Centralized and Collective Neurodynamic Optimization Approaches for Sparse Signal Reconstruction via L_1-Minimization

You Zhao, Xiaofeng Liao, *Fellow, IEEE*, Xing He, and Rongqiang Tang

***Abstract*— This article develops several centralized and collective neurodynamic approaches for sparse signal reconstruction by solving the L_1-minimization problem. First, two centralized neurodynamic approaches are designed based on the augmented Lagrange method and the Lagrange method with derivative feedback and projection operator. Then, the optimality and global convergence of them are derived. In addition, considering that the collective neurodynamic approaches have the function of information protection and distributed information processing, first, under mild conditions, we transform the L_1-minimization problem into two network optimization problems. Later, two collective neurodynamic approaches based on the above centralized neurodynamic approaches and multiagent consensus theory are proposed to address the obtained network optimization problems. As far as we know, this is the first attempt to use the collective neurodynamic approaches to deal with the L_1-minimization problem in a distributed manner. Finally, several comparative experiments on sparse signal and image reconstruction demonstrate that our proposed centralized and collective neurodynamic approaches are efficient and effective.**

***Index Terms*— Centralized and collective neurodynamic approaches, global convergence, L_1-minimization problem, sparse signal reconstruction.**

I. Introduction

SPARSE signal reconstruction is an important problem in compressed sensing (CS) theory [1] and has been widely used in image processing, data analysis, pattern recognition, and so on [2]–[4]. The purpose of sparse signal reconstruction is to recover the sparse solution by solving the following problem:

$$\min \ \|x\|_0, \quad \text{s.t. } Ax = b \tag{1}$$

where $A \in R^{m\times n}$ is a sensing matrix, b is an observation vector, and $\|\cdot\|_0$ is the so-called L_0-norm. However, the L_0-minimization problem is NP-hard [5]. Based on the restricted isometry property (RIP) condition of matrix A, the same sparse solutions of L_0-minimization problem can be obtained by solving the L_1-norm minimization [6]

$$\min \ \|x\|_1, \quad \text{s.t. } Ax = b \tag{2}$$

where $\|\cdot\|_1$ is the L_1-norm. The problem (2) can be transformed into the following unconstrained optimization problem according to the penalty function method:

$$\min \ \frac{1}{2}\|\text{Ax-b}\|^2 + \ell\|x\|_1 \tag{3}$$

where $\ell \in R$ is a tradeoff parameter.

Many polynomial-time optimization approaches have been proposed to deal with problem (2) or (3), such as the interior-point algorithm [10], the augmented Lagrangian method [11], and the orthogonal matching pursuit (OMP) approach [12].

Recently, the neurodynamic approaches have been attractive because they can be implemented by hardware circuit and possess a parallel structure, which can eliminate the influence of high dimension and large density of polynomial-time optimization approaches. The research of neurodynamic approaches can be distinguished roughly by centralized and collective neurodynamic approaches.

A. Centralized Neurodynamic Approaches

Numerous centralized neurodynamic approaches have been studied for reconstructing sparse signals in recent decades (see [13]–[21] and references therein). For example, Balavoine *et al.* [13] proposed a dynamics of locally competitive algorithm (LCA) to solve problem (3). Liu *et al.* [16], Xu *et al.* [17] discussed several L_1-minimization projection neural network algorithms to reconstruct sparse signal. Feng *et al.* [18] studied a modified Lagrange programming neural network based on LCA (LPNN-LCA) for L_1-minimization problems only with local convergence characteristic. In addition, in order to solve the nonconvex,

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You Zhao, Xiaofeng Liao, and Rongqiang Tang are with the Key Laboratory of Dependable Services Computing in Cyber Physical Society-Ministry of Education, College of Computer Science, Chongqing University, Chongqing 400044, China (e-mail: zhaoyou1991sdtz@163.com; xfliao@cqu.edu.cn).

Xing He is with the Chongqing Key Laboratory of Nonlinear Circuits and Intelligent Information Processing, School of Electronics and Information Engineering, Southwest University, Chongqing 400715, China (e-mail: hexingdoc@swu.edu.cn).

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nonsmooth, and non-Lipschitz (L_p-norm) sparse signal reconstruction problem, Bian and Chen [19] proposed a smoothing neural network algorithm based on the parameter smooth approximation technique. Zhao *et al.* [20] proposed a smoothing projection neural network for minimizing L_{p-q} mixed norm problem to reconstruct the sparse signal. Later, a smoothing neural network based on primal–dual method was proposed for minimization l_1–l_p to reconstruct sparse signal in [21]. In recent years, many centralized neurodynamic approaches with finite and fixed-time convergence rates have been investigated based on the sliding model techniques in order to solve certain optimization problems with better characteristics in terms of convergence speed. For example, to solve the linear programming with fixed-time convergence rate, a dynamical system approaches in [40] and an augmented Lagrangian neural network [42] are proposed and investigated. Garg and Panagou [41] proposed a novel gradient-flow scheme, which converges to the optimal point of the convex optimization problem within a fixed time. To solve problem (3) and get the sparse signal, Yu *et al.* [14] proposed two dynamical approaches with finite-time [14] and fixed-time convergence rate by exploiting sliding mode technique in LCA [13]. Garg and Baranwal [43] proposed a CAPPA for sparse recovery with fixed-time convergence rate.

Although the centralized neurodynamic approaches [16]–[19] mentioned above can effectively solve the L_1-minimization problem (2) to reconstruct sparse signal, there still exist some drawbacks, for example, the matrix inversion operation with high complexity is required in [16], there is a difficulty in selecting tradeoff parameter in [17] and [19], and the LPNN-LCA [18] only provides local convergence characteristics. How to effectively overcome these drawbacks motivates us to investigate new and more effective methods of centralized neurodynamic approaches.

B. Collective Neurodynamic Approaches

Recently, a large number of distributed dynamic algorithms are proposed for distributed network optimization problems (see [7]–[9], [22]–[26], [32]). For example, Yan *et al.* [7] first proposed a collective neurodynamic optimization to solve nonconvex optimization problem. Che and Wang [8] proposed a collaborative neurodynamic approach to optimize a combinatorial optimization problem. Later, to deal with the mixed-integer optimization, a collective neurodynamic approach (i.e., two-timescale duplex neurodynamic) was discussed in [9]. Gharesifard and Cortés [22] proposed a distributed dynamic algorithm to deal with a smooth distributed convex problem under a directed graph. Liu and Wang [23] proposed a second-order multiagent network projection approach to deal with the distributed nonsmooth constrained convex optimization problem based on differential inclusion technique. Yang *et al.* [24] studied a continuous-time projection multiagent network algorithm to solve the distributed constrained nonsmooth convex optimization problem. To reduce the traffic cost between agents, He *et al.* [27] developed an average quasi-uniform distributed dynamical approach to deal with the nonsmooth and constrained convex optimization based on pulse communication framework. Le *et al.* [26] proposed a distributed neurodynamic approach to solve the smooth convex optimization problem with global coupled inequality constraints. He *et al.* [28] developed a simple continuous-time distributed approach to general distributed convex problem based on nonautonomous system. In addition, some applications of the distributed continuous-time optimization in recent years are also involved in [29]–[33] and [39].

Considering that sparse signal reconstruction is an important problem that has been widely used in many different fields, including image processing, data analysis, and pattern recognition. Unfortunately, a few results involve collective (fully distributed) neurodynamic approaches for sparse signal reconstruction problems. In our opinion, there are mainly two difficulties, i.e., the first difficulty is to transform problem (2) into a distributed and solvable optimization problem without damaging the signal sparsity since problem (2) is not a standard distributed optimization problem, and the second is how to design effective collaborative neurodynamic approaches to solve the obtained distributed version of problem (2) since the mentioned above-distributed approaches [22]–[32] cannot be used directly. To overcome the above two difficulties effectively motivates us to study how to transform the problem (2) into two solvable network optimization problems and how to design collective neurodynamic approaches to cope with the obtained two network optimization problems based on the above centralized neurodynamic approaches (8) and (9).

This article develops two centralized and collective neurodynamic approaches for sparse signal reconstruction by solving the L_1-norm minimization problem. The main contributions of this article are highlighted as follows.

1) Two centralized neurodynamic approaches based on the augmented Lagrange method and Lagrange method with derivative feedback are proposed, which are modeled by ordinary differential equations (ODE). Different from the existing approaches in [16] and [17], our proposed centralized neurodynamic approaches do not require complicated matrix inversion, i.e., $(AA^T)^{-1}$. This is because the computation of the inverse of AA^T is difficult for large-scale sparse signal reconstruction problems even if the inverse exists.
2) A new Lemma 2 of projection operator is first developed for neurodynamic approaches. Then, a novel Lyapunov function is constructed based on Lemma 2, which can be used to effectively prove the global convergence of our presented centralized neurodynamic approach. Compared with LPNN-LCA in [18], our proposed centralized neurodynamic approaches have global stability in the sense of Lyapunov.
3) Unlike the works in [13]–[15] and [19], our proposed approaches have no difficulty in selecting tradeoff parameters (e.g., if the penalty parameter is too large, the sparsity may be destroyed; if the penalty parameter is too small, the equality constraint may not be satisfied).
4) Based on the RIP condition and coherence condition of matrix A and multiagent consensus theory, we prove that the problem (2) can be transformed into two solvable

network convex optimization problems, i.e., a distributed optimization problem with consensus constraints and a distributed extended monotropic optimization problem.

5) Two collective neurodynamic approaches based on the proposed centralized neurodynamic approaches and multiagent consensus theory are proposed to solve two network optimization problems, they retain the advantages of centralized neurodynamic approaches. In addition, their optimality and global convergence are analyzed. As far as we know, this work is the first attempt to design collective neurodynamic approaches to recover the sparse signal by solving the L_1-minimization problem.

This article is organized as follows. In Section II, several basic definitions and properties are introduced briefly. In Section III, two centralized neurodynamic approaches are proposed. In Section IV, the problem (2) is transformed into equivalent two network optimization problems, and meanwhile, two collective neurodynamic approaches are proposed for solving two network optimization problems in a distributed manner. In Section V, simulation results on sparse signal reconstruction and image reconstruction are carried out to show the effectiveness of our proposed approaches. Finally, Section VI concludes this article.

II. Preliminaries

Let $x \in R^n$ and $y \in R^n$ be column vectors. The superscript T denotes transpose, $x^T y = \sum_{i=1}^n x_i y_i$ is a product of x and y, and x_i is the ith element of x. Let $1_n = (1, \ldots, 1)^T \in R^n$. $|\cdot|$ denotes the absolute value. ${}^s x$ denotes the s-sparse signal. I_n denotes an identity matrix in $R^{n\times n}$. $\|x\| = (\sum_{i=1}^n x_i^2)^{(1/2)}$ denotes the Euclidean norm and $\|x\|_1 = \sum_{i=1}^n |x_i|$ denotes the L_1-norm. The range (B) denotes the range of the matrix B, $\ker(B)$ denotes the kernel of matrix B, and $\otimes$ denotes the Kronecker product. $\sigma_{\max}(A)$ denotes the maximum singular value of matrix A.

A. Restricted Isometry Property Condition

Definition 1 [34]: The matrix $A \in R^{m\times n}$ $(m \ll n)$ satisfies the RIP condition if for all s-sparse signal with an isometry constant δ_s of a matrix A, it has

$$(1-\delta_s)\|{}^s x\|^2 \le \|A{}^s x\|^2 \le (1+\delta_s)\|{}^s x\|^2.$$

Candès [35] asserted that if $\delta_{2s} < \sqrt{2}-1$, the solution of problem (2) is that of (1), i.e., the convex relaxation is accurate. However, it is generally NP-hard to ascertain whether the given matrix A is an RIP matrix. Another concept in CS is called coherence, which is closely related to RIP with the characteristic of easy examination.

Definition 2 [36]: The coherence of a matrix A represents the maximum absolute value of any cross-correlation columns of A, namely

$$\Upsilon(A) = \max_{i\neq j} \frac{|A_i^T A_j|}{\|A_i\|\|A_j\|}$$

specifically, if a matrix satisfies the RIP condition, then it has a small coherence or incoherent. On the contrary, a highly coherent matrix has a small RIP with low probability.

B. Subdifferential

Definition 3 [37]: $f : R^n \to R$ is a convex function if it satisfies

$$f(u) \geqslant f(v) + \eta^T(\text{u-v}) \tag{4}$$

where η is the subgradient of f at v.

Definition 4: Subdifferential $\partial f(u)$ at u is a set of all subgradients

$$\partial f(u) = \{\eta | f(v) - f(u) \geqslant \eta^T(\text{v-u})\ \forall v \in R^n\}. \tag{5}$$

C. Projection Operator

Let $P_\Omega(x)$ be projection operator of $x \in R^n$ on a nonempty, closed convex set Ω, which is defined as follows:

$$P_\Omega(x) = \operatorname*{argmin}_{u\in\Omega} \|\text{u-x}\|.$$

Lemma 1 [16]: Let $\Omega \in R^n$ be a nonempty, closed convex set, such that

$$(u - P_\Omega(u))^T(P_\Omega(u) - v) \geqslant 0 \quad \forall u \in R^n,\ v \in \Omega. \tag{6}$$

Lemma 2: Define the Bregman divergence of $\mathfrak{h}$ at u and v

$$D_{\mathfrak{h}}(u,v) = \mathfrak{h}(u) - \mathfrak{h}(v) - (\text{u-v})^T \nabla \mathfrak{h}(v) \tag{7}$$

where $\mathfrak{h}(u) = (1/2)\|u\|^2 - (1/2)\|u - P_\Omega(u)\|^2$. If the function $\mathfrak{h}(u)$ satisfies $\nabla_u \mathfrak{h}(u) = P_\Omega(u)$, then the following properties hold.

1) $D_{\mathfrak{h}}(u,v)$ is a continuous, differentiable function with $\nabla_u D_{\mathfrak{h}}(u,v) = P_\Omega(u) - P_\Omega(v)$.
2) $D_{\mathfrak{h}}(u,v) \geqslant (1/2)\|P_\Omega(u) - P_\Omega(v)\|^2$.
3) $D_{\mathfrak{h}}(u,v) \leqslant (1/2)\|\text{u-v}\|^2$.

Proof:

1) By taking the derivative of $D_{\mathfrak{h}}(u,v)$ with respect to u, we have

$$\nabla_u D_{\mathfrak{h}}(u,v) = P_\Omega(u) - P_\Omega(v).$$

2) According to (7) and *Lemma 1*, one has

$$\begin{aligned} D_{\mathfrak{h}}(u,v) &\geqslant \frac{1}{2}(P_\Omega(v) - P_\Omega(u))^T(P_\Omega(v) - P_\Omega(u)) \\ &\quad + \frac{1}{2}(P_\Omega(u) - u)^T(P_\Omega(v) - P_\Omega(u)) \\ &\geqslant \frac{1}{2}\|P_\Omega(v) - P_\Omega(u)\|^2. \end{aligned}$$

3) By simple calculation based on (7) and *Lemma 1*, one has

$$\begin{aligned} &\frac{1}{2}\|\text{u-v}\|^2 - D_{\mathfrak{h}}(u,v) \\ &\geqslant \frac{1}{2}\|\text{u-v}\|^2 + \frac{1}{2}\|P_\Omega(u) - P_\Omega(v)\|^2 \\ &\quad + (P_\Omega(u) - P_\Omega(v))^T(\text{v-u}) \\ &= \frac{1}{2}\|u - v - P_\Omega(u) + P_\Omega(v)\|^2 \geqslant 0. \end{aligned}$$

The inequality holds, which comes from *Lemma 1*. Thus, we get $(1/2)\|\text{u-v}\|^2 - D_{\mathfrak{h}}(u,v) \geqslant 0$. ■

D. Algebraic Graph Theory

Consider a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = (\nu_1, \ldots, \nu_n)$ is a vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges. $\mathcal{A} = \{a_{ij}\} \in R^{m\times m}$ represents the weighted adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise in $\mathcal{G}$. The undirected graph $\mathcal{G}$ is connected if every unorder pair $\{\nu_i, \nu_j\}$, $i \neq j \in \{1, \ldots, m\}$ of vertices is connected, i.e., there exists a path between any pair of distinct υ_i and υ_j. Otherwise, it is an undirected unconnected graph. The weighted degree of vertex ν_i is defined as $d(\upsilon_i) = \sum_{j=1}^{n} a_{ij}$. Then, the Laplacian matrix is defined as $L_n = (l_{ij})_{n\times n} = D - A$, where $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\} \in R^{n\times n}$. If $\mathcal{G}$ is connected, then its Laplacian matrix is symmetric with a simple eigenvalue 0 such that $L_n 1_n = 0_n$.

III. Centralized Neurodynamic Approaches for Problem (2)

In this section, we present two centralized neurodynamic approaches for solving problem (2) to reconstruct sparse signal based on the augmented Lagrange method and Lagrange method with derivative feedback and projection operator.

A. Centralized Neurodynamic Approaches

The first centralized neurodynamic approach is designed based on the augmented Lagrange method

$$\begin{cases} \dot{w} = -A^T\lambda - \left(I_n - A^T A\right)P_\Omega(w) + A^T b - A^T A w \\ \dot{\lambda} = A(w - P_\Omega(w)) - b \\ x = w - P_\Omega(w). \end{cases} \tag{8}$$

The second centralized neurodynamic approach is proposed based on the derivative feedback item, which is used to deal with the "troubles" of projection operator

$$\begin{cases} \dot{w} = -P_\Omega(w) - A^T\lambda \\ \dot{\lambda} = A(w - 2P_\Omega(w)) - AA^T\lambda - b \\ x = w - P_\Omega(w). \end{cases} \tag{9}$$

Remark 1: Comparison of neurodynamic approaches (8) and (9), first, the design motivation based on the augmented Lagrangian method (ALM) of neurodynamic approach (8) is easier to understand than the neurodynamic approach (9) on account of derivative feedback $\dot{w}$. Second, based on the condition $\dot{\lambda} = A(w - P_\Omega(w)) - b$, the neurodynamic approach (8) becomes $\dot{w} = -A^T(\lambda + \dot{\lambda}) - P_\Omega(w)$, $\dot{\lambda} = A(w - P_\Omega(w)) - b$. It means that the approach (9) has the same simple structure as neurodynamic approach (9) (similarly, the approach (9) can be transformed into $\dot{w} = -P_\Omega(w) - A^T\lambda, \dot{\lambda} = A(w - P_\Omega(w) + \dot{w}) - b, x = w - P_\Omega(w)$, see Table I). Finally, the applied scenarios are different. For the distributed optimization problem with consensus constraints (26), we can easily design effective collective neurodynamic approach (27) by using neurodynamic methods (8) and distributed consensus theory. However, the collective neurodynamic approach (37) can be easily obtained based on neurodynamic approach (9) to solve the distributed extended monotropic optimization problem (33).

Lemma 3: If $x^* = w^* - P_\Omega(w^*)$ is an optimal solution of problem (2), if and only if there exists an equilibrium point (w^*, λ^*) to (8) and (9).

Proof: According to the KKT condition, it has

$$\theta^* + A^T\lambda^* = 0, \quad \theta^* \in \partial\|x^*\|_1, \; Ax^* = b \tag{10}$$

where $\lambda^* \in R^m$ is a Lagrangian multiplier (dual variable), x^* is an optimal solution, and $\partial\|x^*\|_1 = (\partial|x_1^*|, \ldots, \partial|x_n^*|)^T$ is the subdifferential of $\|x^*\|_1$ and is composed of n elements as follows:

$$\partial\left|x_i^*\right| = \begin{cases} 1, & x_i^* > 0 \\ [-1, 1], & x_i^* = 0 \\ -1, & x_i^* < 0, \quad i = 1, \ldots, n \end{cases} \tag{11}$$

and $P_\Omega(w) = (P_\Omega(w_1), \ldots, P_\Omega(w_n))^T$ with

$$P_\Omega(w_i) = \begin{cases} 1, & w_i > 1 \\ w_i, & -1 \leqslant w_i \leqslant 1 \\ -1, & w_i < -1, \quad i = 1, \ldots, n. \end{cases} \tag{12}$$

Furthermore, based on the projection operator, it has

$$P_\Omega\left(\theta^* + x^*\right) = \theta^*. \tag{13}$$

Define $\theta^* + x^* = w^*$, such that $P_\Omega(w^*) = \theta^* = w^* - x^*$. Then, the KKT condition in (10) becomes

$$\begin{aligned} P_\Omega\left(w^*\right) + A^T\lambda^* = 0, \quad A\left(w^* - P_\Omega\left(w^*\right)\right) = b \\ w^* - P_\Omega\left(w^*\right) = x^*. \end{aligned} \tag{14}$$

From (14), it has

$$\begin{aligned} x^* = w^* - P_\Omega\left(w^*\right), \quad A\left(w^* - P_\Omega\left(w^*\right)\right) - b = 0 \\ P_\Omega\left(w^*\right) + A^T\lambda^* - A^T\left(A\left(w^* - P_\Omega\left(w^*\right)\right) - b\right) = 0 \end{aligned} \tag{15}$$

and

$$\begin{aligned} x^* = w^* - P_\Omega(w^*), \quad P_\Omega(w^*) + A^T\lambda^* = 0 \\ A(w^* - P_\Omega(w^*)) - b - A(P_\Omega(w^*) + A^T\lambda^*) = 0. \end{aligned} \tag{16}$$

Therefore, (w^*, λ^*) is an equilibrium point of (8) and (9).

Conversely, let (w^*, λ^*) be an equilibrium point of (8) and (9), for neurodynamic approach (8), it satisfies

$$\begin{cases} 0 = -A^T\lambda^* - \left(I_n - A^T A\right)P_\Omega\left(w^*\right) + A^T b - A^T A w^* \\ 0 = A\left(w^* - P_\Omega\left(w^*\right)\right) - b \\ x^* = w^* - P_\Omega\left(w^*\right). \end{cases} \tag{17}$$

Furthermore, according to $A(w^* - P_\Omega(w^*)) - b = 0$ and $x^* = w^* - P_\Omega(w^*)$, $Ax^* = b$ and $A^T\lambda^* + P_\Omega(w^*) = 0$ hold. For any $\tilde{x} \in \{x \mid Ax = b\}$, according to $A^T\lambda^* + P_\Omega(w^*) = 0$, one obtains $(\tilde{x} - x^*)^T(A^T\lambda^* + P_\Omega(w^*)) = 0$. Based on the condition $A(\tilde{x} - x^*) = 0$, we further get

$$\left(\tilde{x} - x^*\right)^T P_\Omega\left(w^*\right) = 0. \tag{18}$$

Let $\theta^* = P_\Omega(w^*)$ and then $\theta^* = P_\Omega(\theta^* + x^*)$ on account of $x^* = w^* - P_\Omega(w^*)$, such that $\theta^* \in \partial\|x^*\|_1$. Since $\|x\|_1$ is a convex function, it follows that $\|\tilde{x}\|_1 - \|x^*\|_1 \geqslant (\tilde{x} - x^*)^T\theta^*, \forall\theta^* \in \partial\|x^*\|_1$. The condition (15) implies $\|\tilde{x}\|_1 \geqslant \|x^*\|_1$ on set $\tilde{x} \in \{x \mid Ax = b\}$. Thus, $x^* = w^* - P_\Omega(w^*)$ is an optimal solution to problem (2), which can be obtained.

Based on the similar method above, the equilibrium point (w^*, λ^*) [i.e., $x^* = w^* - P_\Omega(w^*)$] of approach (9) is an optimal solution of problem (2), which can be obtained. The proof is thereby completed. ■

Lemma 4: The neurodynamic approaches (8) and (9) have a unique solution with any initial value $\gamma(t_0) = (w_0^T, \lambda_0^T)^T$.

Proof: If $\gamma_1 = (w_1^T, \lambda_1^T)^T, \gamma_2 = (w_2^T, \lambda_2^T)^T \in R^{m+n}$ are two solutions of neurodynamic approaches (9) with the same initial value $\gamma(t_0)$. If $\gamma_1 \neq \gamma_2$, then there exist $\tau > 0$ and $\varpi > 0$ such that $\gamma_1(t) \neq \gamma_2(t)$ for any $t \in [\tau, \tau + \varpi]$. Define

$$\Psi(\gamma) = \begin{pmatrix} -P_\Omega(w) - A^T\lambda \\ A(w - 2P_\Omega(w)) - AA^T\lambda - b \end{pmatrix}. \tag{19}$$

Since $\Psi(\gamma)$ satisfies the Lipschitz condition on $t \in [0, \tau + \varpi]$, that is,

$$\begin{aligned} &\| \Psi(\gamma_1) - \Psi(\gamma_2)\| \\ &\leqslant \|P_\Omega(w_1) - P_\Omega(w_2)\| \\ &\quad + \left\|A^T(\lambda_2 - \lambda_1)\right\| + \left\|AA^T(\lambda_2 - \lambda_1)\right\| \\ &\quad + \|A(w_1 - w_2)\| + \|2A(P_\Omega(w_1) - P_\Omega(w_2))\| \\ &\leqslant l\|\gamma_1 - \gamma_2\| \quad \forall t \in [0, \tau + \varpi] \end{aligned} \tag{20}$$

where $l = (1 + 4\sigma_{\max}(A) + \sigma_{\max}(AA^T))$ is a Lipschitz constant. One has

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|\gamma_1 - \gamma_2\|^2 &= (r_1 - r_2)^T(\Psi(\gamma_1) - \Psi(\gamma_2)) \\ &\leqslant l\|\gamma_1 - \gamma_2\|^2 \quad \forall t \in [0, \tau + \varpi]. \end{aligned} \tag{21}$$

Integrating the above equation from 0 to $\tau + \varpi$, it follows that $\gamma_1 \neq \gamma_2, \forall t \in [0, \tau + \varpi]$ with the same initial values $r(t_0)$. This leads to a contradiction. Thus, the proof is thereby completed.

By using a similar approach, we can obtain that the neurodynamic approach (8) has a unique solution with a Lipschitz constant $l = 1 + 3\sigma_{\max}(A) + 2\sigma_{\max}(A^TA)$. ■

Theorem 1: The output vector w, i.e., x of neurodynamic approach (8), is globally asymptotically stable and converges to an optimal solution of (2) for any initial value $w_0 \in R^n, \lambda_0 \in R^m$.

Proof: Consider a Lyapunov function as follows:

$$V_1(w, \lambda) = \frac{1}{2}\|w - w^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2 - D_h(w, w^*) \tag{22}$$

where $V_1(w, \lambda) \geqslant (1/2)\|\lambda - \lambda^*\|^2$ from Lemma 2 and is positive definite, continuous, differentiable and radially unbounded.

By Lemma 2, the derivative of $V_1(w, \lambda)$ along the neurodynamic approach (8) is given by

$$\begin{aligned} &\dot{V}_1(w, \lambda) \\ &= \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T \\ &\quad \times \left(-A^T\lambda - \left(I_n - A^TA\right)P_\Omega(w) + A^Tb - A^TAw\right) \\ &\quad + \left(\lambda - \lambda^*\right)^T(A(w - P_\Omega(w)) - b). \end{aligned} \tag{23}$$

By subtracting from both the sides of (8) and (15) and substituting it into formula (23), it follows that

$$\begin{aligned} &\dot{V}_1(w, \lambda) \\ &= \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T A^T\left(\lambda^* - \lambda\right) \\ &\quad + \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T\left(P_\Omega(w^*) - P_\Omega(w)\right) \\ &\quad + \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T \\ &\quad \times A^TA\left(P_\Omega(w) - P_\Omega(w^*)\right) \\ &\quad + \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T\left(-A^TA\left(w - w^*\right)\right) \\ &\quad + \left(\lambda - \lambda^*\right)^T\left(A(w - P_\Omega(w)) - A\left(w^* - P_\Omega(w^*)\right)\right) \\ &= -\left\|A\left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)\right\|^2 \\ &\quad + \left(w - w^* + P_\Omega(w^*) - P_\Omega(w)\right)^T\left(P_\Omega(w^*) - P_\Omega(w)\right) \\ &= -\|\text{Ax-b}\|^2 + (w - P_\Omega(w))^T\left(P_\Omega(w^*) - P_\Omega(w)\right) \\ &\quad + \left(P_\Omega(w^*) - w^*\right)^T\left(P_\Omega(w^*) - P_\Omega(w)\right) \\ &\leqslant 0. \end{aligned} \tag{24}$$

The Lyapunov function $V_1(w, \lambda)$ is nonincreasing as $t \to +\infty$ for any initial value $(w_0^T, \lambda_0^T)^T \in R^{m+n}$. It can be seen from inequality (24) that $\{(w(t), \lambda(t)) : 0 \leqslant t \leqslant \mathcal{T}\} \subset \{(w(t), \lambda(t)) : V_1(w(t), \lambda(t)) \leqslant V_1(w(t_0), \lambda(t_0))\}$. Therefore, $\mathcal{T} = +\infty$ and (w, λ) is bounded. Since $V_1(w, \lambda)$ is a Lyapunov function of neurodynamic approach (8), which implies that the approach (8) is stable in the sense of Lyapunov.

Since $(w(t), \lambda(t))$ is bounded, then there exist an increasing sequence $\{t_s\}_{s=1}^\infty$ and a limit point $(\hat{w}, \hat{\lambda})$ such that $\lim_{s\to\infty} w(t_s) = \hat{w}$ and $\lim_{s\to\infty} \lambda(t_s) = \hat{\lambda}$. Thus, $(\hat{w}, \hat{\lambda})$ is an ω-limit point of $(w(t), \lambda(t))$.

According to LaSalle's invariance principle, $(w(t), \lambda(t))$ converges to a set W as $t \to +\infty$, where W is so-called the largest invariant subset of the following set:

$$W = \left\{(w(t), \lambda(t)) : \dot{V}_1(w, \lambda) = 0\right\}.$$

Obviously, $(w, \lambda) = (w^*, \lambda^*)$, if $\dot{V}_1(w, \lambda) = 0$. Thus, (w, λ) will converge to the equilibrium point set when $t \to +\infty$. From this, ω-limit point $(\hat{w}, \hat{\lambda})$ is an equilibrium point.

From the analysis above, there exists an ω-limit point $(w(t), \lambda(t))$, which is an equilibrium point of the neurodynamic approach (8). Next, we will prove that $(w(t), \lambda(t))$ converges to $(\hat{w}, \hat{\lambda})$ as $t \to +\infty$ with any given initial point (w_0, λ_0).

Define a Lyapunov function as

$$\bar{V}(w, \lambda) = \frac{1}{2}\|w - \hat{w}\|^2 + \frac{1}{2}\|\lambda - \hat{\lambda}\|^2 - D_h(w, \hat{w}).$$

Similar to the above mentioned proof of V_1, we obtain $\dot{\bar{V}}(w, \lambda) \leqslant 0$. Since the function $\bar{V}(w, \lambda)$ is continuous, thus, for any $\varepsilon > 0$, there exists a positive constant $\rho > 0$ such that $\bar{V}(w, \lambda) < \varepsilon$ as $\|(w^T, \lambda^T)^T - (\hat{w}^T, \hat{\lambda}^T)^T\| \leqslant \rho$. Moreover, there exists a positive integer E since $\bar{V}(w, \lambda)$ is monotonically nonincreasing over $[0, +\infty)$ such that $\|(w(t_E)^T, \lambda(t_E)^T)^T - (\hat{w}^T, \hat{\lambda}^T)^T\| < \rho$, $\bar{V}(w(t), \lambda(t)) \leqslant \bar{V}(w(t_E), \lambda(t_E)) < \varepsilon$, for any $t \geqslant t_E$.

In consequence, $\lim_{t\to\infty}(w(t), \lambda(t)) = (\hat{w}, \hat{\lambda})$. Since $\bar{V}(w, \lambda)$ is radially unbounded, thus, $(w(t), \lambda(t))$ of neurodynamic approach (8) globally converges to an equilibrium point. Thus, $x = w - P_\Omega(w)$ is also globally convergent to the optimal solution of problem (2). ■

Theorem 2: The neurodynamic approach (9) is globally asymptotically stable for any initial value $w_0 \in R^n$, $\lambda_0 \in R^m$.

Proof: Consider a candidate Lyapunov function

$$V_2(w,\lambda) = \frac{1}{2}\|w - w^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2 \tag{25}$$

where (w^*, λ^*) is an equilibrium point of centralized neurodynamic approach (9) and is also the optimal solution of problem (2) according to Lemma 3. Obviously, the Lyapunov function $V_2(w,\lambda)$ is positive definite, continuously differentiable and radially unbounded. On the basis of the chain rule, it follows that

$$\begin{aligned}\dot{V}_2(w,\lambda) &= \left(w - w^* - P_\Omega(w) - A^T\lambda\right)^T \times \left(-P_\Omega(w) - A^T\lambda\right) \\ &\quad - \left\|P_\Omega(w) + A^T\lambda\right\|^2 + (A(w - P_\Omega(w)) - b)^T \\ &\quad \times \left(\lambda - \lambda^*\right) + \left(A^T\left(\lambda - \lambda^*\right)\right)^T\left(-P_\Omega(w) - A^T\lambda\right) \\ &= \left(w - w^* - P_\Omega(w) - A^T\lambda + A^T\left(\lambda - \lambda^*\right)\right)^T \\ &\quad \times \left(-P_\Omega(w) - A^T\lambda\right) - \left\|P_\Omega(w) + A^T\lambda\right\|^2 \\ &\quad + \left(\lambda - \lambda^*\right)^T(A(w - P_\Omega(w)) - b) \\ &= \left(w - w^* - P_\Omega(w) + P_\Omega\left(w^*\right)\right)^T \\ &\quad \times \left(-P_\Omega(w) - A^T\lambda\right) - \left\|P_\Omega(w) + A^T\lambda\right\|^2 \\ &\quad + \left(\lambda - \lambda^*\right)^T(A(w - P_\Omega(w)) - b) \\ &= -\|\dot{w}\|^2 + (w - P_\Omega(w))^T\left(P_\Omega\left(w^*\right) - P_\Omega(w)\right) \\ &\quad + \left(w^* - P_\Omega\left(w^*\right)\right)^T\left(-P_\Omega\left(w^*\right) + P_\Omega(w)\right) \\ &\leqslant 0.\end{aligned}$$

Obviously, the function $\dot{V}_2(w,\lambda) < 0$ for any $(w,\lambda) \neq (w^*,\lambda^*)$, and $\dot{V}_2(w,\lambda) = 0$ if and only if $(w,\lambda) = (w^*,\lambda^*)$. Therefore, the neurodynamic approach (9) is globally asymptotically stable in the sense of Lyapunov. In addition, by the similar method in the proof of Theorem 1, the proposed neurodynamic approach (9) is globally convergent to an equilibrium point of problem (2). ■

IV. Collective Neurodynamic Approaches for Problem (2)

In this section, we discuss that the problem (2) can be equivalent to two kinds of network optimization problems and propose two relevant collective neurodynamic approaches to solve the obtained network optimization problems. In addition, some sufficient conditions are needed to reconstruct sparse signal by minimizing problem (2), such as RIP condition and coherence condition.

Theorem 3: The problem (2) can be equivalent to the following network optimization problem, i.e., a distributed optimization problem with consensus constraints:

$$\min\ \frac{1}{h}\sum_{k=1}^{h}\|\boldsymbol{x}_k\|_1, \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = b,\ \ L\boldsymbol{x} = 0 \tag{26}$$

where $L_h \in R^{h\times h}$ is the Laplacian matrix of connected graph $\mathcal{G}$ (which is a positive semidefinite matrix) and $L = L_h \otimes I_n \in R^{hn\times hn}$, $\boldsymbol{x} = (\boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_h^T)^T \in R^{hn}$

$$\boldsymbol{A} = \begin{bmatrix} A_{m_1\times n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{m_h\times n}\end{bmatrix} \in R^{m\times hn}.$$

The corresponding RIP condition is converted to $(1 - (h-1)/h - (\delta_s/h))\|^{hs}\boldsymbol{x}\|^2 \leqslant \|\boldsymbol{A}^{hs}\boldsymbol{x}\|^2 \leqslant (1 - (h-1)/h + (\delta_s/h))\|^{hs}\boldsymbol{x}\|^2$, for any hs-sparse vector ${}^{hs}\boldsymbol{x}$.

Proof: Assume that the undirected graph $\mathcal{G}$ is connected, and according to the algebraic graph theory in Section II, 1_h is the eigenvector of L_h corresponding to the simple zero eigenvalue. Then, define $\bar{x} = 1_h \otimes x \in R^{hn}$ and $Ax = b$, and one has

$$\begin{aligned} L\bar{x} &= (L_h \otimes I_n)\bar{x} = (L_h \otimes I_n)(1_h \otimes x) \\ &= (L_h 1_h) \otimes (I_n x) = 0 \\ \boldsymbol{A}\bar{x} &= \sum_{k=1}^{h}\boldsymbol{A}_{mk}\bar{x}_k = \sum_{k=1}^{h}A_{mk}x = Ax = b.\end{aligned}$$

With respect to the RIP condition, for any hs-sparse signal ${}^{hs}\bar{x} = 1_h \otimes {}^{s}x$, it follows that $\|\boldsymbol{A}^{hs}\bar{x}\|^2 = ({}^{hs}\bar{x}_1)^T(\boldsymbol{A}_{m1}^T\boldsymbol{A}_{m1})^{hs}\bar{x}_1 + \cdots + ({}^{hs}\bar{x}_h)^T(\boldsymbol{A}_{mh}^T\boldsymbol{A}_{mh})^{hs}\bar{x}_h$. Using the RIP condition $(1-\delta_s)\|^{s}x\|^2 \leqslant \|A^{s}x\|^2 = ({}^{s}x_1)^T(A_{m1}^TA_{m1})^{s}x_1 + \cdots + ({}^{s}x_h)^T(A_{mh}^TA_{mh})^{s}x_h \leqslant (1+\delta_s)\|^{s}x\|^2$ with any s-sparse signals ${}^{s}x \in R^n$ in Definition 1, and we have $(1 - (h-1)/h - (\delta_s/h))\|^{hs}\bar{x}\|^2 \leqslant \|\boldsymbol{A}^{hs}\bar{x}\|^2 \leqslant (1 - (h-1)/h + (\delta_s/h))\|^{hs}\bar{x}\|$.

Conversely, based on the properties of Kronecker product, it has that $L\bar{x} = (L_h \otimes I_n)\bar{x} = (L_h \otimes I_n)\text{vec}(\mathcal{X}) = \text{vec}(I_n\mathcal{X}L_h)$ with $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$, $\mathcal{X} = (x_1, \ldots, x_h) \in R^{n\times h}$, and $\bar{x} = \text{vec}(\mathcal{X})$. If $L\bar{x} = 0$, such that $I_n\mathcal{X}L_h = 0$ and $L_h\mathcal{X}^T = 0$, 1_h is the eigenvector of L_h, which corresponds to a unique simple zero eigenvalue since $\mathcal{G}$ is connected. Therefore, $\mathcal{X}^T = \alpha^T \otimes 1_h$, where $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in R^n$ is a column vector. It follows that $\mathcal{X} = \alpha \otimes 1_h^T$. Then, each column vector $\bar{x}_i$ of $\bar{x}$ satisfies $\bar{x}_i = \alpha$. Therefore, we have $\bar{x}_i = \alpha$.

From the above analysis, $L\bar{x} = 0$ and $\boldsymbol{A}\bar{x} = b$ hold for undirected connection diagram $\mathcal{G}$ if and only if $\bar{x} = 1_h \otimes x$, for any $x \in \{x | Ax = b\}$, $(1/h)\|\bar{x}\|_1 = \|x\|_1$ for any $x \in R^n$. Thus, for any hs-sparse vector ${}^{hs}\bar{x} = 1_h \otimes^{s} x$, it follows that $\|^{hs}\bar{x}\|^2 = h\|^{s}x\|^2$ and $\|\boldsymbol{A}^{hs}\bar{x}\|^2 = \|\boldsymbol{A}1_h \otimes^{s} x\|^2 = ({}^{s}x)^T(A_{m1}^TA_{m1})^{s}x + \cdots + ({}^{s}x)^T(A_{m1}^TA_{m1})({}^{s}x)^T = \|A^{s}x\|^2$ and $(1-\delta_s)\|^{s}x\|^2 \leqslant \|A^{s}x\|^2 \leqslant (1+\delta_s)\|^{s}x\|^2$. Thus, the problem (2) can be equivalently transformed into (26). Therefore, the proof is thereby completed. ■

Remark 2: Since problem (2) is not a standard distributed optimization problem, the collective (distributed) neurodynamic approaches cannot be designed directly. In order to solve problem (2) in a distributed way, the first thing that needs to be solved is to obtain the distributed form of problem (2) without destroying the sparsity of signal (that is, the first difficulty introduced in Section I). Thus, based on the row decomposition of matrix A, multiagent consensus theory, and RIP condition, we obtain the problem (26) and its proof in Theorem 3.

Next, a collective neurodynamic approach is proposed to address problem (26) based on multiagent consensus theory and centralized neurodynamic approach (8)

$$\begin{cases} \dot{u} = L(w - P_\Omega(w)) \\ \dot{w} = -P_\Omega(w) - A^T\lambda - L(u + \dot{u}) \\ \dot{\lambda} = A(w - P_\Omega(w)) - b \\ x = w - P_\Omega(w). \end{cases} \tag{27}$$

Lemma 5: If $x^* = w^* - P_\Omega(w^*)$ is an optimal solution to problem (26), then (w^*, λ^*, u^*) is an equilibrium point of (27). Conversely, when (w^*, λ^*, u^*) is an equilibrium of collective neurodynamic approach (28), then $x^* = w^* - P_\Omega(w^*)$ is an optimal solution to problem (2).

Proof: According to the KKT condition, there exist λ^*, u^* and $\theta^* \in (1/h)\partial\|x^*\|_1$ such that

$$\begin{aligned} \theta^* + A^T\lambda^* + Lu^* = 0 \\ Ax^* - b = 0, \quad Lx^* = 0 \end{aligned} \tag{28}$$

where w^* is the optimal solution of problem (26).

According to (13) and $x^* = w^* - P_\Omega(w^*)$ and $P_\Omega(w^*) = \theta^*$, one has

$$P_\Omega(w^*) + A^T\lambda^* + Lu^* = 0 \tag{29a}$$
$$A(w^* - P_\Omega(w^*)) = b \tag{29b}$$
$$L(w^* - P_\Omega(w^*)) = 0. \tag{29c}$$

By adding (29c) to (29a), $P_\Omega(w^*) + A^T\lambda^* + L(u^* + w^* - P_\Omega(w^*)) = 0$ can be obtained. Thus, (w^*, λ^*, u^*) is an equilibrium point of collective neurodynamic approach (28).

To prove the converse, assume that (w^*, λ^*, u^*) is an equilibrium point of collective neurodynamic approach (28). It follows that

$$\begin{cases} 0 = L(w^* - P_\Omega(w^*)) \\ 0 = -P_\Omega(w^*) - A^T\lambda^* - L(u^* + w^* - P_\Omega(w^*)) \\ 0 = A(w^* - P_\Omega(w^*)) - b \\ x^* = w^* - P_\Omega(w^*). \end{cases} \tag{30}$$

Based on $x^* = w^* - P_\Omega(w^*)$, $Ax^* - b = 0$, $Lx^* = 0$, and $P_\Omega(w^*) + A^T\lambda^* + Lu^* = 0$ hold. Let $\theta^* = P_\Omega(w^*)$ and $\theta^* = P_\Omega(\theta^* + x^*)$ on account of $x^* = w^* - P_\Omega(w^*)$, which follows that $\theta^* \in (1/h)\partial\|x^*\|_1$. Define an augment lagrangian function $L(x, \lambda, u) = (1/h)\|x\|_1 + \lambda^T Ax + u^T Lx + (1/2)x^T Lx$, and one has $(x - x^*)(P_\Omega(w^*) + A^T\lambda^* + Lu^*) = 0$, $Lx^* = 0$, and $Ax^* = 0$ for any (x, λ, u). According to the variational analysis theory, the equilibrium point (x^*, λ^*, u^*) is a saddle point of the convex function $\mathcal{L}(x, \lambda, u)$, which confirms that the equilibrium point (w^*, λ^*, u^*), i.e., $x^* = w^* - P_\Omega(w^*)$ of (28), is the optimal solution to problem (27) or (2), which completes the proof of Lemma 4. ■

Theorem 4: The output vector $x = w - P_\Omega(w)$ of collective neurodynamic approach (27) globally converges to the same optimal solution of problem (26) with any initial point $y_0 = (w_0^T, \lambda_0^T, u_0^T)^T$.

Proof: Consider a candidate Lyapunov function

$$V(y) = \frac{1}{2}\|w - w^*\|^2 + \frac{1}{2}\|u - u^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2 - D_h(w, w^*). \tag{31}$$

Note that the Lyapunov function (31) is positive definite. From Lemma 2, one has $V(y) \geqslant (1/2)\|u - u^*\|^2 + (1/2)\|\lambda - \lambda^*\|^2 \geqslant 0$. Furthermore, $V(y) > 0$, if $y \neq y^*$; otherwise, $V(y) = 0$, which means that the Lyapunov function (31) is positive definite.

From condition (27), a straightforward calculation of (31) yields that

$$\begin{aligned} \dot{V}(y) &= (w - w^*)^T\dot{w} + (u - u^*)^T\dot{u} \\ &\quad + (\lambda - \lambda^*)^T\dot{\lambda} + (P_\Omega(w^*) - P_\Omega(w))^T\dot{w} \\ &= (w - w^*)^T(P_\Omega(w^*) - P_\Omega(w)) \\ &\quad + (w - w^*)^T L(u^* - u) + (w - w^*)^T \\ &\quad \times L((w^* - P_\Omega(w^*)) - (w - P_\Omega(w))) \\ &\quad + (w - w^*)^T(A^T(\lambda^* - \lambda)) + (\lambda^* - \lambda)^T A \\ &\quad \times (P_\Omega(w^*) - P_\Omega(w)) + (P_\Omega(w^*) - P_\Omega(w))^T \\ &\quad \times L((w^* - P_\Omega(w^*)) - (w - P_\Omega(w))) \\ &\quad + (P_\Omega(w^*) - P_\Omega(w))^T(P_\Omega(w^*) - P_\Omega(w)) \\ &\quad + (P_\Omega(w^*) - P_\Omega(w))^T L(u^* - u) \\ &\quad + (u - u^*)^T L(w - w^* - P_\Omega(w) + P_\Omega(w^*)) \\ &\quad + (\lambda - \lambda^*)^T A(w - w^* - P_\Omega(w) + P_\Omega(w^*)) \\ &= -(x - x^*)^T L(x - x^*) + \|P_\Omega(w^*) - P_\Omega(w)\|^2 \\ &\quad - (w^* - w)^T(P_\Omega(w^*) - P_\Omega(w)) \\ &\leqslant 0. \end{aligned} \tag{32}$$

For any initial value y_0, we define $\mathcal{Q}(y) = \{(w, \lambda, u) : V(y) \leqslant V(y_0), w \in R^{hn}, \lambda \in R^m, u \in R^{hn}\}$, and thus, $\mathcal{Q}(y)$ is bounded with regard to (w, λ, u); it implies that (w, λ, u) is bounded.

Define

$$\begin{aligned} \Phi(w) = &-(w - P_\Omega(w) - (w^* - P_\Omega(w^*)))^T \\ &\times L(w - P_\Omega(w) - (w^* - P_\Omega(w^*))) \\ &+ \|P_\Omega(w^*) - P_\Omega(w)\|^2 - (w^* - w)^T \\ &\times (P_\Omega(w^*) - P_\Omega(w)). \end{aligned}$$

Since the Laplacian matrix L is positive semidefinite, it confirms that $X^T LX = 0$, $X \in R^{hn}$, if and only if $LX = 0$. On the one hand, if $y = (w^T, \lambda^T, u^T)^T$ satisfies (29), we obtain $\Phi(w) = 0$. On the other hand, $\Phi(w) = 0$ if and only if $w = w^*$, i.e., x^* is an optimal solution to problem (26). According to LaSalle's invariance principle, the collective neurodynamic approach (27) is capable of reaching consensus optimal solution of the problem (26). ■

Next, we show that the same sparse solution of problem (2) can be obtained by minimizing a distributed extended monotropic optimization problem with a complementary variable $\mathbf{y}$.

Theorem 5: Let $L_g \in R^{g\times g}$ be the Laplacian matrix of the connected graph $\mathcal{G}$ and $L = L_g \otimes I_m \in R^{mg\times mg}$. Therefore, there exists a supplementary variable $\mathbf{y} \in R^{mg}$ such that the solution to the problem (2) can be obtained effectively by solving the following optimal problem:

$$\min \sum_{q=1}^{g} \|\mathbf{x}_q\|_1, \quad \text{s.t. } \mathbf{A}\mathbf{x} + L\mathbf{y} = \mathbf{b} \tag{33}$$

where

$$\mathbf{A} = \begin{bmatrix} A_{m\times n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{m\times n_g} \end{bmatrix}_{mg\times n}$$
$$\mathbf{x} = \left(\mathbf{x}_1^T, \ldots, \mathbf{x}_g^T\right)^{\mathrm{T}} \in R^n \quad \text{and}$$
$$\mathbf{b} = \left(\left(\frac{b}{g}\right)^{\mathrm{T}}, \ldots, \left(\frac{b}{g}\right)^{\mathrm{T}}\right)^{\mathrm{T}} \in R^{mg}.$$

The corresponding coherence of a matrix $\mathbf{A}$ satisfies $\Upsilon(\mathbf{A}) = \Upsilon(A)$, which implies that the same optimal solution as the problem (2) can be reconstructed by minimizing the problem (33).

Proof: To prove Theorem 5, we only need to show that the convex problem (33) has the same KKT condition as (2). From the KKT condition, it shows that $\mathbf{w}^*$ be an optimal solution to problem (33) if and only if there exist $\mathbf{\Theta} \in \partial\|\mathbf{x}^*\|_1 \in R^n$, $\mathbf{z}^* \in R^{mg}$, and $\mathbf{y}^* \in R^{mg}$ such that

$$\mathbf{\Theta} + \mathbf{A}^T\mathbf{z}^* = 0, \quad \mathbf{\Theta} \in \partial\|\mathbf{x}^*\|_1 \tag{34a}$$
$$\mathbf{A}\mathbf{x}^* + L\mathbf{y}^* = \mathbf{b} \tag{34b}$$
$$L\mathbf{z}^* = 0. \tag{34c}$$

Let us left multiply both sides of (34b) by $1_g \otimes I_m$, and then, one has

$$\begin{aligned} &\left(1_g \otimes I_m\right)^T\left(\mathbf{A}\mathbf{x}^* + L\mathbf{y}^* - \mathbf{b}\right) = 0 \\ &\Rightarrow \left(\sum_{q=1}^{g}\left(\frac{b}{g} - \mathbf{A}_{m\times n_q}x_q^*\right)\right) - \left(1_g \otimes I_m\right)^T L\mathbf{y}^* = b \\ &\Rightarrow Ax - \left(1_g \otimes I_m\right)^T L\mathbf{y}^* = b. \end{aligned} \tag{35}$$

Based on the properties of the Laplacian matrix of undirected connected graph, it confirms that $(1_g^T \otimes I_m)(L_g \otimes I_m)\mathbf{y}^* = (1_g^T L_g) \otimes I_m\mathbf{y}^* = 0$. Thus, the equation $Ax = b$ holds.

Since $L1_{mg} = 0$ holds, (34c) implies that there exists $\mathbf{z}^* = 1_g \otimes \mathrm{z}^* \in R^{mg}, \mathrm{z}^* \in R^m$, such that $\mathbf{\Theta} + \mathbf{A}^T\mathbf{z}^* = \mathbf{\Theta} + A^T\mathrm{z}^* = 0$. Thus, the KKT optimal condition of (33) is equal to (10), i.e., the KKT condition of (2).

On the other hand, based on the properties of Laplacian matrix $L1_{mg} = 0$, $1_{mg}^T L = 0^T$ and optimal condition in (10), it is easy to get

$$\begin{aligned} &\theta^* + A^T\lambda^* = 0, \quad \theta^* \in \partial\|x^*\|_1 \\ &\Rightarrow \mathbf{\Theta} + \mathbf{A}^T\mathbf{z}^* = 0, \quad L\mathbf{z}^* = 0, \ \mathbf{\Theta} = \theta^*. \end{aligned}$$

For any $\rho \in R^g$ and let $\bar{\rho} = \rho \otimes I_m$, it follows that $L\bar{\rho} = 0$ and $\bar{\rho} \in \ker(L)$. Because $Ax^* = b$ in (10), then $\bar{\rho}^T(\mathbf{A}\mathbf{x}^* - \mathbf{b}) = \rho\sum_{q=1}^{g}((b/g) - \mathbf{A}_{m\times n_q}\mathbf{x}_w^*) = 0$. It implies that $\mathbf{A}\mathbf{x}^* - \mathbf{b} \in \mathrm{range}(L)$ by using the orthogonal decomposition theory. Thus, there exists a vector $\mathbf{y}^*$ such that $\mathbf{A}\mathbf{x}^* - (1_g \otimes I_m)^T L\mathbf{y}^* = \mathbf{b}$, i.e., (34b) holds.

From Definition 2, it shows

$$\begin{aligned} \Upsilon(\mathbf{A}) &= \max_{i\neq j}\frac{|\mathbf{A}_i^T\mathbf{A}_j|}{\|\mathbf{A}_i\|\|\mathbf{A}_j\|} \\ &= \begin{cases} 0, & \text{if } i \in \Delta\{\mathbf{A}_{m\times n_\pi}\} \\ j \in \Delta\{\mathbf{A}_{m\times n_q}\}, & \pi \neq q \\ \max\limits_{i\neq j}\dfrac{|\mathbf{A}_i^T\mathbf{A}_j|}{\|\mathbf{A}_i\|\|\mathbf{A}_j\|}, & \text{if } i, j \in \Delta\{\mathbf{A}_{m\times n_q}\}, \\ & q = 1, \ldots, g \end{cases} \end{aligned} \tag{36}$$

where Δ is a index set, which corresponds to the number of columns $A_{m\times n_q}$, i.e., if $\mathbf{A}_{m\times n_q} = \mathbf{A}_{m\times 2:5}$, then $\{\Delta\mathbf{A}_{m\times n_q}\} = \{2, 3, 4,5\}$. Let $\mathcal{C}(n_q)$ be the number of columns in a matrix, i.e., if $n_q = 2:5$, then $\mathcal{C}(n_q) = 4$.

Recall Definition 2 that it follows that $\Upsilon(A) = \max_{i\neq j}(|A_i^T A_j|)/(\|A_i\|\|A_j\|) \geqslant 0$, for $\forall i, j$. Thus, the coherence of a matrix $\mathbf{A}$ is equivalent to A's coherence of problem (2), i.e., $\Upsilon(\mathbf{A}) = \Upsilon(A)$. Thus, the proof is completed. ■

In order to solve the problem (33) in a distributed manner, we develop the following collective neurodynamic approach based on the neurodynamic approach (9) and multiagent consensus theory

$$\begin{cases} \dot{\mathbf{w}} = -P_\Omega(\mathbf{w}) - \mathbf{A}^T\mathbf{z} \\ \dot{\mathbf{z}} = \mathbf{A}(\mathbf{z} - P_\Omega(\mathbf{z})) - L(\mathbf{u} + \mathbf{z}) - \mathbf{b} + \mathbf{A}\dot{\mathbf{w}} \\ \dot{\mathbf{u}} = L\mathbf{z} \\ \mathbf{x} = \mathbf{w} - P_\Omega(\mathbf{w}). \end{cases} \tag{37}$$

Lemma 6: If $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$ is an optimal solution to problem (33), if and only if there exist $\mathbf{z}^*$ and $\mathbf{u}^*$ such that $(\mathbf{w}^*, \mathbf{z}^*, \mathbf{u}^*)$ is an equilibrium point of (36), where $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$.

Proof: According to the KKT condition, if $\mathbf{x}^*$ is an optimal solution to problem (33), it has

$$\mathbf{\Theta}^* + \mathbf{A}^T\mathbf{z}^* = 0, \quad \mathbf{\Theta}^* \in \partial\|\mathbf{x}^*\|_1 \tag{38a}$$
$$\mathbf{A}\mathbf{x}^* + L\mathbf{y}^* = \mathbf{b} \tag{38b}$$
$$L\mathbf{z}^* = 0. \tag{38c}$$

Since $\mathbf{x}^*$ is an optimal solution of problem (33), such that $\mathbf{\Theta}^* = P_\Omega(\mathbf{\Theta}^* + \mathbf{x}^*)$ on account of $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$, then the condition (40) becomes

$$\begin{cases} 0 = -P_\Omega(\mathbf{w}^*) - \mathbf{A}^T\mathbf{z}^* \\ 0 = \mathbf{A}\left(\mathbf{w}^* - P_\Omega(\mathbf{w}^*)\right) - \mathbf{b} - L\mathbf{u}^* \\ 0 = L\mathbf{z}^* \\ \mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*). \end{cases} \tag{39}$$

Substituting the third formula $0 = L\mathbf{z}^*$ into the second formula $0 = \mathbf{A}(\mathbf{w}^* - P_\Omega(\mathbf{w}^*)) - \mathbf{b} - L\mathbf{u}^*$, one has $\mathbf{A}(\mathbf{w}^* - P_\Omega(\mathbf{w}^*)) - \mathbf{b} - L\mathbf{u}^* - L\mathbf{z}^* = 0$, which confirms that (39) is equivalent to (40). Thus, there exist $\mathbf{z}^*$ and $\mathbf{u}^*$ such that $(\mathbf{w}^*, \mathbf{z}^*, \mathbf{u}^*)$ is an equilibrium point of (36), where $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$.

Conversely, since $(\mathbf{w}^*, \mathbf{z}^*, \mathbf{u}^*)$ is an equilibrium point of collective neurodynamic approach (36), then, one has

$$\begin{cases} 0 = -P_\Omega(\mathbf{w}^*) - \mathbf{A}^T\mathbf{z}^* \\ 0 = \mathbf{A}(\mathbf{w}^* - P_\Omega(\mathbf{w}^*)) - \mathbf{b} - L(\mathbf{u}^* + \mathbf{z}^*) \\ 0 = L\mathbf{z}^* \\ \mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*). \end{cases} \tag{40}$$

Let $\mathbf{\Theta}^* = P_\Omega(\mathbf{w}^*)$, and then, $\mathbf{\Theta}^* = P_\Omega(\mathbf{\Theta} + \mathbf{x}^*)$ on account of $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$, which follows that $\mathbf{\Theta} \in \partial\|\mathbf{x}^*\|_1$. From $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$ and $L\mathbf{z}^* = 0$ in (37), one has $\mathbf{A}\mathbf{x}^* - \mathbf{b} - L\mathbf{u}^* = 0$ and $P_\Omega(\mathbf{w}^*) + \mathbf{A}^T\mathbf{z}^* = 0$. Similar to the proof in Lemma 4, the equilibrium point $(\mathbf{w}^*, \mathbf{z}^*, \mathbf{u}^*)$ satisfies the KKT condition of problem (33). Thus, $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$ is an optimal solution to problem (33). Thus, the proof is completed. ■

Theorem 6: For any initial point $\boldsymbol{\xi}_0 = (\mathbf{w}_0^T, \mathbf{z}_0^T, \mathbf{u}_0^T)^T$, $\mathbf{x} = \mathbf{w} - P_\Omega(\mathbf{w})$ of collective neurodynamic approach (37) globally converges to the optimal solution of problem (33).

Proof: Let $\mathbf{w}^*$ or $\mathbf{x}^* = \mathbf{w}^* - P_\Omega(\mathbf{w}^*)$ be an optimal solution to problem (33). Consider the following candidate Lyapunov function:

$$\mathbb{V}(\xi) = \frac{1}{2}\|\mathbf{w} - \mathbf{w}^*\|^2 + \frac{1}{2}\|\mathbf{z} - \mathbf{z}^*\|^2 + \frac{1}{2}\|\mathbf{u} - \mathbf{u}^*\|^2 + D_h(\mathbf{w}, \mathbf{w}^*). \tag{41}$$

Obviously, the Lyapunov function $\mathbb{V}(\xi) \geqslant (1/2)\|\mathbf{w} - \mathbf{w}^*\|^2 + (1/2)\|\mathbf{z} - \mathbf{z}^*\|^2 + (1/2)\|\mathbf{u} - \mathbf{u}^*\|^2 + \|P_\Omega(\mathbf{w}) - P_\Omega(\mathbf{w}^*)\|^2$. $\mathbb{V}(\xi)$ is positive definite, differentiable and radially unbounded. The derivative of $\mathbb{V}(\xi)$ along with the trajectories of (33) is

$$\begin{aligned} \dot{\mathbb{V}}(\xi) &= (\mathbf{w} - \mathbf{w}^*)^T(P_\Omega(\mathbf{w}^*) + \mathbf{A}^T\mathbf{z}^* - P_\Omega(\mathbf{w}) - \mathbf{A}^T\mathbf{z}) \\ &\quad + (\mathbf{u} - \mathbf{u}^*)^T(L\mathbf{z} - L\mathbf{z}^*) + 2(P_\Omega(\mathbf{w}) - P_\Omega(\mathbf{w}^*))^T \\ &\quad \times (P_\Omega(\mathbf{w}^*) - \mathbf{A}^T\mathbf{z}^* - P_\Omega(\mathbf{w}) + \mathbf{A}^T\mathbf{z}) + (\mathbf{z} - \mathbf{z}^*)^T \\ &\quad \times \mathbf{A}(\mathbf{w} - P_\Omega(\mathbf{w}) - \mathbf{w}^* + P_\Omega(\mathbf{w}^*)) + (\mathbf{z} - \mathbf{z}^*)^T \\ &\quad \times L(\mathbf{u}^* + \mathbf{z}^* - \mathbf{u} - \mathbf{z}) + (\mathbf{z} - \mathbf{z}^*)^T \\ &\quad \times (\mathbf{A}P_\Omega(\mathbf{w}^*) + \mathbf{A}\mathbf{A}^T\mathbf{z}^* - \mathbf{A}P_\Omega(\mathbf{z}) - \mathbf{A}\mathbf{A}^T\mathbf{z}) \\ &= (\mathbf{w} - \mathbf{w}^*)^T(P_\Omega(\mathbf{w}^*) - P_\Omega(\mathbf{w})) + (\mathbf{w} - \mathbf{w}^*)^T \\ &\quad \times \mathbf{A}^T(\mathbf{z}^* - \mathbf{z}) + 2(P_\Omega(\mathbf{w}) - P_\Omega(\mathbf{w}^*))^T \\ &\quad \times \mathbf{A}^T(\mathbf{z} - \mathbf{z}^*) + (\mathbf{z} - \mathbf{z}^*)^T\mathbf{A}(\mathbf{w} - \mathbf{w}^*) \\ &\quad + 2(P_\Omega(\mathbf{w}) - P_\Omega(\mathbf{w}^*))^T(P_\Omega(\mathbf{w}^*) - P_\Omega(\mathbf{w})) \\ &\quad + 2(\mathbf{z} - \mathbf{z}^*)^T\mathbf{A}(P_\Omega(\mathbf{w}^*) - P_\Omega(\mathbf{w})) \\ &\quad + (\mathbf{z} - \mathbf{z}^*)^T(L + \mathbf{A}\mathbf{A}^T)(\mathbf{z}^* - \mathbf{z}) + (\mathbf{z} - \mathbf{z}^*)^T \\ &\quad \times L(\mathbf{u}^* - \mathbf{u}) + (\mathbf{u} - \mathbf{u}^*)^T L(\mathbf{z} - \mathbf{z}^*) \\ &= (\mathbf{w} - \mathbf{w}^*)^T(P_\Omega(\mathbf{w}^*) - P_\Omega(\mathbf{w})) \\ &\quad - 2\|(P_\Omega(\mathbf{w}^*) - P_\Omega(\mathbf{w}))\|^2 \\ &\quad - (\mathbf{z} - \mathbf{z}^*)^T L(\mathbf{z} - \mathbf{z}^*) - \|\mathbf{A}(\mathbf{z}^* - \mathbf{z})\|^2 \\ &\leqslant 0. \end{aligned} \tag{42}$$

Let $Q(\xi_0) = \{(\mathbf{w}, \mathbf{z}, \mathbf{u}) : \mathbb{V}(\xi) \leqslant \mathbb{V}(\xi_0), \mathbf{w} \in R^n, \mathbf{z} \in R^{mg}, \mathbf{u} \in R^{mg}\}$ with for initial value $\xi_0 = (\mathbf{w}_0^T, \mathbf{z}_0^T, \mathbf{u}_0^T)^T$, and $Q(\xi)$ is bounded, which implies that $\mathbf{w}$, $\mathbf{z}$, and $\mathbf{u}$ are always bounded.

Obviously, for any $\mathbf{w} \neq \mathbf{w}^*$ and $\mathbf{z} \notin \text{range}(L)$, the condition (41) holds from Lemma 1. Thus, globally asymptotical stability of the proposed neurodynamic approach (37) can be obtained. $\dot{\mathbb{V}}(\xi) = 0$ if and only if $\mathbf{w} = \mathbf{w}^*$ and $\mathbf{z} = \text{range}(L)$. Similar to the proof in Theorem 1, we can obtain that the neurodynamic approach in (37) globally converges to the equilibrium point, i.e., the optimal solution of problem (33). The proof is completed. ■

Remark 3: Note that problem (2) can be equivalent to two kinds of network optimization problems, i.e., a distributed optimization problem with consensus constraints (26) and a distributed extended monotropic optimization problem (33). The salient features of them are summarized as follows. First, the distributed optimization problem with consensus constraints (26) is constructed based on the row decomposition of matrix A and x consensus method. Different from all existing distributed nonsmooth optimization problems with consensus constraints, the distributed sparse signal reconstruction problem (26) requires a corresponding RIP condition, i.e., $(1 - (h-1)/h - (\delta_s/h))\|^{hs}\boldsymbol{x}\|^2 \leqslant \|A^{hs}\boldsymbol{x}\|^2 \leqslant (1 - (h-1)/h + (\delta_s/h))\|^{hs}\boldsymbol{x}\|^2$, for any hs-sparse vector $^{hs}\boldsymbol{x}$. It is an important sufficient condition for obtaining the same sparse solution as problem (2). Second, the distributed extended monotropic optimization problem (33) is constructed based on the column decomposition of matrix A and z consensus method. Moreover, a supplementary variable $\mathbf{y} \in R^{mg}$ is introduced, which is used to match changed equality constraints in (33). In addition, to obtain the same sparsity of the solution as the problem (2), the coherence of a matrix $\mathbf{A}$ in problem (33) also needs considered. Thus, it is different from the all existing distributed optimization algorithms.

Remark 4: Usually, there are two types of efficient methods for solving convex nonsmooth optimization problems.

1) Smoothing approximation methods in [19]–[21] and [44]. Different from the smoothing approaches mentioned above, we use the projection operators to obtain the exact subgradient without introducing approximation parameters (in this case, only suboptimal solutions with certain accuracy can be obtained) and internal optimization subproblem (such as the calculation of the conjugate function), so the neurodynamic approaches proposed in our article have much higher reconstruction accuracy (see Figs. 3 and 4).
2) Variable decomposition method in [45]. Its main idea is to decompose optimal variables x into its positive and negative parts (i.e., $x = [x]^+ - [-x]^+$), the nonsmooth L_1 objection function becomes $\|x\|_1 = \mathbf{1}^T[x]^+ + \mathbf{1}^T[-x]^+$. Although it can effectively deal with the nonsmooth one-norm problem, it is worth noting that it doubles the dimension of the optimization variable x, which greatly increases the computational cost of the optimization algorithms. In this article, we directly optimize the variable x and do not increase the dimension of the optimization variable.

Remark 5: In this article, we mainly consider the L_1-minimization problem (2) and its distributed variant

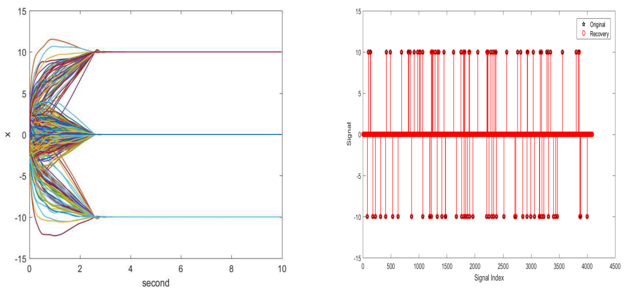

Fig. 1. (Left) States transition curves on (8). (Right) Recovered signal with RelErr $= 1.47e{-}16$.

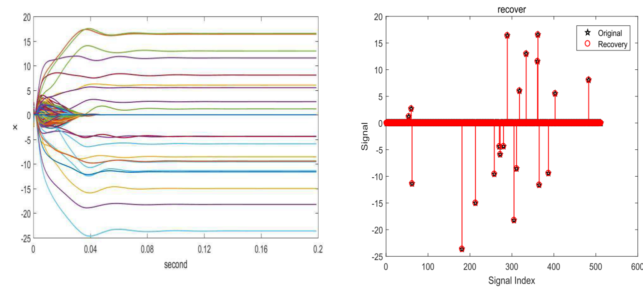

Fig. 2. (Left) States transition curves on (9). (Right) Recovered signal with RelErr $= 2.10e{-}16$.

problems (26) and (33), which can be used to deal with many problems. Although problem (2) can be transformed into an unconstrained optimization problem based on the penalty function method, i.e., problem (3), comparing (2) and (3), problem (2) is more attractive than (3) because the solution of the model usually has a better performance without the tradeoff parameter selection dilemma. The existing works in [13]–[15] and [43] are mainly used to solve the optimization problem (3) with finite and fixed-time convergence rate based on the soft thresholding of one-norm and sliding mode technique. The LCA [13], LCA-finite [14], LCA-fixed [15], and CAPPA [43] have a faster convergence rate than the proposed neurodynamic approaches (8) and (9) since they have finite- or fixed-time convergence properties. However, the proposed neurodynamic approaches (8) and (9) have better accuracy with increasing time than LCA [13], LCA-finite [14], LCA-fixed [15], and CAPPA [43] because they have used a fixed tradeoff parameter, and however, it is difficult to choose the best tradeoff parameter.

V. Numerical Simulations

In this section, Example 1 is carried out to demonstrate the effectiveness and superiority of centralized neurodynamic approaches (8) and (9), and Example 2 is used to illustrate the effectiveness of the proposed collective neurodynamic approaches (27) and (37).

Example 1: We use typical sparse signal and image reconstruction to show the effectiveness and superiority of centralized neurodynamic approaches (8) and (9).

A. Sparse Signal Restoration

In this section, we repeat our experiment 100 times independently and use relative error RelErr $= (\|x - x^*\|)/(\|x\|)$ (x is the original sparse signal and x^* is the recovered signal) to evaluate the validity of centralized neurodynamic approaches (8) and (9). For neurodynamic approach (8), consider a scenario that $s = 100$ (s-sparse), signal length $n = 4096$, the numbers of nonzero values are at ± 10, the dimension of the observed value is 600, that is, $m = 600$, and A is a Gaussian random matrix. The convergence trajectories of $x(t)$ are shown in Fig. 1 (left). Fig. 1 (right) shows that the stationary point of the neurodynamic approach (8) approximates the original sparse signal with a relative error at $1.47e{-}16$. For centralized neurodynamic approach (9), setting $s = 20$, $m = 100$, and $n = 512$, the numbers of nonzero values are generated randomly. The measurement matrix A is also a Gaussian random matrix. As shown in Fig. 2 that the output state $x(t)$ is globally asymptotically stable and the stationary point approximates the original sparse signal with a relative error of $2.10e{-}16$.

1) Contrast Experiments for Sparse Signal Reconstruction: In order to illustrate the feasibility and advantages of centralized neurodynamic approaches (8) and (9), we compare our neurodynmic approaches (8) and (9) with PNNSR [16], PNNBIM [17], and LPNN-LCA [18]. First, let $n = 256$, $m = 128$, and $s = 10$. The experimental results are shown in Fig. 3 (left), where the y-axis denotes the average relative error of x with ten trials and the x-axis is the CPU time (unit: second), while Fig. 3 (middle) shows the result with $n = 512$, $m = 256$, and $s = 10$. In addition, Fig. 3 (right) shows the result when $n = 1024$, $m = 400$, and $s = 10$. From Fig. 3, it is observed that when $n = 512$, the centralized neurodynamic approach (8) is better than others. The PNNSR [16] with different tradeoff parameters $0.00005(2\log(m))^{1/2}$, $0.0005(2\log(m))^{1/2}$, and $0.005(2\log(m))^{1/2}$ performs worst since it is difficult to choose the best tradeoff parameter. As the dimensions n and m increase, the performances of PNNSR [16] and PNNBIM [17] are getting worse and worse since they have higher model complexity (see Table I). Moreover, the centralized neurodynamic approaches (8) and PNNSR [16] are more robust than neurodynamic approaches (9) because they are designed based on the augmented Lagrangian method, which has better robustness. Comparing with others, the proposed centralized neurodynamic approaches (8) and (9) have relatively stable convergence time.

To further demonstrate the effectiveness of centralized neurodynamic approaches (8) and (9), we compare our approaches with LCA [13], LCA-finite [14], and LCA-fixed [15] for reconstructing sparse signal with $n = 128$, $m = 64$, and $s = 8$. Fig. 4 shows the convergence performances. As shown in Fig. 4, the LCA-finite and LCA-fixed have faster convergence rate than neurodynamic approaches (8), (9) and LCA. Moreover, LCA-fixed has a faster convergence rate than LCA-finite. Note that neurodynamic approaches (8), (9) have better accuracy with increasing time because LCA, LCA-finite, and LCA-fixed use a fixed tradeoff parameter; however, it is difficult to choose the best tradeoff parameter. Therefore, in different scenarios of practical applications, different algorithms can be chosen. If a high-precision sparse signal is needed, neurodynamic approaches (8) and (9) can be chosen,

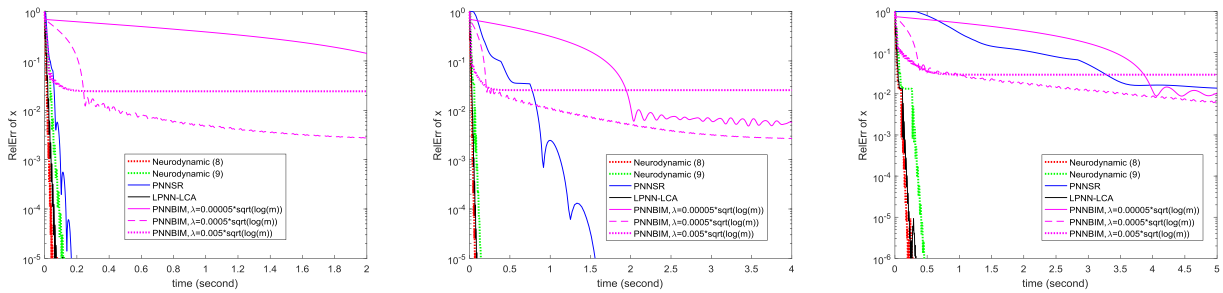

Fig. 3. Relative error of x with respect to the convergence time. (Left) $n = 256$, $m = 128$, and $s = 10$. (Middle) $n = 512$, $m = 128$, and $s = 10$. (Right) $n = 1000$, $m = 400$, and $s = 10$.

TABLE I
MODEL COMPLEXITY OF NEURODYNAMIC APPROACHES

Neurodynamic approach	Per iteration		Additional calculation for $P = A^T\left(AA^T\right)^{-1}A$, $q = A^T\left(AA^T\right)^{-1}b$	
	Multiplications	Additions/Subtractions	Multiplications	Additions/Subtractions
neurodynamic approaches (8)	$2mn$	$2mn+3n+m$	0	0
neurodynamic approaches (9)	$2mn$	$2mn+3n+m$	0	0
LPNN-LCA [18]	$2mn$	$2mn+3n+m$	0	0
PNNSR [16]	$2n^2+n$	$2n^2+n$	$mn\,(2m+1+n)+O(m^3)$	$m^2\,(2n-1)+n^2\,(m-2)-n$
PNNBIM [17]	n^2+2n	n^2+4n	$mn\,(2m+1+n)+O(m^3)$	$m^2\,(2n-1)+n^2\,(m-2)-n$

and if the sparse signal is obtained quickly, neurodynamic LCA-finite and LCA-fixed can be chosen.

2) Complexity Analysis: Let us analyze the model complexity of centralized neurodynamic approaches (8), (9), PNNSR [16], PNNBIM [17], and LPNN-LCA [18] by using the total number of multiplications/divisions and additions/subtractions performed in per iteration, which is used in [38] and the computational complexity of inverse operation of matrix AA^T ($O(m^3)$). The model complexity results are shown in Table I. From Table I, it can be seen that the neurodynamic approaches (8), (9) and LPNN-LCA have the same model complexity, which are less that of PNNSR and PNNBIM, and thus, the neurodynamic approaches (8) and (9) and LPNN-LCA have faster speed (less CPU time) than PNNSR and PNNBIM. In addition, PNNBIM has less model complexity than PNNSR, so it has a faster speed (less CPU time) than PNNBIM. Experimental results in Fig. 4 further demonstrate the effectiveness and correctness of complexity analysis.

B. Image Restoration

Image restoration has a wide range of applications in the field of engineering and science. In this section, we perform the neurodynamic approach (8) on the restoration of 256 × 256 "Lena" image and the neurodynamic approaches (9) on the restoration of 256 × 256 "Fingerprint" image.

Apply the peak signal-to -noise ratio (PSNR) (PSNR= $10\log_{10}(255^2/\text{MSE})$ with MSE $= 1/(n \times n)\sum_{i,j}(\hat{x}(i,j) - x(i,j))^2$, where $\hat{x}(i,j)$, $x(i,j)$ represents the pixels of the original image and the restored image respectively) to evaluate the validity of centralized neurodynamic approach (8) and (9). Fig. 5(a) shows the original image and Fig. 5(b)–(d) shows the reconstructed image by using neurodynamic approaches (8), OMP in [12] and PNNSR in [16]. Under the same condition, the proposed

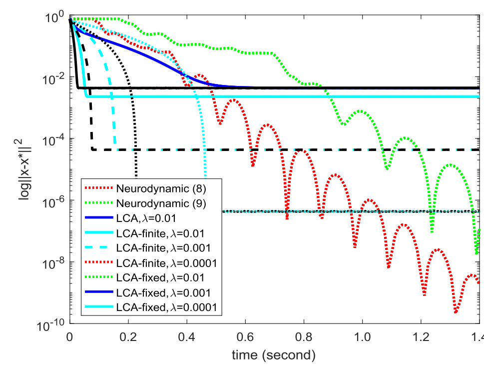

Fig. 4. $\log\|x - x^*\|^2$ with $n = 128$, $m = 64$, and $s = 8$.

neurodynamic approach in (8) has higher PSNR than OMP and PNNSR, which further proves the effectiveness of the approach (8). Fig. 6 shows that the proposed neurodynamic approach (9) is about 1 dB higher than PNNSR and 2 dB higher than OMP, which means that the proposed method is effective and superior.

Example 2: Sparse signal reconstruction tests are used to verify the effectiveness of collective neurodynamic approaches (27) and (37).

1) Sparse Signal Reconstruction Using (27)*:* The proposed collective neurodynamic approach (27) is applied to solve problem (26) to obtain the sparse signal. Taking a matrix $A_{64\times128}$, we divide it into five parts randomly based on row decomposition to obtain $\boldsymbol{A} = \text{block}\{A_{m_1\times n}, A_{m_2\times n}, A_{m_3\times n}, A_{m_4\times n}, A_{m_5\times n}\} \in R^{64\times640}$. Every $A_{m_i\times n}\boldsymbol{x}_i$, $i = 1,\ldots,5$ acts as an agent, and five agents are connected as a ring diagram in Fig. 7 (left). Let $s = 10$ and the initial values be random values. Fig. 7 (middle) visualizes that the output state $\boldsymbol{x}(t)$ is uniformly (i.e., $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{x}_3 = \boldsymbol{x}_4 = \boldsymbol{x}_5 \in R^{128}$) and globally asymptotically stable, which matches the conclusion of Theorem 4. Fig. 7 (right)

Fig. 5. (a) Original image. (b) Restored images with approach (8) (m = 190, PSNR = 35.80 dB). (c) Restored images with OMP [12] (m = 190, PSNR = 30.76 dB, OMP). (d) Restored images with PNNSR [16] (m = 190, PSNR = 34.85 dB, PNNSR).

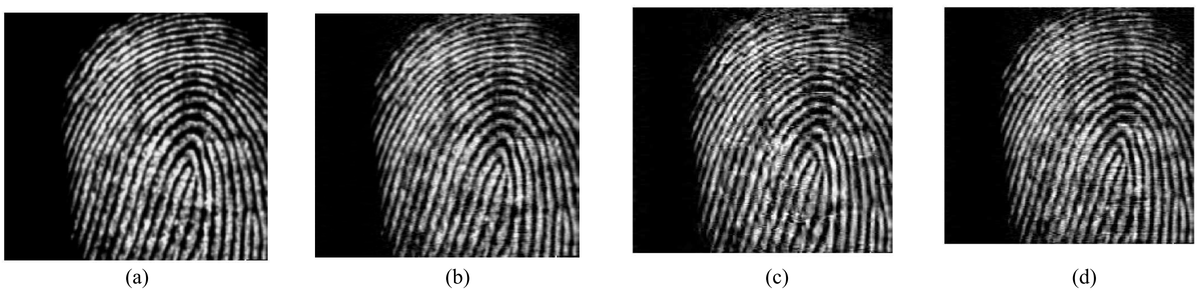

Fig. 6. (a) Original image. (b) Restored images with approach (9) (m = 150, PSNR = 23.79 dB). (c) Restored images with OMP [12] (m = 150, PSNR = 20.12 dB, OMP, OMP). (d) Restored images with PNNSR [16] (m = 150, PSNR = 21.782 dB, PNNSR).

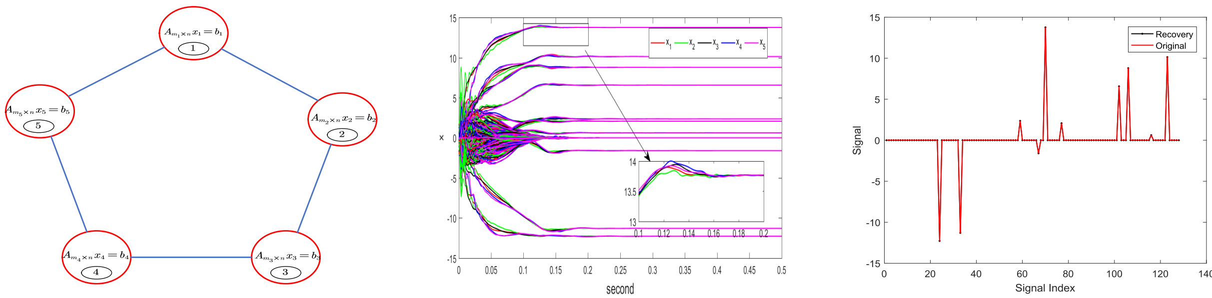

Fig. 7. (Left) Graph of five interacting neurodynamic with decentralized constraints. (Middle) Consensus of the output states transition curves on (27). (right) Recovered signal with RelErr $= 1.92e$–17.

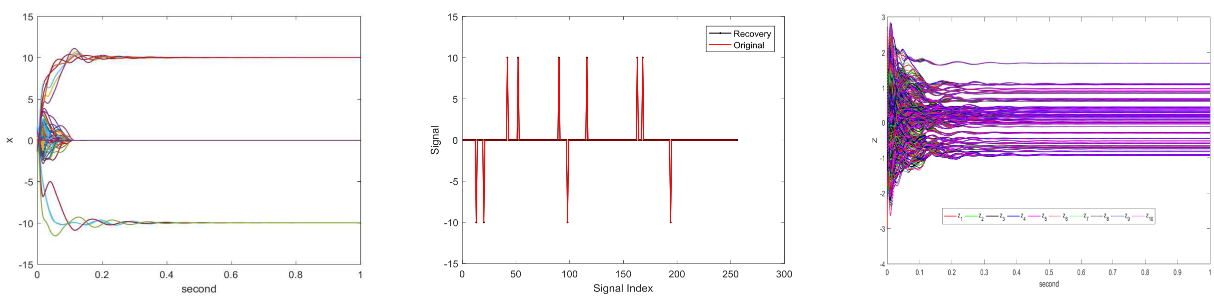

Fig. 8. (Left) States transition curves on (8). (Middle) Recovered signal with RelErr $= 2.47e$–16. (Right) Consensus of states transition curves of $\mathbf{z}$.

shows that its stable point is very close to the original sparse signal with a relative error of $1.92e$–17. Thus, the proposed collective neurodynamic approach (27) can effectively solve problem (26), that is, solving problem (2) in a distributed way. In addition, the results of Fig. 7 further demonstrate the equivalent validity of distributed problem (26) and centralized problem (2).

2) Sparse Signal Reconstruction Using (37)*:* In the example, we validate the effectiveness of collective neurodynamic approach (37) to deal with the problem (33) by a sparse signal reconstruction test. Take $A_{64\times 256}$ a sensing matrix and divide it into ten parts by column ($\mathbf{A} = \text{block}\{A_{m\times n_1}, \ldots, A_{m\times n_{10}}\} \in R^{640\times 256}$). Let every parts be an agent, and ten agents are connected as a ring diagram. Set sparsity $s = 10$, and generate a real signal with an amplitude of ± 10 randomly. We repeat our experiment 100 times with different random initial values and plot the average of them in Fig. 8. According to Fig. 8 (left), the proposed approach (37) is globally asymptotically stable. Fig. 8 (right) plots that the state $\mathbf{z}$ is uniformly convergent and globally asymptotically stable, which means that $\mathbf{z}_1 = \cdots = \mathbf{z}_{10}$. It meets the conclusion of Theorem 6. From Fig. 8 (middle), $\mathbf{x}(t)$'s stable point can effectively approximate the original sparse signal with a relative error of 2.47e–16, and it implies that the approach (37) can effectively solve problem (33) in a distributed manner and further demonstrates the validity of the transformed distributed problem (33).

VI. Conclusion

In this article, we have proposed several centralized and collective neurodynamic approaches for L_1-minimization problem to reconstruct the sparse signal. Then, we have found that the L_1-minimization problem can be converted into two network optimization problems, i.e., a distributed optimization problem with consensus constraints and a distributed extended monotropic optimization problem. Based on the above centralized neurodynamic approaches and multiagent consensus theory, two collective neurodynamic approaches have been developed and their convergence and optimality have been discussed. Simulation results of sparse signal reconstruction and image reconstruction have been done to test the feasibility and effectiveness of our proposed neurodynamic approaches. Considering the finite-time convergence and fixed-time convergence that have remarkable convergence rate, we will try to modify the centralized nueroynamic approaches (8) and (9) for solving problem (2) with finite-, fixed-, and predefined-time convergence in the future work.

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You Zhao received the M.S. degree in signal and information processing from the School of Electronics and Information Engineering, Southwest University, Chongqing, China, in 2018. He is currently pursuing the Ph.D. degree with the College of Computer Science, Chongqing University, Chongqing.

His current research interests include neurodynamic optimization, distributed optimization, compressed sensing, and smart grid.

Xiaofeng Liao (Fellow, IEEE) received the B.S. and M.S. degrees in mathematics from Sichuan University, Chengdu, China, in 1986 and 1992, respectively, and the Ph.D. degree in circuits and systems from the University of Electronic Science and Technology of China, Chengdu, in 1997.

From 1999 to 2012, he was a Professor with Chongqing University, Chongqing, China. From July 2012 to July 2018, he was a Professor and the Dean of the College of Electronic and Information Engineering, Southwest University, Chongqing. He is currently a Professor and the Dean of the College of Computer Science, Chongqing University. He is also a Yangtze River Scholar of the Ministry of Education of China, Beijing, China. From November 1997 to April 1998, he was a Research Associate with the Chinese University of Hong Kong, Hong Kong. From October 1999 to October 2000, he was a Research Associate with the City University of Hong Kong, Hong Kong. From March 2001 to June 2001 and March 2002 to June 2002, he was a Senior Research Associate with the City University of Hong Kong, Hong Kong. From March 2006 to April 2007, he was a Research Fellow with the City University of Hong Kong. He has published four books and over 300 international journal and conference articles. He holds five patents. His current research interests include optimization and control, machine learning, neural networks, bifurcation and chaos, and cryptography.

Prof. Liao currently serves as an Associate Editor for the IEEE TRANSACTIONS ON CYBERNETICS and the IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS.

Xing He received the B.S. degree in mathematics and applied mathematics from the Department of Mathematics, Guizhou University, Guiyang, China, in 2009, and the Ph.D. degree in computer science and technology from Chongqing University, Chongqing, China, in 2013.

From November 2012 to October 2013, he was a Research Assistant with Texas A&M University at Qatar, Doha, Qatar. From December 2015 to February 2016, he was a Senior Research Associate with the City University of Hong Kong, Hong Kong. He is currently a Professor with the School of Electronics and Information Engineering, Southwest University, Chongqing. His research interests include neural networks, bifurcation theory, optimization method, smart grid, and nonlinear dynamical system.

Rongqiang Tang received the M.S. degree in applied mathematics from Chongqing Normal University, Chongqing, China, in 2018. He is currently pursuing the Ph.D. degree with Chongqing University, Chongqing.

His research interests include collective behavior in complex dynamical networks, chaos synchronization, control theory, and neural networks.