Standard Simplices and Pluralities are Not the Most Noise Stable

[Abstract]

Steven Heilman*
Department of Mathematics
UCLA
Los Angeles, CA 90095-1555
heilman@math.ucla.edu

Elchanan Mossel†
Department of Statistics
University of Pennsylvania and
U.C. Berkeley
mossel@wharton.upenn.edu

Joe Neeman‡
Mathematics and Electrical
and Computer Engineering
University of Texas at Austin
Austin, TX 78712
joeneeman@gmail.com

ABSTRACT

The Standard Simplex Conjecture and the Plurality is Stablest Conjecture are two conjectures stating that certain partitions are optimal with respect to Gaussian and discrete noise stability respectively. These two conjectures are natural generalizations of the Gaussian noise stability result by Borell (1985) and the Majority is Stablest Theorem (2004). Here we show that the standard simplex is not the most stable partition in Gaussian space and that Plurality is not the most stable low influence partition in discrete space for every number of parts $k \geq 3$, for every value $\rho \neq 0$ of the noise and for every prescribed measures for the different parts as long as they are not all equal to $1/k$. Our results do not contradict the original statements of the Plurality is Stablest and Standard Simplex Conjectures concerning partitions into sets of equal measure. However, they indicate that if these conjectures are true, their veracity and their proofs will crucially rely on assuming that the sets are of equal measures, in stark contrast to Borell's result, the Majority is Stablest Theorem and many other results in isoperimetric theory.

In other words, the optimal partitions for noise stability are of a different nature than the ones considered for partitions into three parts in isoperimetric theory. In the latter case, the standard simplex is the partition of the plane into three sets of smallest Gaussian perimeter, where the sets are restricted to have Gaussian measures $a_1, a_2, a_3 > 0$ respectively, with $a_1 + a_2 + a_3 = 1$ and $|a_i - 1/3| < .04$ for all $i \in \{1, 2, 3\}$. Thus, we now know that the extension of noise stability theory from two to three or more parts is very much different than the extension of isoperimetric theory from two to three or more parts. Moreover, all existing proofs which optimize noise stability of two sets must fail for more than three sets, since these proofs rely on the fact that a half-space optimizes noise stability with respect to any measure restriction. Given our results it is natural to ask for (conjectured) partitions achieving the optimum noise stability.

The main new ingredient in our work shows that the Ornstein-Uhlenbeck operator applied to the indicator function of a simplicial cone becomes holomorphic when restricted to certain lines. This holomorphicity condition, when combined with a first variation argument (i.e. an infinite dimensional perturbative argument of the first order), then shows that any simplicial cone can be perturbed in a volume-preserving manner to improve its noise stability. Such a holomorphicity argument seems unavailable for the isoperimetric problem, since this argument uses the inherent nonlocality of the Ornstein-Uhlenbeck semigroup.

A full version of the paper is available at arXiv:1403.0885.

Categories and Subject Descriptors

Theory of computation [**Analysis of algorithms and problem complexity**]: Nonnumerical algorithms and problems—*Geometrical problems and computations*

General Terms

Theory

Keywords

Plurality; Standard Simplex; Noise Stability

Acknowledgement. Thanks to Oded Regev, Ronen Eldan, Kostya Makarychev, and Assaf Naor for helpful comments. This work was mostly conducted during the semester on "Real Analysis in Computer Science" at the Simons Institute for Theoretical Computer Science. We are grateful to the institute for its hospitality and support.

*S. H. was supported by NSF Graduate Research Fellowship DGE-0813964 and a Simons-Berkeley Research Fellowship. Part of this work was completed while S. H. visited the Network Science and Graph Algorithms program at ICERM.

†E. M. was supported by NSF grants DMS 1106999 and CCF 1320105, and by grant 328025 from the Simons Foundation.

‡J. N. was supported by NSF grant DMS-1106999 and DOD ONR grant N000141110140. Part of this work was carried out while the authors were visiting the Real Analysis in Computer Science program at the Simons Institute for the Theory of Computing.

ITCS'15, January 11–13, 2015, Rehovot, Israel.
ACM 978-1-4503-3333-7/15/01.
http://dx.doi.org/10.1145/2688073.2688076.