcal{E}_2)| = \dim \Omega_1(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_1(\mathsf{R}(\mathcal{G}_2)) - |\{e_{od}\}| = \dim \Omega_1(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_1(\mathsf{R}(\mathcal{G}_2)) - 1$.

(iii) Note that if $\mathcal{R} \in \{\mathcal{R}_\alpha\}_{\alpha\in[N_j]}$ for $j = 1, 2$, then $\mathcal{R} \in \{\mathcal{R}_\alpha\}_{\alpha\in[N]}$. Thus, if a pair $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$ triangulates $e_{i_0 i_1 i_2}$ in $\mathsf{R}(\mathcal{G}_j)$ for $j \in \{1, 2\}$, then it also triangulates the 2-path in $\mathsf{R}(\mathcal{G})$. It follows that $\Delta_2(\mathsf{R}(\mathcal{G}_j)) \subset \Delta_2(\mathsf{R}(\mathcal{G})) \implies \Omega_2(\mathsf{R}(\mathcal{G}_j)) \subseteq \Omega_2(\mathsf{R}(\mathcal{G}))$ for each j, and thus, (iii) holds.

(iv) Let $j \in \{1, 2\}$ and $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}_j))$. Then $e_{i_{k-1} i_k i_{k+1}} \in \Omega_2(\mathsf{R}(\mathcal{G}_j)) \implies e_{i_{k-1} i_k i_{k+1}} \in \Omega_2(\mathsf{R}(\mathcal{G}))$ for each $k \in [p-1]$ which in turn implies $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}))$. Thus, $\Omega_p(\mathsf{R}(\mathcal{G}_1)) \oplus \Omega_p(\mathsf{R}(\mathcal{G}_2)) \subset \Omega_p(\mathsf{R}(\mathcal{G}))$. On the other hand, if $e_{i_{k-1} i_k i_{k+1}}$ does not have a triangulating pair in $\mathsf{R}(\mathcal{G}_j)$, then it cannot not have one in $\mathsf{R}(\mathcal{G})$ either unless $(i_{k-1}, i_{k+1}) \neq (o, d)$ since then at least one of i_{k-1} and i_{k+1} does not belong $\mathcal{G}_{-j}$ (where $-j \in \{1, 2\}, j \neq -j$). Hence, $\Omega_p(\mathsf{R}(\mathcal{G}_1)) \oplus \Omega_p(\mathsf{R}(\mathcal{G}_2)) = \Omega_p(\mathsf{R}(\mathcal{G}))$ follows. Further, since $p > 2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \{o, d\}$, $\Omega_p(\mathsf{R}(\mathcal{G}_1)) \perp \Omega_p(\mathsf{R}(\mathcal{G}_2))$ and the proposed follows. ■

Proposition 3: Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, o, h)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, h, d)$ be two two-terminal graphs and $\mathcal{G} = \mathcal{G}_1 \to \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, o, d)$ be their series combination. Then, the following hold.

(i) $\dim \Omega_0(\mathsf{R}(\mathcal{G})) = \dim \Omega_0(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_0(\mathsf{R}(\mathcal{G}_2)) - 1$.

(ii) $\dim \Omega_p(\mathsf{R}(\mathcal{G})) = \dim \Omega_p(\mathsf{R}(\mathcal{G}_1)) + \dim \Omega_p(\mathsf{R}(\mathcal{G}_2))\ \forall p > 1$.

Proof: (i) Follows since $\Omega_0(\mathsf{R}(\mathcal{G}))$ is a linear space spanned by all vertices of $\mathsf{R}(\mathcal{G})$: $\dim \Omega_0(\mathsf{R}(\mathcal{G})) = |\mathcal{V}_1 \cup \mathcal{V}_2| = |\mathcal{V}_1| + |\mathcal{V}_2| - 1 = |\dim \Omega_0(\mathsf{R}(\mathcal{G}_1))| + |\dim \Omega_0(\mathsf{R}(\mathcal{G}_2))| - 1$.

(ii) For $p = 1$ follows since $\Omega_1(\mathsf{R}(\mathcal{G}))$ is a linear space spanned by all edges of $\mathsf{R}(\mathcal{G})$: $\dim \Omega_1(\mathsf{R}(\mathcal{G})) = |\mathcal{E}_1 \cup \mathcal{E}_2| = |\mathcal{V}_1| + |\mathcal{V}_2| = |\dim \Omega_1(\mathsf{R}(\mathcal{G}_1))| + |\dim \Omega_1(\mathsf{R}(\mathcal{G}_2))|$. Now notice that no pair of routes can triangulate e_{jhk} for all $j, k \in \mathcal{V}$ with $j < h < k$ since h belongs to every route of $\mathcal{G}$ by definition. Thus $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}))$ requires $h \leq i_0$ or $h \geq i_p$ which is equivalent to $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}_2))$ or $e_{i_0,\ldots i_p} \in \Omega_p(\mathsf{R}(\mathcal{G}_1))$ respectively. Thus, $\Omega_p(\mathsf{R}(\mathcal{G})) = \Omega_p(\mathsf{R}(\mathcal{G}_1)) \oplus \Omega_p(\mathsf{R}(\mathcal{G}_2))$ and the proposed follows. ■

We are now in a position to establish our main result.

Theorem 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, o, d) = \cup_{\alpha \in A} \mathcal{R}_\alpha$ be a two terminal DAG. Then,

(i) If $\mathcal{G}$ is a series-parallel graph, then $\dim \Omega_p(\mathcal{G}) = 0$ for all $p \geq 3$.

(ii) If $\mathcal{G}$ is not a series-parallel graph, then $\dim \Omega_3(\mathsf{R}(\mathcal{G})) > 0$.

Proof: (i) If $\mathcal{G}$ is a series-parallel graph, then using Propositions 2. (iv), 3.(ii), and the edge combinational representation of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, o, d)$, we deduce

$$\dim \Omega_p(\mathsf{R}(\mathcal{G})) = \sum_{e_{ij} \in \mathcal{E}} \dim \Omega_p\big(\mathsf{R}(K_{ij})\big) = 0\ \forall\ p \geq 3. \quad (4)$$

(ii) If $\mathcal{G}$ is not series-parallel, then sequentially decomposing $\mathcal{G}$ serially and/or parallelly one eventually arrives at a two-terminal subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', o', d') \neq K_{o'd'}$ which is not decomposable further. Since $\mathcal{G}' \subseteq \mathcal{G}$, we have an $A' \subseteq A$ such that $\mathcal{G}' = \cup_{\alpha \in A'} \mathcal{R}_\alpha^{o' \to d'}$. Note that $|A'| > 1$ since $|A'| = 1$ implies $\mathcal{G}'$ is a single $o' \to d'$ route which is serially decomposable by definition.

Case 1: Suppose that $\exists\ k \in \mathcal{V}' : e_{o'kd'} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$ and let $(\mathcal{R}_\alpha^{o' \to d'}, \mathcal{R}_\beta^{o' \to d'})$ be the corresponding triangulating pair whereby $\mathcal{R}_\alpha^{o' \to d'} \cap \mathcal{R}_\beta^{o' \to d'} = \{o', d'\}$ (Recall Definition 7). Define a subset $D' \subset A'$: $D' = \{\delta : \mathcal{R}_\delta^{o' \to d'} \cap \mathcal{R}_\alpha^{o' \to d'} = \{o', d'\}$ and $D'^c = A' \backslash D'$. Note that $\beta \in D'$ and $\alpha \in D'^c$ so that D', D'^c are both non-empty. Now consider the following two-terminal graphs induced by the partition $\{D', D'^c\}$:

$$\mathcal{G}'_{D'} = \bigcup_{D'} \mathcal{R}_\delta^{o' \to d'} = \big(\mathcal{V}'_{D'}, \mathcal{E}'_{D'}, o', d'\big);$$

$$\mathcal{G}'_{D'^c} = \bigcup_{D'^c} \mathcal{R}_\delta^{o' \to d'} = \big(\mathcal{V}'_{D'^c}, \mathcal{E}'_{D'^c}, o', d'\big).$$

If $\mathcal{V}'_{D'} \cap \mathcal{V}'_{D'^c} = \{o', d'\}$ then $\mathcal{G}' = \mathcal{G}_{D'} || \mathcal{G}_{D'^c}$ which contradicts the supposition that $\mathcal{G}'$ is not decomposable parallely. Otherwise if $j \in \mathcal{V}'_{D'} \cap \mathcal{V}'_{D'^c}, j \notin \{o', d'\}$ then $j \in \mathcal{R}_{\delta_0}^{o' \to d'}$ for some $\delta_0 \in D'^c$ and $j \notin \mathcal{R}_\alpha^{o' \to d'}$. Further, choose a $\gamma \in D'$ with $j \in \mathcal{R}_\gamma^{o' \to d'}$ (γ exists since $j \in \mathcal{V}_{D'}$). Then, at least one of the following two vertices exist outside $\{o', d', j\}$:

$$\ell_1 = \min \mathcal{R}_{\delta_0}^{j \to d'} \cap \mathcal{R}_\alpha^{o' \to d'},\ \ \ell_2 = \max \mathcal{R}_{\delta_0}^{o' \to j} \cap \mathcal{R}_\alpha^{o' \to d'}.$$

If ℓ_1 exists outside $\{o', d', j\}$ then $e_{o'j\ell_1 d'} \in \Omega_3(\mathsf{R}(\mathcal{G}'))$ as it is 3-triangulated by $\{(\mathcal{R}_\alpha^{o' \to d'}, \mathcal{R}_\gamma^{o' \to j} \cup \mathcal{R}_{\delta_0}^{j \to d'}), (\mathcal{R}_\gamma^{o' \to d'}, \mathcal{R}_{\delta_0}^{j \to \ell_1} \cup \mathcal{R}_\alpha^{\ell_1 \to d'})\}$.

Similarly if ℓ_2 exists outside $\{o', d', j\}$, $e_{o'\ell_2 jd'} \in \Omega_3(\mathsf{R}(\mathcal{G}'))$ as $e_{o'\ell_2 j}$ is 3-triangulated by $\{(\mathcal{R}_\gamma^{o' \to d'}, \mathcal{R}_\alpha^{o' \to \ell_2} \cup \mathcal{R}_{\delta_0}^{\ell_2 \to d'}), (\mathcal{R}_\alpha^{o' \to d'}, \mathcal{R}_{\delta_0}^{o' \to j} \cup \mathcal{R}_\gamma^{j \to d'})\}$. Thus, $\dim \Omega_3(\mathsf{R}(\mathcal{G}')) > 0$ and we arrive at a contradiction since $0 = \dim \Omega_3(\mathsf{R}(\mathcal{G})) > \dim \Omega_3(\mathsf{R}(\mathcal{G}'))$ following $\mathcal{G}' \subset \mathcal{G} \implies \Omega_3(\mathsf{R}(\mathcal{G}')) \subset \Omega_3(\mathsf{R}(\mathcal{G}))$.

Case 2: Now suppose that $\nexists\ k \in \mathcal{V} : e_{o'kd'} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$. If $\mathcal{G}'$ has only two routes, i.e., $A' = \{a, b\}$, then, $\mathcal{R}_a^{o' \to d'} \cap \mathcal{R}_b^{o' \to d'} = \{o', d'\}$ will imply that $(\mathcal{R}_a^{o' \to d'}, \mathcal{R}_b^{o' \to d'})$ triangulate some $e_{o'kd'}$ for some $k \in \mathcal{R}_a^{o' \to d'} \cup \mathcal{R}_b^{o' \to d'}$, a contradiction to the supposition of Case 2. So let $k \in \mathcal{R}_a^{o' \to d'} \cap \mathcal{R}_b^{o' \to d'}$. This allows $\mathcal{G}'$ the serial decomposition $\mathcal{G}' = \mathcal{R}_a^{o' \to k} \cup \mathcal{R}_b^{o' \to k} \to \mathcal{R}_a^{k \to d'} \cup \mathcal{R}_b^{k \to d'}$.

Now let $|A'| > 2$. If there is a vertex k common across all routes, i.e., $\exists k \in \mathcal{V}'$ such that $k \in \mathcal{R}_a^{o' \to d'}$ for all $a \in A'$, $\mathcal{G}'$ admits the serial decomposition $\mathcal{G}' = \bigcup_{A'} \mathcal{R}_a^{o' \to k} \to \bigcup_{A'} \mathcal{R}_a^{k \to d'}$ and we arrive at a contradiction. So suppose otherwise, that for each $k \in \mathcal{V}' \setminus \{o', d'\}$, there is a route in $\mathcal{G}'$ that excludes k. We will show that this supposition contradicts at least one of the two: (i) $\dim \Omega_3(\mathsf{R}(\mathcal{G}')) = 0$ or (ii) $\nexists j : e_{o'jd'} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$, the supposition of Case 2.

We begin by establishing the existence of a pair $j_0, k_0 \in \mathcal{V}'$ such that $e_{o'j_0k_0} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$. Pick an arbitrary $\beta \in A'$ and take j_0 as the second vertex in the route $\mathcal{R}_\beta^{o' \to d'}$ i.e. $j_0 = \min \mathcal{R}_\beta^{o' \to d'}$ such that $j_0 \neq o'$. Then there is a route $\mathcal{R}_\alpha^{o' \to d'}$ that excludes j_0. Take $k_0 = \min \mathcal{R}_\alpha^{o' \to d'} \cap \mathcal{R}_\beta^{o' \to d'}, k_0 > o'$. Then $k_0 > j_0$ since j_0 is the second smallest vertex in $\mathcal{R}_\beta^{o' \to d'}$. This selection ensures $e_{o'j_0k_0} \in \Omega_2(\mathsf{R}(\mathcal{G}))$ with the triangulating pair $(\mathcal{R}_\alpha^{o' \to d'}, \mathcal{R}_\beta^{o' \to d'})$. By supposition, there exists a route $\mathcal{R}_\gamma^{o' \to d'}$ that excludes k_0.

– Let $\ell_\alpha = \min \mathcal{R}_\alpha^{o' \to d'} \cap \mathcal{R}_\gamma^{o' \to d'}$ such that $\ell_\alpha \neq o'$ and note that if $\ell_\alpha > k_0$, then $e_{o'k_0\ell_\alpha} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$ with triangulating pair $(\mathcal{R}_\gamma^{o' \to d'}, \mathcal{R}_\alpha^{o' \to d'})$.

– Let $\ell_\beta = \min \mathcal{R}_\beta^{o' \to d'} \cap \mathcal{R}_\gamma^{o' \to d'}$ such that $\ell_\beta \neq o'$ and note that if $\ell_\beta > k_0$, then $e_{o'k_0\ell_\beta} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$ with triangulating pair $(\mathcal{R}_\gamma^{o' \to d'}, \mathcal{R}_\beta^{o' \to d'})$.

– If both $\ell_\alpha < k_0$ and $\ell_\beta < k_0$, then the intersection graphs $\mathcal{R}_\gamma^{o' \to d'} \cap \mathcal{R}_\alpha^{o' \to k_0}$ and $\mathcal{R}_\gamma^{o' \to d'} \cap \mathcal{R}_\beta^{o' \to k_0}$ contain other vertices in addition to o' so let p_α and p_β be the maximal vertices in the above two intersection graphs and note that $p_\alpha < k_0$ and $p_\beta < k_0$ hold by definition.

– Now if $p_\alpha < p_\beta$, then $e_{o'p_\alpha p_\beta k_0} \in \Omega_3(\mathsf{R}(\mathcal{G}'))$ as it is 3-triangulated by $\{(\mathcal{R}_\beta^{o' \to d'}, \mathcal{R}_\alpha^{o' \to p_\alpha} \cup\ \mathcal{R}_\gamma^{p_\alpha \to d'}), (\mathcal{R}_\alpha^{o' \to d'}, \mathcal{R}_\beta^{o' \to p_\beta} \cup \mathcal{R}_\gamma^{p_\beta \to d'})\}$.

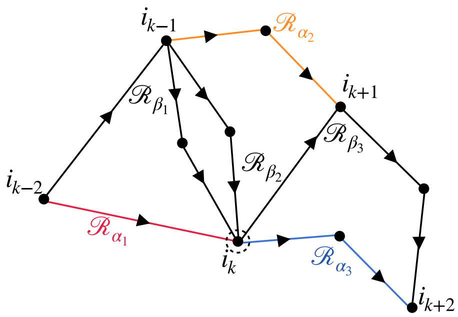

Fig. 4. Representative Topologies for each case in the proof of the Proposition 1.

– Otherwise if $p_\alpha > p_\beta$, then $e_{o'p_\beta p_\alpha k_0} \in \Omega_3(\mathsf{R}(\mathcal{G}'))$ as it is 3-triangulated by $\{(\mathcal{R}_\alpha^{o'\to d'}, \mathcal{R}_\beta^{o'\to p_\beta} \cup \mathcal{R}_\gamma^{p_\beta\to d'}), (\mathcal{R}_\beta^{o'\to d'}, \mathcal{R}_\alpha^{o'\to p_\alpha} \cup \mathcal{R}_\gamma^{p_\alpha\to d'})\}$.

This long line of reasoning thus brings us to the following conclusion. If there is a route for every vertex in $\mathcal{V}' \setminus \{o', d'\}$ that excludes it, then there exists at least one 2-path $e_{o'jk} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$. Further, existence of a 2-path $e_{o'jk} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$ implies one of the following two implications:

– Either $\dim \Omega_3(\mathsf{R}(\mathcal{G}')) \neq 0$ which is a contradiction to the presupposition on $\mathcal{G}'$

– Or there is a pair $j', k' \in \mathcal{V}'$ with $k' > k$ such that $e_{oj'k'} \in \Omega_2(\mathsf{R}(\mathcal{G}'))$. One can then set $j = j', k = k'$ and inductively run the same line of arguments again to obtain either $\dim \Omega_3(\mathsf{R}(\mathcal{G}')) \neq 0$ or $k' = d'$ which is a contradiction to the presupposition of Case 2. The proof rests here. ■

We have shown that series-parallel graphs associate with a chain complex truncated at order two, that is, robust paths of length three and above are absent in the associated chain complex. The topological simplicity of series-parallel graphs is thus mapped onto an algebraic simplicity in the graph homology of robust paths.

C. Robust 3-Paths and the Braess Paradox

We say that a graph $\mathcal{H}$ is embedded in a graph $\mathcal{G}$ if upon deletion of suitable edges and vertices in $\mathcal{G}$, and subsequent merging of edges e_{ij} and e_{jk} in the graph obtained upon the deletion into a single edge e_{ik}, the result is a graph $\mathcal{G}'$ that is isomorphic to $\mathcal{H}$. For example, for $\mathcal{G}$ in Fig. 3, deleting e_{35}, e_{5d} and merging the pairs e_{o1}, e_{12} to e_{o2} and e_{o3}, e_{34} to e_{o4} yields a graph isomorphic to the Braess embedding in Fig. 2(b) with $(i_0, i_1, i_2, i_3) \cong (o, 2, 4, d)$. Any robust 3-path $e_{i_{k-2}i_{k-1}i_k i_{k+1}}$ in $\mathcal{G}$ induces the Braess embedding $(i_{k-2}, i_{k-1}, i_k, i_{k+1})$, as can be seen in Fig. 4. A robust p-path $e_{i_0,\ldots i_p}$ with $p > 3$ contains $\binom{p}{4}$ robust 3-paths within itself and hence identifies a large collection of interacting Braess-susceptible sites. Conversely, if (i_0, i_1, i_2, i_3) induce a Braess embedding in $\mathcal{G}$, then reintroduction of all deleted edges and vertices in the embedding reconstructs $\mathcal{G}$ that contains the robust 3-path $e_{i_0 i_1 i_2 i_3}$ with the structure shown by Case 1 in Fig. 4.

IV. Conclusion

In this letter, we introduced the notion of a k-robust path in a DAG $\mathcal{G}$, which localizes the deviation of a graph $\mathcal{G}$ from a series-parallel topology for, with larger k signifying larger deviation. We showed that the association of the $\mathbb{K}$-linear spaces of robust k-paths with k-chains in a chain complex sets up a consistent graph homology. We established that the topological simplicity of series-parallel graphs translates into a triviality of k-chains in the induced complex for $k \geq 3$, and any non-triviality therein deviates the graph from the simple topology. We further discussed the resulting correspondence between the space of 3-chains and Braess-susceptible sites within a network. With this discussion serving as an illustrative example, we believe that the graph homology developed with robust paths as its basis will be a useful tool for the systematic characterization of complex behavior in flow networks and analyze combinatorial optimization problems on them, which remains a direction for future study.

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