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EFFICIENTLY FILLING SPACE

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We show that for each $n = 3, 4, \dots$ there is a space-filling curve $f : [0, 1] \to [0, 1]^n$ such that f is at most $(n+1)$-to-1 at every point of $[0, 1]^n$. The fact that any such dimension raising continuous function is at least $(n+1)$-to-1 has been known since the 1930s, so the examples we provide here are, in that sense, the best possible. The classic space-filling curves due to first Peano and a year later, Hilbert, that map $[0, 1]$ onto $[0, 1]^2$ are both 4-to-1 at a dense set of points and their generalizations to $[0, 1]^n$ are known to be 2^n-to-1 at a dense set of points. Flaten, Humke, Olson and Vo (*J. Math. Anal. Appl.* **500**:2 (2021), art. id. 125113) gave an example, $f : [0, 1] \to [0, 1]^2$ based on the Hilbert linear ordering of somewhat altered Hilbert partitions which is at most 3-to-1 at every point of $[0, 1]^2$, but there are technical difficulties with generalizing that example to higher dimensions. In a sense, this paper represents an overcoming of those difficulties.

1. Introduction and definitions

From the introduction of [2]:

> In the late 19th century the nature of "dimension" was thought to be rather intuitive so that no definition was needed. This changed quite abruptly in 1878 when Georg Cantor crisply proved that any two finite-dimensional manifolds have the same cardinality. In particular, there is a 1-to-1 mapping from the line segment $[0, 1]$ onto the square $[0, 1]^2$. Shortly thereafter E. Netto showed there is no continuous bijection from $[0, 1]$ onto the square $[0, 1]^2$. However, in 1890 G. Peano proved there was a continuous function mapping $[0, 1]$ onto the square, $[0, 1]^2$, and shortly thereafter many such curves were published. In his comprehensive book on space-filling Hans Sagan includes a short but illuminating history of space filling curves.

Peano's original paper caused the entire mathematics community of 1890 to come to an abrupt pause (if not halt) to try and understand how such a curve could possibly exist. Peano's proof was complicated and not entirely clear, and although he employed a variant of a technique used by Cantor, his proof was entirely nongeometric. It was Hilbert, in 1891, who unmasked the geometry lying behind Peano's construction and it is Hilbert's geometric analysis that is now almost universally used to describe Peano's curve. In a sense, it is a tribute to Peano's insight and persistence that he was able to define the very first space-filling curve without reference to any underlying geometry. And so, it is Peano's name that is rightfully and almost universally attached to space-filling curves. In fact, the term "Peano curve" is now considered almost synonymous with "space-filling curve" whether it be a function from $[0, 1]$ to $[0, 1]^2$

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or from [0, 1] to $[0, 1]^n$ for some natural number $n \geq 2$. This paper is not about applications of Peano curves, but they are numerous and quite varied; see [5].

The purpose of this paper is to show that for each $n \in \mathbb{N}$ with $n \geq 2$, there are space-filling functions, $f : [0, 1] \to [0, 1]^n$ which are at most $(n+1)$-to-1 at every point and to do this we rely on the geometry of space-filling that Hilbert introduced. The focus of this paper concerns continuous functions, so unless stated otherwise, all functions will be assumed continuous. Suppose $f : A \to B$ is surjective. Then f is said to be n-to-1 at a point $y \in B$ if $\operatorname{card}(f^{-1}(y)) = n$ and n-to-1 at a point $x \in A$ if f is n-to-1 at $f(x)$. If for each $y \in B$, f is m-to-1 for some $m \leq n$, then the function f is said to be *at most* n-to-1 on B (or dually, on A). Finally, f is called *exactly* n-to-1 if it is n-to-1 at every point of B (dually, of A), and f will be simply called n-to-1 if it is at most n-to-1 at every point and exactly n-to-1 at least one point.

In [2] we give a fairly complete arithmetization of the Hilbert curve [4], and in [3] we investigate the structure of general planar space-filling curves in more detail referring to and analyzing four historically important examples of Peano curves: the original Peano curve [13], the Hilbert curve [4], the Lebesgue curve [8; 9], and the Schoenberg curve [16]. In particular, all four of these classic space-filling curves are at least 4-to-1 at infinitely many points. It is of some interest that it has become folklore that the Hilbert curve, f_H is actually at most 3-to-1 at every point, but a close analysis shows this to be incorrect. For example, one can see by direct computation that

$$f_H^{-1}\left(\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\right) = \left\{\tfrac{5}{48}, \tfrac{7}{48}, \tfrac{41}{48}, \tfrac{43}{48}\right\}.$$

These numbers are most easily found using the arrow diagrams introduced by Hilbert and the base 4 arithmetic on [0, 1] induced by those diagrams. The somewhat delicate details of this arithmetic and a fairly complete analysis of the k-to-1 points, for $k = 1, 2, 3, 4$ can be found in [2]. Interestingly, the original Peano curve is at most 4-to-1 at every point, but has no 3-to-1 points whatsoever; an analysis of the k-to-1 points for Peano's original curve and for $k = 1, 2, 4$ can be found in [5].

However, in [3, Theorem 10] an example of a 3-to-1 planar space-filling is constructed and it is shown that, for the plane, this is optimal. The purpose of this paper is to construct an optimal $(n+1)$-to-1 space-filling curve for every subsequent $n \in \mathbb{N}$. The construction is based on a Hilbert style sequence of nested partitions of $[0, 1]^n$ and then linearly ordering each partition so as to be compatible with the previous partitions and satisfying the conditions Hilbert used to insure that those partitions define a continuous limit function. The elegance of this Hilbert approach was referred to as "luminous to the geometric imagination" by the American set-theorist E. H. Moore in [11, p. 73]. We begin with a theorem of Witold Hurewicz who proved that any space-filling curve $f : [0, 1] \to [0, 1]^n$ must have a dense set of points at which it is $(n+1)$-to-1. Rather than simply quote his theorem, which Hurewicz proved in greater generality, we take an extra few lines to give the essence of his remarkable proof for $\mathbb{R}^n$. In the third section we describe a general version of Hilbert's partitioning method as in pertains to $[0, 1]^n$, and thereafter, introduce the "Lebesgue partitions" we will use in order to adopt Hilbert's method for construction we'll make in the final section. It is this "adoption" that is somewhat delicate as not every sequence of Lebesgue partitions is compatible with the linearization required by Hilbert's method.

2. Space-filling curves are at least $(k+1)$-to-1

There is quite a lively history of *dimension raising* maps, and in 1933 Witold Hurewicz [6] proved the following theorem.

Hurewicz 1933 theorem. *Suppose that $n, k \in \mathbb{N}$ and the dimension of X is n and the dimension of Y is $n+k$. If $f : X \to Y$ is continuous and surjective, then f is at least k-to-1 at some $x \in X$. Moreover, if X is a subset of a Euclidean space, then f is at least $(k+1)$-to-1 at some $x \in X$.*

An immediate consequence is the following.

Theorem 1. *If $E \subset \mathbb{R}^n$ is k-dimensional, and $f : [0, 1] \to E$ is surjective, then f is at least $(k+1)$-to-1 at a countably dense set of points of E.*

For general interest and completeness, we give a very short proof of this theorem using some more recent notation and a classic result from dimension theory, all of which can be found in [7, Chapter 6]. If $k \geq 1$, a compact k-dimensional space is called a *k-dimensional Cantor manifold* if it cannot be disconnected by an m-dimensional subset for any $m \leq k-2$. Theorem 1 is an almost direct consequence of the theorem [7, Theorem VI.8] stated below as the Cantor manifold theorem.

Cantor manifold theorem. *Any compact k-dimensional space, X, contains a subset which is a k-dimensional Cantor manifold.*

As is pointed out in [7], it is easy to see that a k-dimensional Cantor manifold is connected and k-dimensional at each of its points.

Proof of Theorem 1. Since E is k-dimensional, E contains a k-dimensional Cantor manifold, $E^* \subset E$, and consequently, there is a compact $K^* \subset K$ such that $f(K^*) = E^*$. Hence, according to the Hurewicz 1933 theorem, f is at least $(k+1)$-to-1 at some $x \in K^*$. Moreover, this same argument holds for any portion of E^*. This completes the proof. □

3. Three conditions to filling space and Lebesgue blocks

By a *domain* in $\mathbb{R}^n$ we will mean a connected open set, U for which $\operatorname{int}(\overline{U}) = U$; a *closed domain* will mean the closure of a domain, or equivalently a closed set D for which $\overline{\operatorname{int}(D)} = D$. If $K \subset \mathbb{R}^n$ is compact and $D \subset [0, 1]^n$ is a domain with $D \cap K \neq \varnothing$, then $D \cap K$ is called a portion of K, and $\overline{D} \cap K$ is referred to as a *closed portion* of K.

Based on Hilbert's original proof, in [3, Theorem 9] a general geometric construction was formulated and it was used to define space-filling curves $f : [0, 1] \to R^2$ with various properties. However, that theorem was stated only for functions mapping to compact domains in R^2, and in this paper our goal is to define functions mapping into higher-dimensional spaces. As a consequence, we develop the notation and terminology for the R^k case, but will not repeat the proof as the proof in the instance we define below is almost identical to the one published in [3].

Now, let $k \in \mathbb{N}$ and $D \subset [0, 1]^k$ be a fixed closed domain and for each $n = 1, 2, \ldots, N$ let U_n be a closed domain such that $\bigcup_{n=1}^{N} U_n = D$. Suppose too that if $1 \leq n \leq N-1$, then the following *adjacency condition* holds:

$$\varnothing \neq U_n \cap U_{n+1} \subset \partial U_n \cap \partial U_{n+1}, \tag{1}$$

where ∂ denotes the boundary operator. In this case, $\mathfrak{U} = \{U_n : n = 1, 2, \ldots, N\}$ is called a *linked partition* of D. The *norm* of $\mathfrak{U}$, denoted by $|\mathfrak{U}|$, is defined as

$$|\mathfrak{U}| = \max\{\operatorname{diam}(U) : U \in \mathfrak{U}\}.$$

In what follows we use "arrow bracket" notation for intervals of natural numbers to distinguish them from intervals of real numbers, so for example, $\langle N, M\rangle = \{n \in \mathbb{N} : N \le n \le M\}$.

Suppose that $\mathfrak{C} = \{\mathfrak{U}_m\}$ is a finite or countable sequence of linked partitions of D, where $\mathfrak{U}_m = \{U_1^m, U_2^m, \dots, U_{n_m}^m\}$. Then $\mathfrak{C}$ is called *proper* provided the following compatibility conditions hold:

(1) (*the diameter condition*) $\{|\mathfrak{U}_m|\} \to 0$ if $\mathfrak{U}$ is infinite or $|\mathfrak{U}_m| < \frac{1}{m}$ if $\mathfrak{U}$ is finite.

(2) (*the nesting condition*) For each $m = 2, 3, \dots$, the interval $\langle 1, n_m\rangle \subset \mathbb{N}$ can be partitioned into n_{m-1} blocks $B_i = \langle \ell_{i-1}^m + 1, \ell_i^m\rangle$ where $\ell_0^m = 0$ and $\ell_{n_{m-1}}^m = n_m$ such that

$$\{U_n^m : n \in B_i\} \text{ is a linked partition of } U_i^{m-1}.$$

Now, suppose that $\mathfrak{C} = \{\mathfrak{U}_m : m \in \mathbb{N}\}$ is a proper sequence of linked partitions. Then, there are two sequences of $\mathbb{N}$ that determine the nature of the nesting condition, namely $\{n_m : m \in \mathbb{N}\}$ and $\{\ell_i^m : m \in \mathbb{N} \text{ and } 0 \le i \le n_m\}$ and we refer to these two sequences as the *scaffold* of the proper sequence $\mathfrak{C}$. [1]

Theorem 2. *Suppose $\mathfrak{C} = \{\mathfrak{U}_m : m \in \mathbb{N}\}$ is a proper sequence of linked partitions of a domain $D \subset \mathbb{R}^k$ and $\mathfrak{P} = \{\mathfrak{P}_m : m \in \mathbb{N}\}$ is a proper sequence of linked partitions of* $[0, 1]$ *with the same scaffold as $\mathfrak{C}$. Then there is a continuous surjective map, $f : [0, 1] \to D$ such that $f(I_n^m) = U_n^m$ for every $m = 1, 2, \dots, n_m$ and $n \in \mathbb{N}$.*

Proof. This proof is nearly identical to the proof of Theorem 9 of [3] and as such is omitted here. □

When using Hilbert's methodology, a point of the range that is in exactly one partition block U_n^m for every $n \in \mathbb{N}$ is necessarily a $1-1$ point of the resulting space-filling function. However, points that lie on distinct ordered blocks, $U_n^{m_1} \cap U_n^{m_2}$ where $m_1 \neq m_2$ are candidates for 2-to-1 points and so on. Points common to $k+2$ are candidates for $(k+2)$-to-1 points so to construct the function we claim, we simply avoid such points altogether when partitioning. The fact that such points can be avoided was established very early by Henri Lebesgue.

In 1911, while attempting to establish that dimension was an invariant distinguishing the Euclidean spaces, Lebesgue proved that $[0, 1]^2$ could be partitioned into finitely many closed "bricks" of arbitrarily small size, and if the bricks were sufficiently small, no point was in more than 3 of them. This generalizes to partitions of $[0, 1]^k$ and such a partition has come to be called a *Lebesgue partition* of $[0, 1]^k$. Lebesgue then conjectured the following theorem; see [10].

Lebesgue covering theorem. *Suppose a k-dimensional cube is the union of a finite number of closed sets, none of which contains points of two opposite faces. Then at least $k+1$ of these closed sets have a common point.*

This was proved by Brouwer later that same year; see [1]. The Lebesgue covering theorem is closely related to Hurewicz 1933 theorem in a way we exploit in the next section.

[1]The conditions of adjacency, nesting and diameter generalize Hilbert's original description in [4] of what has become called the Hilbert function. Those original conditions of Hilbert were first named by Rose in [14]

4. Main results

A necessarily perfect set, K, is called a *space-filling core* for a space-filling function $f : [0, 1] \to [0, 1]^k$ provided $f(K) = [0, 1]^k$ and there is no proper subset $A \subset K$ such that $f(A) = [0, 1]^k$. It follows that if K is a space-filling core for f, then $f(P)$ is n-dimensional for every portion $P \subset K$; see [3] for details.

The purpose of this section is to prove the following general theorem; the case when $k = 2$ was proved in [3, Theorem 10] using different Lebesgue partitioning.

Theorem 3. *For every $k \in \mathbb{N}$ there is a space-filling function, $f : [0, 1] \to [0, 1]^k$, such that:*

(1) $[0, 1]$ *is the sf-core for f.*

(2) *f is at most $(k+1)$-to-1 at every $x \in [0, 1]$.*

(3) *f is exactly $(k+1)$-to-1 at a countable dense subset of* $[0, 1]$.

(4) *f is* 1*-to-*1 *on a residual subset of* $[0, 1]$.

The proof consists of defining a specific nested sequence of ordered Lebesgue partitions of $[0, 1]^k$ and applying Theorem 2. We'll use the term *brick* to mean an oriented k-dimensional interval in $[0, 1]^k$. All partitions we use will be comprised of bricks and if $\mathscr{P}$ is any partition of a brick, B, then $H(\mathscr{P})$ will denote the real numbers determining the hyperplanes bounding the bricks in $\mathscr{P}$ with the exception of the hyperplanes bounding B itself.

In anticipation of applying Theorem 2, we will begin the proof of Theorem 3 by defining a proper sequence of nested partitions of $[0, 1]^k$. Points of $[0, 1]^k$ that lie on the boundary of no brick from any of those partitions will turn out to be 1-to-1 points of the space-filling function assured by Theorem 2, while points lying on a common boundary of exactly two bricks will be either 2-to-1 points or 1-to-1 points and so on. The construction will entail that all partitions are Lebesgue partitions of $[0, 1]^k$ so that no point is common to more than $k+1$ bricks, and so no point will be more than a $(k+1)$-to-1 point of the derived space-filling function. It then will follow from the Hurewicz 1933 theorem that there are $(k+1)$-to-1 points in every portion and that will complete the proof. First, however, we introduce a bit more notation and prove a preliminary theorem.

An ordered partition, $\mathscr{P} = \{C_i : i = 1, 2, \ldots, M\}$ of $[0, 1]^k$ is said to be *linearized* if $\mathscr{P}$ satisfies the adjacency condition, (1) and for each $1 \le i \le M$ an initial vertex, $v_0(C_i)$ and a terminal vertex, $v_1(C_i)$ of C_i has been designated such that $v_1(C_i) = v_0(C_{i+1})$ for $1 \le i \le M-1$ with $v_0(C_0) = \vec{0}$ and $v_1(C_M) = \vec{1}$.

Theorem 4. *For every $k \in \mathbb{N}$ there are infinitely many ordered and linearized Lebesgue partitions of* $[0, 1]^k$ *such that no number is a hyperplane number for two of them.*

Proof. Set $A = (.3, .4)$ and $B = (.6, .7)$; these sets will be used to define all hyperplanes other than the bounding hyperplanes of the unit cubes, $[0, 1]^k$. To begin, let $a \in A$ and $b \in B$ be fixed and define

$$\mathscr{P}^1(a, b) = \{[0, a], [a, b], [b, 1]\} = \{C_i^1 : i = 1, 2, 3\},$$

and denote $v_0(C_i^1)$ and $v_0(C_i^1)$ as the left and right endpoints of C_i^1 respectively for $i = 1, 2, 3$. Our first infinite set of partitions is any infinite collection $\mathscr{T}^1$ for which if $\mathscr{P}^1(a_1, b_1), \mathscr{P}^1(a_2, b_2) \in \mathscr{T}^1$ then $a_1 = a_2$ and $b_1 = b_2$.

Now suppose $k \in \mathbb{N}$ and a countable collection, $\mathscr{T}^k$ of ordered linearized partitions of $[0, 1]^k$ is given such that each partition in $\mathscr{T}^k$ contains 3^k k-dimensional bricks. Suppose too that other than the bounding hyperplanes for $[0, 1]^k$ no hyperplane bounds bricks from two different partitions. Let $\mathscr{P}^i = \{C^i_j : j = 1, 2, \dots, 3^k\}$, where $i = 1, 2, 3$, be any three such partitions. We use these three partitions of $[0, 1]^k$ to define a single partition of $[0, 1]^{k+1}$ as follows.

(1) First, select $a \in A \setminus \bigcup_{i=1}^3 H(\mathscr{P}^i)$ and $b \in B \setminus \bigcup_{i=1}^3 H(\mathscr{P}^i)$. Then for $j = 1, 2, \dots, 3^k$ define

$$C^*_j = C^1_j \times [0, a],$$
$$v_0(C^*_j) = \begin{cases} (v_0(C^1_j), 0) & \text{if } j \text{ is odd,} \\ (v_0(C^1_j), a) & \text{if } j \text{ is even,} \end{cases}$$
$$v_1(C^*_j) = \begin{cases} (v_1(C^1_j), a) & \text{if } j \text{ is odd,} \\ (v_1(C^1_j), 0) & \text{if } j \text{ is even.} \end{cases}$$

(2) For $j = 3^k + 1, 3^k + 1, \dots, 2 \cdot 3^k$, define

$$C^*_j = C^2_{j-2\cdot 3^k+1} \times [a, b],$$
$$v_0(C^*_j) = \begin{cases} (v_1(C^2_{2\cdot 3^k-j+1}), a) & \text{if } j \text{ is even,} \\ (v_1(C^2_{2\cdot 3^k-j+1}), b) & \text{if } j \text{ is odd,} \end{cases}$$
$$v_1(C^*_j) = \begin{cases} (v_0(C^2_{2\cdot 3^k-j+1}), b) & \text{if } j \text{ is even,} \\ (v_0(C^2_{2\cdot 3^k-j+1}), a) & \text{if } j \text{ is odd.} \end{cases}$$

(3) For $j = 2 \cdot 3^k + 1, 2 \cdot 3^k + 2, \dots, 3^{k+1}$, define

$$C^*_j = C^3_{j-2\cdot 3^k} \times [b, 1],$$
$$v_0(C^*_j) = \begin{cases} (v_0(C^3_{j-2\cdot 3^k}), b) & \text{if } j \text{ is odd,} \\ (v_0(C^3_{j-2\cdot 3^k}), 1) & \text{if } j \text{ is even,} \end{cases}$$
$$v_0(C^*_j) = \begin{cases} (v_1(C^3_{j-2\cdot 3^k}), 1) & \text{if } j \text{ is odd,} \\ (v_1(C^3_{j-2\cdot 3^k}), b) & \text{if } j \text{ is even.} \end{cases}$$

That is, three ordered linearized layers that have no hyperplane boundaries in common other than the boundary hyperplanes of $[0, 1]^k$. Except for the last brick, and the transition bricks between layers, it is easy to see that each C^*_j shares a face with C^*_{j+1}; for example, $C^*_1 \cap C^*_2 = (C^1_1 \cap C^1_2) \times [0, a]$. As for the transition bricks,

$$C^*_{3^k} \cap C^*_{3^k+1} = (C^1_{3^k} \cap C^2_{3^k}) \times \{a\} \quad \text{and} \quad C^*_{2\cdot 3^k} \cap C^*_{2\cdot 3^k+1} = (C^2_1 \cap C^3_1) \times \{b\}.$$

Similarly, $v_1(C^*_i) = v_0(C^*_{i+1})$ for $j = 1, 2, \dots, 3^{k+1}$.

This, then completes the induction showing that for every triple of the infinite set of given partitions of $[0, 1]^k$ there is a linearized and ordered partition of $[0, 1]^{k+1}$ satisfying the given properties, particularly the unique hyperplane property. It is now an easy matter to extract an infinite such set of partitions of $[0, 1]^{k+1}$ so that no two partitions have a common hyperplane number. This completes the proof. □

To define a full proper sequence of ordered partitions, we insert copies of partitions defined in the previous theorem into predefined bricks in the following sense. Let $\mathscr{P}$ be any partition of $[0, 1]^n$ and $B \subset [0, 1]^n$ be any brick for which an initial vertex, $v_0(B)$ and diagonally opposite terminal vertex, $v_1(B)$

have been designated. Then there is a unique affine transformation $t : [0, 1]^k \to B$ such that $t(\vec{0}) = v_0(B)$, and $t(\vec{1}) = v_1(B)$ and as a result, $t([0, 1]^k) = B$. Define $\mathscr{P}(B) = \{t(C) : C \in \mathscr{P}\}$. If $\mathscr{P}$ is ordered or linearized then we infer that $\mathscr{P}(B)$ inherits that order or linearization in the obvious way.

Proof of Theorem 3. Fix $k \in \mathbb{N}$ and let $\mathscr{T}$ denote the infinite set of ordered, linearized Lebesgue partitions of $[0, 1]^k$ guaranteed by Theorem 4. We use $\mathscr{T}$ to define a proper sequence of linked partitions of $[0, 1]^k$ as follows.

Begin with any $\mathfrak{P}_1 = \{C^1_j : j = 1, 2, \ldots, 3^k\} \in \mathscr{T}$. Now assume that for each $s \le m$, $\mathfrak{P}_s = \{C^s_j : j = 1, 2, \ldots 3^{sk}\}$ is an ordered, linearized, Lebesgue partition of $[0, 1]^k$ and that $\{\mathfrak{P}_s : s \le m\}$ is a proper finite sequence of partitions. In particular we assume that the diameter of any brick comprising $\mathfrak{P}_s$ is smaller than $\frac{1}{2^s}$ for every $1 \le s \le m$.

We define $\mathfrak{P}_{m+1}$ inductively on the ordering of the partition. We first partition C^m_1 using a partition from $\mathscr{T}$ as a pattern, resulting in an ordered linearized partition, $\{B_1, B_2, \ldots, B_{3^k}\}$ of C^m_1. This ensures that $v_0(C^m_1) = v_0(B_1)$ and $v_1(C^m_1) = v_1(B_{3k})$ and, consequently, that there will be a smooth transition to a similar partitioning of C^m_2 and so on. Moreover, since we're using partitions from $\mathscr{T}$ as a pattern, the diameters of the newly minted bricks, the B_i, will be less than half the diameter of C^m_1. The more delicate part is to ensure that the newly formed partition is again a Lebesgue partition of $[0, 1]^k$ and for that we use the properties of $\mathscr{T}$ afforded by Theorem 4. A more formal description follows.

As $\mathscr{T}$ is infinite, there is a partition $\mathscr{P}_1 \in \mathscr{T}$ so that $H(C^m_1(\mathscr{P}_1))$ has no number in common with $H(\mathfrak{P}_s)$ for $1 \le s \le m$. Now, if $\mathscr{P}_j \in \mathscr{T}$ has been identified for $1 \le j < J$, let $\mathscr{P}_J \in \mathscr{T}$ be such that

$$H(C^m_J(\mathscr{P}_J)) \cap \Big(H(\mathfrak{P}_s) \cup \bigcup_{j=1}^{J-1} H(C^m_j(\mathscr{P}_j))\Big).$$

This is possible since $\mathscr{T}$ is an infinite set of partitions no two of which share a hyperplane number, and $H(\mathfrak{P}_s) \cup \bigcup_{j=1}^{J-1} H(C^m_j(\mathscr{P}_j))$ is a finite set. This process terminates when each of the bricks from $\mathfrak{P}_m$ have themselves been partitioned, or when $J = 3^{mk}$ so that $|\mathfrak{P}_J| = 3^{(m+1)k}$. Define

$$\mathfrak{P}_{m+1} = \bigcup_{j=1}^{3^{mk}} C^m_j(\mathscr{P}_j). \tag{2}$$

As the $C^m_j(\mathscr{P}_j)$ have disjoint hyperplane sets that are also disjoint from those of the previous partitions, $\mathfrak{P}_s$ for $s \le m$, it follows that $\mathfrak{P}_{m+1}$ is a Lebesgue partition of $[0, 1]^k$. Moreover, if $\mathfrak{P}_{m+1}$ is ordered lexicographically then the fact that for each $1 \le j \le 3^{mk}$, if B and B^* are the initial and terminal bricks of any $C^m_j(\mathscr{P}_j)$, then $v_0(B) = v_0(C^m_j)$ and $v_0(B^*) = v_0(C^m_j)$.

Finally, the diameter of each of the bricks from $\mathfrak{P}_{m+1}$ is less than half that or the brick of $\mathfrak{P}_m$ that contains it. From this, and the induction hypothesis it follows that $\|\mathfrak{P}_{m+1}\| < \frac{1}{2^{m+1}}$. This completes the inductive definition of the sequence $\{\mathfrak{P}_m : m = 1, 2, \ldots\}$ and we are in a position to apply Theorem 2.

For each $k \in \mathbb{N}$ and using the same regular $\frac{1}{3^k}$ scaffold for $[0, 1]$ as was used above, Theorem 2 yields a space-filling function $f : [0, 1] \to [0, 1]^k$ for which $[0, 1]$ is the core. Moreover, individual scaffold defined intervals are mapped onto the corresponding scaffold defined bricks in $[0, 1]^k$. The set of points in the range $[0, 1]^k$ that are on the boundary of no brick form a dense $\mathscr{G}_\delta$ set (residual) in the range as do their preimages in $[0, 1]$; these points are all 1-to-1 points of f; see [2]. Points of the range that lie on the boundary of exactly one brick lie on the boundary of $[0, 1]^k$ and are also 1-to-1 points of f. Points of the range that lie on the boundary of exactly two bricks are either 2-to-1 points or 1-to-1 points and in general, if $2 \leq m \leq k+1$ then points that lie on the boundary of exactly m bricks are all at most m-to-1 points, depending on the inherent arithmetic defined as in [2]. Hurewicz's 1933 theorem [6] guarantees there is at least one $(k+1)$-to-1 point in the range. However, the fact that when the function f is restricted to a scaffold interval, the restricted function is again space-filling. Hurewicz's 1933 theorem [6] guarantees a $(k+1)$-to-1 point in that scaffold interval so in all, such points are dense. This completes the proof. □

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