Non-Euclidean Mixture Model for Social Network Embedding

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Abstract

It is largely agreed that social network links are formed due to either homophily or social influence. Inspired by this, we aim at understanding the generation of links via providing a novel embedding-based graph formation model. Different from existing graph representation learning, where link generation probabilities are defined as a simple function of the corresponding node embeddings, we model the link generation as a mixture model of the two factors. In addition, we model the homophily factor in spherical space and the influence factor in hyperbolic space to accommodate the fact that (1) homophily results in cycles and (2) influence results in hierarchies in networks. We also design a special projection to align these two spaces. We call this model Non-Euclidean Mixture Model, i.e., **NMM**. We further integrate **NMM** with our non-Euclidean graph variational autoencoder (VAE) framework, **NMM-GNN**. **NMM-GNN** learns embeddings through a unified framework which uses non-Euclidean GNN encoders, non-Euclidean Gaussian priors, a non-Euclidean decoder, and a novel space unification loss component to unify distinct non-Euclidean geometric spaces. Experiments on public datasets show **NMM-GNN** significantly outperforms state-of-the-art baselines on social network generation and classification tasks, demonstrating its ability to better *explain* how the social network is formed.

1 Introduction

Social networks are omnipresent because they are used for modeling interactions among users on social platforms. Social network analysis plays a key role in several applications, including detecting underlying communities among users [1], classifying people into meaningful social classes [2], and predicting user connectivity [3]. Most existing embedding models are designed based on the ***homophily*** aspect of social networks [4, 5]. They utilize the intuition that associated nodes in a social network imply feature similarity, and an edge is usually generated between similar nodes. Prior works have used shallow embedding models to represent homophily, like matrix factorization and random-walk (Section 2), which are parameter intensive and do not employ message passing. As an improvement, graph neural network (GNN) models (Section 2) have been proposed to more effectively capture homophily by representing a node through its local neighborhood context.

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However, research of **RaRE** [6] and work of [7] show homophily is insufficient, and ***social influence*** is also critical in forming connections. This is due to popular nodes having direct influence in forming links [8]. For example, in Twitter network, users tend to follow celebrities *in addition to* users who share similar interests [9]. Though **RaRE** jointly models both factors, it has limitations in modeling capabilities. Specifically, it is parameter intensive as each node embedding is fully parameterized through a Bayesian framework. Further, **RaRE** assumes graphs are transductive, limiting its performance in the practical inductive setting where new links not seen during training must be predicted. Moreover, nearly all works embedding social networks utilize a single zero-curvature Euclidean space, when in reality, network factors may create different topologies. Specifically, edges generated by homophily tend to form cycles [10], while edges generated by social influence tend to form tree structures [11, 12]. From Riemannian geometry, people have found that networks with cycles are best represented by spherical space embeddings [13], while tree structured networks are best represented by hyperbolic space embeddings [14]. Thus, an end-to-end model to bridge social network embeddings of distinct non-Euclidean geometric spaces is a promising direction.

Our motivation is two-fold: (1) We aim to *understand* how the social network is generated e.g., which factors affect node connectivity and what topological patterns emerge in the network as a result. (2) Using our learning from (1), we aim to design a more realistic deep learning model to *explain* how the network is generated (inferring new connections). We summarize our contributions as follows:

- We propose Graph-based Non-Euclidean Mixture Model (**NMM**) to explain social network generation. **NMM** represents nodes via joint influence by homophily (modeled in spherical space) and social influence (modeled in hyperbolic space), while seamlessly unifying embeddings via our space unification loss.
- To our knowledge, we are also the first to couple **NMM** with a graph-based VAE learning framework, **NMM-GNN**. Specifically, we introduce a novel non-Euclidean VAE framework where node embeddings are learned with a powerful encoder of GNNs using spherical and hyperbolic spaces, non-Euclidean Gaussian priors, and unified non-Euclidean optimization.
- Extensive experiments on several real-world datasets on large-scale social networks, Wikipedia networks, and attributed graphs demonstrate effectiveness of **NMM-GNN** in social network generation and classification, which outperforms state-of-the-art (SOTA) network embedding models.

2 Preliminary and Related Work

We provide an overview of social network embedding models and discuss advancements in non-Euclidean graph learning.

2.1 Social Network Embedding

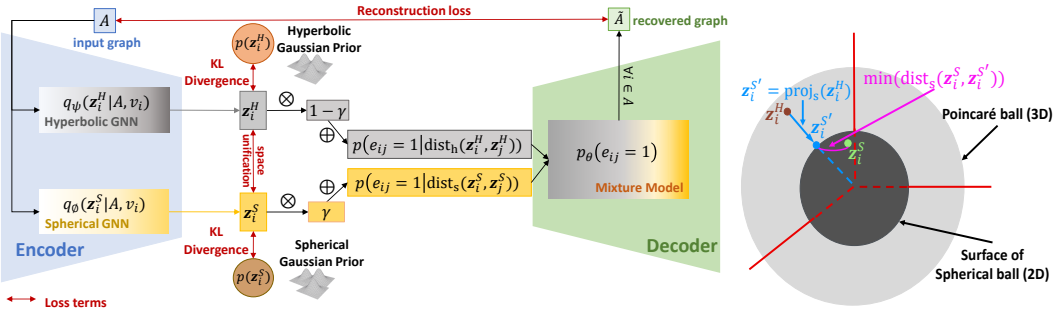
Several works that embed social networks merely model homophily [15], capturing node-node similarity, without also considering a node's social influence or node popularity e.g., a celebrity. Homophily-based models include shallow embedding models and GNN embedding models. Most shallow embedding models are either based on matrix factorization [16] or random-walk [17]. Though GNN models [18] effectively learn on large networks and in inductive settings, they still fail to model the social influence factor in the network. Further, even the model that captures both homophily and social influence, **RaRE** [6], has limitations in that its fully-parameterized node embeddings require large parameter size to be learned, and it models all nodes in Euclidean space, an approach not effective in capturing different topologies (e.g., cycles and hierarchy) in the social network.

2.2 Non-Euclidean Geometry for Graphs

Non-Euclidean geometric spaces, commonly used to model surfaces in mathematics and physics, are curved geometries that include spherical spaces with positive curvature, and hyperbolic spaces with negative curvature [19]. Works have recently found Euclidean space modeling to be insufficient for non-Euclidean graph-structured data [20]. Namely, spherical spaces have been shown to effectively embed graphs with cyclic structure due to their positive curvature [21, 13], while hyperbolic spaces have been shown to effectively embed graphs with hierarchical structure due to their negative curvature and exponential or "tree-like" growth of the space [22, 23]. HGCN [24] and [25] are critical works exploring GNNs in non-Euclidean spaces. Both these hyperbolic GNN methods have shown

significant improvement on benchmark datasets by preserving hierarchical structure of graphs. In knowledge graphs, [26] has achieved notable performance by modeling graphs in non-Euclidean spaces of various curvatures, using both hyperbolic and spherical space.

3 Methodology



(a) (b)

Figure 1: Model Architecture. (a) Architecture overview of **NMM-GNN**, a non-Euclidean Mixture Model with non-Euclidean VAE framework. (b) Illustration of space unification loss of **NMM-GNN** . $\boldsymbol{z}_i^H$ from the hyperbolic space is projected to its corresponding point in the spherical space, $\boldsymbol{z}_i^{S'}$, such that its geodesic distance is ensured to be close to v_i's existing spherical space representation, $\boldsymbol{z}_i^S$.

A social network $G = (V, A)$ consists of a set of vertices $V = \{v_i\}_{i=1}^N$, and associated adjacency matrix $e_{ij} \in A$, where e_{ij} is an edge from v_i to v_j. We aim to design a model to jointly learn both node homophily and social influence representation, denoted $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_i^H$ respectively, that can best explain the social network in terms of link reconstruction. In this section, we describe our architecture. First, we introduce a novel non-Euclidean mixture model, called **NMM**, to model the probability of a new link. Second, we add a non-Euclidean GNN encoder to enrich **NMM**, called **NMM-GNN**, which is connected to the variational autoencoder framework. Source code is in the Appendix. For details on future directions, the reader is also referred to the Limitations section in the Appendix.

3.1 Overview

To model both factors of homophily and social influence that affect graph connectivity, we model each node v_i with homophily regulated representation, $\boldsymbol{z}_i^S$, and social influence regulated representation, $\boldsymbol{z}_i^H$ [1]. A unified framework is designed such that $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_i^H$ influence each other bi-directionally and are seamlessly merged through **NMM**. **NMM** utilizes non-Euclidean geometric spaces to better represent homophily and social influence components which produce curved structures like cycles and trees. To improve **NMM**, we enrich its encoder by GNNs, $q_\psi(\boldsymbol{z}_i^H|G)$ and $q_\phi(\boldsymbol{z}_i^S|G)$. To ensure generated embeddings are in spherical and hyperbolic spaces respectively, non-Euclidean GNNs are adopted. The enhanced model is **NMM-GNN**, which has a clear connection to the VAE framework.

Figure 1(a) illustrates the architecture overview of **NMM-GNN**. Specifically, the encoder component maps nodes into homophily based embedding, $\boldsymbol{z}^S$, in spherical space (for homophily generated cycles) and social influence based embedding, $\boldsymbol{z}^H$, in hyperbolic space (for social influence generated trees), which follow non-Euclidean prior distributions. The embeddings are passed into our mixture model decoder, which models the probability of a link as a mixture of a homophily based distribution component and a social influence based distribution component. The objective is to maximize the likelihood to observe the links, or equivalently to minimize the link reconstruction loss. In addition, the two geometric spaces are ensured to be aligned together via a space unification regularization term, to make sure the two embeddings of the same node are corresponding to each other.

3.2 Modeling via Non-Euclidean Mixture Model (NMM)

Geometrically, the space of **NMM** is a spherical surface inside a unit Poincaré ball. Each node v_i corresponds to (1) a point in the Poincaré ball, $\boldsymbol{z}_i^H$, and (2) a point on the spherical surface,

[1] The geometric space community conventionally uses $\boldsymbol{S}$ to represent spherical space and $\boldsymbol{H}$ to represent hyperbolic space.

$\boldsymbol{z}_i^S$, where both spaces are embedded inside a Euclidean space. These two points are aligned by enforcing projection of $\boldsymbol{z}_i^H$ onto the spherical surface to be close to $\boldsymbol{z}_i^S$, as projecting a star onto earth's atmosphere, shown in Fig. 1(b). We note that the space alignment does not mean embeddings are enforced to be the same in two spaces[2]. Rather, we require the projection of $\boldsymbol{z}_i^H$ is close to i's embedding in spherical space $\boldsymbol{z}_i^S$. In this case, the two distances of spherical and hyperbolic spaces are also different from each other (norm space difference vs. spherical geodesic distance). Without the space alignment, $\boldsymbol{z}_i^H$ has too much degree of freedom, which can move freely as long its norm is kept the same. Lastly, the probability of generating a link is a mixture of the probability of generating the link in each non-Euclidean space.

3.2.1 Modeling for homophily

Representation of homophily regulated nodes. We embed homophily based representation of node v_i, or $\boldsymbol{z}_i^S$, on the surface of the spherical ball in spherical space, $\mathbb{S}^d$. Formally, $\mathbb{S}^d = \{\boldsymbol{z}_i^S \in \mathbb{R}^d \big| \|\boldsymbol{z}_i^S\| = w^S\}$ is the d-dimensional w^S-norm ball, where $\|\cdot\|$ is the Euclidean norm and $w^S \in [0, 1)$ is a constant to ensure the spherical surface is inside the unit Poincaré ball. To better capture homophily distribution, $p(\boldsymbol{z}_i^S)$, we use spherical Gaussian distribution $G_S(\cdot)$ as spherical prior, described in Table 1. β is the axis or *direction* of the lobe controlling where the lobe is located on the sphere, and points towards the center of the lobe; λ is the *sharpness* of the lobe such that as this value increases, the lobe will become narrower in width; and a is the amplitude or *intensity* of the lobe, corresponding to the height of the lobe at its peak.

Link prediction using homophily based distribution. The probability of link $e_{ij} = 1$ between nodes v_i and v_j, also described in Table 1, is determined by the geodesic distance between $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_j^S$. $\mathrm{dist}_{\mathrm{s}}(\boldsymbol{z}_i^S, \boldsymbol{z}_j^S) = \arccos(\langle \boldsymbol{z}_i^S, \boldsymbol{z}_j^S \rangle)$ is the geodesic distance between $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_j^S$; $J > 0$ and $B \geq 0$ are model parameters; and $\langle \cdot, \cdot \rangle$ denotes the inner product. The probabilistic model is designed so nodes greatly dissimilar (exhibiting low homophily), or $\mathrm{dist}_{\mathrm{s}}(\boldsymbol{z}_i^S, \boldsymbol{z}_j^S) \to +\infty$, have low probability of a link being generated, or $p_{\mathrm{hom}}(e_{ij} = 1) \to 0$. In the case of nodes exhibiting high homophily, they either (1) may be connected due to highly similar characteristics, or (2) may not be connected simply because the individuals do not know each other. Our distribution models both scenarios. Note when $\mathrm{dist}_{\mathrm{s}}(\boldsymbol{z}_i^S, \boldsymbol{z}_j^S) \to 0$, $p_{\mathrm{hom}}(e_{ij} = 1) \to \frac{1}{1+e^B}$, which can be interpreted as a factor to control sparsity of the network.

Table 1: Homophily regulated nodes: Distribution Prior and Link Generation Probability

(a) Spherical Distribution Prior

$$\begin{aligned} p(\boldsymbol{z}_i^S) &= G_S(\boldsymbol{z}_i^S; \beta, \lambda, a) \\ &= a e^{\lambda(\beta \cdot \boldsymbol{z}_i^S - 1)} \end{aligned}$$

(b) Link Generation Probability

$$\begin{aligned} p_{\mathrm{hom}}(e_{ij} = 1) &= p(e_{ij} = 1 | \mathrm{dist}_{\mathrm{s}}(\boldsymbol{z}_i^S, \boldsymbol{z}_j^S)) \\ &= \frac{1}{1 + e^{J \times \mathrm{dist}_{\mathrm{s}}(\boldsymbol{z}_i^S, \boldsymbol{z}_j^S) + B}} \end{aligned}$$

3.2.2 Modeling for social influence

Representation of social influence regulated nodes. We embed social influence based representations of node v_i, or $\boldsymbol{z}_i^H$, on the Poincaré ball from hyperbolic space, $\mathbb{H}^{d+1}$, to better capture resulting hierarchical structures. Social influence regulated nodes are represented as points, $\boldsymbol{z}_i^H$, belonging inside the Poincaré (open) ball in $\mathbb{H}^{d+1}$. Formally, $\mathbb{H}^{d+1} = \{\boldsymbol{z}_i^H \in \mathbb{R}^{d+1} \big| \|\boldsymbol{z}_i^H\| = w_{z_i}^H\}, w_{z_i}^H \in [0, 1)$, is $(d+1)$-dimensional $w_{z_i}^H$-norm ball, where $\|\cdot\|$ is Euclidean norm. We assume center of the Poincaré ball is aligned with center of the sphere, and is one dimension larger than the sphere to ensure the spherical surface is inside the Poincaré ball. To better capture a social influence regulated distribution, $p(\boldsymbol{z}_i^H)$, we use Hyperbolic Gaussian distribution $G_H(\cdot)$ as non-Euclidean Gaussian prior, described in Table 2. $\overline{\boldsymbol{z}_i^H}$ is the origin of (r, ω) for radius r and angle ω in polar coordinates. $\overline{\boldsymbol{z}_i^H}$ is the center of mass and $\zeta > 0$ is the dispersion parameter, where the dispersion dependent normalization constant $Z(\zeta)$ accounts for the underlying non-Euclidean geometry. $Z(\zeta)$ is partitioned into angular ω and radial r components. $\Gamma(\cdot)$ is Euler's gamma function, and $\mathrm{dist}_{\mathrm{H}}(\cdot)$ is the hyperbolic distance between two hyperbolic space node embeddings, $\boldsymbol{x}$ and $\boldsymbol{y}$, where $\|\cdot\|$ denotes Euclidean norm.

[2]Note that it is impossible to equate these two embeddings directly as they are in different geometric spaces.

Link prediction using social influence based distribution. We model existence of an edge, $e_{ij} = 1$ between nodes v_i and v_j, also described in Table 2, as a function of norm space difference, $\mathrm{dist_h}(\cdot)$, between two hyperbolic space node embeddings, $\boldsymbol{z}_i^H$ and $\boldsymbol{z}_j^H$. We utilize norm space difference as opposed to hyperbolic distance, since nodes of similar social influence status may have large distance in the Poincaré (open) ball due to nodes possibly being placed towards the ball's boundary. This is because, at the boundary of the ball, nodes become infinitely distanced apart. Thus, to allow for numerical stability and to capture social influence difference (in which the higher the social influence of nodes indicated by large in-degree and smaller out-degree, the closer they are embedded towards the center of the ball), we utilize norm space as indicator. Consistent with the notation of [6], a node with higher social influence is associated with a smaller norm value. C and D are learned model parameters and norm space of a vector is $\mathrm{norm}(\boldsymbol{x}) = \|\boldsymbol{x}\|$. $\mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H) = |\mathrm{norm}(\boldsymbol{z}_i^H) - \mathrm{norm}(\boldsymbol{z}_j^H)|$ is the norm difference between $\boldsymbol{z}_i^H$ and $\boldsymbol{z}_j^H$; $C > 0$ and $D \geq 0$ are model parameters; and $\mathrm{norm}(\cdot)$ denotes the L1 normalization function. The probabilistic model is designed such that nodes largely different in popularity (exhibiting high social influence), or $\mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H) \to +\infty$, have high probability of a link being generated, or $p_{\mathrm{rank}}(e_{ij} = 1) \to 1$. In the case both nodes exhibit low social influence (such as having similar social rank), they either (1) may be connected due to highly similar characteristics, or (2) may not be connected simply because individuals do not know each other. Our distribution models both scenarios. When $\mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H) \to 0$, $p_{\mathrm{rank}}(e_{ij} = 1) \to \frac{e^D}{1+e^D}$, which can be interpreted as another factor to control sparsity of the network.

Table 2: Social influence regulated nodes: Distribution Prior and Link Generation Probability

(a) Hyperbolic Distribution Prior

$$p(\boldsymbol{z}_i^H) = G_H(\boldsymbol{z}_i^H; \overline{\boldsymbol{z}_i^H}, \zeta) \tag{1}$$
$$= \frac{1}{Z(\zeta)} e^{-\frac{\mathrm{dist_H}(\boldsymbol{z}_i^H, \overline{\boldsymbol{z}_i^H})^2}{2\zeta^2}}$$
$$Z(\zeta) = Z_\omega(\zeta) Z_r(\zeta) \tag{2}$$
$$Z_\omega(\zeta) = \mathrm{Vol}(\mathbb{H}^d) \tag{3}$$
$$= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$$
$$Z_r(\zeta) = \int_0^{+\infty} e^{-\frac{r^2}{2\zeta^2}} \sinh^d(r)\, dr \tag{4}$$
$$= \frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (-1)^k \sqrt{\frac{\pi}{2}} \zeta e^{\frac{(2k-d)^2\zeta^2}{2}} \mathrm{erfc}\big(\frac{(2k-d)\zeta}{\sqrt{2}}\big)$$
$$\mathrm{dist_H}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{arrcosh}\big(1 + \frac{2\|\boldsymbol{x}-\boldsymbol{y}\|^2}{(1-\|\boldsymbol{x}\|^2)(1-\|\boldsymbol{y}\|^2)}\big) \tag{5}$$

(b) Link Generation Probability

$$p_{\mathrm{rank}}(e_{ij} = 1) = p(e_{ij} = 1 | \mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H)) = \frac{e^{C \times \mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H) + D}}{1 + e^{C \times \mathrm{dist_h}(\boldsymbol{z}_i^H, \boldsymbol{z}_j^H) + D}}$$

3.2.3 Non-Euclidean Mixture Model

Link Prediction Using Mixed Space Distribution. Since both homophily and social influence affect the connectivity structure of social networks, we model existence of a new link between nodes, $p_\theta(e_{ij} = 1)$, as a weighted combination distribution of these factors. Specifically, our non-Euclidean mixture model is a weighted combination of homophily based distribution, $p_{\mathrm{hom}}(e_{ij} = 1)$, and social influence based distribution, $p_{\mathrm{rank}}(e_{ij} = 1)$, with learned weight γ:

$$\begin{aligned} p_\theta(e_{ij} = 1) &= p_\theta(e_{ij} = 1 | \boldsymbol{z}_i^S, \boldsymbol{z}_j^S, \boldsymbol{z}_i^H, \boldsymbol{z}_j^H) \\ &= \gamma \cdot p_{\mathrm{hom}}(e_{ij} = 1) + (1-\gamma) \cdot p_{\mathrm{rank}}(e_{ij} = 1) \end{aligned} \tag{6}$$

where $p_{\mathrm{hom}}(e_{ij} = 1)$ is modeled in positively curved spherical space since homophily based links may form cycles due to similarity connections between node clusters. $p_{\mathrm{rank}}(e_{ij} = 1)$ is modeled in negatively curved hyperbolic space since social influence based links may form tree-like structures due to popularity-based social hierarchy between node clusters. We would like to highlight that the link between nodes i and j is a mixture model because each link is a weighted combination of influence from both spherical and hyperbolic spaces (not one or the other) as evidenced in Equation 6. As shown in Figure 1b, the same node has two representations – one in the spherical space and one in the hyperbolic space, and because they represent the same underlying node, they need to be aligned. Therefore, these two network factors do not contradict each other, but rather *work together* to explain how links are formed between users.

3.3 Modeling via Non-Euclidean VAE on Graphs

We enrich the encoder of **NMM** to generate better embeddings. To do so, we explore GNN methods which have been shown to be more effective than shallow embedding methods (see Experiments and Ablation Studies). We refer to this enriched **NMM** model as **NMM-GNN**. **NMM-GNN** uses non-Euclidean VAE as its learning framework. The framework integrates a mixture of different non-Euclidean geometric spaces e.g., hyperbolic and spherical spaces, for learning of encoder, decoder, and node prior distributions, and ensures geometric spaces are unified during training.

Encoder model. The encoder learns two corresponding embedding representations per node v_i to produce spherical embedding $\boldsymbol{z}_i^S$ and hyperbolic embedding $\boldsymbol{z}_i^H$. For homophily regulated nodes in spherical space, $\mathbb{S}^d$, any spherical space GNN (SGNN) can be applied, and for social influence regulated nodes in hyperbolic space, $\mathbb{H}^{d+1}$, any hyperbolic space GNN (HGNN) can be applied. The general framework for SGNN and HGNN are shown in Tables 3 and 4. $\boldsymbol{z}_i^{S^{(l)}} \in \mathbb{R}^{d^{(l)}}$ and $\boldsymbol{z}_i^{H^{(l)}} \in \mathbb{R}^{d+1^{(l)}}$ are spherical and hyperbolic feature representations of node v_i at layer l, with dimensionality d and $(d+1)$ respectively. f is a message-specific neural network function of incoming messages to v_i from neighborhood context N_i, and activation function $\sigma(\cdot)$, typically $\mathrm{ReLU}(\cdot)$ for all layers but the last one being $\mathrm{softmax}(\cdot)$.

Table 3: General Framework and Example Model for Spherical Graph Neural Network.

(a) General Framework: Spherical GNN

$$\begin{aligned}\boldsymbol{z}_i^{S^{(l+1)}} &= q_{\mathrm{SGNN}}(\boldsymbol{z}_i^S|A, v_i)\\ &= q_\phi(\boldsymbol{z}_i^S|A, v_i)\\ &= \sigma\big(\sum_{j\in N_i} f(\boldsymbol{z}_i^{S^{(l)}}, \boldsymbol{z}_j^{S^{(l)}})\big)\end{aligned}$$

(b) Example: Spherical GCN

$$\begin{aligned}\boldsymbol{z}_i^{S^{(l+1)}} &= q_\phi(\boldsymbol{z}_i^S|A, v_i)\\ &= \exp_0^S\Big(\sigma\Big(\boldsymbol{W}_l^T\big(\sum_{j\in N_i\cup\{i\}} \frac{e_{j,i}}{\sqrt{m_j m_i}}\log_0^S(\boldsymbol{z}_j^{S^{(l)}})\big)\Big)\Big)\end{aligned}$$

Table 4: General Framework and Example Model for Hyperbolic Graph Neural Network.

(a) General Framework: Hyperbolic GNN

$$\begin{aligned}\boldsymbol{z}_i^{H(l+1)} &= q_{\mathrm{HGNN}}(\boldsymbol{z}_i^H|A, v_i)\\ &= q_\psi(\boldsymbol{z}_i^H|A, v_i)\\ &= \sigma\big(\sum_{j\in N_i} f(\boldsymbol{z}_i^{H^{(l)}}, \boldsymbol{z}_j^{H^{(l)}})\big)\end{aligned}$$

(b) Example: Hyperbolic GCN

$$\begin{aligned}\boldsymbol{z}_i^{H(l+1)} &= q_\psi(\boldsymbol{z}_i^H|A, v_i)\\ &= \exp_0^H\Big(\sigma\Big(\boldsymbol{W}_l^T\big(\sum_{j\in N_i\cup\{i\}} \frac{e_{j,i}}{\sqrt{m_j m_i}}\log_0^H(\boldsymbol{z}_j^{H^{(l)}})\big)\Big)\Big)\end{aligned}$$

Example Non-Euclidean Encoder models. We illustrate Non-Euclidean Graph Convolutional Neural Network (Non-Euclidean GCN) as an example GNN. Tables 3 and 4 describe the model architectures for both Spherical GCN and Hyperbolic GCN respectively. Node features are initialized by sampling each embedding dimension from a distribution uniformly at random for all nodes where $\boldsymbol{z}_i^{S^{(0)}} \in \mathbb{R}^{d^{(l)}} \leftarrow \mathrm{Unif}([0,1))^d$ and $\boldsymbol{z}_i^{H^{(0)}} \in \mathbb{R}^{d+1^{(l)}} \leftarrow \mathrm{Unif}([0,1))^{d+1}$ respectively. The retraction operator, $\mathcal{R}(\cdot)$, involves mapping between spaces. For non-Euclidean spaces, retraction is performed between non-Euclidean space and approximate tangent Euclidean space using logarithmic and exponential map functions. Specifically, $\log_0^H(\boldsymbol{z}_i^H) = \tanh^{-1}(\mathrm{i}\cdot\|\boldsymbol{z}_i^H\|)\frac{\boldsymbol{z}_i^H}{\mathrm{i}\cdot\|\boldsymbol{z}_i^H\|}$ is a logarithmic map at center $\mathbf{0}$ from hyperbolic space to Euclidean tangent space, and $\exp_0^H(\boldsymbol{z}_i^H) = \tanh(\mathrm{i}\cdot\|\boldsymbol{z}_i^H\|)\frac{\boldsymbol{z}_i^H}{\mathrm{i}\cdot\|\boldsymbol{z}_i^H\|}$ is an exponential map at center $\mathbf{0}$ from Euclidean tangent space to hyperbolic space. $\log_0^S(\boldsymbol{z}_i^S) = \tanh^{-1}(\|\boldsymbol{z}_i^S\|)\frac{\boldsymbol{z}_i^S}{\|\boldsymbol{z}_i^S\|}$ is a logarithmic map at center $\mathbf{0}$ from spherical space to Euclidean tangent space and $\exp_0^S(\boldsymbol{z}_i^S) = \tanh(\|\boldsymbol{z}_i^S\|)\frac{\boldsymbol{z}_i^S}{\|\boldsymbol{z}_i^S\|}$ is an exponential map at center $\mathbf{0}$ from Euclidean tangent space to spherical space. where $\boldsymbol{z}_i^{S^{(l)}}$, $\boldsymbol{z}_i^{H^{(l)}}$ are embeddings of node v_i at layer $l \in [0, L)$, $L = 2$; $\boldsymbol{W}_l$ is a layer-specific learnable weight matrix; N_i is the set of nodes in the neighborhood context of v_i; $e_{j,i}$ is the edge-weight between nodes $v_j \to v_i$, with default edge weight being 1.0 if an edge exists. m_i,m_j are entries of the degree matrix, with $m_i = 1 + \sum_{j\in N_i} e_{j,i}$.

Decoder model. **NMM** can be considered as a probabilistic decoder for link generation, which maps embeddings of two nodes into the probablity to generate a link between them.

Joint loss function. The training loss involves components of reconstruction loss (to ensure the generated graph is consistent with the original graph), KL divergence loss (to ensure predicted embeddings $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_i^H$ closely match their non-Euclidean Gaussian distributions), and space unification loss (to ensure $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_i^H$ map to the same node v_i).

Reconstruction Loss. Table 5 shows reconstruction loss, which minimizes the upper bound on the negative log-likelihood. λ_A is a hyperparameter; $A' = XAX^T$, given $X \in 0, 1^{k \times d}$ where $X_{a,i} = 1$ only if node $a \in \tilde{G}$ is assigned to $i \in G$ and $X_{a,i} = 0$ otherwise, where $\tilde{G}$ is the predicted graph.

Table 5: Description for Reconstruction Loss.

$$p(A'|\boldsymbol{z}^S, \boldsymbol{z}^H) = \frac{1}{k(k-1)} \sum_{a \neq b} A'_{a,b} \log \tilde{A}_{a,b} + (1 - A'_{a,b}) \log(1 - \tilde{A}_{a,b}) \quad (7)$$

$$-\log p_\theta(G|\boldsymbol{z}_i^S, \boldsymbol{z}_j^S, \boldsymbol{z}_i^H, \boldsymbol{z}_j^H) = -\lambda_A \log p(A'|\boldsymbol{z}_i^S, \boldsymbol{z}_j^S, \boldsymbol{z}_i^H, \boldsymbol{z}_j^H) \quad (8)$$

$$L^S_{\text{recon}}(\phi, \theta; G) = \mathbb{E}_{q_\phi(\boldsymbol{z}_i^S|G)}[-\log p_\theta(G|\boldsymbol{z}_i^S, \boldsymbol{z}_j^S, \boldsymbol{z}_i^H, \boldsymbol{z}_j^H)] \quad (9)$$

$$L^H_{\text{recon}}(\psi, \theta; G) = \mathbb{E}_{q_\psi(\boldsymbol{z}_i^H|G)}[-\log p_\theta(G|\boldsymbol{z}_i^S, \boldsymbol{z}_j^S, \boldsymbol{z}_i^H, \boldsymbol{z}_j^H)] \quad (10)$$

KL Divergence Loss. The KL divergence loss is formed by minimizing the equations described in Table 6. Minimizing the KL divergence loss ensures that the homophily regulated nodes and social influence regulated nodes closely align to their underlying non-Euclidean distribution priors. As described in Section 3, these distributions are designed to appropriately capture the distinct topologies that emerge as a result of the respective social network factors.

Space Unification Loss. The space unification loss is formed by minimizing the equations described in Table 6. Minimizing the space unification loss ensures that the hyperbolic space representation of node v_i in spherical space, $\text{proj}_S(\boldsymbol{z}_i^H)$, is close to the corresponding learned representation of node v_i in the spherical space, $\boldsymbol{z}_i^S$. Note that using a normalized hyperbolic disk is not a substitute for the projection operator from the social influence hyperbolic space onto the homophily spherical space. The projection operation *solely* projects out-of-sphere nodes onto the w^S norm space, or the norm at the surface of the spherical ball. A normalization operator would instead change the embedding values of all nodes. More importantly, the space unification loss component ensures minimal spherical geodesic distance between $\boldsymbol{z}_i^H$'s representation on the spherical ball and $\boldsymbol{z}_i^S$, illustrated in Figure 1(b).

Table 6: Description of KL Divergence Loss and Space Unification Loss.

(a) KL Divergence Loss

$$L^S_{\text{KL}}(\phi; G) = \text{KL}[q_\phi(\boldsymbol{z}_i^S|G)||p(\boldsymbol{z}_i^S)] \quad (11)$$

$$L^H_{\text{KL}}(\psi; G) = \text{KL}[q_\psi(\boldsymbol{z}_i^H|G)||p(\boldsymbol{z}_i^H)] \quad (12)$$

(b) Space Unification Loss

$$\text{proj}_S(\boldsymbol{x}) = \begin{cases} w^S \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} & \text{if } \|\boldsymbol{x}\| \neq w^S \\ \boldsymbol{x} & \text{otherwise} \end{cases} \quad (13)$$

$$L_{\text{unify}}(G) = \text{dist}_{\text{s}}(\text{proj}_S(\boldsymbol{z}_i^H), \boldsymbol{z}_i^S) \quad (14)$$

Total Loss. The overall loss function for homophily regulated and social influence regulated nodes respectively is then a summation of the above loss components given by:

$$L^S = L^S_{\text{recon}}(\phi, \theta; G) + L^S_{\text{KL}}(\phi; G) + L_{\text{unify}}(G) \quad (15)$$

$$L^H = L^S_{\text{recon}}(\psi, \theta; G) + L^H_{\text{KL}}(\psi; G) + L_{\text{unify}}(G) \quad (16)$$

3.4 Training

This section details **NMM-GNN**'s training framework, using non-Euclidean VAE, for representing social networks. We describe the optimization method for variables from Sections 3.2 and 3.3.

Embedding Initialization. We randomly initialize all embeddings of $\boldsymbol{z}_i^S$ and $\boldsymbol{z}_i^H$. For homophily regulated nodes, we choose a value for norm w^S, that is sampled from uniform distribution: $w^S : w^S \in [0, 1) \rightarrow \text{Unif}([0, 1))$ for all nodes, and for social influence regulated nodes, we choose a value for norm $w^H_{z_i}$, assigned uniformly at random per node. We set curvature values of spherical and hyperbolic spaces as $K_S = 1$ and $K_H = -1$ respectively. We leave the non-trivial problem of learning optimal curvatures as future work.

Training procedure for homophily regulated nodes. Parameter optimization for learning node embeddings is performed using Riemannian stochastic gradient descent (RSGD) for the spherical space as shown in Table 7. To ensure the updated node embeddings remain in norm-w^S space, we perform a rescaling operation, proj_S, to project out-of-boundary embeddings back to the surface of

Table 7: Training Procedure for homophily regulated nodes and social influence regulated nodes.

(a) Homophily Regulated Nodes

$$r(\boldsymbol{z}^S_{i,t}, L^S) = (1 + \frac{\boldsymbol{z}^{S^T}_{i,t} \nabla L^S(\boldsymbol{z}^S_{i,t})}{\|\nabla L^S(\boldsymbol{z}^S_{i,t})\|})(I - \boldsymbol{z}^S_{i,t}\boldsymbol{z}^{S^T}_{i,t}) \quad (17)$$

$$\boldsymbol{z}^S_{i,t+1} \leftarrow \text{proj}_S\big(- \eta_t \cdot r(\boldsymbol{z}^S_{i,t}, L^S)\nabla L^S(\boldsymbol{z}^S_{i,t})\big) \quad (18)$$

$$\text{SGD}^S(x) : x_{t+1} \leftarrow x_t - \eta_t \nabla L^S(x_t) \quad (19)$$

(b) Social Influence Regulated Nodes

$$\boldsymbol{z}^H_{i,t+1} \leftarrow \boldsymbol{z}^H_{i,t} - \eta_t (\frac{1 - \|\boldsymbol{z}^H_{i,t}\|^2}{2})^2 \nabla L^H(\boldsymbol{z}^H_{i,t}) \quad (20)$$

$$\text{SGD}^H(x) : x_{t+1} \leftarrow x_t - \eta_t \nabla L^H(x_t) \quad (21)$$

Table 8: Dataset statistics for evaluation datasets.

Dataset	# Vertices	# Edges	Type	# Classes
BlogCatalog	10.3K	334.0K	undirected	39
LiveJournal	4.8M	69.0M	directed	10
Friendster	65.6M	1.8B	undirected	–

the w^S-ball. We further update scalar parameters J and B (from the homophily regulated distribution) and β, λ, a (from the spherical gaussian prior) through stochastic gradient descent (SGD) as defined below via $\text{SGD}^S(J), \text{SGD}^S(B), \text{SGD}^S(\beta), \text{SGD}^S(\lambda), \text{SGD}^S(a)$.

Training procedure for social influence regulated nodes. Parameter optimization for learning node embeddings is performed using RSGD for the hyperbolic space as shown in Table 7. The corresponding norm space, $w^H_{z_i}$, is also learned through RSGD by updating embeddings of $\boldsymbol{z}^H_i$. We further update scalar parameters C and D (from the social influence regulated distribution) and ζ (from the hyperbolic gaussian prior) through SGD as defined below via $\text{SGD}^H(C), \text{SGD}^H(D), \text{SGD}^H(\zeta)$.

Training procedure for NMM-GNN weights. Parameter optimization for γ uses SGD as follows, where $L_{\text{total}} = L^S + L^H$:

$$\gamma_{t+1} \leftarrow \gamma_t - \eta_t \nabla L_{\text{total}}(\gamma_t) \quad (22)$$

4 Experiments

We comprehensively evaluate **NMM-GNN** on social network generation for popular large-scale social networks through multi-label classification and link prediction tasks against competitive SOTA baselines in various categories. The Appendix section further details Ablation Studies where we test quality of using (1) a mixture model, (2) distinct non-Euclidean geometric spaces, (3) non-Euclidean GNN-based encoders and non-Euclidean GraphVAE framework (through the inductive setting), and (4) space unification loss component. All experiments and each of the ablation studies consistently show **NMM-GNN** outperforms baseline network embedding models on all metrics for all datasets.

4.1 Datasets

For comprehensive evaluation, we assess our models on real-world datasets from well-known social media venues: *BlogCatalog (BC)* [27], *LiveJournal (LJ)* [28], and *Friendster (F)* [29] which are friendship networks among bloggers. Table 8 provides statistics of the datasets. In the Appendix, we also include experiments for Wikipedia datasets, to show that our model can also benefit other networks. Our research goal is to design an embedding model to better explain how the social network is formed. Thus, to separate model contribution from the learned representation, we focus on the setting of featureless graphs since quality of node features can be a confounding factor in determining quality of our model. However, since our model can handle feature graphs, we also provide these experiments in the Appendix, with our model outperforming all baselines on attributed networks.

4.2 Models

Baselines. We compare **NMM-GNN** to SOTA network embedding models, in Table 9, and report results in Table 10. We omit comparison to prior models e.g., **LINE** [30] due to lower performance. We also highlight that unlike prior works like κ**-GCN**, our model overcomes limitations of the product space, e.g., where the entire model belongs to a Cartesian product of non-Euclidean geometric spaces by default. Our work is in a category called *mixed space* model that uses a multi-geometric space framework where different portions of the graph may possibly belong to different spaces (based on the amount of impact each of homophily and social influence has for that personalized pair of

Table 9: Category and description of baseline models.

Category	Description
Structural Embedding Models	**GraRep** [31], shallow embedding integrating global structural information
	RolX [32], unsupervised learning approach using structural role based similarity
	GraphWave [33], shallow embedding model using spectral graph wavelet diffusion patterns
GNN Embedding Models (Euclidean space)	**GraphSAGE** [34], inductive framework using node features and neighbor aggregation
	GCN [35], semi-supervised learning model via graph convolution on local neighborhoods
	GAT [36], graph attention model using mask self-attention layers on local neighborhoods
	GIN [37], graph embedding model based on the Weisfeiler-Lehman (WL) graph isomorphism test
	GRAPHCL [38], graph contrastive learning framework for unsupervised graph data
Homophily-based Embedding Models	**GELTOR** [17], embedding method using learning-to-rank with AdaSim* similarity metric
	NRP [27], embedding model using pairwise personalized PageRank on the global graph
GNN Embedding Models (non-Euclidean space)	**HGCN** [24], hyperbolic GCN model utilizing Riemannian geometry and hyperboloid model
	κ-**GCN** [39], GCN model using product space e.g., product of constant curvature spaces
Mixture Models (homophily and social influence)	**RaRE** [6], Bayesian probabilistic modelfor node proximity/popularity via posterior estimation
	NMM, our non-Euclidean mixture model (see Eqn. 6), without use of GraphVAE framework
	NMM-GNN , our non-Euclidean mixture model with non-Euclidean GraphVAE framework

nodes). In the extreme case (Case 1) where only social influence is at play, e.g., weight of homophily representation is learned close to 0, the hyperbolic space will be used. On the other hand if only homophily is at play (Case 2), e.g., weight of homophily representation is learned close to 0, the spherical space will be used. In the normal case of both factors at play (Case 3), then both spaces will be used and can be jointly aligned with our space alignment mechanism. When using product space, Cases 1, 2, and 3 will all not be distinguished from each other as all cases will be modeled by one complex non-Euclidean geometric space as a Cartesian product of spherical and hyperbolic spaces.

Time Complexity of NMM-GNN. Regarding our model, **NMM** is highly efficient, with time complexity $O(ed + nd)$, where n is number of nodes, e is number of edges, and d is dimension size. In comparison, the time complexity analyses for the remaining baseline models are as follows: the mixture model of **RaRE** is $O(ed + nd)$ which is comparable to the **NMM** mixture model, and the GNN embedding models of **GCN**, **GAT** (with one-head attention), and κ-**GCN** (for $\kappa = 0$) are $O(ed + nd^2)$. κ-**GCN** (for $\kappa \neq 0$) and **HGCN**'s time complexities are $O(ed + a \cdot nd^2)$, where a is the filter length, and **NMM-GNN** is $O(ed + nd^2)$ which is comparable to GNN embedding models. As our work focuses on improving accuracy of learned embeddings, we further use GraphVAE training with **NMM** to achieve SOTA performance. We would like to point out that GraphVAE (of **NMM-GNN**) training is also designed to be highly parallelizable, which allows for scalability. Moreover, our model is capable of learning on real-world, highly large-scale graphs on the order of millions of nodes and billion of edges, e.g., Friendster, while achieving the best performance, which attests to its practical value to the network science community.

4.3 Evaluation

We detail our evaluation procedure for multi-label classification and link prediction. For all experiments, for fairness of comparison to baselines, we utilize the experiment procedure of [6]. Specifically, 90% of links are randomly sampled as training data. We do not perform cross-validation, since it may cause overfitting to occur as our framework uses learnable parameters e.g., $\mathbf{z}^S$, $\mathbf{z}^H$, J, B, C, D, γ, β, λ, α, $\mathbf{W}_l$, and ζ which is a function of $\mathbf{z}^H$ equivalently interpreted as mean square error. Per dataset, we choose hyperparameter values for λ_A in reconstruction loss: {0, 1, 2, 4, 8, 16, 32, 64}, step sizes η_t: {0.005, 0.001, 0.01, 0.05, 0.1}, and experiments are performed on AWS cluster (8 Nvidia GPUs).

4.3.1 Classification

Evaluation results are in Table 10. We observe that mixture models (homophily and social influence), achieve better performance on all datasets for all metrics. Specifically, it improves over structural embedding models (**GraphWave**), GNNs (**GAT**, **HGCN**), and homophily-based models (**GELTOR**, **NRP**). We also see learning embeddings in non-Euclidean geometric spaces helps better represent structures in social networks (**HGCN** vs. **GAT**, **RaRE** vs. **NMM**). We further observe using GNN-based encoders with GraphVAE learning yields additional improvement (**NMM** vs. **NMM-GNN**).

4.3.2 Link Prediction

As from [6], we measure quality of link prediction by sorting probability scores of every pair of nodes per model and evaluating them using area under the ROC curve (AUC) score. Specifically, 10% of existing edges and non-existing edges are hidden from training set, and probabilities are examined by

Table 10: Results of social network classification and link prediction for **Jaccard Index (JI)**, **Hamming Loss (HL)**, **F1 Score (F1)**, and **AUC** in % using embedding dimension 64. Our **NMM** and its variants are in gray shading. For each group of models, the best results are bold-faced. The overall best results on each dataset are underscored. †Ablation study variant models using distinct non-Euclidean geometric spaces for **NMM** (homophily/social influence) where $\mathbb{E}$, $\mathbb{S}$, and $\mathbb{H}$ denote Euclidean, Spherical, and Hyperbolic spaces.

Datasets	BlogCatalog				LiveJournal				Friendster			
Metrics	JI	HL	F1	AUC	JI	HL	F1	AUC	JI	HL	F1	AUC
GraRep	36.0	28.2	45.6	87.9	40.1	41.1	35.2	56.7	53.6	34.2	40.6	89.8
RolX	37.2	25.4	48.7	90.4	40.9	38.0	35.6	**60.1**	58.8	33.9	40.9	90.3
GraphWave	**39.5**	**22.8**	**48.9**	**92.3**	**42.2**	**37.6**	**35.9**	**60.1**	**59.0**	**31.5**	**41.1**	**90.5**
GraphSAGE	45.4	20.1	49.3	**92.0**	45.5	34.7	34.1	59.0	64.1	28.7	43.4	90.5
GCN	47.3	19.5	55.1	91.6	46.7	31.2	47.8	62.6	66.5	28.0	47.2	91.9
GAT	**47.9**	**19.3**	54.5	91.4	47.4	28.5	**49.0**	65.3	66.3	28.0	46.8	92.0
GIN	47.1	19.7	**56.2**	91.5	48.6	28.3	48.1	67.2	66.0	27.7	48.1	92.3
GRAPHCL	47.5	19.4	55.8	91.3	**49.7**	**27.9**	**49.0**	**69.4**	**68.1**	**25.5**	**49.9**	**92.8**
GELTOR	47.4	**19.3**	54.9	92.0	51.0	28.9	48.6	65.3	66.7	27.9	47.5	91.7
NRP	**61.6**	20.4	**65.2**	**95.5**	**69.7**	**24.5**	**64.0**	**78.7**	**72.2**	**22.6**	**52.8**	**92.2**
HGCN	56.7	**19.2**	60.9	92.7	58.8	**27.1**	**57.7**	68.5	**69.9**	24.3	49.9	**93.3**
κ-GCN	**61.6**	20.7	**65.4**	**95.3**	**63.6**	27.3	57.2	**69.1**	69.4	**24.1**	**50.3**	93.1
RaRE	61.4	20.6	65.6	95.1	74.2	23.8	65.1	79.9	75.7	22.5	55.0	94.4
NMM($\mathbb{H}^d/\mathbb{S}^d$)†	56.6	19.8	62.3	95.1	74.0	28.4	55.5	68.8	74.6	26.9	50.6	93.0
NMM($\mathbb{S}^d/\mathbb{S}^d$)†	57.1	19.6	65.9	94.0	74.7	27.6	57.1	69.0	75.3	26.2	52.5	93.4
NMM($\mathbb{E}^d/\mathbb{E}^d$)†	57.9	19.5	66.3	95.4	75.1	25.0	58.4	71.2	77.0	24.7	52.8	94.5
NMM($\mathbb{S}^d/\mathbb{E}^d$)†	59.2	19.2	67.1	95.5	75.3	24.4	59.3	74.5	77.5	23.3	54.3	94.5
NMM($\mathbb{H}^d/\mathbb{H}^d$)†	58.4	19.0	66.7	95.3	75.6	24.6	61.9	76.0	78.8	23.3	55.0	94.7
NMM($\mathbb{E}^d/\mathbb{H}^d$)†	60.3	19.1	67.8	95.7	76.2	23.2	64.4	79.2	79.1	22.6	55.4	94.5
NMM (ours)	**62.7**	19.0	70.9	95.8	76.5	22.7	**67.3**	84.2	79.8	22.1	56.3	94.8
NMM-GNN (ours)	62.6	**17.3**	**78.8**	**96.9**	**78.6**	**20.4**	**67.3**	**86.8**	**83.3**	**21.8**	**57.7**	**94.9**

the model. Further, 10% of non-training edges are used for validation. For fairness against baselines on undirected networks, we treat all directed networks as undirected. Table 10 shows evaluation results for AUC score. Comparing relative score differences between best performing homophily embedding model (**NRP**) to **RaRE**, we observe that *LiveJournal* and *Friendster* datasets contain relatively more node social influence than *BlogCatalog*, which homophily-based models do not capture. Due to the above observation, it is likely that *LiveJournal* and *Friendster* (which are also larger datasets) show more realistic heterogeneity in network structure compared to *BlogCatalog* dataset e.g., cyclic structures produced by homophily based nodes and tree-like structures produced by social influence based nodes. Thus, modeling these structures in non-Euclidean spaces (**RaRE** vs. **NMM**) also shows more improvement. Moreover, in our **NMM** variant models, *every* node is influenced by *both* factors of homophily and social influence through our non-Euclidean mixture model. It is a *weighted combination* personalized per node of these factors that influence the links formed, rather than being generated solely through homophily vs. social influence.

5 Conclusions

We are among the first to explore a Graph-based non-Euclidean mixture model for social networks. As social networks are influenced by homophily and social influence, we design a model to represent both factors jointly for nodes. Further, we model resulting unique network topologies (cycles and trees) using distinct non-Euclidean geometric spaces and introduce a GNN-based non-Euclidean variational autoencoder framework for our model, to effectively learn embeddings. The resulting model, **NMM-GNN**, significantly outperforms various state-of-the-art models for social networks. As future work, we hope to explore alternatives to graph-based VAE methods for improved learning.

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Appendix

A Limitations

NMM is highly efficient, with time complexity $O(ed + nd)$, where n is number of nodes, e is number of edges, and d is dimension size. As our work focuses on improving accuracy of learned embeddings, we further use GraphVAE training with **NMM** to achieve SOTA performance, called **NMM-GNN**. Using the GraphVAE training pipeline may be seen as a limitation as it is less efficient compared to GNNs for training time. However, in the worst case scenario, **NMM-GNN** still achieves efficiency comparable to popular heterogeneous graph models including GraphRNN [40], and training is designed to be highly parallelizable, which allows for scalability. Moreover, our model is capable of learning on real-world, highly large-scale graphs on the order of millions of nodes and billion of edges, e.g., *Friendster*, while achieving the SOTA performance, which attests to its practical value to the network science community.

Our model also makes the assumption the homophily regulated nodes lie on the surface of the spherical ball and the social influence regulated nodes lie on the open Poincare ball, which are both natural representations for modeling nodes. People have observed that more popular celebrity nodes tend to have smaller norm space and are embedded towards the center of the ball (similar to higher order concepts in the knowledge graph space), while less popular nodes (containing less social influence) are embedded towards the boundary of the ball (similar to entities in the knowledge graph space). To ensure that we can compute an intersection space (for the space unification component to unify the homophily and social influence representation for the same node), we enforce that the spherical surface norm is in the Poincare ball e.g., any learned soft value norm space between 0 and 1. In compute, while we evaluate on a cluster of 8 GPUs, at least one GPU is necessary for running our experiments (which may been seen as compute limitation).

Regarding ethical considerations and fairness, privacy and fairness are often topics of focus regarding social networks. Our model uses information like node popularity based on graph structure (in degree vs. out degree ratio), as well as attributed features if available. Further it also infers links through neighborhood context by looking at connected nodes. In general, a network embedding model uses personal information from the user which may impinge on their privacy. However, we make every attempt in our model to allow the user to select their granular choice of model personalization (e.g., we provide the option to choose from whether or not users want to enable their attributed information like profile interest and history being learned).

B A Note on Graph-Level Learning

Our work can be generalized to the graph-level because our method learns to represent social science network factors based on topologies in the graph on clusters of nodes and edges. Thus, if the cluster of nodes and edges comprised of the entire graph and we subsequently applied graph pooling per node embedding (that we currently learn), we can reduce the embedding from node level to graph level. In this way, homophily and social influence can be modeled at the graph level. That said, it is unclear whether graph-level modeling would be specifically useful or interpretable for social network embedding models as compared to node-level modeling. This is due to the social network setting requiring links to be generated that are per node and not at the graph level, because a user (modeled as a node) is recommended to another specific user in the practical social network setting. This is different from other network domains like molecular classification where the entire graph represents one molecule e.g., atoms form individual nodes and chemical bonds form the edges. For this reason, node-level learning is in fact consistent with the recent state-of-the-art NN methods in the network science community though **NMM-GNN** can still be generalized to learning at the graph-level.

C Additional Experiments

In this section, we provide experiment results on Wikipedia networks and on attributed graphs. As in the case for social networks, we also evaluate on multi-label classification and link prediction for Wikipedia networks and attributed graphs. For all experiments, for fairness of comparison to baselines, we utilize experiment procedure of [6]. Specifically, 90% of links are randomly sampled

as training data. We do not perform cross-validation, since it may cause overfitting to occur as our framework uses learnable parameters e.g., $\mathbf{z}^S$, $\mathbf{z}^H$, J, B, C, D, γ, β, λ, α, $\mathbf{W}_l$, and ζ which is a function of $\mathbf{z}^H$ equivalently interpreted as mean square error. Per dataset, we choose hyperparameter values for λ_A in reconstruction loss: {0, 1, 2, 4, 8, 16, 32, 64}, and step sizes η_t: {0.005, 0.001, 0.01, 0.05, 0.1}.

Evaluation on Wikipedia networks. In Table 12, we additionally present multi-label classification and link prediction experiments on Wikipedia datasets to show the benefits of our model on other networks. These include *Wikipedia Clickstream* [6], which contains counts of (referrer, resource) pairs extracted from the request logs of Wikipedia, and *Wikipedia Hyperlink* [6], which contains edges as hyperlinks from one page to another. Dataset statistics for the Wikipedia datasets are summarized in Table 11. Our **NMM** model variants consistently achieves the best performance over all the competitive baseline models belonging to categories of (1) structural embedding models, (2) GNN embedding models (Euclidean space), (3) homophily-based embedding models, (4) GNN embedding models (non-Euclidean space), and (5) mixture models. This shows that **NMM**'s model is generalizable and widely applicable because it can learn effective representations on online information networks that go beyond social networks.

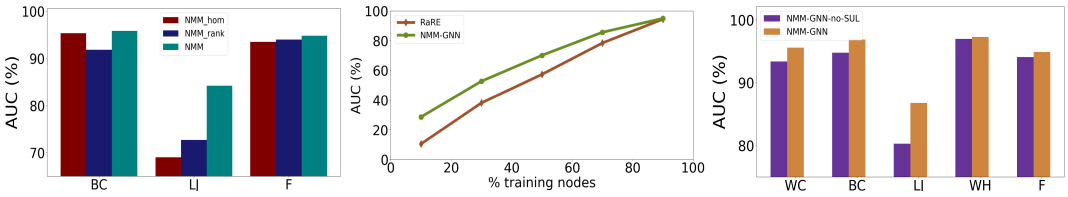
Table 11: Dataset statistics for evaluation datasets.

Dataset	# Vertices	# Edges	Type	# Classes
Wikipedia Clickstream	2.4M	15.0M	directed	6
Wikipedia Hyperlink	488K	5.5M	directed	6

Table 12: Results of social network classification and link prediction for **Jaccard Index (JI)**, **Hamming Loss (HL)**, **F1 Score (F1)**, and **AUC** in % using embedding dimension 64. Our **NMM** and variants are in gray shading. For each group of models, best results are bold-faced. The overall best results on each dataset are underscored.† Ablation study variant models using distinct non-Euclidean geometric spaces for **NMM** (homophily/social influence) where $\mathbb{E}$, $\mathbb{S}$, and $\mathbb{H}$ denote Euclidean, Spherical, and Hyperbolic spaces.

Datasets	Wikipedia Clickstream				Wikipedia Hyperlink			
Metrics	JI	HL	F1	AUC	JI	HL	F1	AUC
GraRep	31.9	28.3	44.4	79.6	48.3	22.7	50.2	76.8
RolX	32.2	28.1	**44.9**	85.4	**57.1**	**17.6**	**56.0**	**82.6**
GraphWave	**32.8**	**27.0**	44.6	**86.1**	55.3	18.1	54.8	81.8
GraphSAGE	33.1	26.6	45.1	89.3	62.8	8.3	68.7	89.4
GCN	33.1	**24.1**	50.7	89.5	63.7	**6.5**	**76.4**	92.0
GAT	33.4	24.2	**51.2**	89.0	64.2	**6.5**	76.2	92.1
GIN	**33.7**	24.4	50.6	**91.2**	**65.1**	6.9	76.1	92.3
GRAPHCL	33.3	**24.1**	51.0	90.9	64.5	6.6	**76.4**	**92.7**
GELTOR	33.2	24.1	50.9	91.4	64.0	**6.8**	76.7	92.3
NRP	**46.5**	**20.9**	**57.1**	**91.8**	**75.5**	7.1	**81.1**	**96.9**
HGCN	**38.1**	**20.6**	54.3	89.9	69.8	**6.0**	78.4	94.1
κ-GCN	37.6	20.9	**55.0**	**90.2**	**75.0**	7.1	**81.6**	**96.9**
RaRE	47.8	20.7	58.0	93.0	75.7	6.8	82.3	97.5
NMM($\mathbb{H}^d/\mathbb{S}^d$)†	44.8	27.8	44.2	86.5	76.0	8.2	76.4	92.1
NMM($\mathbb{S}^d/\mathbb{S}^d$)†	45.1	22.9	55.7	90.3	78.4	7.1	79.7	97.0
NMM($\mathbb{E}^d/\mathbb{E}^d$)†	46.2	21.7	56.0	90.6	79.2	6.8	82.0	97.5
NMM($\mathbb{S}^d/\mathbb{E}^d$)†	47.7	20.9	56.2	91.0	79.9	6.6	82.3	97.6
NMM($\mathbb{H}^d/\mathbb{H}^d$)†	47.4	20.9	56.8	91.2	81.0	6.6	81.6	97.3
NMM($\mathbb{E}^d/\mathbb{H}^d$)†	48.1	20.7	57.9	91.7	80.5	6.2	82.4	97.8
NMM (ours)	49.2	18.5	59.0	92.8	80.3	5.8	82.5	**98.0**
NMM-GNN (ours)	**49.7**	**16.5**	**60.8**	**95.6**	**81.7**	**5.5**	**83.0**	97.3

Evaluation on attributed graphs. As our model can also handle attributed graphs, we provide experiments against SOTA attributed network embedding models that are summarized below. Since these models require the presence of network attributes, we do not include them in our experiments on featureless graphs for fairness of comparison. We conduct evaluation on well-known large-scale attributed graph datasets which include *Facebook* [9] and *Google+* [9] social networks where consistent with [27], we treat each ego-network as a label and extract attributes from their user profiles. Dataset statistics are in Table 13, and results are reported in Table 14. It can be seen that our **NMM-GNN** model consistently achieves the best performance in both the large-scale graph datasets, indicating that our model's inherent learning ability is effective to learn *graph structure* and *topology*, which goes beyond simply exploiting information from node and edge attributes. Below is a summary of the baseline attributed network embedding models:



(a) (b)

Figure 2: Ablation studies. (a) Quality of mixture model, where $\mathbf{NMM_{hom}}$ and $\mathbf{NMM_{rank}}$ are homophily-only and social influence-only deconstructed **NMM** components. (b) Inductive reasoning for **NMM-GNN** and **RaRE** on LiveJournal. *% nodes* $\{10, 30, 50, 70, 90\}$ are sampled ensuring no overlap with test. (c) Ablation study on quality of using space unification loss (SUL) component.

- **PANE** [41], random-walk based attirubted network embedding (ANE) model
- **NRP** [27], described in Table 9 of the main paper
- **CONN** [42], GNN for attributed networks via collaborative aggregation on bipartite graphs

Table 13: Dataset statistics for evaluation datasets.

Dataset	# Vertices	# Edges	#Attributes	Type	# Classes
Facebook	4.0K	88.2K	1.3K	undirected	193
Google+	107.6K	13.7M	15.9K	directed	468

Table 14: Results of social network classification and link prediction for **Jaccard Index (JI)**, **Hamming Loss (HL)**, **F1 Score (F1)**, and **AUC** in % using embedding dimension 64. Our **NMM** and variants are in gray shading. The overall best results on each dataset are bold-faced.

Datasets	Facebook				Google+			
Metrics	JI	HL	F1	AUC	JI	HL	F1	AUC
NRP	48.0	19.7	55.8	96.1	40.7	26.5	51.1	81.3
PANE	49.3	15.9	64.3	96.4	45.1	23.8	60.2	92.2
CONN	51.2	16.4	61.2	**96.8**	44.9	23.3	61.3	90.8
NMM-GNN (ours)	**52.6**	**14.2**	**67.1**	**96.8**	**47.3**	**20.4**	**66.0**	**93.9**

The source code and datasets for our work can be found at: `https://github.com/roshnigiyer/nmm`.

D Ablation Studies

This section details ablation studies where we test quality of using (1) a mixture model, (2) distinct non-Euclidean geometric spaces, (3) non-Euclidean GNN-based encoders and non-Euclidean GraphVAE framework (through the inductive setting), and (4) space unification loss component. Each of the ablation studies provides insight to motivate our architecture design choices, as well as consistently shows **NMM-GNN** outperforms baseline network embedding models.

Quality of using a mixture model architecture. We deconstruct our mixture model of **NMM**, to observe the effect different network factors have on learning embeddings, where α and β are learnable. We report results on embedding dimension 64, evaluated on AUC score, with the models summarized below:

- $\mathbf{NMM_{hom}}$, deconstructed homophily component: $p_\theta(e_{ij} = 1) = \alpha \cdot p_{\text{hom}}(e_{ij} = 1)$
- $\mathbf{NMM_{rank}}$, deconstructed social influence component: $p_\theta(e_{ij} = 1) = \beta \cdot p_{\text{rank}}(e_{ij} = 1)$
- **NMM**, which is our mixture model defined by Eqn. 6

As shown in Figure 2(a), the mixture model of **NMM** outperforms that of its subcomponents $\mathbf{NMM_{hom}}$ and $\mathbf{NMM_{rank}}$ on all datasets for link prediction. This validates the effectiveness of using a mixture model architecture for modeling both homophily and social influence factors jointly.

Quality of using distinct non-Euclidean geometric spaces. We study combinations of geometric spaces to model **NMM** (homophily/social influence) to observe the effect it has on learning topological structure, denoted with $\mathbf{NMM}(\cdot)^{\dagger}$ in Table 10 (main paper). As shown, the choice of modeling homophily based nodes in spherical space and modeling social influence based nodes in hyperbolic space leads to the best performance. Further, **NMM** outperforms its Euclidean space counterpart showing that the social network exhibits structures (cycles and hierarchy) that need to be appropriately represented in curved geometric spaces. There is also evidence of non-Euclidean topologies in the datasets. For example, the average node on LiveJournal has in-degree of 17 but outdegree of 25, showing several hierarchical structures present, and 17.7% of LiveJournal data contains cycles [43].

Link prediction on unseen nodes (inductive task). We study ability of **NMM-GNN** to learn on the inductive setting (in addition to the standard transductive setting of Table 10 of the main paper). To do so, we randomly sample % of *nodes* being $\{10, 30, 50, 70, 90\}$ and their corresponding links as our training set. Test nodes are ensured to have no overlap with training nodes, to allow for link prediction on unseen graphs. Figure 2(b) reports results on **NMM-GNN** and **RaRE** for *LiveJournal* on embedding dimension of 64 for AUC score. **NMM-GNN** outperforms **RaRE** on all settings of training nodes. This shows the effectiveness of **NMM-GNN** in using non-Euclidean GNN-based encoders and a non-Euclidean GraphVAE training framework during the learning process, two components that **RaRE** lacks. Further, as less training nodes are observed, **NMM-GNN** outperforms **RaRE** by larger margins (e.g., 10% vs. 70% training nodes), showing **NMM-GNN** better generalizes to unseen graphs.

Quality of using space unification loss. We test quality of using our proposed space unification loss (SUL) in our **NMM-GNN**, by conducting experiments for with the loss component (**NMM-GNN**) and without it (**NMM-GNN-no-SUL**). We report results on embedding dimension 64, evaluated on AUC score in Figure 2(c), for datasets *Wikipedia Clickstream (WC)*, *BlogCatalog (BC)*, *LiveJournal (LJ)*, *Wikipedia Hyperlink (WH)*, and *Friendster (F)*. As shown, using SUL improves performance on all datasets, indicating importance of bridging together representations of distinct non-Euclidean spaces (spherical and hyperbolic space) at node level.

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   Our model novelly represents both homophily and social influence factors when modeling the social network, an advancement over the recent network embedding methods. Further, the novel utilization of non-Euclidean geometric spaces to model the resulting topologies due to network factors through appropriate positive and negative curvatures naturally exibiting properties of cycles and hierarchy (the observed topological heterogeneity), tremendously improves the representation capability of network embedding modes. This is evidenced by the empirical results as ablation studies for our work. Moreover, **NMM-GNN** not only improves against SOTA models in the performance metrics (Jaccard Index, Hamming Loss, , F1 score, AUC Score), but is also applicable to the Inductive Setting, other large-scale information networks like Wikipedia networks, and attributed networks (Facebook and Google+), while consistently achieving the best performance indicating model generalizability.

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   Regarding ethical considerations and fairness, privacy and fairness are often topics of focus regarding social networks. Our model uses information like node popularity based on graph structure (in degree vs. out degree ratio), as well as attributed features if available. Further it also infers links through neighborhood context by looking at connected nodes. In general, a network embedding model uses personal information from the user which may impinge on their privacy. However, we make every attempt in our model to allow the user to select their granular choice of model personalization (e.g., we provide the option to choose from

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Justification: The paper provides clear motivation of the importance of the research problem of investigation and further identifies key challenges in the network science community in state-of-the-art embedding models, which thus motivates our model architecture. Further, our paper provides clear intuition and justification for each design choice decision in our model components. Additionally, impact of our model is discussed with respect to the social network community and more broadly for other online information networks e.g., Wikipedia datasets and attributed networks (in the Appendix section on Ablation studeies). Lastly, we also discuss Limitations of our model in Appendix Section A.

This model effectively also learns without needed data attributes, which could beneficially influence privacy considerations in social networks (that users do not need to have their profile information and history being stored). Of course, any network science model could have potential for misuse in areas like surveillance or manipulation of online communities, so litigation and action should be taken to safeguard against misuse.

Guidelines:

- The answer NA means that there is no societal impact of the work performed.
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