

---

# Fair Minimum Labeling: Efficient Temporal Network Activations for Reachability and Equity

---

**Lutz Oettershagen**

University of Liverpool

lutz.oettershagen@liverpool.ac.uk

**Othon Michail**

University of Liverpool

othon.michail@liverpool.ac.uk

## Abstract

Balancing resource efficiency and fairness is critical in networked systems that support modern learning applications. We introduce the *Fair Minimum Labeling* (FML) problem: the task of designing a minimum-cost temporal edge activation plan that ensures each group of nodes in a network has sufficient access to a designated target set, according to specified coverage requirements. FML captures key trade-offs in systems where edge activations incur resource costs and equitable access is essential, such as distributed data collection, update dissemination in edge-cloud systems, and fair service restoration in critical infrastructure. We show that FML is NP-hard and  $\Omega(\log |V|)$ -hard to approximate, where  $V$  is the set of nodes, and we present probabilistic approximation algorithms that match this bound, achieving the best possible guarantee for the activation cost. We demonstrate the practical utility of FML in a fair multi-source data aggregation task for training a shared model. Empirical results show that FML enforces group-level fairness with substantially lower activation cost than baseline heuristics, underscoring its potential for building resource-efficient, equitable temporal reachability in learning-integrated networks.

## 1 Introduction

Resource efficiency and fairness represent dual challenges in designing modern machine learning and networked systems. From large-scale federated learning managing communication budgets across heterogeneous devices [19], to recommender systems scheduling updates and notifications to diverse user groups [18], and distributed sensor networks operating under stringent energy constraints [17], the underlying theme is constant: interactions are dynamic, resources are limited, and equity across participants or data sources is paramount [8, 12, 26]. In such systems, the graph structure captures potential interactions, but the temporal dimension, i.e., when connections are active, governs the actual flow of information, updates, and influence.

Optimizing these temporal interactions solely for aggregate performance or minimal static infrastructure cost often yields suboptimal or unfair outcomes. Simple heuristics might overload communication bottlenecks, drain energy reserves rapidly, or systematically neglect harder-to-reach nodes or groups representing minority populations, specific event types, or distinct data distributions. This can lead to biased data collection pipelines, skewed model training, feedback loops that amplify inequity, and inefficient use of limited resources like energy or bandwidth.

Therefore, principled mechanisms are needed that explicitly co-optimize efficiency and fairness in these dynamic network settings. We consider systems where a central controller has the ability to decide when communication links are active. Our focus is on designing temporal activation patterns over an underlying static graph  $G = (V, E)$  that satisfy strong temporal reachability guarantees for distinct node groups, while minimizing the total number of activations. Nodes may represent users, sensors, or agents belonging to specific populations  $(V_1, \dots, V_k)$ , and fairness dictates that

each group must maintain adequate connectivity over time to designated target nodes. Minimizing the number of edge activations directly translates to reduced energy consumption, communication overhead, and computational load, which is central for sustainable system design [26, 38].

To address this, we introduce the **Fair Minimum Labeling (FML)** problem framework. Conceptually, FML seeks to determine *which* edges in the graph need to be activated and *when*, such that the required number of nodes from *each specified group* can reach the target nodes via paths formed by these time-ordered activations. The primary objective is to achieve this guaranteed fair reachability using the minimum total number of edge activations over time. FML thus provides a formal basis for optimizing resource sparsity while enforcing equitable temporal access.

The introduction of FML as a novel optimization problem is motivated by both practical relevance and theoretical gaps. First, it applies to a wide range of systems where temporal resource allocation directly affects group equity. In distributed data gathering for learning applications, such as those involving environmental sensors, mobile agents, or geographically diverse data streams [31, 43, 42], FML enables minimal-cost activation plans that ensure all groups are fairly represented in the collected data. In edge-cloud systems, it helps decide when communication links should be active so that all device groups can receive timely model updates under bandwidth constraints. Similarly, in infrastructure restoration, FML can guide activation sequences that ensure fair reactivation of services. Second, FML fills a critical gap in the literature. Existing fairness-aware graph methods, such as fair clustering [24], node embeddings [37], and influence maximization [40, 44], are primarily designed for static settings and do not model temporal interactions or activation cost. Fairness has been studied in wireless systems [17], and temporal connectivity has been explored in dynamic networks [22, 27], but fairness constraints remain absent in algorithmic temporal graph design. FML addresses this gap by explicitly minimizing temporal edge activations, treated as a proxy for resource cost, while enforcing group-level temporal reachability. In contrast to decentralized approaches such as classical federated learning [19], where participation is uncoordinated, FML assumes centralized control over link activations, allowing the enforcement of fairness guarantees over time. This design objective distinguishes FML from prior work focused on static structures or continuous flows.

**Our main contributions can be summarized as follows:**

- We introduce the *Fair Minimum Labeling (FML)* problem, a new framework for designing temporally sparse interaction patterns that guarantee group-wise temporal reachability under global resource constraints. We show that FML generalizes classical temporal connectivity problems and prove that it is NP-hard and  $\Omega(\log |V|)$ -hard to approximate.
- We develop a probabilistic approximation framework based on tree embeddings, yielding two algorithms: (i) an  $\mathcal{O}(\log |V|)$ -approximation for activation cost, and (ii) a highly efficient bicriteria approximation that guarantees  $\mathcal{O}(\log |V|)$  expected cost and a fairness violation factor of  $(1 + \varepsilon)^H$ , where  $H \in \mathcal{O}(\log |V|)$  denotes the height of the embedding tree. Our algorithms address the case of two distinct groups temporally reaching a single terminal.
- We empirically evaluate our algorithms on multi-source learning tasks. The results show that FML-based methods reliably enforce fairness with significantly lower activation cost than fairness-agnostic baselines, while remaining computationally efficient.

The omitted proofs are provided in Appendix A.

## 2 Preliminaries

We denote the set of positive integers (natural numbers without zero) by  $\mathbb{N} = \{1, 2, 3, \dots\}$ . For  $\ell \in \mathbb{N}$ , we use  $[\ell]$  to denote the set  $\{1, 2, \dots, \ell\}$ . The powerset of a set  $X$  is denoted by  $\mathcal{P}(X)$ .

A *static graph* is an ordered pair  $G = (V, E)$ , where  $V$  is a finite set of *nodes* (or vertices) and  $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$  is a finite set of undirected *edges*. We assume static graphs are simple (no self-loops, no multiple edges between the same pair of nodes). A *colored static graph*  $\bar{G} = (V, E, c)$  with colors  $\mathcal{C}$  has an additional function  $c : V \rightarrow \mathcal{P}(\mathcal{C})$  assigning a (possibly empty) set of colors  $c(u)$  to each node  $u \in V$  where  $\mathcal{C} = [c_{\max}]$  is a set of  $c_{\max}$  colors. We use  $V_c = \{v \in V \mid c(v) = c\}$ . Let  $n = |V|$  and  $m = |E|$ .

A *temporal graph* is an ordered pair  $\mathcal{G} = (V, \mathcal{E})$ , where  $V$  is a finite set of nodes and  $\mathcal{E}$  is a finite set of *temporal edges*. Each temporal edge is a pair  $e = (\{u, v\}, \tau)$ , where  $\{u, v\}$  is an undirected pair

of distinct nodes from  $V$ , and  $\tau \in \mathbb{N}$  is the *timestamp* indicating when the interaction between  $u$  and  $v$  occurs.

Given a static graph  $G = (V, E)$ , a *temporal labeling* is a function  $\lambda : E \rightarrow \mathcal{P}(\mathbb{N})$  that assigns a set of timestamps (possibly empty) to each static edge  $e \in E$ . A labeling  $\lambda$  induces a temporal graph  $G_\lambda = (V, E_\lambda)$ , where the node set is the same as the static graph's, and the set of temporal edges is defined as:  $E_\lambda = \{(\{u, v\}, \tau) \mid e = \{u, v\} \in E \text{ and } \tau \in \lambda(e)\}$ . The *size* (or *cost*) of a labeling  $\lambda$ , denoted by  $|\lambda|$ , is the total number of assigned timestamps, which corresponds to the number of temporal edges in the induced temporal graph, that is,  $|\lambda| = |E_\lambda| = \sum_{e \in E} |\lambda(e)|$ .

A *temporal path* (or *time-respecting path*) in a temporal graph  $\mathcal{G} = (V, \mathcal{E})$  from node  $u$  to node  $v$  is a sequence of temporal edges  $(\{v_0, v_1\}, \tau_1), (\{v_1, v_2\}, \tau_2), \dots, (\{v_{p-1}, v_p\}, \tau_p)$  from  $\mathcal{E}$  such that  $v_0 = u$ ,  $v_p = v$ , and the timestamps are strictly increasing, i.e.,  $\tau_1 < \tau_2 < \dots < \tau_p$ . A node  $v$  is *temporally reachable* from a node  $u$  in  $\mathcal{G}$  if such a temporal path exists from  $u$  to  $v$ .

We define the *reachability indicator function*  $r(u, v)$  with respect to a given temporal graph  $\mathcal{G} = (V, \mathcal{E})$ . For any two nodes  $u, v \in V$ :

$$r(u, v) = \begin{cases} 1 & \text{if } v \text{ is temporally reachable from } u \text{ in } \mathcal{G}, \\ 0 & \text{otherwise.} \end{cases}$$

By convention, we consider every node to be temporally reachable from itself (via a path of length zero), so  $r(v, v) = 1$  for all  $v \in V$ . The specific temporal graph  $\mathcal{G}$  (often an induced graph  $G_\lambda$ ) for which reachability is considered will always be clear from the context. Finally, for  $U, W \subseteq V$ , let  $r(U, W) = \sum_{u \in U, v \in W} r(u, v)$  be the sum of pairwise reachability of nodes in  $U$  to nodes in  $W$ .

### 3 Related Work

The Fair Minimum Labeling (FML) problem intersects several research areas, primarily temporal graph algorithms, network optimization with fairness considerations, and resource-efficient AI.

**Temporal Graph Connectivity and Labeling.** A core component of FML is finding a minimum-cost set of temporal edge activations. Mertzios et al. [27] (and its journal version [28]) introduced the study of temporal network design in multi-labeled temporal graphs, where the goal is to optimize some global cost measure of the graph labeling, such as the total number of timestamps, while satisfying specific temporal reachability properties. The *Minimum Labeling (ML)* problem is such a temporal network design problem, seeking the minimum number of labels to make an entire static graph temporally connected. For its directed version on strongly connected digraphs, [27] had provided bounds on the number of labels needed, proving that 2 labels per arc and at most  $4(|V| - 1)$  total labels are always sufficient. Using similar constructions on undirected graphs, Akrida et al. [2] proved that  $2|V| - 3$  total labels are sufficient, later refined by Klobas et al. [25] to  $2|V| - 4$  under certain conditions, who also showed ML is polynomially solvable. While ML aims for global temporal connectivity, FML focuses on targeted reachability for specific node groups to a designated terminal set, under explicit fairness (coverage) constraints for each group, and a cost minimization objective. Klobas et al. [25] also introduced extensions like *Minimum Steiner Labeling (MSL)*, where the goal is to pairwise temporally connect a predefined set of terminal nodes. Our FML generalizes ML and MSL. Other research has explored concepts like temporally-connected spanning subgraphs [3], temporal core decompositions [15, 33], and various notions of temporal paths and reachability [22, 7, 29, 16, 34, 35, 30]. However, these works generally do not incorporate group fairness constraints or the specific cost model of minimizing temporal edge activations as FML does.

**Fairness in Graph Mining and Network Optimization.** There is a significant body of work on incorporating fairness into static graph algorithms and network optimization, driven by the need for equitable outcomes in various applications. This includes, e.g., fair clustering [1, 10, 24], fair node embeddings [37, 6], fair influence maximization [40, 44], and graph learning tasks [9, 11, 32, 45, 36]. While these contributions are crucial for advancing fairness in graph-based AI, they predominantly operate on static graphs and often define fairness differently (e.g., proportionality in output, balanced representation in a static structure). For example, a related problem in static graphs is the *Balanced Connected Subgraph (BCS)* problem [5] which seeks a connected subgraph with balanced representation from different node colors. FML, in contrast, addresses fairness through *temporal path-based accessibility* for predefined groups, directly optimizing the dynamic activation of interactions over time to meet these fairness goals while minimizing resource usage. Moreover,

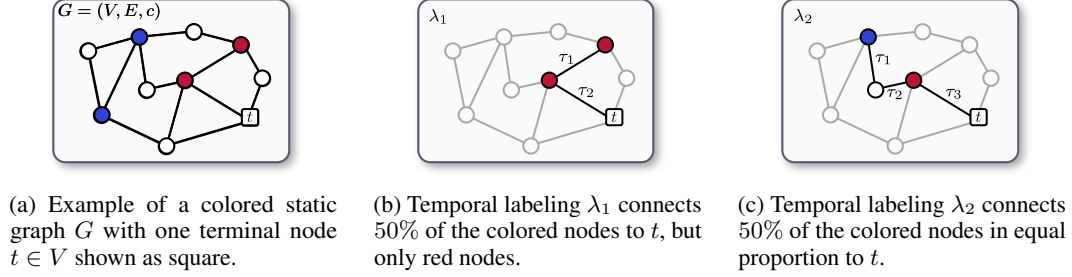


Figure 1: A simple toy example showing a non-equitable and an equitable labeling of a small colored graph. The labelings  $\lambda_1$  and  $\lambda_2$  assign time stamps  $\tau_1 < \tau_2 < \tau_3$  to edges. Both connect 50% of the colored nodes, but only  $\lambda_2$  connects both colors resulting in higher costs of three. Therefore, under a color requirement of 50% for each color, only  $\lambda_2$  is a valid solution for the FML problem.

in networked systems, fairness is a long-standing concern in resource allocation. This includes fair bandwidth sharing in communication networks [21], fair scheduling in wireless systems [17, 41], and equitable service provisioning. In edge-cloud environments, fair scheduling of updates (e.g., for recommender systems [18]) or computation offloading is critical. FML contributes to this line of work by providing a formal model for achieving group-level temporal reachability fairness specifically through the lens of minimizing discrete edge activations, a key proxy for communication cost, energy, or computational overhead, aligning with sustainability principles [26, 38]. Unlike other problems that focus on flow rates or latency targets, FML provides hard guarantees on temporal path existence for specified quotas from diverse groups.

**Connection to Combinatorial Optimization.** FML resembles Set Cover and Steiner Tree [20], with its goal of covering group nodes via minimal-cost temporal activations. Its  $\Omega(\log |V|)$  approximation hardness matches Set Cover [14], but FML is more complex due to temporal constraints, labeling-based edge induction, and fairness across groups. Standard algorithms fall short; our approximation algorithms use probabilistic tree embeddings to address FML’s structural challenges.

## 4 The fair minimum labeling problem

We now formalize the central optimization problem studied in this work: designing temporally sparse activation patterns over a static graph to guarantee group-level access to designated target nodes under strict resource constraints. In many learning-integrated systems, such as data aggregation, distributed monitoring, or update dissemination, ensuring both efficiency and equitable access over time is essential. Existing frameworks, however, lack mechanisms to enforce fairness through temporal reachability while minimizing total activation effort.

The *Fair Minimum Labeling (FML)* problem addresses this gap by integrating three key aspects: (i) cost-aware edge activation planning, (ii) per-group temporal reachability guarantees, and (iii) flexible, group-specific coverage requirements. The goal is to compute a temporal edge labeling that minimizes total activation cost while satisfying fairness constraints across all groups (see Appendix B for an application example).

**Definition 1** (Fair Minimum Labeling (FML)). *Given a colored static graph  $G = (V, E, c)$  with node colors  $\mathcal{C}$ , a set of terminals  $\mathcal{T} \subseteq V$ ,  $\rho \in \mathbb{N}$ , and a requirement function  $f_c : V \rightarrow \mathbb{R}$  for each color  $c \in \mathcal{C}$ , the goal is to find a temporal labeling  $\lambda : E \rightarrow \mathcal{P}(\mathbb{N})$  that minimizes the total number of assigned time labels  $|\lambda| = \sum_{e \in E} |\lambda(e)|$  such that for each color  $c \in \mathcal{C}$ , the number of nodes in  $V_c$  that can temporally reach at least  $\rho$  terminals  $t \in \mathcal{T}$  under  $\lambda$  is at least  $f_c(V_c)$ . Reachability is defined with respect to the temporal subgraph induced by  $\lambda$ .*

Figure 1 shows a simple example. In the corresponding decision version, we are additionally given a budget  $k \in \mathbb{N}$  and asked whether there exists a temporal labeling  $\lambda$  such that  $|\lambda| \leq k$  while satisfying the fairness constraints for all groups. We also note that a symmetric variant of the problem can be defined by requiring temporal reachability *from* (rather than *towards*) the terminal set  $\mathcal{T}$ .

The FML problem generalizes both the Minimum Labeling (ML) and the Minimum Steiner Labeling (MSL) problems (see Observation 1 in the appendix). Since MSL is NP-complete [25], it immediately

follows that FML is NP-complete. Moreover, we show that FML remains NP-hard and hard to approximate within a factor better than  $\Omega(\log n)$  even in the special case where the terminal set is a singleton, i.e.,  $\mathcal{T} = \{t\}$ .

**Theorem 1.** *The FML problem with a single terminal is NP-complete, and no  $((1 - \epsilon) \log n)$ -approximation algorithm exists unless  $NP \subseteq DTIME[n^{\mathcal{O}(\log \log n)}]$ .*

## 5 Approximation

Given the hardness results in Theorem 1, we now present approximation algorithms that are tight in expectation with respect to cost and solve the FML problem in instances with a single terminal. We focus on the case where each node has colors  $B$  (blue),  $R$  (red), or no color, i.e.,  $\mathcal{C} = \{B, R, \emptyset\}$ . For convenience, we use the set notation  $B \subseteq V$  and  $R \subseteq V$  to directly refer to the nodes of each group of colored nodes. Furthermore, we define the group-specific requirement functions as  $f_B(B) = \alpha|B|$  and  $f_R(R) = \alpha|R|$  for a fixed parameter  $\alpha \in \mathbb{R}$  with  $0 \leq \alpha \leq 1$ , indicating that an equal fraction  $\alpha$  of both  $B$  and  $R$  nodes must reach the terminal. This two-group setting captures a common fairness use case and provides a natural foundation for future extensions to multiple groups or terminals.

Our approach leverages the framework of *probabilistic tree embeddings*, a powerful technique for approximating arbitrary metric spaces with simpler tree metrics. The concept of probabilistic tree embeddings was first introduced by Bartal [4], who showed that any finite metric space can be probabilistically approximated by a distribution over tree metrics, incurring only a logarithmic distortion in expectation. This result was later significantly improved by Fakcharoenphol, Rao, and Talwar (FRT) [13], who established that any  $n$ -point metric space can be embedded into a distribution over dominating tree metrics with an expected stretch of  $\mathcal{O}(\log n)$ , and that this bound is tight.

**Definition 2** (Probabilistic Tree Embedding). *Let  $(X, d)$  be a finite metric space. A probabilistic tree embedding is a randomized mapping from  $(X, d)$  into a distribution  $\mathcal{D}$  over tree metrics  $(X, d_T)$  such that for all  $x, y \in X$ :*

1. *For every  $(X, d_T) \in \mathcal{D}$ ,  $d(x, y) \leq d_T(x, y)$*
2. *The expected stretch is  $\mathbb{E}_{d_T \sim \mathcal{D}}[d_T(x, y)] \leq \mathcal{O}(\log n) \cdot d(x, y)$  for all  $x, y \in X$ .*

---

### Algorithm 1: Approximation Framework for FML with Single Terminal

---

**Input:** Graph  $G = (V, E)$ ,  $B \subseteq V$ ,  $R \subseteq V$ , terminal  $t \in V$ ,  $\alpha \in [0, 1]$

**Output:** Temporal graph labeling  $\lambda_G$

- 1 Compute shortest path metric  $d_G$  on  $G$ .
  - 2 Generate a probabilistic tree embedding for  $(V, d_G)$  using FRT [13], sample a single weighted tree  $T = (V, E_T, w)$  from the distribution  $\mathcal{D}$ . Let  $w(u, v) = d_T(u, v)$ .
  - 3 Root the tree  $T$  at the terminal  $t$ .
  - 4 Compute an FML solution  $\lambda_T$  for the weighted tree  $T$  (rooted at  $t$ ) with groups  $R, B$  and parameter  $\alpha$ , minimizing the weighted cost  $C_T = \sum_{(\{u, v\}, \tau) \in \lambda_T} w(u, v)$ , using either the exact (Section 5.1) or approximate (Section 5.2) algorithm.
  - 5 Project the tree solution  $\lambda_T$  back to a solution  $\lambda_G$  in the original graph  $G$  (Section 5.3).
  - 6 **return**  $\lambda_G$
- 

Algorithm 1 outlines our general framework for approximating the Fair Minimum Labeling (FML) problem using probabilistic tree embeddings. We begin with the input graph  $G = (V, E)$  and the FML parameters  $(R, B, t, \alpha)$ . We define the standard shortest path metric  $d_G$  on  $G$ , where  $d_G(u, v)$  is the minimum number of edges on a path between  $u$  and  $v$ . We then employ the probabilistic tree embedding algorithm of Fakcharoenphol, Rao, and Talwar [13]. In practice, we first sample a single weighted tree  $T = (V, E_T, w)$  from the distribution  $\mathcal{D}$  and then solve an adapted version of the FML problem on the sampled weighted tree  $T$ . We use the original node set  $V$  with colors  $B$  and  $R$ , terminal  $t$ , and the parameter  $\alpha$ . The objective, however, is modified to align with the structure provided by the embedding and the goal of minimizing the final cost in  $G$ . We aim to find a temporal labeling  $\lambda_T$  for the tree  $T$  that minimizes the total weighted cost

$$C_T = \sum_{(\{u, v\}, \tau) \in \lambda_T} w(u, v) = \sum_{(\{u, v\}, \tau) \in \lambda_T} d_T(u, v)$$

subject to the coverage constraints being satisfied with respect to temporal reachability within the tree  $T$ , requiring that an  $\alpha$ -fraction of the nodes in each of  $B$  and  $R$  can temporally reach the terminal  $t$ .

To achieve this, we use dynamic programming algorithms presented in the following sections:

- An exact algorithm (Section 5.1) finds the minimum weighted cost  $C_{T,\text{opt}}$  satisfying the constraints exactly on  $T$ .
- An approximation algorithm (Section 5.2) finds a labeling  $\lambda_T$  with weighted cost  $C_T \leq C_{T,\text{opt}}$  that satisfies the constraints approximately, introducing an error factor  $\xi \leq (1+\varepsilon)^{H+1}$  on the counts of blue and red nodes, where  $H$  is the height of  $T$  and  $\varepsilon > 0$ .

Finally, in Section 5.3, we discuss how the solution obtained in the tree can be projected back to the graph  $G$  and  $\lambda_G$  can be obtained.

### 5.1 An exact algorithm for FML in weighted trees

We compute labels at the nodes of the tree  $T$  in a bottom-up fashion starting from the leafs. The edges of the tree have weights  $w(u, v) \geq 1$  corresponding the shortest paths distance between  $u, v \in V$  in  $G$ . Without loss of generality the maximum distance (in terms of edge weights) from any leaf to the root is at most  $k$ . Furthermore, all leafs are colored blue or red. We use labels of the form  $(b, r, c)$  where  $b$  and  $r$  are the numbers of blue and red nodes in the subtree, and  $c$  is the sum of edge weights used to connect the  $b + r$  nodes to the node having the label. For example, a blue node gets a label  $(1, 0, 0)$  and a red node  $(0, 1, 0)$ . Let  $v$  be an inner node with  $\ell$  children  $u_1, \dots, u_\ell$ . We construct the following labels based on the labels of the children. Let  $\{L_1, \dots, L_\ell\}$  be the sets of labels at  $u_1, \dots, u_\ell$ . We only choose at most one label per child that determines the (possibly empty) subtree to include. Therefore, we construct at most  $\prod_{i=1}^{\ell} (1 + |L_i|)$  labels at node  $v$ . At each node we keep at most one label  $(b, r, c)$  for each possible pair  $(b, r)$  with the minimum value of  $c$ . Because  $r$  and  $b$  are at most  $n$  there are  $\mathcal{O}(n^2)$  possibilities of  $(b, r)$  which is also an upper bound of the number of (non-dominated) labels at each node. Algorithm 2 in the appendix shows the pseudocode.

**Theorem 2.** *The FML problem on a tree with a single terminal can be solved in  $\mathcal{O}(n^5)$ .*

### 5.2 A bicriterial approximation

The running time of  $\mathcal{O}(n^5)$  of the exact algorithm for solving the FML problem in trees limits its scalability. The bottleneck is the linear number of labels per node, and the pairwise merging costing  $\mathcal{O}(n^4)$  per node. So the idea is to reduce the number of labels per node in a principled way. A classic method to reduce dynamic programming overhead is label sparsification by grid rounding. The key is to group labels with similar  $(b, r)$  values into a common bucket and keep only the best, i.e., lowest cost  $c$ , label in that bucket.

Specifically our approach partitions the integer range  $[0, n]$  into *geometric intervals*, or buckets, defined as follows: Bucket  $i$  corresponds to the interval  $((1 + \varepsilon)^{i-1}, (1 + \varepsilon)^i]$  for  $i \geq 1$  until the upper bound  $n$  is reached. This results in  $\mathcal{O}(\log_{1+\varepsilon}(|B|) \cdot \log_{1+\varepsilon}(|R|)) = \mathcal{O}(\frac{1}{\varepsilon^2} \log^2 n)$  buckets for possible  $(b, r)$ -bucket pairs. Once the label set  $L_v$  is constructed at a node  $v$ , each label  $(b, r, c)$  is mapped to a pair of bucket indices  $(\text{BucketIndex}(b), \text{BucketIndex}(r))$ , and only the label with the minimum cost  $c$  is retained per bucket. All other labels in the same bucket are discarded, thereby reducing the label set to at most  $\mathcal{O}(\frac{1}{\varepsilon^2} \log^2 n)$  entries.

**Theorem 3.** *Let  $\varepsilon > 0$ . Let  $T$  be the input tree with root  $t$  and height  $H$ . Let  $\xi = (1 + \varepsilon)^{H+1}$ . Let  $\ell_{\text{opt}} = (b_{\text{opt}}, r_{\text{opt}}, c_{\text{opt}})$  be an optimal exact label at the root  $t$  with minimum weighted cost  $c_{\text{opt}}$  that satisfies  $b_{\text{opt}} \geq \alpha|B|$  and  $r_{\text{opt}} \geq \alpha|R|$ . Then the geometric bucketing algorithm finds a label  $\ell' = (b', r', c')$  in the final set  $L'_t$  at the root with*

- (i)  $c' \leq c_{\text{opt}}$ ,
- (ii)  $b' \geq b_{\text{opt}}/\xi \geq (\alpha|B|)/\xi$  and  $r' \geq r_{\text{opt}}/\xi \geq (\alpha|R|)/\xi$ .

### 5.3 Projecting the solution to the graph $G$

Let  $\lambda_T$  be the temporal labeling obtained by one of these algorithms on the tree  $T$ . The tree labeling  $\lambda_T$  must be translated into a valid temporal labeling  $\lambda_G$  for the original graph  $G$ . This projection

ensures that the reachability achieved in the tree is preserved in the graph. For each temporal tree edge activation  $(\{u, v\}, \tau) \in \lambda_T$  we first identify a shortest path  $P_{uv}$  between  $u$  and  $v$  in the original graph  $G$ . Let this path consist of the sequence of  $p = d_G(u, v)$  edges  $e_1, e_2, \dots, e_p$ . To mimic the activation of the tree edge  $\{u, v\}$  at time  $\tau$ , we activate the path  $P_{uv}$  in  $G$  using a sequence of timestamps. The final graph labeling  $\lambda_G$  is the union of all temporal edges generated by projecting all  $(\{u, v\}, \tau) \in \lambda_T$ .

The projection mechanism ensures that if node  $w$  is temporally reachable from node  $u$  in the tree  $T$  using the labeling  $\lambda_T$ , then  $w$  is also temporally reachable from  $u$  in the graph  $G$  using the labeling  $\lambda_G$ . This holds because each temporal tree edge in a path is replaced by a corresponding temporal path segment in  $G$ , and the timestamping ensures correct sequencing and avoids conflicts. Therefore, if  $\lambda_T$  satisfies the coverage constraints of Definition 1 (either exactly or approximately with factor  $\xi$ ), the projected solution  $\lambda_G$  will also satisfy these constraints either exactly or approximately with the same factor  $\xi$  in the original graph  $G$ . Note that we potentially include additional colored nodes that will be able to reach the terminal  $t$ , however, this can only improve the approximation.

**Theorem 4.** *Algorithm 1, combined with the appropriate weighted tree algorithm, yields a randomized algorithm for FML with one terminal on general graphs with two colors.*

- *If the exact weighted tree algorithm (Section 5.1) is used, the framework produces a solution  $\lambda_G$  that satisfies the fairness coverage constraints exactly, and its expected cost  $\mathbb{E}[|\lambda_G|]$  is within an  $\mathcal{O}(\log n)$  factor of the optimal cost  $k_G^*$ . The expected running time is in  $\mathcal{O}(n^5)$ .*
- *If the bicriteria weighted tree algorithm (Section 5.2) is used, the framework yields a solution  $\lambda_G$  with expected cost  $\mathbb{E}[|\lambda_G|] \leq \mathcal{O}(\log n) k_G^*$  that approximately satisfies the coverage and fairness constraints, with violation bounded by the factor  $\xi = (1 + \varepsilon)^{H+1}$ , where  $H \in \mathcal{O}(\log n)$ . This results in a randomized bicriteria  $(\mathcal{O}(\log n), \xi)$ -approximation.*

*The expected running time is in  $\mathcal{O}(n^2 + n\varepsilon^{-4} \log^4 n)$ .*

*Both variants use  $\mathcal{O}(n)$  space for the tree embedding and  $\mathcal{O}(n)$  (exact) or  $\mathcal{O}(n\varepsilon^{-2} \log^2 n)$  (bicriteria) additional space for the DP tables.*

## 5.4 Generalizations

The algorithms and the specific complexity analysis presented in the previous sections are for the two-group (plus no-color) case. However, the framework is generalizable. For  $k$  distinct groups, the dynamic programming label becomes  $(g_1, g_2, \dots, g_k, c)$ , where  $g_i$  is the count of covered nodes from group  $i$ . The state space for the counts becomes  $\mathcal{O}(n^k)$ . The pairwise merge operation takes  $\mathcal{O}((n^k)^2) = \mathcal{O}(n^{2k})$  time. With  $n$  nodes, the total runtime for the exact DP is  $\mathcal{O}(n^{2k+1})$ .

Our framework can also be extended to support non-uniform activation costs  $w(e)$ , while preserving the  $\mathcal{O}(\log n)$  approximation guarantee, by defining the metric for the FRT embedding using these costs. First, we compute a shortest-path metric  $d(u, v)$  where the distance between nodes is defined as the minimum cost of a path between them, using the activation costs  $w(e)$  as the edge weights. We then use FRT to embed this weighted metric into a tree  $T$ . Our DP algorithm then runs on the tree  $T$ , and because the tree distances being optimized are already low-distortion approximations of the minimum activation costs, the  $\mathcal{O}(\log n)$  guarantee on the final, total weighted cost holds directly.

Finally, in dynamic environments (e.g., node arrivals, failures or departures), partial recomputation of our labeling is feasible. The DP can be rerun locally on affected subtrees, and minor changes in the graph structure often preserve the validity of the tree embedding. While full dynamic adaptation is beyond the scope of our current work, the methods lend themselves to such extensions.

## 6 Experiments

In this section, we evaluate the performance of our algorithms. We compare the following methods:

- FMLAPPROX, our approximation algorithm based on Algorithm 1 that uses the exact tree dynamic programming subroutine described in Section 5.1.
- FMLBIAPPROX, our bicriteria approximation algorithm based on Algorithm 1 that uses the approximate tree dynamic programming subroutine described in Section 5.2.

We designed the following greedy baselines for comparison as there are no direct competitors to our approach, where we start with two natural starting points, (i) what happens if you ignore fairness entirely and focus only on cost? (ii) what are the simplest, most direct ways to enforce fairness? Leading to the following baselines:

- **GREEDY** is a fairness-agnostic baseline that activates shortest paths to the closest colored nodes until  $\alpha \cdot (|B| + |R|)$  total colored nodes are covered, irrespective of group balance.
- **CLOSEST** greedily activates unweighted shortest paths from the terminal to the closest uncovered colored node (regardless of group) until  $\alpha$ -coverage is achieved for both groups.
- **ALTERNATING** proceeds similar to Closest, but alternates between activating the closest uncovered node from group  $B$  and group  $R$ .

We implemented all algorithms in Python 3.9 and PyTorch 2.5.1. All experiments ran on a computer cluster. Each experiment ran exclusively on a node with an Intel(R) Xeon(R) Gold 6130 CPU @ 2.10GHz, 96 GB of RAM. We used a time limit of 12 hours. Our source code is available at <https://gitlab.com/tgdesign/fml>.

## 6.1 Case study: Fair and efficient multi-source data collection

We evaluate our FML framework in a multi-source learning context, where data must be collected from distributed sources under strict resource constraints. Such sources, e.g., sensors, mobile agents, or geographically distributed streams, often differ by group and face limitations like bandwidth or energy. Ensuring fair representation of each group during training is essential to prevent biased models. However, naive approaches, such as connecting all sources, minimizing total cost, or favoring proximity, either waste resources or neglect important groups. Our method computes a single temporal edge labeling that guarantees sufficient group-wise temporal reachability to a central node while minimizing the number of activations. This enables efficient and fair data collection without the need for repeated adaptation or full connectivity, outperforming the baselines.

### 6.1.1 Network topology and data sources

We simulate a networked learning scenario over a random geometric graph  $G = (V, E)$  with  $|V| = 1024$  nodes and connection radius  $r = 0.2$ . The designated terminal node  $t \in V$  collects data from a subset of data sources  $B \cup R \subseteq V \setminus \{t\}$ . Each of the data sources belongs to one of two demographic groups  $B$  or  $R$ , assigned based on spatial proximity to the terminal (near vs. far).

Each data source generates synthetic data  $(\mathbf{x}, y)$  with  $\mathbf{x} \in \mathbb{R}^2$ ,  $y \in \{0, 1\}$  by sampling  $\mathbf{x}$  from a mixture of two Gaussians and setting  $y$  deterministically:

- Group  $B$ :  $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2 I)$  with  $\sigma = 0.55$ ,  $\mu \in \{(-2, 1), (2, 1)\}$ , and  $y = \mathbf{1}\{x_1 > 0\}$ .
- Group  $R$ :  $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2 I)$  with  $\sigma = 0.55$ ,  $\mu \in \{(1, -2), (1, 2)\}$ , and  $y = \mathbf{1}\{x_2 < 0\}$ .

Thus, the two groups encode orthogonal classification rules, making it impossible to achieve high accuracy on both without training on both.

The algorithms activate edges  $A \subseteq E$  to connect data sources to the terminal. For the baselines, the activation cost is defined as the sum of hop-distances from all covered data sources to the terminal. The collected data is used to train a shared MLP classifier. We use a fully connected MLP using an input layer of size  $d = 2$ , followed by two hidden layers with 32 neurons each and ReLU activations, and a final output layer with a single neuron and sigmoid activation for binary classification. We train the model using the Adam optimizer [23] with learning rate 0.01, batch size 64, and weight decay  $10^{-2}$  for  $\ell_2$  regularization. The loss is binary cross-entropy. Training runs for 100 epochs with early stopping based on validation loss. The model is trained using the combined dataset collected from data sources temporally connected to the terminal. Accuracy is reported separately on held-out test sets for each group. We also report the coverage of nodes in  $B$  and  $R$  that can temporally reach  $t$ . We report the mean values and standard deviations over ten independent runs.

The results in Table 1 demonstrate the trade-offs between fairness, cost, and accuracy across different methods. The straightforward GREEDY, while achieving the low activation cost (88.3), completely neglects group  $R$  (Coverage $_R = 0$ ), leading to severe disparity in model accuracy between groups with



Table 1: Multi-source learning results (mean and standard deviations over ten independent runs).

Algorithm	Time (s)	Cost	Coverage <sub>B</sub>	Coverage <sub>R</sub>	Acc <sub>B</sub>	Acc <sub>R</sub>
GREEDY	0.49 ± 0.08	88.3 ± 6.25	64.0 ± 0.00	0.0 ± 0.00	1.00 ± 0.00	0.27 ± 0.17
CLOSEST	0.73 ± 0.13	224.0 ± 0.00	44.3 ± 2.31	32.0 ± 0.00	0.97 ± 0.04	0.94 ± 0.08
ALTERNATING	0.71 ± 0.09	212.8 ± 1.81	34.2 ± 1.87	32.0 ± 0.00	0.96 ± 0.04	0.94 ± 0.07
FMLAPPROX	69.20 ± 18.11	74.3 ± 4.62	33.5 ± 0.53	32.3 ± 0.48	0.98 ± 0.03	0.93 ± 0.08
FMLBIAPPROX	8.54 ± 6.36	73.9 ± 3.48	33.1 ± 0.99	31.3 ± 0.48	0.97 ± 0.03	0.94 ± 0.08

$\text{Acc}_B = 1.00$  and  $\text{Acc}_R = 0.27$ . This highlights the risk of fairness-agnostic optimization. CLOSEST and ALTERNATING heuristics ensure coverage for both groups, but incur significantly higher costs with 224.0 and 212.8, respectively. In contrast, both our FML-based algorithms, FMLAPPROX and FMLBIAPPROX, achieve near-perfect group-level fairness ( $\text{Coverage}_B \approx 33$ ,  $\text{Coverage}_R \approx 32$ ) with substantially lower cost of around 74, and maintain high accuracy for both groups with  $\text{Acc}_B \geq 0.97$  and  $\text{Acc}_R \geq 0.93$ . FMLBIAPPROX performs comparably to the exact method in terms of fairness and accuracy, while being significantly faster (8.54s vs. 69.20s on average). These results confirm the effectiveness of FML in achieving resource-efficient and equitable temporal reachability.

We also ran additional experiments where we explicitly placed the terminal at either a central (high-degree) or peripheral (low-degree) node. Our bicriteria algorithm (FMLBIAPPROX) proved highly effective in both scenarios, consistently achieving over 98% of the required coverage.

## 6.2 Running time and approximation quality

We additionally compare the performance of our approximations FMLBIAPPROX and FMLAPPROX. To this end, we generated Barabási–Albert graphs with  $n \in \{256, 512, 1024, 2048, 4096\}$  and fixed attachment parameter  $\bar{m} = 3$  (i.e., each new node connects to  $\bar{m}$  existing nodes). Moreover, we chose  $\varepsilon \in \{0.1, 0.01, 0.001\}$ . For each parameter combination, we used ten independent runs. Figure 2a reports the average running time and standard deviation across all in the time limit completed runs. For the largest graphs, FMLAPPROX failed to complete five instances for  $n = 2048$  and seven for  $n = 4096$ , while FMLBIAPPROX timed out on three and five runs, respectively. Among the runs that finished, FMLBIAPPROX consistently outperformed FMLAPPROX, achieving up to an order-of-magnitude speedup. Importantly, decreasing  $\varepsilon$  led to only modest increases in running time, yet produced significant gains in solution quality. Figure 2b shows the mean ratio of  $B$  ( $R$ , resp.) colored nodes selected by FMLBIAPPROX compared to FMLAPPROX. Smaller values of  $\varepsilon$ , particularly  $\varepsilon = 0.001$ , yield ratios close to 1, indicating near-perfect agreement with the exact baseline. These results demonstrate the effectiveness of FMLBIAPPROX in achieving high-quality solutions with substantially reduced running time. Finally, the majority of the running time, more than 99.9% across all settings, is spent on solving the dynamic program on the FRT tree. The time spent on tree embedding is negligible in comparison.

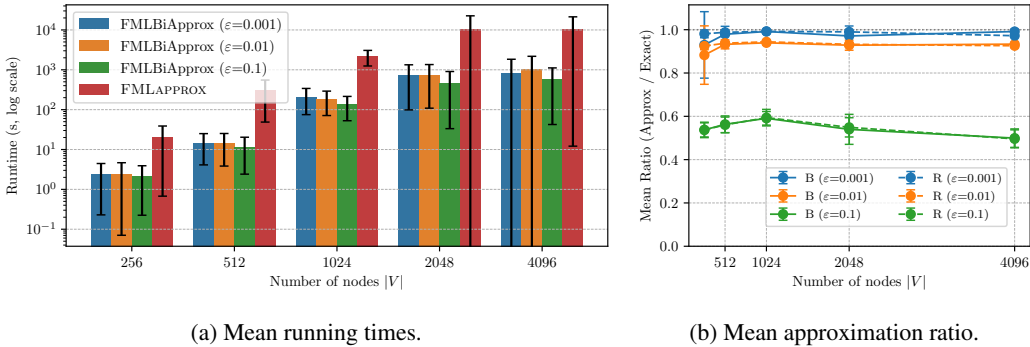


Figure 2: Comparison of FMLBIAPPROX and FMLAPPROX. The results show the mean and standard deviation as error bars over ten independent runs.

Table 2: Results on the real-world network.

Epsilon	Runtime (s)	Accuracy (%)
0.01	3625.0 $\pm$ 420.7	93.1
0.001	4661.5 $\pm$ 991.3	99.3

### 6.3 Practical scalability

We sampled a connected 20,000-node induced subgraph (201,900 edges) from the Pokec dataset<sup>1</sup> [39]. We randomly selected one node as the terminal and colored 10% of the remaining nodes (balanced 50/50) using the binary gender attribute (Blue = male, Red = female). We then ran FMLBIAPPROX with full coverage requirement ( $\alpha = 1.0$ ) and two precision settings ( $\varepsilon = 0.01$  and  $\varepsilon = 0.001$ ) in 10 independent runs. Table 2 shows the results. Accuracy represents the ratio of achieved coverage to the required coverage. This real-world experiment confirms the practical scalability and effectiveness of our FMLBIAPPROX algorithm, which achieves high accuracy under both precision settings.

We investigated the bicriteria guarantee’s dependency on tree height  $H$  on our new Pokec dataset. Over 10 runs, the average tree height was exceptionally small,  $H \approx 6.68$ . Our theoretical analysis provides a worst-case coverage guarantee of  $(1 + \varepsilon)^{-(H+1)}$ . For  $\varepsilon = 0.01$ , this guarantees at least  $(1.01)^{-7.68} \approx 92.6\%$  coverage. Our empirically measured accuracy was 93.1%. This demonstrates that our algorithm robustly meets its formal theoretical guarantee in a real-world setting.

## 7 Limitations

**Conceptual limitations.** While the FML framework captures a fundamental trade-off between temporal resource efficiency and fairness, it assumes centralized control over the activation of communication links. This limits its applicability in decentralized or asynchronous settings such as classical federated learning [19]. Moreover, the current formulation requires predefined group membership and fixed coverage thresholds, which may be restrictive in adaptive or data-driven fairness scenarios. Finally, FML treats fairness as group-level temporal reachability, but does not model finer-grained objectives such as individual fairness or fairness over time.

**Technical limitations.** Our approximation algorithms currently focus on the case of a single terminal and two groups. While this setting already models common use cases, generalizing to arbitrary numbers of terminals and group types remains open. The exact dynamic programming algorithm has a running time complexity in  $\mathcal{O}(n^5)$ , limiting scalability. The bicriteria approximation improves efficiency, reducing runtime to  $\mathcal{O}(n^2 + n\varepsilon^{-4} \log^4 n)$ , but trades off exact fairness for scalability: its guarantees degrade with the tree height, leading to a  $(1 + \varepsilon)^{\log n}$  factor in group coverage.

## 8 Conclusion and future work

We introduced the *Fair Minimum Labeling* (FML) problem, which formalizes the design of temporal activation schedules that ensure fair group-wise reachability under resource constraints. FML bridges a gap between fairness-aware graph optimization and temporal connectivity models. We proved that FML is NP-hard and  $\Omega(\log n)$ -hard to approximate, even under restricted settings. To address this challenge, we developed a randomized approximation framework based on probabilistic tree embeddings, achieving tight  $\mathcal{O}(\log n)$  expected-cost bounds. A bicriteria variant further scales to large networks while guaranteeing bounded fairness violations. Experiments on multi-source learning tasks confirmed that FML-based methods achieve near-optimal fairness and accuracy at substantially lower cost than competing baselines. Future work includes exploiting structural properties such as low treewidth or planarity, extending the model to weighted and online settings, and designing distributed variants for decentralized or privacy-sensitive deployments. Moreover, extending FML to multiple terminals is a promising yet non-trivial direction. This would require the dynamic program to maintain terminal-reachability sets or richer state representations, thereby substantially increasing both the state space and the complexity of the merge step.

<sup>1</sup><https://snap.stanford.edu/data/soc-Pokec.html>

## Acknowledgments

This work was undertaken on Barkla, part of the High Performance Computing facilities at the University of Liverpool, UK.

## References

- [1] Sara Ahmadian, Alessandro Epasto, Ravi Kumar, and Mohammad Mahdian. Clustering without over-representation. In *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 267–275, 2019.
- [2] Eleni C. Akrida, Leszek Gasieniec, George B. Mertzios, and Paul G. Spirakis. The complexity of optimal design of temporally connected graphs. *Theory of Computing Systems*, 61:907–944, 2017.
- [3] Kyriakos Axiotis and Dimitris Fotakis. On the size and the approximability of minimum temporally connected subgraphs. In *Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming*, pages 149:1–149:14, 2016.
- [4] Yair Bartal. Probabilistic approximation of metric spaces and its algorithmic applications. In *Proceedings of the 37th Conference on Foundations of Computer Science*, pages 184–193, 1996.
- [5] Sujoy Bhore, Sourav Chakraborty, Satyabrata Jana, Joseph S. B. Mitchell, Supantha Pandit, and Sasanka Roy. The balanced connected subgraph problem. *Discrete Applied Mathematics*, 319:111–120, 2022.
- [6] Avishek Bose and William Hamilton. Compositional fairness constraints for graph embeddings. In *Proceedings of the 36th International Conference on Machine Learning*, pages 715–724, 2019.
- [7] Arnaud Casteigts, Paola Flocchini, Walter Quattrociocchi, and Nicola Santoro. Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, 27(5):387–408, 2012.
- [8] Simon Caton and Christian Haas. Fairness in machine learning: A survey. *ACM Computing Surveys*, 56(7), 2024.
- [9] April Chen, Ryan A. Rossi, Namyoung Park, Puja Trivedi, Yu Wang, Tong Yu, Sungchul Kim, Franck Dernoncourt, and Nesreen K Ahmed. Fairness-aware graph neural networks: A survey. *ACM Transactions on Knowledge Discovery from Data*, 18(6):1–23, 2024.
- [10] Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, and Sergei Vassilvitskii. Fair clustering through fairlets. In *Advances in Neural Information Processing Systems*, volume 30, 2017.
- [11] Enyan Dai, Tianxiang Zhao, Huaisheng Zhu, Junjie Xu, Zhimeng Guo, Hui Liu, Jiliang Tang, and Suhang Wang. A comprehensive survey on trustworthy graph neural networks: Privacy, robustness, fairness, and explainability. *Machine Intelligence Research*, 21(6):1011–1061, 2024.
- [12] Yushun Dong, Jing Ma, Song Wang, Chen Chen, and Jundong Li. Fairness in graph mining: A survey. *IEEE Transactions on Knowledge and Data Engineering*, 35(10):10583–10602, 2023.
- [13] Jittat Fakcharoenphol, Satish Rao, and Kunal Talwar. A tight bound on approximating arbitrary metrics by tree metrics. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing*, pages 448–455, 2003.
- [14] Uriel Feige. A threshold of  $\ln n$  for approximating set cover. *Journal of the ACM (JACM)*, 45(4):634–652, 1998.
- [15] Edoardo Galimberti, Martino Ciaperoni, Alain Barrat, Francesco Bonchi, Ciro Cattuto, and Francesco Gullo. Span-core decomposition for temporal networks: Algorithms and applications. *ACM Transactions on Knowledge Discovery from Data (TKDD)*, 15(1):1–44, 2020.
- [16] Aristides Gionis, Lutz Oettershagen, and Ilie Sarpe. Mining temporal networks. In *Companion Proceedings of the ACM on Web Conference 2024*, pages 1260–1263, 2024.

- [17] SHI Huaizhou, R. Venkatesha Prasad, Ertan Onur, and I.G.M.M. Niemegeers. Fairness in wireless networks: Issues, measures and challenges. *IEEE Communications Surveys & Tutorials*, 16(1):5–24, 2013.
- [18] Zeinab S. Jalali, Qilan Chen, Shwetha M. Srikanta, Weixiang Wang, Myunghwan Kim, Hema Raghavan, and Sucheta Soundarajan. Fairness of information flow in social networks. *ACM Transactions on Knowledge Discovery from Data*, 17(6):1–26, 2023.
- [19] Peter Kairouz, H. Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and trends® in machine learning*, 14(1–2):1–210, 2021.
- [20] Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of Complexity of Computer Computations*, pages 85–103, 1972.
- [21] Frank Kelly. Charging and rate control for elastic traffic. *European transactions on Telecommunications*, 8(1):33–37, 1997.
- [22] David Kempe, Jon Kleinberg, and Amit Kumar. Connectivity and inference problems for temporal networks. In *Proceedings of the 32nd Annual ACM Symposium on Theory of Computing*, pages 504–513, 2000.
- [23] Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.
- [24] Matthäus Kleindessner, Samira Samadi, Pranjal Awasthi, and Jamie Morgenstern. Guarantees for spectral clustering with fairness constraints. In *Proceedings of the 36th International Conference on Machine Learning*, pages 3458–3467, 2019.
- [25] Nina Klobas, George B. Mertzios, Hendrik Molter, and Paul G. Spirakis. The complexity of computing optimum labelings for temporal connectivity. *Journal of Computer and System Sciences*, 146:103564, 2024.
- [26] Yuyi Mao, Xianghao Yu, Kaibin Huang, Ying-Jun Angela Zhang, and Jun Zhang. Green edge AI: A contemporary survey. *Proceedings of the IEEE*, 112(7):880–911, 2024.
- [27] George B. Mertzios, Othon Michail, Ioannis Chatzigiannakis, and Paul G. Spirakis. Temporal network optimization subject to connectivity constraints. In *Proceedings of the 40th International Colloquium on Automata, Languages, and Programming*, pages 657–668, 2013.
- [28] George B. Mertzios, Othon Michail, and Paul G. Spirakis. Temporal network optimization subject to connectivity constraints. *Algorithmica*, 81(4):1416–1449, 2019.
- [29] Othon Michail. An introduction to temporal graphs: An algorithmic perspective. *Internet Mathematics*, 12(4):239–280, 2016.
- [30] Petra Mutzel and Lutz Oettershagen. On the enumeration of bicriteria temporal paths. In *International Conference on Theory and Applications of Models of Computation*, pages 518–535. Springer, 2019.
- [31] Omar Nassef, Wenting Sun, Hakimeh Purmehdi, Mallik Tatipamula, and Toktam Mahmoodi. A survey: Distributed machine learning for 5G and beyond. *Computer Networks*, 207:108820, 2022.
- [32] Madeline Navarro, Samuel Rey, Andrei Buciulea, Antonio G. Marques, and Santiago Segarra. Fair glasso: Estimating fair graphical models with unbiased statistical behavior. In *Advances in Neural Information Processing Systems*, volume 37, pages 139589–139620, 2024.
- [33] Lutz Oettershagen, Athanasios L. Konstantinidis, and Giuseppe F. Italiano. An edge-based decomposition framework for temporal networks. In *Proceedings of the Eighteenth ACM International Conference on Web Search and Data Mining*, pages 735–743, 2025.

- [34] Lutz Oettershagen, Nils M. Kriege, and Petra Mutzel. A higher-order temporal h-index for evolving networks. In *Proceedings of the 29th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, pages 1770–1782, 2023.
- [35] Lutz Oettershagen, Petra Mutzel, and Nils M. Kriege. Temporal walk centrality: ranking nodes in evolving networks. In *Proceedings of the ACM Web conference 2022*, pages 1640–1650, 2022.
- [36] Lutz Oettershagen, Honglian Wang, and Aristides Gionis. Finding densest subgraphs with edge-color constraints. In *Proceedings of the ACM Web Conference 2024*, pages 936–947, 2024.
- [37] Tahleen Rahman, Bartłomiej Surma, Michael Backes, and Yang Zhang. Fairwalk: Towards fair graph embedding. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence*, page 3289–3295, 2019.
- [38] Roy Schwartz, Jesse Dodge, Noah A. Smith, and Oren Etzioni. Green AI. *Communications of the ACM*, 63(12):54–63, 2020.
- [39] Lubos Takac and Michal Zabovsky. Data analysis in public social networks. In *International scientific conference and international workshop present day trends of innovations*, volume 1, 2012.
- [40] Alan Tsang, Bryan Wilder, Eric Rice, Milind Tambe, and Yair Zick. Group-fairness in influence maximization. *arXiv preprint arXiv:1903.00967*, 2019.
- [41] Nitin H. Vaidya, Paramvir Bahl, and Seema Gupta. Distributed fair scheduling in a wireless LAN. In *Proceedings of the 6th Annual International Conference on Mobile Computing and Networking*, pages 167–178, 2000.
- [42] Jana Vatter, Ruben Mayer, and Hans-Arno Jacobsen. The evolution of distributed systems for graph neural networks and their origin in graph processing and deep learning: A survey. *ACM Computing Surveys*, 56(1):1–37, 2023.
- [43] Joost Verbraeken, Matthijs Wolting, Jonathan Katzy, Jeroen Kloppenburg, Tim Verbelen, and Jan S. Rellermeyer. A survey on distributed machine learning. *ACM Computing Surveys*, 53(2):1–33, 2020.
- [44] Dennis Robert Windham, Caroline J. Wendt, Alex Crane, Madelyn J. Warr, Freda Shi, Sorelle A. Friedler, Blair D. Sullivan, and Aaron Clauset. Fast algorithms to improve fair information access in networks. *arXiv preprint arXiv:2409.03127*, 2024.
- [45] Zhuoping Zhou, Davoud Ataee Tarzanagh, Bojian Hou, Qi Long, and Li Shen. Fairness-aware estimation of graphical models. In *Advances in Neural Information Processing Systems*, volume 37, pages 17870–17909, 2024.

## NeurIPS Paper Checklist

### 1. Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [\[Yes\]](#)

Justification: The abstract clearly states the contributions made in the paper.

Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals are not attained by the paper.

### 2. Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [\[Yes\]](#)

Justification: We added a section discussing conceptual and technical limitations of the work (see Section 7).

Guidelines:

- The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
- The authors are encouraged to create a separate "Limitations" section in their paper.
- The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
- The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
- The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
- The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
- If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
- While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

### 3. Theory assumptions and proofs

Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

Answer: [\[Yes\]](#)

Justification: All theoretical results have proofs in Appendix A.

Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and cross-referenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

#### 4. Experimental result reproducibility

Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

Answer: [\[Yes\]](#)

Justification: We clearly describe the algorithms used in the experiments (see Section 6).

Guidelines:

- The answer NA means that the paper does not include experiments.
- If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
- If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
- Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general, releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
- While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
  - (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
  - (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
  - (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).
  - (d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

#### 5. Open access to data and code

Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: [Yes]

Justification: Data, source code, and scripts are available in an anonymous public repository. The link is available in Section 6.

Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so “No” is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (<https://nips.cc/public/guides/CodeSubmissionPolicy>) for more details.
- The authors should provide instructions on data access and preparation, including how to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

## 6. Experimental setting/details

Question: Does the paper specify all the training and test details (e.g., data splits, hyper-parameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: [Yes]

Justification: All details are provided in the experiments section (Section 6).

Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

## 7. Experiment statistical significance

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: [Yes]

Justification: We use mean and standard deviation in experiments and clearly describe this fact.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).
- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).



- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

#### 8. Experiments compute resources

Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [Yes]

Justification: We clearly describe the experimental setup and the used computer resources at the beginning of Section 6.

Guidelines:

- The answer NA means that the paper does not include experiments.
- The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
- The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
- The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

#### 9. Code of ethics

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics <https://neurips.cc/public/EthicsGuidelines>?

Answer: [Yes]

Justification: We have read the NeurIPS Code of Ethics and have followed it.

Guidelines:

- The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
- If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

#### 10. Broader impacts

Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

Answer: [Yes]

Justification: We discuss the broader impacts on society in the introduction (Section 1) and in the limitations section (Section 7).

Guidelines:

- The answer NA means that there is no societal impact of the work performed.
- If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.
- Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.

- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

## 11. Safeguards

Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

Answer: [NA]

Justification: Our work does not poses such risk and safeguards are not necessary.

Guidelines:

- The answer NA means that the paper poses no such risks.
- Released models that have a high risk for misuse or dual-use should be released with necessary safeguards to allow for controlled use of the model, for example by requiring that users adhere to usage guidelines or restrictions to access the model or implementing safety filters.
- Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
- We recognize that providing effective safeguards is challenging, and many papers do not require this, but we encourage authors to take this into account and make a best faith effort.

## 12. Licenses for existing assets

Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

Answer: [NA]

Justification: Our work does not use existing assets.

Guidelines:

- The answer NA means that the paper does not use existing assets.
- The authors should cite the original paper that produced the code package or dataset.
- The authors should state which version of the asset is used and, if possible, include a URL.
- The name of the license (e.g., CC-BY 4.0) should be included for each asset.
- For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.
- If assets are released, the license, copyright information, and terms of use in the package should be provided. For popular datasets, [paperswithcode.com/datasets](https://paperswithcode.com/datasets) has curated licenses for some datasets. Their licensing guide can help determine the license of a dataset.
- For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.

- If this information is not available online, the authors are encouraged to reach out to the asset’s creators.

### 13. **New assets**

Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

Answer: [NA]

Justification: Our paper does not release new assets.

Guidelines:

- The answer NA means that the paper does not release new assets.
- Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
- The paper should discuss whether and how consent was obtained from people whose asset is used.
- At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

### 14. **Crowdsourcing and research with human subjects**

Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

Answer: [NA]

Justification: Our work does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

### 15. **Institutional review board (IRB) approvals or equivalent for research with human subjects**

Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

Answer: [NA]

Justification: Our paper does not involve crowdsourcing nor research with human subjects.

Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.
- We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
- For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.

### 16. **Declaration of LLM usage**

Question: Does the paper describe the usage of LLMs if it is an important, original, or non-standard component of the core methods in this research? Note that if the LLM is used only for writing, editing, or formatting purposes and does not impact the core methodology, scientific rigorousness, or originality of the research, declaration is not required.

Answer: [NA]

Justification: The core method development in this research does not involve LLMs as any important, original, or non-standard components.

Guidelines:

- The answer NA means that the core method development in this research does not involve LLMs as any important, original, or non-standard components.
- Please refer to our LLM policy (<https://neurips.cc/Conferences/2025/LLM>) for what should or should not be described.

## Appendix

---

### Algorithm 2: Bottom-up Label Computation on Tree $T$

---

**Input:** Rooted tree  $T = (V_T, E_T)$  with edge weights  $w(u, v)$ , coloring  $c : V_T \rightarrow \{B, R, \emptyset\}$ .  
**Output:** At each node  $v \in V_T$ , a set of non-dominated labels  $(b, r, c)$ .

```

1 foreach node  $v \in V_T$  in post-order do
2   Initialize  $\text{Labels}(v) \leftarrow \emptyset$ 
3   if  $c(v) = B$  then add  $(1, 0, 0)$ 
4   if  $c(v) = R$  then add  $(0, 1, 0)$ 
5    $\mathcal{C} \leftarrow \{(0, 0, 0)\}$ 
6   foreach child  $u$  of  $v$  do
7     // Merge labels  $\mathcal{C} \leftarrow \mathcal{C} \cup \text{Labels}(u)$  and keep track of edge weights
8      $\mathcal{C}' \leftarrow \emptyset$ 
9     foreach  $(b_1, r_1, c_1) \in \mathcal{C}$  do
10      Add  $(b_1, r_1, c_1)$  to  $\mathcal{C}'$ 
11      foreach  $(b_2, r_2, c_2) \in \text{Labels}(u)$  do
12        Add  $(b_1 + b_2, r_1 + r_2, c_1 + c_2 + w(v, u))$  to  $\mathcal{C}'$ 
13   Keep only the label with minimum  $c$  for each  $(b, r)$  in  $\text{Labels}(v)$ 

```

---

## A Omitted Proofs

**Observation 1.** *The FML problem generalizes both the Minimum Labeling (ML) and the Minimum Steiner Labeling (MSL) problems.*

*Proof of Observation 1.* In ML, the goal is to make the entire graph temporally connected. We model this by setting the terminal set  $\mathcal{T} = V$ , and using a single color  $c_1$  for all nodes, i.e.,  $\mathcal{C} = \{c_1\}$  with  $c(v) = c_1$  for all  $v \in V$ . Additionally, we set  $\rho = |\mathcal{T}|$ , and we define the requirement function  $f_{c_1}(V) = |V|^2$ , which requires full pairwise temporal reachability among all nodes. Then, any labeling that satisfies the FML condition  $r(V, V) \geq |V|^2$  must realize all-pairs temporal connectivity, which is exactly the ML objective. In MSL, the goal is to ensure temporal connectivity among a given subset of terminals  $\mathcal{T} \subseteq V$ . We use one color  $c_1$  and assign it only to the nodes in  $\mathcal{T}$  while all other nodes are not assigned any color, i.e.,  $\mathcal{C} = \{c_1, \emptyset\}$  and  $V_{c_1} = \mathcal{T}$ . Again, we set  $\rho = |\mathcal{T}|$ . We then define the requirement function  $f_{c_1}(\mathcal{T}) = |\mathcal{T}|^2$  to require full temporal connectivity within the terminal set. The resulting FML instance enforces exactly the MSL condition, and minimizing the cost corresponds to MSL.  $\square$

*Proof of Theorem 1.* Regarding the membership in NP, given an instance of FML with graph  $G = (V, E, c)$ , color set  $\mathcal{C}$ , requirement functions  $\{f_c\}_{c \in \mathcal{C}}$ , terminal set  $\mathcal{T} \subseteq V$ , budget  $k \in \mathbb{N}$ , and a candidate temporal labeling  $\lambda : E \rightarrow \mathcal{P}(\mathbb{N})$ , we first check whether  $|\lambda| \leq k$ . This is a simple summation over the labeled edge sets and takes polynomial time. Next, for each color class  $c \in \mathcal{C}$ , we compute the total number of temporally reachable terminal nodes from nodes in  $V_c$ . For each  $u \in V_c$  and each  $v \in \mathcal{T}$ , we check if there exists a temporal path from  $u$  to  $v$  in the temporal graph induced by  $\lambda$ . This can be done using a modified BFS per source node, each with time complexity polynomial in  $|V| + |\lambda|$ . As we have at most  $|V| \cdot |\mathcal{T}|$  such checks across all color classes, the total verification time is polynomial. For each  $c \in \mathcal{C}$ , we verify that  $r(V_c, \mathcal{T}) \geq f_c(V_c)$ . If this holds for all colors, the labeling is valid.

For hardness, we reduce from the SET COVER problem, which is NP-hard and admits no  $((1 - \epsilon) \log n)$ -approximation unless  $\text{NP} \subseteq \text{DTIME}[n^{O(\log \log n)}]$ .

Let  $(U, \mathcal{S}, k_{SC})$  be a Set Cover instance, where  $U = \{u_1, \dots, u_m\}$  is the universe and  $\mathcal{S} = \{S_1, \dots, S_n\}$  is a collection of subsets  $S_i \subseteq U$ . We construct an FML instance with one terminal and one color class as follows. The graph  $G = (V, E, c)$  includes a terminal node  $r$ , one node  $s_i$  for each set  $S_i \in \mathcal{S}$ , one node  $t_j$  for each element  $u_j \in U$ , and a chain of  $L$  intermediate nodes  $w_{i,1}, \dots, w_{i,L}$  for each  $s_i$ , which simulate a labeling cost. Edges are added between each  $t_j$  and

every  $s_i$  such that  $u_j \in S_i$ , and each  $s_i$  is connected to  $r$  through its respective chain. Thus, the vertex set is  $V = \{r\} \cup \{t_1, \dots, t_m\} \cup \{s_1, \dots, s_n\} \cup \{w_{i,\ell} \mid 1 \leq i \leq n, 1 \leq \ell \leq L\}$ , and the edge set includes  $\{t_j, s_i\}$  if  $u_j \in S_i$  and the path edges  $\{s_i, w_{i,1}\}, \{w_{i,1}, w_{i,2}\}, \dots, \{w_{i,L}, r\}$ .

We define a single color class  $c_1$  and assign it to all  $t_j$  nodes, i.e.,  $V_{c_1} = \{t_1, \dots, t_m\}$ , and set the terminal set as  $\mathcal{T} = \{r\}$  and  $\rho = 1$ . The requirement function for color  $c_1$  is defined as  $f_{c_1}(V_{c_1}) = m$ , requiring that all  $t_j$  nodes reach  $r$  via temporal paths. The budget is set to  $k = m + L \cdot k_{SC}$ .

Suppose there exists a cover  $\mathcal{S}' = \{S_{i_1}, \dots, S_{i_\ell}\}$  with  $\ell \leq k_{SC}$  such that  $\bigcup_{S \in \mathcal{S}'} S = U$ . Then we can construct a temporal labeling as follows. For each chosen set  $S_{i_p} \in \mathcal{S}'$ , we assign labels with timestamps  $2, 3, \dots, L+2$  along its path from  $s_{i_p}$  to  $r$ . For each element  $u_j$ , we select some  $S_{i_p} \in \mathcal{S}'$  containing  $u_j$  and assign timestamp 1 to the edge  $\{t_j, s_{i_p}\}$ . Every  $t_j$  now has a valid temporal path to  $r$  via  $s_{i_p}$ , and the total number of labels used is  $m + L \cdot \ell \leq m + L \cdot k_{SC}$ , so the fairness requirement is satisfied.

Conversely, suppose there is a valid labeling  $\lambda$  of cost at most  $k = m + L \cdot k_{SC}$  such that every  $t_j$  is temporally reachable from some  $s_i$  via a path to  $r$ . Each of the  $m$  element nodes must use at least one label on an edge to a set node  $s_i$ , accounting for  $m$  labels. For any  $s_i$  to enable reachability to  $r$ , the entire chain of  $L$  edges must be labeled with consecutive timestamps. Therefore, the number of such activated chains is at most  $k_{SC}$ . The corresponding sets  $\mathcal{S}'$  form a set cover of  $U$  of size at most  $k_{SC}$ .

To show the inapproximability, we use a gap-preserving reduction from Set Cover. We choose  $L = \lceil m/\delta \rceil$ , where  $\delta > 0$  is the gap parameter. In the YES case, the optimal FML labeling cost is at most  $m + L \cdot k_{SC}$ , and in the NO case, the cost must exceed  $m + L \cdot (1 - \delta) \ln m \cdot k_{SC}$ . Since  $L$  is significantly larger than  $m$ , the additive term  $m$  becomes negligible, and the approximation gap approaches  $(1 - \delta) \log m$ . Because the total number of nodes in the FML instance is  $n = \Theta(n_{SC} \cdot L)$  and  $L = \Theta(m)$ , we have  $n = \Theta(n_{SC} \cdot m)$ , and hence  $\log n = \Theta(\log m)$ . Therefore, the logarithmic hardness gap in  $m$  for SET COVER translates directly to a  $\Theta(\log n)$  gap for the FML instance.  $\square$

*Proof of Theorem 2.* We can represent not including subtrees using an additional label  $(0, 0, 0)$  at each node. Suppose a node  $v$  has children  $u_1, u_2, \dots, u_\ell$  with label sets  $L_1, L_2, \dots, L_\ell$ . It holds  $|L_i| \leq C$  with  $C = \mathcal{O}(n^2)$ . We can combine  $L_1$  and  $L_2$  into a new set  $L_{1,2}$  of size at most  $C$  keeping only minimal-cost labels for each pair  $(b, r)$ . Then combine  $L_{1,2}$  with  $L_3$  to get  $L_{1,2,3}$ , and so forth. Each pairwise merge of two sets of size at most  $C$  can be done in  $\mathcal{O}(C^2)$  time by considering all pairs of labels. We sum up  $(b, r)$ , add the costs plus weights of the edges connecting children, and then discard dominated labels. Because we immediately discard duplicates and dominated labels, the intermediate result remains of size at most  $C$ . Since, each merge corresponds to one child, or equivalently to one edge, and there are  $n - 1$  edges, the total costs of merging is  $\mathcal{O}(n^5)$  for all nodes. Therefore, the total running time is in  $\mathcal{O}(n^5)$ .  $\square$

Before we prove Theorem 3, we show the following result.

**Lemma 1.** *Let  $\varepsilon > 0$ . Let  $T$  be a tree with root  $t$ , height  $H$ , and edge weights  $w(u, v) \geq 1$ . For any node  $v$  in  $T$  with height  $h_v$  in its subtree,  $h_v \leq H$  and any exact label  $\ell = (b, r, c)$  at  $v$ , there exists a label  $\ell' = (b', r', c')$  kept in the approximate set  $L'_v$  at node  $v$  with*

- (i)  $c' \leq c$ ,
- (ii)  $b' \geq b/(1 + \varepsilon)^{h_v+1}$  and  $r' \geq r/(1 + \varepsilon)^{h_v+1}$ .

*Proof.* The proof proceeds by induction on the height  $h_v$  of node  $v$ . For the base case consider a leaf  $v$  with  $h_v = 0$ . Let  $b_{\text{color}}(v)$  and  $r_{\text{color}}(v)$  be indicator functions returning one if  $v$  is blue or red, resp., and zero otherwise. The only possible exact labels are  $\ell = (b_{\text{color}}(v), r_{\text{color}}(v), 0)$ , with weighted cost  $c = 0$ . The algorithm generates and keeps this exact label  $\ell' = \ell$ . It holds  $c' = 0 \leq c = 0$  and  $b' = b$ . We require  $b \geq b/(1 + \varepsilon)^{0+1} = b/(1 + \varepsilon)$ . This holds since  $b \in \{0, 1\}$  and  $1 + \varepsilon \geq 1$ . Similarly,  $r' \geq r/(1 + \varepsilon)$ . Therefore, the claim holds for leaves.

Now, for the inductive step consider an inner node  $v$  with height  $h_v > 0$ . Assume the claim holds for all children  $u_i$  of  $v$ . Let  $h_{u_i}$  be the height of child  $u_i$ , so  $h_{u_i} \leq h_v - 1$ . Consider an exact label  $\ell_v = (b_v, r_v, c_v)$  formed at node  $v$  by combining exact labels  $\ell_{u_i} = (b_{u_i}, r_{u_i}, c_{u_i})$  from a subset

$S \subseteq \{1, \dots, \ell\}$  of its children, plus the node  $v$ 's own color. The exact counts and weighted cost are

$$b_v = \sum_{i \in S} b_{u_i} + b_{\text{color}}(v) \quad \text{and} \quad r_v = \sum_{i \in S} r_{u_i} + r_{\text{color}}(v)$$

$$c_v = \sum_{i \in S} (c_{u_i} + w(v, u_i)).$$

By the inductive hypothesis applied to each child  $u_i$  (for  $i \in S$ ), there exists a label  $\ell'_{u_i} = (b'_{u_i}, r'_{u_i}, c'_{u_i})$  kept in  $L'_{u_i}$  such that:  $c'_{u_i} \leq c_{u_i}$ ,  $b'_{u_i} \geq b_{u_i}/(1+\varepsilon)^{h_{u_i}+1}$ , and  $r'_{u_i} \geq r_{u_i}/(1+\varepsilon)^{h_{u_i}+1}$ .

The approximation algorithm at node  $v$  generates a candidate label  $\ell''_v = (b''_v, r''_v, c''_v)$  by combining these kept labels  $\ell'_{u_i}$  from the children with

$$b''_v = \sum_{i \in S} b'_{u_i} + b_{\text{color}}(v) \quad \text{and} \quad r''_v = \sum_{i \in S} r'_{u_i} + r_{\text{color}}(v)$$

$$c''_v = \sum_{i \in S} (c'_{u_i} + w(v, u_i)).$$

Now, for the costs we have  $c''_v = \sum_{i \in S} (c'_{u_i} + w(v, u_i)) \leq \sum_{i \in S} (c_{u_i} + w(v, u_i)) = c_v$ . Thus,  $c''_v \leq c_v$ . Moreover, for  $b''_v$  it follows

$$b''_v = \sum_{i \in S} b'_{u_i} + b_{\text{color}}(v) \geq \sum_{i \in S} \frac{b_{u_i}}{(1+\varepsilon)^{h_{u_i}+1}} + b_{\text{color}}(v).$$

Since  $h_{u_i} \leq h_v - 1$ , we have  $(1+\varepsilon)^{h_{u_i}+1} \leq (1+\varepsilon)^{(h_v-1)+1} = (1+\varepsilon)^{h_v}$ .

$$b''_v \geq \sum_{i \in S} \frac{b_{u_i}}{(1+\varepsilon)^{h_v}} + \frac{b_{\text{color}}(v)}{(1+\varepsilon)^{h_v}} \quad (\text{since } (1+\varepsilon)^{h_v} \geq 1)$$

$$\geq \frac{1}{(1+\varepsilon)^{h_v}} \left( \sum_{i \in S} b_{u_i} + b_{\text{color}}(v) \right) = \frac{b_v}{(1+\varepsilon)^{h_v}}.$$

Similarly, we have  $r''_v \geq r_v/(1+\varepsilon)^{h_v}$ .

Now, this candidate label  $\ell''_v = (b''_v, r''_v, c''_v)$  falls into a specific  $(b, r)$ -bucket defined by the geometric grid. The algorithm selects one label  $\ell'_v = (b'_v, r'_v, c'_v)$  from all candidates falling into this bucket to keep in the final set  $L'_v$ , specifically the one with the minimum cost  $c'_v$ .

Since  $\ell''_v$  was a candidate in this bucket with cost  $c''_v$ , the chosen label  $\ell'_v$  must satisfy  $c'_v \leq c''_v$ . Combined with  $c''_v \leq c_v$ , we get  $c'_v \leq c_v$ , proving part (i) of the claim for node  $v$ .

Because  $\ell'_v$  and  $\ell''_v$  are in the same  $(b, r)$ -bucket defined by ratio  $(1+\varepsilon)$ , their counts are related by the bucket definition:  $b'_v \geq b''_v/(1+\varepsilon)$  and  $r'_v \geq r''_v/(1+\varepsilon)$ . (If  $b''$  or  $r''$  is 0, the inequality holds trivially).

Combining the bounds we prove part (ii) with

$$b'_v \geq \frac{b''_v}{1+\varepsilon} \geq \frac{b_v/(1+\varepsilon)^{h_v}}{1+\varepsilon} = \frac{b_v}{(1+\varepsilon)^{h_v+1}} \quad \text{and} \quad r'_v \geq \frac{r''_v}{1+\varepsilon} \geq \frac{r_v}{(1+\varepsilon)^{h_v+1}}.$$

□

We are now ready to prove Theorem 3.

*Proof of Theorem 3.* Let  $\ell_{\text{opt}} = (b_{\text{opt}}, r_{\text{opt}}, c_{\text{opt}})$  be an optimal exact label at the root  $t$  (node height  $H$ ) satisfying  $b_{\text{opt}} \geq \alpha|B|$  and  $r_{\text{opt}} \geq \alpha|R|$ . By Lemma 1 applied at the root node  $t$ , there exists a label  $\ell' = (b', r', c')$  kept in the final approximate set  $L'_t$  with  $c' \leq c_{\text{opt}}$  which immediately verifies part (i). Moreover, it holds with  $\xi = (1+\varepsilon)^{H+1}$  that

$$b' \geq b_{\text{opt}}/(1+\varepsilon)^{H+1} = b_{\text{opt}}/\xi \quad \text{and} \quad r' \geq r_{\text{opt}}/(1+\varepsilon)^{H+1} = r_{\text{opt}}/\xi$$

For part (ii), from  $b_{\text{opt}} \geq \alpha|B|$  and  $r_{\text{opt}} \geq \alpha|R|$  it follows  $b' \geq (\alpha|B|)/\xi$  and  $r' \geq (\alpha|R|)/\xi$ . □

*Proof of Theorem 4.* The cost of the final solution in the graph  $G$  is the total number of temporal edge activations, i.e.,

$$|\lambda_G| \leq \sum_{(\{u,v\},t) \in \lambda_T} |P_{uv}| = \sum_{(\{u,v\},t) \in \lambda_T} d_G(u,v).$$

We can relate this cost to the optimal cost  $k_G^*$  of the original FML problem in  $G$  using the properties of the embedding and the weighted tree cost  $C_T = \sum d_T(u,v)$  optimized by our tree algorithm.

By non-contraction ( $d_G(u,v) \leq d_T(u,v)$  for all  $u,v$ ), we have

$$|\lambda_G| \leq \sum_{(\{u,v\},t) \in \lambda_T} d_G(u,v) \leq \sum_{(\{u,v\},t) \in \lambda_T} d_T(u,v) = C_T.$$

Let  $\lambda_G^*$  be an optimal solution in  $G$  with cost  $k_G^* = |\lambda_G^*|$ . This solution induces some temporal structure on the tree  $T$ . Let  $C_T(\lambda_G^*)$  be the corresponding weighted cost on the tree required to support the reachability of  $\lambda_G^*$ .

The expected stretch property implies (summing over the paths corresponding to  $\lambda_G^*$ ):

$$\mathbb{E}_{T \sim \mathcal{D}}[C_T(\lambda_G^*)] \leq \mathcal{O}(\log n) \sum_{(\{u,v\},t) \in \lambda_G^*} d_G(u,v) = \mathcal{O}(\log n) k_G^*.$$

The optimal weighted cost on the tree,  $C_{T,\text{opt}}$ , must satisfy  $C_{T,\text{opt}} \leq C_T(\lambda_G^*)$ , as  $\lambda_G^*$  provides one way to achieve the required reachability on  $T$ .

Our tree algorithm finds a solution  $\lambda_T$  with weighted cost  $C_T$  such that  $C_T \leq C_{T,\text{opt}}$ , either exactly, or via the approximation. Therefore, it follows  $|\lambda_G| \leq C_T \leq C_{T,\text{opt}} \leq C_T(\lambda_G^*)$ .

Finally, by taking the expectations over the random choice of the tree  $T \sim \mathcal{D}$ , we have

$$\mathbb{E}[|\lambda_G|] \leq \mathbb{E}[C_T(\lambda_G^*)] \leq \mathcal{O}(\log n) k_G^*.$$

Let  $G = (V, E)$  with  $|V| = n$ ,  $|E| = m$ . The distances in  $G$  and the FRT embedding can be computed in  $\mathcal{O}(n^2)$ . Using the exact DP of Section 5.1, the overall expected running time is

$$T_{\text{exact}}(n, m) = \mathcal{O}(n^5)$$

due to the exact tree DP, see Theorem 2. The projection back to  $G$  needs  $\mathcal{O}(|\lambda_T| + m) \subseteq \mathcal{O}(n^2)$  time.

Using the bicriteria  $(\mathcal{O}(\log n), \xi)$ -approximation the expected running time is

$$T_{\text{approx}}(n, m, \varepsilon) = \mathcal{O}(n^2 + n \varepsilon^{-4} \log^4 n)$$

where the second term is due to the  $n \cdot \left(\frac{1}{\varepsilon^2} \log^2 n\right)^2$  pairwise merges of label buckets. The projection step again fits in the same bound.  $\square$

## B Application Example

We give a concrete high-impact application scenario: Consider a medical AI initiative deployed in a developing region to collect patient data for training a diagnostic model. The setting involves remote rural villages with limited connectivity and urban health hubs with better infrastructure. Devices operate under tight energy constraints and are duty-cycled, i.e., radios are only active during coordinated upload windows to conserve battery.

- **Nodes  $V$ :** Participants' smartphones, community health hubs, and the central server.
- **Groups  $V_B, V_R$ :** For example, urban (near, well-connected) vs. rural (remote) participants; we must connect an  $\alpha$ -fraction of each. Alternative groupings could capture other fairness dimensions, e.g., gender or ethnic communities (demographic fairness), users with high-end vs. low-end devices (resource fairness), or solar-powered vs. grid-powered nodes (energy fairness).



- **Terminal  $t$ :** The central server.
- **Edges  $E$ :** Potential short-range (Bluetooth/Wi-Fi Direct) links to hubs and long-range links from hubs to  $t$ .
- **Temporal labeling  $\lambda$ :** Each activated edge is assigned a few global upload rounds, i.e., the only times radios are awake.
- **Cost:** The total number of edge-time activations (each consumes energy when radios leave sleep mode).

**Why temporality matters.** Data must traverse time-respecting paths: if a rural phone  $u$  forwards to hub  $v$  in round  $\tau_1$ , then  $v$  must forward towards  $t$  in a later round  $\tau_2 > \tau_1$ . Simply picking a static path is insufficient; we need an ordered activation plan consistent with duty-cycling.

**Outcome.** FML computes a sparse, globally scheduled activation plan with a few rounds and few active links that guarantees at least  $\alpha|V_B|$  and  $\alpha|V_R|$  nodes can reach  $t$  temporally, while minimizing total wake-ups (edge-time labels). This aligns with real duty-cycled systems where communication is batch-scheduled to conserve energy and bandwidth.