Translation-invariant extrapolation in frequency using adaptive multipliers

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***Abstract*—Resolving the fine-scale details of a signal from coarse-scale measurements is a classical problem in signal processing. This problem is usually formulated in terms of *extrapolation in frequency*, i.e., as extrapolating the Fourier transform of the signal from a set of low-frequencies to a larger set.**

An approach to perform extrapolation in frequency is to use a multiplier, or a filter, that minimizes a suitable approximation error metric over a known collection of signals. However, one of the drawbacks of this approach is that this multiplier is not able to exploit the relations between the signals in the collection. In this work, we propose a formulation that is translation-invariant, finding both the optimal multipliers and the optimal centering for the signals in the collection. A consequence of our formulation is that the optimal centering does not correspond to a usual choice such as the center of mass. We perform numerical experiments supporting our claims.

***Index Terms*—Spectral extrapolation, adaptive filters, super-resolution, translation invariance.**

I. Introduction

Resolving the fine-scale details of a signal $u : \mathbb{R}^d \to \mathbb{C}$ from coarse-scale measurements is a classical problem in signal processing. Typically, the coarse-scale measurements correspond to the restriction $\widehat{u}|_{\Omega_0}$ of the Fourier transform $\widehat{u}$ of u to a set $\Omega_0 \in \mathbb{R}^d$ assumed to be compact with non-empty interior. While Ω_0 represents the *low-frequencies* the restriction $\widehat{u}|_{\Omega_0}$ represents the *low-frequency content* of u.

Resolving the fine-scale details of u implies extrapolating $\widehat{u}|_{\Omega_0}$ to a set containing Ω_0. For instance, given an *extrapolation factor* $\alpha > 1$ we can seek a map T for which

$$T(\widehat{u}|_{\Omega_0}) = \widehat{u}|_{\alpha\Omega_0} = (D_\alpha \widehat{u})|_{\Omega_0} \tag{1}$$

where D_α is the *dilation operator* $(D_\alpha \widehat{u})(\xi) = \widehat{u}(\alpha\xi)$. This is reminiscent of the *refinement equation* in multiresolution analysis [1]–[3]. Observe that when $\widehat{u}|_{\Omega_0}$ does not vanish, then there is at least one map, in fact, a *multiplier* or *transfer function*, that satisfies (1), namely,

$$T(\widehat{u}|_{\Omega_0}) = m_{\widehat{u},\alpha}\widehat{u}|_{\Omega_0} \quad \text{for} \quad m_{\widehat{u},\alpha} = \frac{(D_\alpha \widehat{u})|_{\Omega_0}}{\widehat{u}|_{\Omega_0}}.$$

We call $m_{\widehat{u},\alpha}$ an *exact multiplier*. Inspired by this observation, we look for a multiplier that performs well over a collection $u_1, \ldots, u_n$, i.e., we seek a multiplier m such that

$$v \in \operatorname{span}\{\widehat{u}_1, \ldots, \widehat{u}_n\} : \ (mv)|_{\Omega_0} \approx (D_\alpha v)|_{\Omega_0} \tag{2}$$

in a suitable sense. Such a multiplier must *adapt* to the elements in the collection to perform extrapolation in frequency. Furthermore, when the elements of the collection are related by *group actions*, e.g., translations and rotations, it is of interest to determine the extent to which such a multiplier can adapt to both the elements in the collection and to the group actions relating them.

A. Previous work and contributions

There has been an increasing interest in signal analysis and processing methods that can exploit structure in the data by being either invariant or equivariant with respect to specific group actions as this enables the identification or extraction of *intrinsic information* about the signals. Examples of such methods are wave scattering [4], translation-invariant image denoising [5] and implicit neural representations [6]. Extrapolation in frequency using multipliers builds upon previous work by the authors in [7] showing that, under suitable conditions and for a suitable choice of approximation error, there exist optimal multipliers, all of which have a canonical structure, satisfying (2).

In this work, we address one limitation of the results in [7], namely, the inability of the optimal multiplier to exploit relations in the collection of signals, in particular, when some elements are related by an unknown transformations arising from a group action. To overcome this limitation in the case of translations, we propose an alternative formulation that is translation-invariant. A consequence of our proposed approach is that the centering of the signals that achieves the minimum error does not correspond to any the classical choices, e.g., the center of mass of the signals.

II. Preliminaries

The *real* Hilbert space of complex-valued square integrable functions on a domain X is denoted as $L^2(X)$. Its *real* inner

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product is

$$\langle u, v\rangle_{L^2(X)} = \mathrm{Re}\left(\int_X u(x)^* v(x)\, dx\right).$$

We assume that $u_1, \ldots, u_n \in L^2(\mathbb{R}^d)$ and that they are compactly supported. To simplify the notation, we denote their Fourier transforms as $f_1, \ldots, f_n$. Any linear combination of these functions is real-analytic on $\mathbb{R}^d$ and thus it cannot vanish on an open subset of $\mathbb{R}^d$ [8]–[10]. Let $\Omega = \Omega_0 \setminus \alpha^{-1}\Omega_0$ and denote as $L^2(\Omega)$ the subspace of $L^2(\Omega_0)$ consisting of functions supported on Ω. It follows that for any linear combination v of $f_1, \ldots, f_n$ and $m \in L^2(\Omega)$ we have that $mv \in L^2(\Omega)$. Finally, for any $x_0 \in \mathbb{R}^d$ we let $\tau_{x_0} u$ be translation of u by x_0, i.e., $\tau_{x_0} u(x) = u(x - x_0)$.

III. Optimal multipliers

A. Multipliers minimizing the mean-squared error

To quantify the error incurred by a multiplier in (2) we proceed as follows. Define the random function

$$f = \sum\nolimits_{k=1}^n \beta_k f_k \quad \text{where} \quad \beta_1, \ldots, \beta_k \overset{\text{iid}}{\sim} N(0,1).$$

The *approximation error* is the random variable

$$\|D_\alpha f - mf\|^2_{L^2(\Omega)} = \int_\Omega |D_\alpha f(\xi) - m(\xi) f(\xi)|^2\, d\xi.$$

The *mean-squared error (MSE)* becomes

$$\mathbf{E}[\|D_\alpha f - mf\|^2_{L^2(\Omega)}] = \sum\nolimits_{k=1}^n \|D_\alpha f_k - m f_k\|^2_{L^2(\Omega)}.$$

The natural way to choose an optimal multiplier is then to solve

$$\min_{m \in L^2(\Omega)} \quad \sum\nolimits_{k=1}^n \|D_\alpha f_k - m f_k\|^2_{L^2(\Omega)}. \tag{3}$$

Proposition 1 (cf. Proposition 4.3. in [7])**.** *If* $|f_1|^2 + \ldots + |f_n|^2 > 0$ *on* Ω *then*

$$m_{\mathrm{MSE}} = \frac{\sum_{k=1}^n f_k^* D_\alpha f_k}{\sum_{k=1}^n |f_k|^2} \tag{4}$$

is a minimizer for (3).

We call m_I the *MSE optimal multiplier (MSE-OM)*. Its closed-form shows how it achieves the minimum MSE by adapting to the collection. By writing (4) equivalently as

$$m_{\mathrm{MSE}} = \sum\nolimits_{k=1}^n \left(\frac{|f_k|^2}{\sum_{j=1}^n |f_j|^2}\right) m_{f_k,\alpha} \tag{5}$$

it becomes apparent that m_{MSE} does not satisfy (1) for any single u_k. Instead, it finds a compromise between the collection of exact multipliers $m_{f_k,\alpha}$ and it will typically generate artifacts when extrapolating in frequency each one of the signals in the collection.

B. The case of a collection of translates

The closed-form in (5) suggests that the multiplier cannot leverage any structured relation between the elements of the collection. These relations arise, e.g., when the elements of the collection are related by an unknown transformation arising from a group action. We focus specifically on *translations*. In particular, suppose that $u_k = \tau_{x_k} u_0$ for some $u_0 \in L^2(\mathbb{R}^d)$ and some $x_1, \ldots, x_n \in \mathbb{R}^d$. In this case, all the elements of the collections are translates of a single function and$f_k = e_{-x_k} f_0$ for $e_x(\xi) = e^{2\pi i \xi \cdot x}$. If $|f_0| > 0$ on Ω we can apply (5) to deduce that

$$m_{\mathrm{MSE}} = \frac{1}{n} m_{f_0,\alpha} \sum\nolimits_{k=1}^n e_{-(\alpha-1)x_k} \tag{6}$$

for this collection. Therefore, the MSE optimal multiplier is the product of the exact multiplier for f_0 and an average of complex exponentials. Remark that the effect of using this multiplier to extrapolate in frequency is that it attempts to resolve the details of *all* the signals in the collection *simultaneously*. The details are *misplaced* as they are centered at $(\alpha - 1)x_k$ instead of x_k.

C. The effect of centering

A strategy to improve the performance of the MSE optimal multiplier in the previous example is to *center* the signals in the collection first. If we replace f_k by $e_{-c_k} f_k$ for some choice of $c_1, \ldots, c_k \in \mathbb{R}^d$ then (5) yields

$$m_{\mathrm{MSE}} = \sum\nolimits_{k=1}^n \left(\frac{|f_k|^2}{\sum_{j=1}^n |f_j|^2}\right) e_{-(\alpha-1)c_k} m_{f_k,\alpha}.$$

A typical choice are the *centers of mass* of each u_k, i.e.,

$$c_k := \int_{\mathbb{R}^d} x |u_k(x)|^2\, dx \Big/ \int_{\mathbb{R}^d} |u_k(x)|^2\, dx \tag{7}$$

However, it is not clear whether this choice improves the performance of the multiplier. Our goal is thus to determine the centering of the signals that yields the minimum MSE.

IV. Translation-invariant optimal multipliers

We extend the criterion in (3) and solve

$$\min_{\substack{m \in L^2(\Omega) \\ c_1, \ldots, c_n \in \mathbb{R}^d}} \quad \sum\nolimits_{k=1}^n \|D_\alpha(e_{-c_k} f_k) - m(e_{-c_k} f_k)\|^2_{L^2(\Omega)}$$

to find both the the optimal centers and the optimal multiplier. The above is equivalent to

$$\min_{\substack{m \in L^2(\Omega) \\ c_1, \ldots, c_n \in \mathbb{R}^d}} \quad \sum\nolimits_{k=1}^n \|D_\alpha f_k - e_{(\alpha-1)c_k} m f_k\|^2_{L^2(\Omega)}. \tag{8}$$

While the former formulation can be interpreted as finding the centering that yields a multiplier achieving the minimum MSE, the latter can be interpreted as finding a multiplier together with the translations that achieve the minimum MSE when it is used to perform extrapolation in frequency for each one of the elements in the collection. Both formulations are *translation-invariant*: if $m^\star$ and $c_1^\star, \ldots, c_n^\star$ are minimizers then $m^\star$ and $c_1^\star -' x_1', \ldots, c_n^\star - x_n'$ will remain minimizers if we replace f_k

by $e_{-x'_k} f_k$ for *any* choice of $x'_1, \dots, x'_k \in \mathbb{R}^d$. Said otherwise, $m^\star$ can adapt to the translation-invariant structure of the collection. Therefore, we call any such $m^\star$ the *translation-invariant MSE optimal multiplier (TI-MSE-OM)*.

A. Adapting to a collection of translates

Suppose once again that $f_k = e_{-x_k} f_0$ for some $f_0 \in L^2(\mathbb{R}^d)$ that does not vanish on Ω and for some $x_1, \dots, x_n \in \mathbb{R}^d$. By choosing $c_k = -x_k$ and $m = m_{f_0,\alpha}$ each term in the sum in (8) becomes

$$\begin{aligned}&\|D_\alpha f_k - e_{(\alpha-1)c_k} m_{f_0,\alpha} f_k\|^2_{L^2(\Omega)} = \\ &\quad \|D_\alpha (e_{-x_k} f_0) - e_{(\alpha-1)c_k} m_{f_0,\alpha} (e_{-x_k} f_0)\|^2_{L^2(\Omega)} = \\ &\qquad \|D_\alpha f_0 - e_{(\alpha-1)(c_k + x_k)} m_{f_0,\alpha} f_0\|^2_{L^2(\Omega)} = 0.\end{aligned}$$

Since 0 is a global minimum for (8) we conclude that this choice must be a global minimizer for (8). Interestingly, this shows that the exact multiplier $m_{f_0,\alpha}$ for f_0 is not translated to x_k as we would expect, but instead to $(\alpha-1)x_k$, showing a subtle relation between the optimal centering and the extrapolation factor α.

B. Optimal centering for a pair of signals in one-dimension

We now characterize the optimal centering in the case of two signals $u_1, u_2 \in L^2(\mathbb{R})$. Assume that $|f_1|, |f_2| > 0$ on Ω and let $y_k = (\alpha-1)c_k$. The partial minimization problem

$$\min_{m \in L^2(\Omega)} \sum_{k=1}^n \|D_\alpha f_k - e_{y_k} m f_k\|^2_{L^2(\Omega)}$$

has as solution

$$m_{y_1,y_2} = \frac{e_{-y_1} f_1^* D_\alpha f_1 + e_{-y_2} f_2^* D_\alpha f_2}{|f_1|^2 + |f_2|^2}.$$

The MSE of this multiplier as a function of $z = y_1 - y_2$ is

$$E_{\mathrm{MSE}}(z) = \|v_1 - e_z w_1\|^2_{L^2(\Omega)} + \|v_1 - e_{-z} w_1\|^2_{L^2(\Omega)}$$

where

$$v_1 = \frac{|f_2|^2 D_\alpha f_1}{|f_1|^2 + |f_2|^2} \quad \text{and} \quad v_2 = \frac{|f_1|^2 D_\alpha f_2}{|f_1|^2 + |f_2|^2}$$

and

$$w_1 = \frac{f_2^* f_1 D_\alpha f_2}{|f_1|^2 + |f_2|^2} \quad \text{and} \quad w_2 = \frac{f_1^* f_2 D_\alpha f_1}{|f_1|^2 + |f_2|^2}.$$

The proof of the following proposition is in Appendix A.

Proposition 2. *Let $g \in L^2(\mathbb{R})$ be such that*

$$\widehat{g} = -\chi_\Omega \frac{|f_1|^2 |f_2|^2}{|f_1|^2 + |f_2|^2} m^*_{f_1,\alpha} m_{f_2,\alpha}. \tag{9}$$

Then

$$\min_{z \in \mathbb{R}} E_{\mathrm{MSE}}(z) = \min_{z \in \mathbb{R}} \mathrm{Re}(g(z)).$$

On one hand, from $\Omega \subset \alpha\Omega$ it follows that $\widehat{g}$ is supported on a bounded interval. On the other, the origin cannot belong to Ω if it belongs to Ω_0 and thus $\widehat{g}(0) = 0$. We conclude that g is a bandlimited, and thus smooth, function of mean zero that decays to zero at infinity. In particular, it must attain a global minimum on $\mathbb{R}$.

Although minimizing g is often an intractable problem, the closed-form of its Fourier transform in (9) allows us to perform numerical experiments in one dimension. Furthermore, the form of g strongly suggests that, except in some simple cases, the optimal difference $y_1 - y_2 = (\alpha-1)(c_1 - c_2)$ will not correspond to using the center of mass to center u_1 and u_2.

V. Experiments

To illustrate our results in Section IV-B we perform numerical experiments[1] in one dimension using two signals. One is a scaling and a translate of the trapezoidal signal

$$u_\delta(x) = \left(1 \wedge \frac{1 - |x|}{1 - \delta}\right)_+$$

and the other of the triangle signal

$$v_\delta(x) = ((1 + x) \wedge (1 - \delta^{-1} x))_+$$

where $(\cdot)_+$ denotes the positive part. The signal u_1 is a trapezoidal signal with $\delta = 0.7$, scaling 0.7 and center -0.3; the signal u_2 is a triangle signal with $\delta = 0.25$, scaling 1.28 and center 0.738 (Fig. 1a). The Fourier transforms f_1, f_2 can be computed in closed-form. We use $\Omega_0 = [-1.5, 1.5]$ and $\alpha = 6$. In this case $\alpha\Omega_0 = [-3, 3]$ and the *high-frequency approximation (HFA)* is obtained by convolving the signal with the inverse Fourier transform of $\chi_{\alpha\Omega_0}$.

In Figs. 1b and 1f we show the approximations obtained using the MSE optimal multiplier (MSE-OM) in (4). In both cases the approximations have noticeable Gibbs artifacts compared to the HFA. In Figs. 1c and 1g we show the approximations obtained using the MSE optimal multiplier after centering (MSE-OM+C) the signals according to their centers of mass (7) $c_1 = -0.3$ and $c_2 = 0.49$. There are no significant improvements compared to the approximations obtained using the MSE-OM. In fact, in some cases the amplitude of the spurious oscilations increase (Fig. 1g). Finally, in Fig. 1e we show the surrogate function g in Proposition 2. In this case it is real-valued. Although it is oscillatory, it attains a global minimum at $z^\star = 1.920$. This yields$|x_1 - x_2| \approx 11.51$ which does not match $|c_1 - c_2| \approx 0.79$. In Figs. 1b and 1f we show the approximations obtained using the translation-invariant MSE optimal multiplier (TI-MSE-OM). Remark that the artifacts are substantially reduced. However, the oscillations to the right of the triangular signal (Fig. 1h) increase slightly. To compare the above methods, we show some suitable error metrics in Table I. Remark that using TI-MSE-OM outperforms the other methods.

VI. Conclusion

In this work we leveraged the results in [7] to find a multiplier to perform extrapolation in frequency over a finite collection of signals that achieves the minimum MSE. Although this multiplier adapts to the collection, it does not exploit any structured relation between its elements. In particular, it does not exploit the fact that some of the signals may be translates

[1]The code to reproduce the results can be found in the GitHub repository `csl-lab/adaptiveExtrapolationInFrequency`

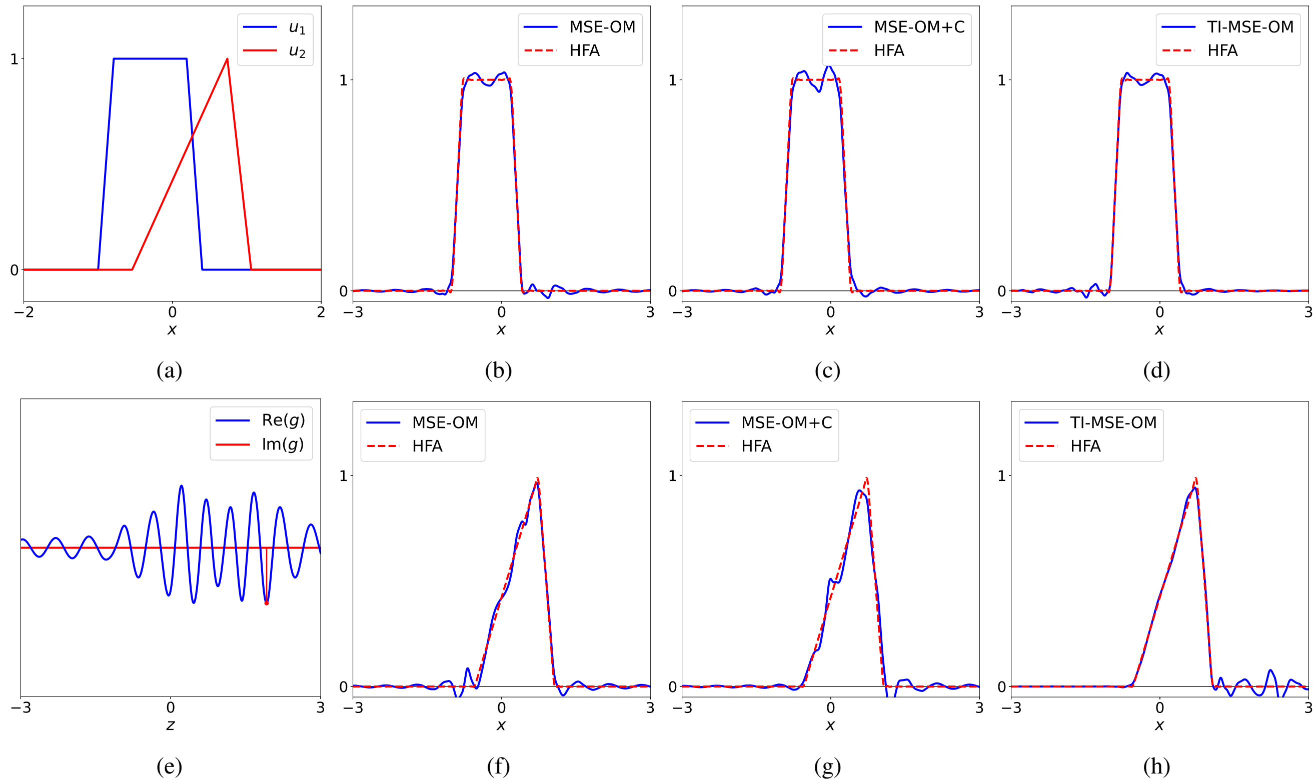

Fig. 1: (a) Signal u_1 and u_2. (e) Surrogate function g. (b, c, d, f, g, h) Comparison between the high-frequency approximation (red) and the approximation (blue) using: (b, f) the MSE optimal multiplier, (c, g) the MSE optimal multiplier after centering with the center of mass, and (d, h) the translation-invariant MSE optimal multiplier; for (b, c, d) signal u_1 and (f, g, h) signal u_2.

		MSE-OM	MSE-OM+C	TI-MSE-OM
u_1	MSE	1.570×10^{-3}	3.330×10^{-3}	$\mathbf{1.323\times 10^{-3}}$
	SNR	28.534	25.268	**29.275**
	PSNR	28.106	24.840	**28.847**
u_2	MSE	3.185×10^{-3}	6.949×10^{-3}	$\mathbf{2.648\times 10^{-3}}$
	SNR	22.239	18.850	**23.041**
	PSNR	24.863	21.475	**25.666**

TABLE I: Error metrics. SNR and PSNR are in dB.

of each other. By adding translations as variables in the MSE we were able to characterize a multiplier that is able to exploit this structure. Interestingly, this criterion centers the functions in a way that is optimal for extrapolation in frequency. Our numerical experiments in one dimension confirm this finding. As future work, we will extend these results to Euclidean motions, and we will identify the properties that allows us to extend these results to general group actions.

Appendix

A. Proof of Proposition 2

To compute the derivative of E_{MSE} define the auxiliary differentiable functions $J_k : L^2(\Omega) \to \mathbb{R}$ as

$$J_k(\varphi) = \|v_k - \varphi w_k\|^2_{L^2(\Omega)} \text{ where } J_k'(\varphi) = 2w_k^*(v_k - \varphi w_k).$$

By the chain rule

$$E'_{\mathrm{MSE}}(z) = \langle J_1'(e_z),\, e_z'\rangle_{L^2(\Omega)} - \langle J_2'(e_z),\, e_{-z}'\rangle_{L^2(\Omega)} = \\ -2\pi\,\mathrm{Re}\left(i\int_\Omega \xi(|w_1|^2 + |w_2|^2 + e_z w_2^* v_2 - e_{-z} w_1^* v_1)(\xi)\, d\xi\right).$$

Remark that the integral of any real integrand vanishes due to the factor i and taking the real part. Define the auxiliary function $\gamma = e_z f_1 f_2^* D_\alpha f_1^* D_\alpha f_2$. Then

$$E_A'(z) = -2\pi\,\mathrm{Re}\left(i\int_\Omega \xi(e_z w_2^* v_2 - e_{-z} w_1^* v_1)(\xi) d\xi\right) = \\ -2\pi\,\mathrm{Re}\left(i\int_\Omega \xi\frac{(|f_1|^2\gamma - |f_2|^2\gamma^*)(\xi)}{(|f_2|^2+|f_1|^2)^2(\xi)} d\xi\right) = \\ 2\pi\,\mathrm{Im}\left(\int_\Omega \xi\frac{(|f_1|^2\gamma + |f_2|^2\gamma)(\xi)}{(|f_2|^2+|f_1|^2)^2(\xi)} d\xi\right) = \\ 2\pi\,\mathrm{Im}\left(\int_\Omega \xi\frac{\gamma(\xi)}{(|f_2|^2+|f_1|^2)(\xi)} d\xi\right) = \\ -\mathrm{Re}\left(\int_\Omega (2\pi i)\xi\frac{(|f_1|^2|f_2|^2)(\xi)}{(|f_2|^2+|f_1|^2)(\xi)}(m^*_{f_1,\alpha} m_{f_2,\alpha} e_z)(\xi) d\xi\right)$$

from where the proposition follows by identifying the real part of the derivative of an inverse Fourier transform.

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