## **Learning with Statistical Equality Constraints**

#### **Aneesh Barthakur**

University of Stuttgart aneesh.barthakur@simtech.uni-stuttgart.de

## Luiz F. O. Chamon

École polytechnique, Institut Polytechnique de Paris luiz.chamon@polytechnique.edu

#### **Abstract**

As machine learning applications grow increasingly ubiquitous and complex, they face an increasing set of requirements beyond accuracy. The prevalent approach to handle this challenge is to aggregate a weighted combination of requirement violation penalties into the training objective. To be effective, this approach requires careful tuning of these hyperparameters (weights), involving trial-anderror and cross-validation, which becomes ineffective even for a moderate number of requirements. These issues are exacerbated when the requirements involve parities or equalities, as is the case in fairness and boundary value problems. An alternative technique uses constrained optimization to formulate these learning problems. Yet, existing approximation and generalization guarantees do not apply to problems involving equality constraints. In this work, we derive a generalization theory for equality-constrained statistical learning problems, showing that their solutions can be approximated using samples and rich parametrizations. Using these results, we propose a practical algorithm based on solving a sequence of unconstrained, empirical learning problems. We showcase its effectiveness and the new formulations enabled by equality constraints in fair learning, interpolating classifiers, and boundary value problems.

#### 1 Introduction

Across a wide range of domains, machine learning (ML) is becoming the core technology driving entire systems rather than specific components. It therefore increasingly faces multi-faceted problems involving not only accuracy, but also requirements such as fairness [1], robustness [2, 3], privacy [4], and safety [5, 6]. The standard approach to handling multiple criteria is to represent them as alternate loss functions (*penalties*) aggregated into a single training objective [7–9, 3]. Designing effective penalties and aggregation weights, however, is often a time consuming process involving trial-and-error, hyperparameter search, and cross-validation data. Hence this approach becomes unwieldy even for a moderate number of criteria.

Constrained learning offers an alternative by framing each requirement as a constraint rather than a penalty [10–12]. While in convex settings these approaches are equivalent [13], this is not the case for the non-convex optimization problems arising in ML. Nevertheless, recent work has shown that such a duality still holds for rich parametrizations under mild conditions. This result has been used to establish generalization guarantees similar to those of unconstrained learning theory and develop practical algorithms based on (primal-)dual methods [11, 12]. Yet, these results hold only for inequality constraints and do *not* account for equality-constrained learning tasks.

This gap is significant. Indeed, many important requirements are expressed as equality constraints, including group fairness [14–16], invariance [17–19], calibration [20, 21], interpolation [22], distribution matching [23], and independence [24]. These are fundamentally different from inequalities since they impose a specific target value rather than a bound, making their feasibility set and sensitivity more intricate to analyze. As such, reformulations of these requirements as inequalities or relaxations based on penalties can substantially change the solution of an ML task. In fact, none of the duality or generalization results obtained for inequality-constrained learning apply to these problems (see Remark 2.1).

In this paper, we address this knowledge gap by developing a generalization theory for equality-constrained learning tasks. To do so, we derive additional regularity conditions under which we obtain duality and sensitivity results for non-convex equality-constrained optimization problems (Section 3). Based on these results, we put forward an empirical, unconstrained (saddle-point) problem and characterize the approximation and generalization error of its solutions, showing a trade-off between model capacity ("bias"), sample size ("variance"), and constraint difficulty (Theorem 3.1). This problem is amenable to a practical dual ascent algorithm (Section 4) whose effectiveness we illustrate in learning problems involving fairness, boundary value problems, and interpolating classifiers (Section 5).

## 2 Problem formulation

#### 2.1 Equality-constrained learning

Consider data pairs  $(\mathbf{x},y) \in \mathcal{X} \times \mathcal{Y}$  and a model  $f_{\theta}: \mathcal{X} \to \mathbb{R}^K$  parametrized by a finite dimensional vector  $\theta \in \Theta \subset \mathbb{R}^p$  and denote by  $\mathcal{H} = \{f_{\theta}: \mathcal{X} \to \mathcal{Y} \mid \theta \in \Theta\}$  the hypothesis class induced by these models. The goal of classical (unconstrained) learning is to use  $f_{\theta}$  to map a *feature vector*  $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^d$  to its *target*  $y \in \mathcal{Y}$ , which can be continuous  $(\mathcal{Y} \subseteq \mathbb{R}^K, \text{ e.g., regression})$  or discrete  $(\mathcal{Y} = \{1, \dots, K\}, \text{ e.g., classification})$ . This is usually done by minimizing the expected value of *one* loss function  $\ell : \mathbb{R}^K \times \mathcal{Y} \to \mathbb{R}$  describing a top-line objective, e.g., accuracy, with respect to *one* distribution  $\mathbb{D}$ . In contrast, *constrained learning* accounts for additional loss functions  $g_i, h_j$  and distributions  $\mathbb{P}_i, \mathbb{Q}_j$  by tackling

$$P^* = \inf_{\theta \in \Theta} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right]$$
subject to 
$$\mathbb{E}_{\mathbb{P}_i} \left[ g_i(f_{\theta}(\mathbf{x}), y) \right] \leq 0, \quad \text{for } i = 1, \dots, I,$$

$$\mathbb{E}_{\mathbb{Q}_j} \left[ h_j(f_{\theta}(\mathbf{x}), y) \right] = 0, \quad \text{for } j = 1, \dots, J.$$
(P)

We consider homogeneous constraints (whose right-hand side is zero) without loss of generality since any constant can be absorbed into the losses  $g_i, h_j$ . We denote the set of optimal solutions for (P) by Opt(P) and the set of feasible solutions (i.e., those that satisfy the constraints of (P)) by Feas(P). These additional expected losses can be used to impose a rich class of constraints and ancillary requirements based on data, such as fairness [10, 11], robustness [25], and invariance [19]. These works and the majority of the constrained learning literature considers problems with inequality constraints [J=0 in (P)]. In contrast, this paper is concerned with problems explicitly involving equality constraints. Before proceeding, we illustrate a few instances in which this additional expressiveness is beneficial (see also Section 5).

**Fairness.** Statistical definitions of fairness in ML are naturally formulated as *equality constraints* [26, 27]. Consider, for instance, demographic parity (DP) in binary classification, where the score  $f_{\theta}(\mathbf{x}) \in [0,1]$  is thresholded to determine a positive  $(f_{\theta}(\mathbf{x}) > 0.5)$  or negative  $(f_{\theta}(\mathbf{x}) \leq 0.5)$  outcome. DP requires that the prevalence of positive (or negative) outcomes within the protected groups  $\{\mathcal{G}_j\}$  be the same as that of the whole population [28, 29, 14]. Protected groups are often based on a sensitive attribute of the feature vector  $\mathbf{x}$ , although we impose no such restrictions. DP-constrained binary classification can be cast as (P), explicitly,

where  $\ell$  is a classification loss and  $\mathbb{I}\left[\mathcal{E}\right]=1$  on the event  $\mathcal{E}$  and 0 otherwise. Note that (P-DP) indeed has the form (P), for  $h_j(f_{\theta}(\mathbf{x}),y)=\frac{1}{\mathbb{P}(\mathbf{x}\in\mathcal{G}_j)}\mathbb{I}\left[f_{\theta}(\mathbf{x})>0.5\right]\mathbb{I}\left[\mathbf{x}\in\mathcal{G}_j\right]-\mathbb{I}\left[f_{\theta}(\mathbf{x})>0.5\right]$  and

 $\mathbb{Q}_j = \mathbb{P}$ . Other statistical definitions of fairness, such as equality of opportunity, can be similarly formulated [15]. While the equality in (P-DP) is sometimes approximated by an inequality with some slack [10, 11], this can change the solution in non-trivial ways (see Figure 1). Equality constraints also enable *prescriptive* forms of equity that enforce specific rates  $r_j > 0$  of positive outcomes for each group  $\mathcal{G}_j$ , which can be cast as

**Boundary value problems.** Boundary value problems (BVPs) arise in many scientific applications. While they can be solved with classical methods such as the finite element methods [30, 31], they have recently also been tackled using learning methods by directly parametrizing their solution with a neural network (e.g. PINNs [32]). Indeed, let  $\Omega \subseteq \mathbb{R}^d$  be a bounded connected region with boundary  $\partial\Omega$  and define the domain  $\mathcal{D}=\Omega\times(0,T]$  (where T>0) and let  $\mathcal{H}=\{f_\theta\mid\theta\in\Theta\}$  be a set of functions defined on  $\mathcal{D}$ . A BVP is typically posed as

find 
$$f_{\theta} \in \mathcal{H}$$
  
subject to  $D[f_{\theta}](\mathbf{x}, t) = \tau_{f}(\mathbf{x}, t), \quad \forall (\mathbf{x}, t) \in \mathcal{D},$   
 $f_{\theta}(\mathbf{x}, t) = \tau_{b}(\mathbf{x}, t), \quad \forall \mathbf{x} \in \partial \Omega, t \in (0, T],$   
 $f_{\theta}(\mathbf{x}, 0) = \tau_{i}(\mathbf{x}, 0), \quad \forall \mathbf{x} \in \Omega,$ 
(BVP)

where D is a differential operator defined over a superset of  $\mathcal{H}$ ,  $\tau_f$  is called the forcing function, and  $\tau_b$ ,  $\tau_i$  describe the boundary and initial conditions (BC and IC) respectively. It turns out that (BVP) can be solved with an instance of (P) as in

$$\begin{aligned} & \underset{\theta \in \Theta}{\text{minimize}} & & \frac{\alpha}{2} \|\theta\|_2^2 \\ & \text{subject to} & & \mathbb{E}_{\mathbb{P}_p}[(D[f_{\theta}](\mathbf{x},t) - \tau(\mathbf{x},t))^2] = 0, \\ & & & \mathbb{E}_{\mathbb{P}_b}[(f_{\theta}(\mathbf{x},t) - \tau_b(\mathbf{x},t))^2] = 0, \\ & & & & \mathbb{E}_{\mathbb{P}_i}[(f_{\theta}(\mathbf{x},0) - \tau_i(\mathbf{x},0))^2] = 0, \end{aligned}$$
 (P-BVP)

where  $\mathbb{P}_p, \mathbb{P}_b$ , and  $\mathbb{P}_i$  are arbitrary distributions (usually uniform) over  $\mathcal{D}$ ,  $\partial\Omega \times (0,T]$ , and  $\Omega$  respectively [33]. Thus, the constraints enforce *mean squared error* versions of the PDE together with the BC, and the IC. This is in contrast to PINNs that aggregate the errors into a single weighted loss, leading to solutions sensitive to the choice of weights [34, 35].

**Interpolating classifiers.** Modern ML models are typically overparametrized and often trained to perfectly fit (interpolate) the training data. Several works [36, 37, 22] have found that interpolating models performs well in practice, contrary to conventional statistical wisdom on overfitting. However, overparametrization leads to problems with multiple optimal solutions that do not all share the same performance. Hence, the quality of the interpolating model is to a large extent determined by the training algorithm. Alternate formulations of the prediction problem can therefore lead to different interpolating algorithms with beneficial properties.

Explicitly, consider a multi-class classification problem  $(\mathcal{Y} = \{1, \dots, K\})$  with a non-negative loss function  $\ell$ , vanishing when  $f_{\theta}(\mathbf{x}) = y$  (e.g., cross entropy). Instead of directly minimizing the expected loss  $J(\theta) = \mathbb{E}_{\mathbb{P}}[\ell(f_{\theta}(\mathbf{x}), y)]$ , we can use (P) to formulate a *classwise interpolation* problem, namely,

As shown by our experiments in Section 5, (P-CI) measures and exploits the heterogeneous difficulty of fitting each class— information that is hard to obtain from the training data given that it is interpolated.

## 2.2 Empirical dual formulation

In ML, the objective/constraints of (P) are non-convex functions of  $\theta$ , either because the parametrization  $f_{\theta}$  is a complex nonlinear function (e.g. a neural network) or because the losses themselves are

non-convex (as in, e.g., (P-DP)). This hinders the use of constrained optimization methods based on projections, conditional gradients, or barrier functions [38–40]. What is more, we only have access to the distribution  $\mathbb{D}, \mathbb{P}_i, \mathbb{Q}_j$  through samples, so that the expectations in (P) must be estimated empirically. This leads to errors that affect  $P^*$  and Feas(P) in non-trivial ways. To overcome these issues, we turn to duality-based methods [13]. Explicitly, define the Lagrangian of (P) as

$$L(f_{\theta}, \lambda, \mu) = \mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta}(\mathbf{x}), y)\right] + \sum_{i=1}^{I} \lambda_{i} \mathbb{E}_{\mathbb{P}_{i}}\left[g_{i}(f_{\theta}(\mathbf{x}), y)\right] + \sum_{j=1}^{J} \mu_{j} \mathbb{E}_{\mathbb{Q}_{j}}\left[h_{j}(f_{\theta}(\mathbf{x}), y)\right], \quad (1)$$

for  $\lambda \in \mathbb{R}_+^I$  and  $\mu \in \mathbb{R}^J$  collecting the  $\lambda_i$  and  $\mu_j$  respectively. The dual problem of (P) is then

$$D^* = \sup_{\lambda \in \mathbb{R}^I, \ \mu \in \mathbb{R}^J} \inf_{\theta \in \Theta} L(f_{\theta}, \lambda, \mu).$$
 (D)

In this context, the weights  $\lambda_i$ ,  $\mu_j$  are called *dual variables* and the set Opt(D) of all solutions  $(\lambda^*, \mu^*)$  of (D) is known as the set of *Lagrange multipliers*. The dual (D) is a relaxation of (P), i.e.  $D^* \leq P^*$  (weak duality) and  $P^* - D^*$  is called the duality gap. Convex optimization problems are strongly dual, i.e.  $P^* = D^*$ , under certain regularity conditions such as Slater's condition (see [13, Proposition 5.3.1]). Of particular interest in this work is the empirical version of (D),

$$\hat{D}^{\star} = \sup_{\lambda \in \mathbb{R}^{I}_{+}, \, \mu \in \mathbb{R}^{J}} \, \hat{q}(\lambda, \mu) \triangleq \inf_{\theta \in \Theta} \, \hat{L}(f_{\theta}, \lambda, \mu), \tag{\hat{D}}$$

where  $\hat{q}(\lambda, \mu)$  is the *empirical dual function* defined based on the *empirical Lagrangian* 

$$\hat{L}(f_{\theta}, \lambda, \mu) = \frac{1}{M_0} \sum_{m_0=1}^{M_0} \ell\left(f_{\theta}(\mathbf{x}_{m_0}), y_{m_0}\right) + \sum_{i=1}^{I} \lambda_i \left[\frac{1}{M_i} \sum_{m_i=1}^{M_i} g_i(f_{\theta}(\mathbf{x}_{m_i}), y_{m_i})\right] + \sum_{j=1}^{J} \mu_j \left[\frac{1}{N_j} \sum_{n_j=1}^{N_j} h_j(f_{\theta}(\mathbf{x}_{n_j}), y_{n_j})\right],$$
(2)

that uses independently drawn samples  $(\mathbf{x}_{m_0}, y_{m_0}) \sim \mathbb{D}, (\mathbf{x}_{m_i}, y_{m_i}) \sim \mathbb{P}_i$ , and  $(\mathbf{x}_{n_i}, y_{n_i}) \sim \mathbb{Q}_i$ .

For inequality-constrained learning problems [J=0 in (P)], prior works [11,41,10] have shown that  $(\hat{D})$  provides an effective way of solving (P) by showing that the solutions of  $(\hat{D})$  approximate those of (P) (generalization) and deriving practical algorithms to do so. The goal of this paper is to extend these results to the more general (P). While  $(\hat{D})$  remains an empirical, unconstrained program amenable to be solved using stochastic optimization techniques (see Section 4), it is no longer clear that its solutions generalize to (approximate) those of (P). Indeed, non-convexity hinders the use of classical duality theory to show  $P^* = D^*$  and the presence of equalities invalidate the results from [11]. What is more, the errors introduced by using empirical, finite sample estimates of the expectations in (P) pose challenges even before considering equality constraints [11, Example 1]. We address these concerns in the sequel after a pertinent remark.

**Remark 2.1.** While equality constraints can be written as inequalities, these reformulations pose theoretical and numerical challenges. For instance, the feasibility set of (P) does not change when replacing its equalities with  $(\mathbb{E}_{\mathbb{Q}_j}[h_j(f_{\theta}(\mathbf{x}),y)])^2 \leq 0$ . This formulation, however, invalidates current duality and generalization guarantees for constrained learning (see, e.g., [11, 12]), potentially even for convex problems (see, e.g., [42, Theorem 5.5, 5.11]). We may also consider approximating each equality by the pair  $-\epsilon \leq \mathbb{E}_{\mathbb{Q}_j}[h_j(f_{\theta}(\mathbf{x}),y)] \leq \epsilon$ . Interestingly, taking  $\epsilon = 0$  yields the same (empirical) Lagrangians and dual problems as in (2) and  $(\hat{D})$  (see Appendix D.2), though the feasibility set of the resulting problem has once again no interior. For  $\epsilon > 0$ , it is not straightforward to determine the effect of this relaxation on the solution of (P) and the use of very small  $\epsilon$  can lead to ill-conditioned problems even in the convex case (see Appendix D.1).

## 3 Generalization error

In this section, we quantify how well the empirical dual problem  $(\hat{D})$  approximates the constrained learning problem (P). We do so by bounding the generalization error  $|P^* - \hat{D}^*|$  (Theorem 3.1), which can be decomposed as

$$|P^* - \hat{D}^*| \le |P^* - D^*| + |D^* - \hat{D}^*|.$$
 (3)

The first term is the *duality gap* of (P) and the second is the *dual estimation error*. The first assumption we make ensures that the dual problems (D) and ( $\hat{D}$ ) are well-posed. Explicitly, define the constraint value epigraph as

$$C = \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^I \times \mathbb{R}^J \mid \exists \theta \in \Theta \quad \text{s.t.} \quad \mathbb{E}_{\mathbb{P}_i} \left[ g_i(f_{\theta}(\mathbf{x}), y) \right] \leq u_i, \quad \text{for } i = 1, \dots, I, \right\}. \quad (4)$$

$$\text{and} \quad \mathbb{E}_{\mathbb{Q}_j} \left[ h_j(f_{\theta}(\mathbf{x}), y) \right] = v_j \quad \text{for } j = 1, \dots, J \right\}.$$

Similarly, we define  $\hat{C}$  by replacing the expectations in (4) with empirical averages as in (2). We will now discuss the assumptions under which Theorem 3.1 is derived, starting with the following.

**Assumption 1.** There exists  $\xi > 0$  such that  $B(0^{I+J}, \xi) = \{c \in \mathbb{R}^{I+J} \mid ||c|| \le \xi\} \subseteq int \ (\mathcal{C}) \cap int \ (\hat{\mathcal{C}})$  where int denotes the interior of the set and  $0^{I+J}$  is the origin of  $\mathbb{R}^{I+J}$ .

Assumption 1 ensures that (D) and ( $\hat{D}$ ) have non-empty and compact solution sets (see e.g. [13, Proposition 4.4.1, 4.4.2]). It can be seen as a stronger version of Slater's condition [43] used in convex optimization, and in particular, implies that  $P^*$  exists and is finite.

**Duality gap.** The bound on the duality gap is based on analyzing the properties of (P) when  $\mathcal{H}$  is large enough to approximate benign function classes that have the property of *decomposability*. We say that a set  $\Phi$  is *decomposable* [44], if for any  $\phi_1, \phi_2 \in \Phi$  and measurable subset  $\mathcal{Z}$ ,

$$\phi_3(\mathbf{x}) = \begin{cases} \phi_1(\mathbf{x}), & \mathbf{x} \in \mathcal{Z} \\ \phi_2(\mathbf{x}), & \mathbf{x} \in \mathcal{Z}^c \end{cases}$$

is also a member of  $\Phi$ . The  $L^p$  spaces and their analogue for vector valued functions, Bochner spaces [45] (see also Appendix B.1.1), are decomposable function spaces. We now introduce the functional version of (P),

$$\begin{split} P_{\phi}^{\star} &= \inf_{\phi \in \Phi} \quad \mathbb{E}_{\mathbb{D}} \big[ \ell(\phi(\mathbf{x}), y) \big] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_{i}} \big[ g_{i}(\phi(\mathbf{x}), y) \big] \leq 0, \quad \text{for } i = 1, \dots, I, \\ \quad \mathbb{E}_{\mathbb{Q}_{i}} \big[ h_{j}(\phi(\mathbf{x}), y) \big] = 0, \quad \text{for } j = 1, \dots, J. \end{split} \tag{$\mathbf{P}_{\phi}$}$$

Notice that the Lagrangian for  $(P_{\phi})$  is also defined as in (1). Similarly, its dual problem is given by

$$D_{\phi}^{\star} = \sup_{\lambda \in \mathbb{R}_{+}^{I}, \, \mu \in \mathbb{R}^{J}} \inf_{\phi \in \Phi} L(\phi, \lambda, \mu). \tag{D_{\phi}}$$

When the distributions  $\mathbb{D}, \mathbb{P}_i, \mathbb{Q}_j$  are atomless probability measures and  $\Phi$  is decomposable,  $(P_{\phi})$  is strongly dual, i.e.,  $P_{\phi}^{\star} = D_{\phi}^{\star}$  (see Appendix B.2 Proposition B.1). We can therefore bound the duality gap in (3) by the decomposition

$$P^* - D^* = P^* - P_\phi^* + D_\phi^* - D^*. \tag{5}$$

The following assumptions allow (P) to inherit similarly favourable duality properties based on (5).

**Assumption 2.** Let  $\mathbb{P}_+ = \mathbb{D} + \sum_{i=1}^{I} \mathbb{P}_i + \sum_{j=1}^{J} \mathbb{Q}_j$ , and let  $L^p(\mathbb{R}^K; \mathbb{P}_+)$  denote the Bochner space corresponding to  $(\mathbb{R}^K, \mathbb{P}_+)$ . Assume that

- 1. the distributions  $\mathbb{D}, \mathbb{P}_i, \mathbb{Q}_i$  are atomless probability measures, and,
- 2. there exists a decomposable set  $\Phi \subseteq L^p(\mathbb{R}^K; \mathbb{P}_+)$  such that  $\mathcal{H} \subseteq \Phi$  and a solution  $\phi^* \in Opt(P_\phi)$  such that for some  $\theta \in \Theta$  and  $\nu > 0$ ,  $\|\phi^* f_\theta\|_{L^p(\mathbb{R}^K; \mathbb{P}_+)} \leq \nu$ .

**Assumption 3.** The loss functions  $\ell$ ,  $g_i$ ,  $h_j : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$  are L-Lipschitz continuous and the model  $f_{\theta}(\mathbf{x})$  is  $L_{\theta}$ -Lipschitz continuous with respect to  $\theta$  at every  $\mathbf{x} \in \mathcal{X}$ .

Atomlessness is satisfied, for instance, if a measure has a density with respect to the Lebesgue measure (Appendix B.1 Lemma B.2). Several works have studied the approximation of decomposable sets like the  $L^p$  spaces by parametrized model classes like neural networks [46–48] and support vector machines [49]. A concrete value of  $\nu$  can be found in some cases, for example in [48]. The regularity assumptions on the model and loss functions are mild. Loss functions only need to be Lipschitz continuous on the range of  $f_{\theta}(\mathbf{x})$ , as is the case for the Huber loss, square loss, and hinge loss when

the range of  $f_{\theta}(\mathbf{x})$  and  $\mathcal{Y}$  is bounded. An example of a model that is Lipschitz with respect to its parameters is a feedforward neural network with Lipschitz activations when the data and parameters are bounded in norm.

To bound the terms in (5), we also use the following quantity that measures the sensitivity of the feasibility set Feas(P), namely,

$$R(\nu) = \sup_{\theta \in \Theta_{\nu}} \inf_{\theta_{0} \in \operatorname{Feas}(P)} \|\theta - \theta_{0}\|_{2},$$
where  $\Theta_{\nu} = \left\{ \theta \in \Theta \middle| \begin{array}{c} \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(f_{\theta}(\mathbf{x}), y) \right] \leq L\nu, & \text{for } i = 1, \dots, I \\ \text{and} & \left| \mathbb{E}_{\mathbb{Q}_{j}} \left[ h_{j}(f_{\theta}(\mathbf{x}), y) \right] \right| \leq L\nu & \text{for } j = 1, \dots, J \end{array} \right\}.$  (6)

 $R(\nu)$  measures the maximum distance between  $\Theta_{\nu}$  and Feas(P). Hence, it is a decreasing function of  $\nu$  and vanishes when  $\nu=0$ . However, if relaxing the constraints causes the feasibility set to change radically,  $R(\nu)$  could be unbounded for  $\nu>0$ . This can be avoided by ensuring that Feas(P<sub>\theta\nu</sub>) is bounded, by e.g., regularizing  $\Theta$  by enforcing  $\|\theta\|_2 \leq W$  for some W>0.

**Dual estimation error.** The next assumption allows us to bound the dual estimation error  $|D^{\star} - \hat{D}^{\star}|$ .

**Assumption 4.** Let  $\mathbb{P} \in \{\mathbb{D}, \mathbb{P}_1, \dots, \mathbb{Q}_J\}$  and let  $(\mathbf{x}^{(k)}, y^{(k)})_{k=1}^N$  be an i.i.d sample of size N drawn from  $\mathbb{P}$ . Denote  $\mathbb{L} = \{\ell, g_1, \dots, g_I, h_1, \dots, h_J\}$ . Then, assume that there exists a function  $\zeta^{UC}(N, \delta) : \mathbb{N} \times [0, 1] \to \mathbb{R}$ , such that, for all  $\delta \in (0, 1)$ , (a)  $\zeta^{UC}$  is strictly decreasing in N, (b) satisfies  $\lim_{N \to \infty} \zeta^{UC}(N, \delta) = 0$ , and (c) for any  $\mathcal{L} \in \mathbb{L}$ , the following is true:

$$\mathbb{P}\left(\sup_{\theta\in\Theta}\left|\mathbb{E}_{\mathbb{P}}\left[\mathcal{L}(f_{\theta}(\mathbf{x}),y)\right] - \frac{1}{N}\sum_{k=1}^{N}\mathcal{L}(f_{\theta}(\mathbf{x}_{k}),y_{k})\right| \leq \zeta^{UC}(N,\delta)\right) \geq 1 - \delta.$$
 (7)

Assumption 4 is called uniform convergence, and is satisfied if e.g.  $\mathcal{H}$  has a finite VC-dimension or Rademacher complexity (see [50][Corollary 3.19, Theorem 3.3] and also [11][Proposition III.1]). Bounds on the VC dimension and Rademacher complexity, which are measures of *model complexity*, are available for many model classes [51, 52].

**Main Result.** The generalization error in (3) can be bounded under these assumptions by combining (5) and the uniform convergence bound in Assumption 4. We next use  $\gamma$  to denote the tuple of dual variables for inequalities and equalities, i.e.,  $(\lambda, \mu)$ .

**Theorem 3.1.** Let  $N_{min} = \min \{M_0, M_1, \dots, M_I, N_1, \dots, N_J\}$  and assume that  $R(\nu) < \infty$  and  $P_{\phi}^{\star} > -\infty$ . Under Assumptions 1-3 there exist  $\gamma_{\phi}^{\star} \in Opt(D_{\phi}), \gamma^{\star} \in Opt(D)$ , and  $\hat{\gamma}^{\star} \in Opt(\hat{D})$ . Moreover, for any  $\delta \in (0,1)$ , it holds with probability at least  $1 - (1 + I + J)\delta$ , that

$$|P^{\star} - \hat{D}^{\star}| \le \left(1 + \|\gamma_{\phi}^{\star}\|_{1}\right) L\nu + LL_{\theta}R(\nu) + \left(1 + \max\left\{\|\gamma^{\star}\|_{1}, \|\hat{\gamma}^{\star}\|\right\}_{1}\right) \zeta^{UC}(N_{\min}, \delta). \tag{8}$$

The proof of Theorem 3.1 is deferred to Appendix B. We next discuss the main factors driving the generalization error bound.

The main difference between constrained and unconstrained generalization bounds is that (8) is driven by the *constraint sensitivity*. This is represented by the Lagrange multipliers of (P) and its empirical and functional versions, which appear in the bound through their norms. More specifically, it is well known that for strongly dual problems, Lagrange multipliers are sensitivity measures (subgradients) of the optimal value with respect to the constraint constants. Explicitly, suppose that we were to perturb the constraints of (P) to obtain

$$\begin{split} P^{\star}(\mathbf{c}, \mathbf{d}) &= \inf_{\theta \in \Theta} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(f_{\theta}(\mathbf{x}), y) \right] \leq c_{i}, \quad \text{for } i = 1, \dots, I, \\ \mathbb{E}_{\mathbb{Q}_{i}} \left[ h_{j}(f_{\theta}(\mathbf{x}), y) \right] &= d_{j}, \quad \text{for } j = 1, \dots, J. \end{split} \tag{P-pert}$$

The problem (P-pert) defines the (perturbation) function  $P^*(\mathbf{c}, \mathbf{d})$ . If (P) were strongly dual (e.g. a convex program) and  $P^*(\mathbf{c}, \mathbf{d})$  is differentiable, then

$$\lambda_j^{\star} = -\frac{\partial P^{\star}(0^I, 0^J)}{\partial c_i} \quad \text{and} \quad \mu_j^{\star} = -\frac{\partial P^{\star}(0^I, 0^J)}{\partial d_j}, \tag{9}$$

for  $i=1,\ldots,I$  and  $j=1,\ldots,J$  (see [40, Equation 5.58]). Though differentiability and strong duality of (P-pert) does not generally hold in our setting, *approximate* relations can be obtained in terms of the bound on the duality gap  $P^\star - D^\star$  (see Remark B.6). Thus,  $\|\gamma^\star\|_1 = \|\lambda^\star\|_1 + \|\mu^\star\|_1$ , that appears in (8), measures how much the objective changes when the constraints are perturbed. Constraint sensitivity also appears in terms of  $R(\nu)$ , which marks a clear difference with the generalization bounds for inequality constraints J=0 that depend solely on  $\|\lambda^\star\|_1$  (see Remark 3.1). The bound in (8) also depends on the sensitivity of the loss functions and the model (with respect to its parameters) through their Lipschitz constants L and  $L_\theta$  respectively.

While these factors are dictated by the problem formulation, Theorem 3.1 shows us that they can be mitigated by using richer parametrizations (i.e., reducing  $\nu$ ) and larger datasets (i.e., increasing  $N_{\rm min}$ ). Indeed, as  $\nu \to 0$ , the first set of terms (relating to the duality gap) vanish. However, the sample complexity  $\zeta^{\rm UC}(N_{\rm min},\delta)$  typically increases with larger model classes, i.e., as  $\nu$  decreases. Hence, we find a trade-off between approximation error and estimation error that mirrors the trade-off in unconstrained learning [53, Section 5.2].

To summarise, Theorem 3.1 reveals *four* key drivers of the generalization error: (a) constraint sensitivity, (b) sensitivity of the losses and parametrization, (c) model capacity, and (d) sample size. The bound also mirrors the classical decomposition of the unconstrained learning error into an approximation error (here, the duality gap  $P^* - D^*$ ) and the estimation error (here, the dual estimation error  $|D^* - \hat{D}^*|$ ). We conclude this section with a few remarks.

**Remark 3.1** (Comparison with results for inequality constraints). Notably, the bounds for problems involving only inequality constraints from [12, 11] do not depend on the smoothness of the parametrization  $L_{\theta}$  (Assumption 3). This fundamental distinction is rooted in the fact that in the absence of equality constraints (J=0) going from  $\Phi$  to  $\mathcal{H}$  is akin to contracting the functional feasibility set  $Feas(P_{\phi})$  by tightening the functional inequality constraints. Consequently,  $(P_{\phi})$  and its parametrized formulation (P) remain closely aligned, and the latter approximately inherits the duality properties of the former. The equality constraints in (P) however make it so that the feasibility sets are no longer nested. Changing the targets lead to more intricate changes in the set of feasible parameters  $\theta$ . The sensitivity of the parametrization  $L_{\theta}$  therefore affects generalization. This is also the reason why Assumption 1 reduces to the existence of a strictly feasible solution when J=0. Indeed, the presence of equalities requires the stronger regularity conditions in Assumption 1 to ensure that (P) is feasible for any small perturbation. The reader is referred to Appendix B.5 for additional technical distinctions.

**Remark 3.2** (Functional strong duality). The core result underpinning Theorem 3.1 is the strong duality of the functional problem  $(P_{\phi})$  (Proposition B.1). The crux of the proof lies in proving that the cost-constraint epigraph,

$$\mathcal{M}_{\phi} = \left\{ (f, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^{I} \times \mathbb{R}^{J} \middle| \begin{array}{l} \exists \phi \in \Phi \quad \textit{s.t.} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right] = f, \\ & \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(\phi(\mathbf{x}), y) \right] \leq u_{i}, \quad \textit{for } i = 1, \dots, I, \\ \textit{and} \quad \mathbb{E}_{\mathbb{Q}_{j}} \left[ h_{j}(\phi(\mathbf{x}), y) \right] = v_{j} \quad \textit{for } j = 1, \dots, J, \end{array} \right\}$$

is convex under Assumption 2. Thus, while  $(P_{\phi})$  is not a convex optimization problem, a classical result from convex optimization [13, Proposition 4.4.1] can be applied to the previous fact to show that  $(P_{\phi})$  is strongly dual. The convexity of  $\mathcal{M}_{\phi}$  is established using Lyapunov's theorem on the range of atomless measures [54, Chaper IX Corollary 5], closely related to the bang-bang principle in control theory [55]. Proposition B.1 has appeared with minor variations in [56–58, 11], though our proof improves on [11] for the regression case (continuous  $\mathcal{Y}$ ), which we prove without additional assumptions.

#### 4 Algorithm

Theorem 3.1 establishes that solving  $(\hat{D})$  provides (approximate) solutions to (P). Next, we propose an algorithm to tackle  $(\hat{D})$  based on the traditional dual ascent method. This algorithm assumes access to an oracle for unconstrained problems as in [10, 11], a fact formalised in the following assumption.

## Algorithm 1 Primal-dual constrained learning algorithm

```
1: Inputs: Loss functions \ell, g_i, h_j, samples (\mathbf{x}_{m_0}, y_{m_0}) \sim \mathbb{D}, (\mathbf{x}_{m_i}, y_{m_i}) \sim \mathbb{P}_i, (\mathbf{x}_{n_j}, y_{n_j}) \sim \mathbb{Q}_j for i = 1, \ldots, I, j = 1, \ldots, J, iterations T \in \mathbb{N}, dual learning rate \eta > 0.

2: Initialize: \lambda^{(0)} \leftarrow \mathbf{0}^I, \mu^{(0)} \leftarrow \mathbf{0}^J

3: for t = 1, \ldots, T do

4: \theta^{(t)} \leftarrow \mathcal{O}(\hat{L}(f, \lambda^{(t-1)}, \mu^{(t-1)})) \rhd Assumption 5

5: \lambda_i^{(t)} \leftarrow \max\left\{0, \lambda_i^{(t-1)} + \eta \sum_{m_i=1}^{M_i} g_i(f_{\theta^{(t)}}(\mathbf{x}_{m_i}), y_{m_i})\right\} \rhd for i = 1, \ldots, I

6: \mu_j^{(t)} \leftarrow \mu_j^{(t-1)} + \eta \sum_{n_j=1}^{N_j} h_j(f_{\theta^{(t)}}(\mathbf{x}_{n_j}), y_{n_j}) \rhd for j = 1, \ldots, J

7: end for
```

**Assumption 5.** The solution  $\theta^*(\lambda, \mu) = \mathcal{O}(\hat{L}(f_\cdot, \lambda, \mu))$  returned by the oracle in Algorithm 1 approximately minimizes the empirical Lagrangian in (2), i.e., for  $\rho \geq 0$  it holds that  $\hat{L}(f_{\theta^*(\lambda, \mu)}, \lambda, \mu) \leq \hat{q}(\lambda, \mu) + \rho$  for all  $\lambda, \mu$ .

In Algorithm 1, this oracle is used to update the model  $\theta$  (primal variable) in Step 4, while the empirical constraint violations (or slacks) are used to update the dual variables  $\lambda$ ,  $\mu$  (Steps 5-6). If the oracle is optimal (i.e.,  $\rho=0$ ), Line 6-7 constitute a projected *subgradient* ascent with respect to the dual objective  $\hat{q}(\lambda,\mu)$  [59]. Since the dual objective is always concave, subgradient ascent converges to a global optimum for certain reducing step size rules [59, Chapter 2]. However, the descent is generally non-monotonic, i.e., the last iterate is not necessarily the best iterate. Hence, guarantees are often of the following form. Note that Theorem 4.1 relates the empirical dual value  $\hat{D}^{\star}$  to the *average* of the Lagrangian iterates.

**Theorem 4.1.** Suppose Assumptions 1 and 5 hold and let the loss functions  $\ell$ ,  $g_i$ ,  $h_j$  be B-bounded. Let  $U_0 = \inf_{\gamma^* \in Opt(\hat{D})} \|\gamma^{(0)} - \gamma^*\|$ . Then, for any  $T \in \mathbb{N}$ , it holds that,

$$\left| \hat{D}^{\star} - \frac{1}{T} \sum_{t=0}^{T-1} \left( \hat{L}(\theta^{(t)}, \gamma^{(t)}) \right) \right| \le \rho + \frac{U_0}{2\eta T} + \frac{1}{2} (I+J)\eta B^2.$$
 (10)

If  $\eta \leq \frac{\rho}{(I+J)B^2}$  and  $T \geq \frac{U_0}{\eta\rho}$ , then the bound is equal to  $2\rho$ .

Theorem 4.1 establishes results for the dual iterates, modulo the averaging, but does not characterise the feasibility or optimality of  $\theta^{(t)}$ . Since the set of primal variables defined by the Lagrange multipliers  $(\hat{\lambda}^*, \hat{\mu}^*)$  need not be unique, or all feasible, recovering a feasible near-optimal model can be difficult. This is not a substantial issue in convex optimization, where averaging solves this problem (e.g. [59–62]). Non-convex settings often rely on randomization to overcome this challenge (see, e.g. [11, 10, 63]), although there is empirical and theoretical evidence that this is not a substantial issue in ML [10, 11, 64, 65].

In practice, we do not have access to the oracle from Assumption 5. Our experiments in Section 5 show that replacing line 5 with a single (stochastic) gradient descent step can produce feasible solutions (without additional primal recovery techniques) that also perform well with respect to the objective loss.

## 5 Experiments

In this section, we demonstrate the empirical performance of Algorithm 1 on instances of (P) presented in Section 5. Detailed descriptions of the experiments can be found in Appendix E.

**Exact vs. approximate fairness.** Figure 1 compares models trained on the COMPAS dataset [66] using the fairness formulation in (P-DP) against models trained with a double-sided inequality approximation with tolerance parameter  $\epsilon$  (see Remark 2.1) and an unconstrained baseline. The indicator functions that appear in the constrained formulations are replaced by sigmoid functions to

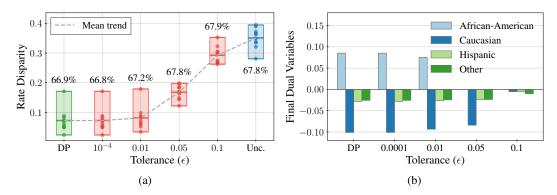


Figure 1: Exact vs approximate fairness. (a) Comparison between (P-DP) and the inequality relaxation described in Remark 2.1 with parameter  $\epsilon>0$  (10 random splits). Mean accuracy (across splits) is reported for each method/tolerance. (b) Final (effective) dual variables for Algorithm 1. Indeed, since the inequality relaxation uses two constraints for each group (upper and lower bound), we show only the difference between upper and lower dual variables.

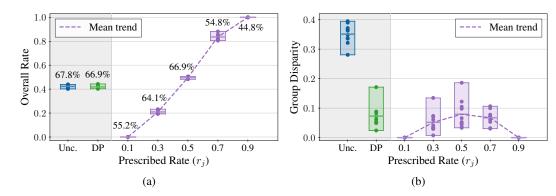


Figure 2: Prescribed rates. Solutions of (P-F) for different  $r_j$  (10 random splits). (a) Average rate of positive outcomes across population, annotated with the mean accuracy (across splits); (b) Rate disparity across different groups.

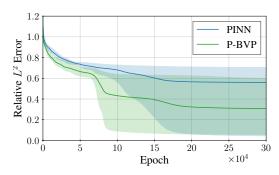
enable the use of gradient descent to replace the oracle in Step 4 of Algorithm 1. Explicitly, we use

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{I}\left[f_{\theta}(\mathbf{x})>0.5\right]|\mathbf{x}\in\mathcal{G}_{j}\right]\approx\mathbb{E}_{\mathbb{P}}\left[\sigma\left(\alpha\left(f_{\theta}(\mathbf{x})-0.5\right)\right)|\mathbf{x}\in\mathcal{G}_{j}\right]$$

and similarly for the overall rate  $\mathbb{E}_{\mathbb{P}}\left[\mathbb{I}\left[f_{\theta}(\mathbf{x})>0.5\right]\right]$ , where  $\sigma$  denotes the sigmoid function. We split the dataset into a training (70%) and test (30%) set 10 times and report the results.

Figure 1(a) shows that the equality formulation achieves the lowest *disparity* in group rates (measured as the difference between the maximum and minimum group rates) comparable to the inequality-based model with small tolerances (e.g.,  $\epsilon=10^{-4}$ ). Figure 1(b) shows that the final (effective) dual variables for  $\epsilon=10^{-4}$  are also indistinguishable from those of the equality formulation. However, we see that a looser tolerance ( $\epsilon=10^{-2}$ ) introduces noticeable differences, both in the dual solution and the *group disparity*. This highlights the challenge of selecting an appropriate tolerance, a difficulty circumvented by directly enforcing the equality constraints.

**Prescribed rates.** In Figure 2, we showcase the results of imposing specific group rates using (P-F) and compare them to unconstrained and DP-constrained [i.e., (P-DP)] problems. With the exception of extreme rates (0.1 and 0.9), which yield nearly constant classifiers, models with intermediate targets  $r_j$  achieve low group disparities, comparable to those of the DP-constrained model. However, while (P-DP) maintains an overall rate of positive outcomes close to the unconstrained model, (P-F) allows this rate to be adjusted more granularly. This flexibility comes at a negligible difference in accuracy for prescribed rates close to the DP rate. For example,  $r_j = 0.5$  achieves the same test accuracy as (P-DP) (Figure 2(a)) and similar (test) group disparities (Figure 2(b)).



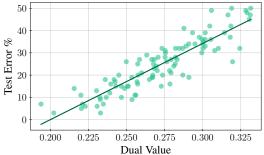


Figure 3: Relative  $L^2$  error for convection BVP ( $\beta=50$ ). The lines show the mean curve across 5 runs, and the shaded region indicates the max and min curves.

Figure 4: Comparison of classwise test errors and dual values for CIFAR-100 trained with (P-CI) (along with the best fit line). The correlation is 0.89 indicating a strong linear relationship.

Table 1: Relative  $L^2$  error for convection

Tuble 1. Relative E circl for convection				
β	PINN	(P-BVP)		
30	$2.46 \pm 0.99 \%$	$0.62 \pm 0.17 \%$		
50	$56.0 \pm 25.8 \%$	$30.7 \pm 24.2 \%$		

Table 2: Test Accuracy on CIFAR-10/100

Dataset	ERM	(P-CI)
CIFAR-10	$95.03 \pm 0.21 \%$	$95.01 \pm 0.10 \%$
CIFAR-100	$76.11 \pm 0.28 \%$	$75.17 \pm 0.21 \%$

**Boundary value problems.** Table 1 shows that taking a constrained approach to solving a convection BVP with sinusoidal initial condition (see Appendix E.2 for details) outperforms the unconstrained approach with fixed multipliers for the boundary and initial conditions (PINN). The mean and standard deviation of 5 seeds have been reported. The primary challenge of solving this BVP is propagating the initial condition through time since the solution itself is very regular. Thus, the improvement may be explained by larger dual variables for the boundary conditions (as seen in Appendix F Figure 9).

Interpolating classifiers. Table 2 (also computed over 5 seeds) shows that (P-CI) has worse test accuracy compared to the unconstrained problem (ERM) on CIFAR-100—though the gap is not large. On the other hand, solving (P-CI) using Algorithm 1 yields dual variables that exhibit an approximately linear relationship with the mean test error of the class. Since models are trained to interpolation on the train set, this information is not generally available without cross-validation. It can be used as a confidence measure for the performance of the model or to detect biases in the model. Meanwhile, (P-CI) performs virtually the same as ERM on CIFAR-10. This relationship between test error and dual variables may be partially explained by 9, which suggests that a large dual variable indicates a constraint on the class-wise error that significantly contributes to the model complexity  $(\|\theta\|_2^2)$ —since relaxing the constraint would lead to a large decrease in the parameter norm. Though this norm has been tied to generalization error in [67, 68], a definitive answer would require a more detailed analysis that is beyond the scope of this work.

## 6 Conclusion

In this paper, we studied equality-constrained learning problems, i.e., statistical optimization problems with equality constraints. We extended the existing generalization theory for inequality-constrained problems, showing that equalities are also tractable through Lagrangian methods as long as the parametrization is rich enough. Nevertheless, they demand stronger assumptions than inequalities. We also introduced a practical algorithm based on dual ascent to solve problems with both equality and inequality constraints. We illustrated the behavior of this algorithm in a fair learning problem, showing results for both classical problems, involving demographic parity, as well as new formulations enabled by these equality-constrained problems, namely, learning tasks that enforce specific prediction rates for each group. We also showcase results for solving BVPs and fitting interpolating classifiers.

## Acknowledgements

The work of Aneesh Barthakur is funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC 2075 - 390740016. The work of L.F.O. Chamon is supported by the Agence Nationale de la Recherche (ANR) project ANR-25-CE23-3477-01 as well as a chair from Hi!PARIS, funded in part by the ANR AI Cluster 2030 and ANR-22-CMAS-0002. The authors thank the Stuttgart Center for Simulation Science (SimTech) and the International Max Planck Research School for Intelligent Systems (IMPRS-IS) for supporting Aneesh Barthakur and acknowledge the computing time provided on the high-performance computer HoreKa by the National High-Performance Computing Center at KIT (NHR@KIT). This center is supported by the Federal Ministry of Education and Research and the Ministry of Science, Research and the Arts of Baden-Württemberg as well as the DFG.

#### References

- [1] Solon Barocas, Moritz Hardt, and Arvind Narayanan. Fairness and Machine Learning: Limitations and Opportunities. MIT Press, 2023.
- [2] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards Deep Learning Models Resistant to Adversarial Attacks. In *International Conference on Learning Representations*, 2018.
- [3] Hongyang Zhang, Yaodong Yu, Jiantao Jiao, Eric P. Xing, Laurent El Ghaoui, and Michael I. Jordan. Theoretically Principled Trade-off between Robustness and Accuracy. In *International Conference on Machine Learning*, pages 7472–7482, 2019.
- [4] Cynthia Dwork and Aaron Roth. The Algorithmic Foundations of Differential Privacy. *Foundations and Trends*® *in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [5] Javier García and Fernández. A Comprehensive Survey on Safe Reinforcement Learning. *Journal of Machine Learning Research*, 16(42):1437–1480, 2015.
- [6] Santiago Paternain, Miguel Calvo-Fullana, Luiz F. O. Chamon, and Alejandro Ribeiro. Safe Policies for Reinforcement Learning via Primal-Dual Methods. *IEEE Transactions on Automatic Control*, 68(3):1321–1336, 2023.
- [7] Ian J. Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and Harnessing Adversarial Examples. In *International Conference on Learning Representations*, 2015.
- [8] Richard Berk, Hoda Heidari, Shahin Jabbari, Matthew Joseph, Michael J. Kearns, Jamie Morgenstern, Seth Neel, and Aaron Roth. A Convex Framework for Fair Regression. CoRR, 2017.
- [9] Irina Higgins, Loic Matthey, Arka Pal, Christopher Burgess, Xavier Glorot, Matthew Botvinick, Shakir Mohamed, and Alexander Lerchner. beta-VAE: Learning Basic Visual Concepts with a Constrained Variational Framework. In *International Conference on Learning Representations*, 2017.
- [10] Andrew Cotter, Heinrich Jiang, Maya R. Gupta, Serena Lutong Wang, Taman Narayan, Seungil You, and Karthik Sridharan. Optimization with Non-Differentiable Constraints with Applications to Fairness, Recall, Churn, and Other Goals. *Journal of Machine Learning Research*, 20:172:1–172:59, 2019.
- [11] Luiz F. O. Chamon, Santiago Paternain, Miguel Calvo-Fullana, and Alejandro Ribeiro. Constrained Learning With Non-Convex Losses. *IEEE Transactions on Information Theory*, 69(3):1739–1760, 2023.
- [12] Luiz F. O. Chamon and Alejandro Ribeiro. Probably Approximately Correct Constrained Learning. In *Advances in Neural Information Processing Systems*, 2020.
- [13] Dimitri Bertsekas. Convex Optimization Theory. Athena Scientific, 2009.
- [14] Muhammad Bilal Zafar, Isabel Valera, Manuel Gomez-Rodriguez, and Krishna P. Gummadi. Fairness Constraints: Mechanisms for Fair Classification. In *International Conference on Artificial Intelligence and Statistics*, pages 962–970, 2017.
- [15] Moritz Hardt, Eric Price, and Nati Srebro. Equality of Opportunity in Supervised Learning. In Advances in Neural Information Processing Systems, pages 3315–3323, 2016.
- [16] Matt J. Kusner, Joshua R. Loftus, Chris Russell, and Ricardo Silva. Counterfactual Fairness. In *Advances in Neural Information Processing Systems*, pages 4066–4076, 2017.
- [17] Taco Cohen and Max Welling. Group Equivariant Convolutional Networks. In *International Conference on Machine Learning*, pages 2990–2999, 2016.
- [18] Risi Kondor and Shubhendu Trivedi. On the Generalization of Equivariance and Convolution in Neural Networks to the Action of Compact Groups. In *International Conference on Machine Learning*, pages 2752–2760, 2018.

- [19] I. Hounie, L. F. O. Chamon, and A. Ribeiro. Automatic Data Augmentation via Invariance-Constrained Learning. In *International Conference on Machine Learning*, pages 13410–13433, 2023.
- [20] Geoff Pleiss, Manish Raghavan, Felix Wu, Jon M. Kleinberg, and Kilian Q. Weinberger. On Fairness and Calibration. In Advances in Neural Information Processing Systems, pages 5680– 5689, 2017.
- [21] Chuan Guo, Geoff Pleiss, Yu Sun, and Kilian Q. Weinberger. On Calibration of Modern Neural Networks. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70, pages 1321–1330. Proceedings of Machine Learning Research, 2017.
- [22] Mikhail Belkin. Fit without fear: remarkable mathematical phenomena of deep learning through the prism of interpolation, 2021.
- [23] Shengjia Zhao, Jiaming Song, and Stefano Ermon. The Information Autoencoding Family: A Lagrangian Perspective on Latent Variable Generative Models, 2018.
- [24] Xi Chen, Yan Duan, Rein Houthooft, John Schulman, Ilya Sutskever, and Pieter Abbeel. InfoGAN: Interpretable Representation Learning by Information Maximizing Generative Adversarial Nets. In *Advances in Neural Information Processing Systems*, 2016.
- [25] A. Robey, L. F. O. Chamon, G. J. Pappas, H. Hassani, and A. Ribeiro. Adversarial Robustness with Semi-Infinite Constrained Learning. In *Conference on Neural Information Processing Systems*, pages 6198–6215, 2021. \* equal contribution.
- [26] Sam Corbett-Davies, Emma Pierson, Avi Feller, Sharad Goel, and Aziz Huq. Algorithmic Decision Making and the Cost of Fairness. In *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 797–806, 2017.
- [27] Alexandra Chouldechova. Fair Prediction with Disparate Impact: A Study of Bias in Recidivism Prediction Instruments. *Big Data*, 5(2):153–163, 2017.
- [28] Sam Corbett-Davies and Sharad Goel. The Measure and Mismeasure of Fairness: A Critical Review of Fair Machine Learning. *arXiv:1808.00023*, 2018.
- [29] Michael Feldman, Sorelle A. Friedler, John Moeller, Carlos Scheidegger, and Suresh Venkatasubramanian. Certifying and Removing Disparate Impact. In *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 259–268, 2015.
- [30] Lloyd N. Trefethen. Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations. 1996. Unpublished text.
- [31] Hans Petter Langtangen and Kent-Andre Mardal. *Introduction to Numerical Methods for Variational Problems*. Springer International Publishing, 2019.
- [32] M. Raissi, P. Perdikaris, and G.E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2019.
- [33] Viggo Moro and Luiz F. O. Chamon. Solving Differential Equations with Constrained Learning. In *International Conference on Learning Representations*, 2025.
- [34] Sifan Wang, Yujun Teng, and Paris Perdikaris. Understanding and Mitigating Gradient Flow Pathologies in Physics-Informed Neural Networks. *SIAM Journal on Scientific Computing*, 43(5):A3055–A3081, 2021.
- [35] Aditi S. Krishnapriyan, Amir Gholami, Shandian Zhe, Robert M. Kirby, and Michael W. Mahoney. Characterizing possible failure modes in physics-informed neural networks. In *Advances in Neural Information Processing Systems*, pages 26548–26560, 2021.
- [36] Robert E. Schapire, Yoav Freund, Peter Barlett, and Wee Sun Lee. Boosting the margin: A new explanation for the effectiveness of voting methods. In *ICML*, pages 322–330, 1997.

- [37] Mikhail Belkin, Daniel J. Hsu, and Partha Mitra. Overfitting or perfect fitting? Risk bounds for classification and regression rules that interpolate. In Samy Bengio, Hanna M. Wallach, Hugo Larochelle, Kristen Grauman, Nicolò Cesa-Bianchi, and Roman Garnett, editors, *NeurIPS*, pages 2306–2317, 2018.
- [38] Sébastien Bubeck. Convex Optimization: Algorithms and Complexity, 2015.
- [39] Gábor Braun, Alejandro Carderera, Cyrille W. Combettes, Hamed Hassani, Amin Karbasi, Aryan Mokhtari, and Sebastian Pokutta. Conditional Gradient Methods, 2025.
- [40] Stephen Boyd and Lieven Vandenberghe. Convex Optimization. Cambridge University Press, 2004.
- [41] Juan Elenter, Luiz F. O. Chamon, and Alejandro Ribeiro. Near-Optimal Solutions of Constrained Learning Problems. In *International Conference on Learning Representations*, 2024.
- [42] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczynski. Lectures on Stochastic Programming: Modeling and Theory, Third Edition. Society for Industrial and Applied Mathematics, 2021.
- [43] Morton Slater. Lagrange Multipliers Revisited. Cowles Foundation Discussion Papers 80, Cowles Foundation for Research in Economics, Yale University, 1959.
- [44] Andrzej Fryszkowski. Fixed Point Theory for Decomposable Sets. Springer, 2004.
- [45] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces*. Springer, 2016.
- [46] Kurt Hornik, Maxwell Stinchcombe, and Halbert White. Multilayer feedforward networks are universal approximators. *Neural Networks*, 2(5):359–366, 1989.
- [47] Boris Hanin and Mark Sellke. Approximating Continuous Functions by ReLU Nets of Minimal Width, 2017.
- [48] Zuowei Shen, Haizhao Yang, and Shijun Zhang. Deep Network Approximation Characterized by Number of Neurons. *Communications in Computational Physics*, 28(5):1768–1811, 2020.
- [49] Ingo Steinwart and Andreas Christmann. Support Vector Machines. Springer, 2008. Information Science and Statistics.
- [50] Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. Foundations of Machine Learning. MIT Press, 2012.
- [51] Peter L. Bartlett, Nicholas J. A. Harvey, Christopher Liaw, and Abbas Mehrabian. Nearly-tight VC-dimension and Pseudodimension Bounds for Piecewise Linear Neural Networks. *Journal of Machine Learning Research*, 20(63):1–17, 2019.
- [52] Taeyoung Kim and Myungjoo Kang. Bounding the Rademacher Complexity of Fourier Neural Operator. *Machine Learning*, 113(5):2467–2498, 2024.
- [53] Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: From Theory to Algorithms*. Cambridge University Press, 2014.
- [54] J. Diestel and J. J. Uhl. Vector Measures. Mathematical Surveys and Monographs. American Mathematical Society, 1977.
- [55] M. Ali Khan and Nobusumi Sagara. The Bang-Bang, Purification and Convexity Principles in Infinite Dimensions: Additional Characterizations of the Saturation Property. Set-Valued and Variational Analysis, 22(4):721–746, 2014.
- [56] Alejandro Ribeiro. Optimal resource allocation in wireless communication and networking. *EURASIP Journal on Wireless Communications and Networking*, 2012(1), 2012.
- [57] Luiz F. O. Chamon, Yonina C. Eldar, and Alejandro Ribeiro. Strong Duality of Sparse Functional Optimization. In *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4739–4743, 2018.

- [58] Dionysis Kalogerias and Spyridon Pougkakiotis. Strong Duality in Risk-Constrained Nonconvex Functional Programming, 2023.
- [59] Naum Zuselevich Shor. *Minimization Methods for Non-Differentiable Functions*. Springer, 1985. Springer Series in Computational Mathematics.
- [60] Hanif D. Sherali and Choi Gyunghyun. Recovery of primal solutions when using subgradient optimization methods to solve lagrangian duals of linear programs. *Operations Research Letters*, 19(3):105–113, 1996.
- [61] Torbjörn Larsson, Michael Patriksson, and Ann-Brith Strömberg. Ergodic, primal convergence in dual subgradient schemes for convex programming. *Mathematical Programming*, 86(2):283–312, 1999.
- [62] Angelia Nedić and Asuman Ozdaglar. Approximate Primal Solutions and Rate Analysis for Dual Subgradient Methods. *SIAM Journal on Optimization*, 19(4):1757–1780, 2009.
- [63] Michael Kearns, Seth Neel, Aaron Roth, and Zhiwei Steven Wu. Preventing Fairness Gerrymandering: Auditing and Learning for Subgroup Fairness. In *International Conference on Machine Learning*, volume 80, pages 2564–2572, 2018.
- [64] Tianyi Lin, Chi Jin, and Michael I. Jordan. On Gradient Descent Ascent for Nonconvex-Concave Minimax Problems. In *International Conference on Machine Learning*, volume 119, pages 6083–6093, 2020.
- [65] Tanner Fiez, Lillian J. Ratliff, Eric Mazumdar, Evan Faulkner, and Adhyyan Narang. Global Convergence to Local Minmax Equilibrium in Classes of Nonconvex Zero-Sum Games. In Advances in Neural Information Processing Systems, pages 29049–29063, 2021.
- [66] ProPublica. Compas data analysis, 2016. https://github.com/propublica/ compas-analysis/.
- [67] Behnam Neyshabur, Ryota Tomioka, and Nathan Srebro. Norm-Based Capacity Control in Neural Networks. volume 40 of *Proceedings of Machine Learning Research*, pages 1376–1401, 2015.
- [68] Noah Golowich, Alexander Rakhlin, and Ohad Shamir. Size-Independent Sample Complexity of Neural Networks. In *COLT*, volume 75 of *Proceedings of Machine Learning Research*, pages 297–299, 2018.

## **NeurIPS Paper Checklist**

#### 1. Claims

Question: Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope?

Answer: [Yes]

Justification: Our contributions are listed in the abstract/introduction and accurately describe the contributions of the paper, namely a generalization theory for equality-constrained learning problem based on new regularity assumptions (Section 3), characterization of approximation and generalization error of solutions (Theorem 3.1), and analysis of dual ascent algorithm (Sections 4, 5).

#### Guidelines:

- The answer NA means that the abstract and introduction do not include the claims made in the paper.
- The abstract and/or introduction should clearly state the claims made, including the contributions made in the paper and important assumptions and limitations. A No or NA answer to this question will not be perceived well by the reviewers.
- The claims made should match theoretical and experimental results, and reflect how much the results can be expected to generalize to other settings.
- It is fine to include aspirational goals as motivation as long as it is clear that these goals
  are not attained by the paper.

#### 2. Limitations

Question: Does the paper discuss the limitations of the work performed by the authors?

Answer: [Yes]

Justification: The limitations of the theory developed here are clearly listed throughout the paper, e.g., the issues of primal recovery, non-monotonicity of dual ascent (Section 4), and the need for stronger regularity assumptions than inequality-constrained problems (Section 3).

- The answer NA means that the paper has no limitation while the answer No means that the paper has limitations, but those are not discussed in the paper.
- The authors are encouraged to create a separate "Limitations" section in their paper.
- The paper should point out any strong assumptions and how robust the results are to violations of these assumptions (e.g., independence assumptions, noiseless settings, model well-specification, asymptotic approximations only holding locally). The authors should reflect on how these assumptions might be violated in practice and what the implications would be.
- The authors should reflect on the scope of the claims made, e.g., if the approach was only tested on a few datasets or with a few runs. In general, empirical results often depend on implicit assumptions, which should be articulated.
- The authors should reflect on the factors that influence the performance of the approach. For example, a facial recognition algorithm may perform poorly when image resolution is low or images are taken in low lighting. Or a speech-to-text system might not be used reliably to provide closed captions for online lectures because it fails to handle technical jargon.
- The authors should discuss the computational efficiency of the proposed algorithms and how they scale with dataset size.
- If applicable, the authors should discuss possible limitations of their approach to address problems of privacy and fairness.
- While the authors might fear that complete honesty about limitations might be used by reviewers as grounds for rejection, a worse outcome might be that reviewers discover limitations that aren't acknowledged in the paper. The authors should use their best judgment and recognize that individual actions in favor of transparency play an important role in developing norms that preserve the integrity of the community. Reviewers will be specifically instructed to not penalize honesty concerning limitations.

#### 3. Theory assumptions and proofs

Question: For each theoretical result, does the paper provide the full set of assumptions and a complete (and correct) proof?

Answer: [Yes]

Justification: All assumptions are clearly listed and justified (namely, Assumptions 1–5). Proofs to all claims stated in the paper are available in the appendices. Explicitly, Theorem 3.1 (Appendix B), and Theorem 4.1 (Appendix C). Supporting results referred to in the paper are proven and discussed in Appendix D.

#### Guidelines:

- The answer NA means that the paper does not include theoretical results.
- All the theorems, formulas, and proofs in the paper should be numbered and crossreferenced.
- All assumptions should be clearly stated or referenced in the statement of any theorems.
- The proofs can either appear in the main paper or the supplemental material, but if they appear in the supplemental material, the authors are encouraged to provide a short proof sketch to provide intuition.
- Inversely, any informal proof provided in the core of the paper should be complemented by formal proofs provided in appendix or supplemental material.
- Theorems and Lemmas that the proof relies upon should be properly referenced.

## 4. Experimental result reproducibility

Question: Does the paper fully disclose all the information needed to reproduce the main experimental results of the paper to the extent that it affects the main claims and/or conclusions of the paper (regardless of whether the code and data are provided or not)?

Answer: [Yes]

Justification: All experimental details needed to reproduce the experimental results, including optimizer and hyperparameters, are described in Section 5 and Appendix E.

- The answer NA means that the paper does not include experiments.
- If the paper includes experiments, a No answer to this question will not be perceived well by the reviewers: Making the paper reproducible is important, regardless of whether the code and data are provided or not.
- If the contribution is a dataset and/or model, the authors should describe the steps taken to make their results reproducible or verifiable.
- Depending on the contribution, reproducibility can be accomplished in various ways. For example, if the contribution is a novel architecture, describing the architecture fully might suffice, or if the contribution is a specific model and empirical evaluation, it may be necessary to either make it possible for others to replicate the model with the same dataset, or provide access to the model. In general, releasing code and data is often one good way to accomplish this, but reproducibility can also be provided via detailed instructions for how to replicate the results, access to a hosted model (e.g., in the case of a large language model), releasing of a model checkpoint, or other means that are appropriate to the research performed.
- While NeurIPS does not require releasing code, the conference does require all submissions to provide some reasonable avenue for reproducibility, which may depend on the nature of the contribution. For example
  - (a) If the contribution is primarily a new algorithm, the paper should make it clear how to reproduce that algorithm.
  - (b) If the contribution is primarily a new model architecture, the paper should describe the architecture clearly and fully.
  - (c) If the contribution is a new model (e.g., a large language model), then there should either be a way to access this model for reproducing the results or a way to reproduce the model (e.g., with an open-source dataset or instructions for how to construct the dataset).

(d) We recognize that reproducibility may be tricky in some cases, in which case authors are welcome to describe the particular way they provide for reproducibility. In the case of closed-source models, it may be that access to the model is limited in some way (e.g., to registered users), but it should be possible for other researchers to have some path to reproducing or verifying the results.

#### 5. Open access to data and code

Question: Does the paper provide open access to the data and code, with sufficient instructions to faithfully reproduce the main experimental results, as described in supplemental material?

Answer: [Yes]

Justification: The python code used to reproduce all experiments in the camera-ready will be uploaded to GitHub.

#### Guidelines:

- The answer NA means that paper does not include experiments requiring code.
- Please see the NeurIPS code and data submission guidelines (https://nips.cc/public/guides/CodeSubmissionPolicy) for more details.
- While we encourage the release of code and data, we understand that this might not be possible, so "No" is an acceptable answer. Papers cannot be rejected simply for not including code, unless this is central to the contribution (e.g., for a new open-source benchmark).
- The instructions should contain the exact command and environment needed to run to reproduce the results. See the NeurIPS code and data submission guidelines (https://nips.cc/public/guides/CodeSubmissionPolicy) for more details.
- The authors should provide instructions on data access and preparation, including how
  to access the raw data, preprocessed data, intermediate data, and generated data, etc.
- The authors should provide scripts to reproduce all experimental results for the new proposed method and baselines. If only a subset of experiments are reproducible, they should state which ones are omitted from the script and why.
- At submission time, to preserve anonymity, the authors should release anonymized versions (if applicable).
- Providing as much information as possible in supplemental material (appended to the paper) is recommended, but including URLs to data and code is permitted.

#### 6. Experimental setting/details

Question: Does the paper specify all the training and test details (e.g., data splits, hyper-parameters, how they were chosen, type of optimizer, etc.) necessary to understand the results?

Answer: [Yes]

Justification: All experimental details needed to reproduce the experimental results, including the data preprocessing and splits, are described in Section 5 and Appendix E.

## Guidelines:

- The answer NA means that the paper does not include experiments.
- The experimental setting should be presented in the core of the paper to a level of detail that is necessary to appreciate the results and make sense of them.
- The full details can be provided either with the code, in appendix, or as supplemental material.

## 7. Experiment statistical significance

Question: Does the paper report error bars suitably and correctly defined or other appropriate information about the statistical significance of the experiments?

Answer: [Yes]

Justification: Error bars are provided on all experiments across different seeds and train/test splits (see Figures 1 and 2).

- The answer NA means that the paper does not include experiments.
- The authors should answer "Yes" if the results are accompanied by error bars, confidence intervals, or statistical significance tests, at least for the experiments that support the main claims of the paper.
- The factors of variability that the error bars are capturing should be clearly stated (for example, train/test split, initialization, random drawing of some parameter, or overall run with given experimental conditions).
- The method for calculating the error bars should be explained (closed form formula, call to a library function, bootstrap, etc.)
- The assumptions made should be given (e.g., Normally distributed errors).
- It should be clear whether the error bar is the standard deviation or the standard error of the mean.
- It is OK to report 1-sigma error bars, but one should state it. The authors should preferably report a 2-sigma error bar than state that they have a 96% CI, if the hypothesis of Normality of errors is not verified.
- For asymmetric distributions, the authors should be careful not to show in tables or figures symmetric error bars that would yield results that are out of range (e.g. negative error rates).
- If error bars are reported in tables or plots, The authors should explain in the text how they were calculated and reference the corresponding figures or tables in the text.

#### 8. Experiments compute resources

Question: For each experiment, does the paper provide sufficient information on the computer resources (type of compute workers, memory, time of execution) needed to reproduce the experiments?

Answer: [Yes]

Justification: The experiments do not require any particularly intensive computer resources. We describe all the resources used during our experiments in Appendix E.

## Guidelines:

- The answer NA means that the paper does not include experiments.
- The paper should indicate the type of compute workers CPU or GPU, internal cluster, or cloud provider, including relevant memory and storage.
- The paper should provide the amount of compute required for each of the individual experimental runs as well as estimate the total compute.
- The paper should disclose whether the full research project required more compute than the experiments reported in the paper (e.g., preliminary or failed experiments that didn't make it into the paper).

## 9. Code of ethics

Question: Does the research conducted in the paper conform, in every respect, with the NeurIPS Code of Ethics https://neurips.cc/public/EthicsGuidelines?

Answer: [Yes]

Justification: We have reviewed the NeurIPS Code of Ethics and found that our work presents none of the issues raised therein.

## Guidelines:

- The answer NA means that the authors have not reviewed the NeurIPS Code of Ethics.
- If the authors answer No, they should explain the special circumstances that require a deviation from the Code of Ethics.
- The authors should make sure to preserve anonymity (e.g., if there is a special consideration due to laws or regulations in their jurisdiction).

## 10. Broader impacts

Question: Does the paper discuss both potential positive societal impacts and negative societal impacts of the work performed?

Answer: [Yes]

Justification: The primary contribution of this work is theoretical and presents no major direct societal impact, be they positive or negative. That being said, its developments can be used to address issues of, e.g., fairness, as the work illustrates. This application is clearly presented and discussed in the manuscript.

#### Guidelines:

- The answer NA means that there is no societal impact of the work performed.
- If the authors answer NA or No, they should explain why their work has no societal impact or why the paper does not address societal impact.
- Examples of negative societal impacts include potential malicious or unintended uses (e.g., disinformation, generating fake profiles, surveillance), fairness considerations (e.g., deployment of technologies that could make decisions that unfairly impact specific groups), privacy considerations, and security considerations.
- The conference expects that many papers will be foundational research and not tied to particular applications, let alone deployments. However, if there is a direct path to any negative applications, the authors should point it out. For example, it is legitimate to point out that an improvement in the quality of generative models could be used to generate deepfakes for disinformation. On the other hand, it is not needed to point out that a generic algorithm for optimizing neural networks could enable people to train models that generate Deepfakes faster.
- The authors should consider possible harms that could arise when the technology is being used as intended and functioning correctly, harms that could arise when the technology is being used as intended but gives incorrect results, and harms following from (intentional or unintentional) misuse of the technology.
- If there are negative societal impacts, the authors could also discuss possible mitigation strategies (e.g., gated release of models, providing defenses in addition to attacks, mechanisms for monitoring misuse, mechanisms to monitor how a system learns from feedback over time, improving the efficiency and accessibility of ML).

#### 11. Safeguards

Question: Does the paper describe safeguards that have been put in place for responsible release of data or models that have a high risk for misuse (e.g., pretrained language models, image generators, or scraped datasets)?

Answer: [NA]

Justification: This work poses no such risks.

#### Guidelines:

- The answer NA means that the paper poses no such risks.
- Released models that have a high risk for misuse or dual-use should be released with
  necessary safeguards to allow for controlled use of the model, for example by requiring
  that users adhere to usage guidelines or restrictions to access the model or implementing
  safety filters.
- Datasets that have been scraped from the Internet could pose safety risks. The authors should describe how they avoided releasing unsafe images.
- We recognize that providing effective safeguards is challenging, and many papers do
  not require this, but we encourage authors to take this into account and make a best
  faith effort.

## 12. Licenses for existing assets

Question: Are the creators or original owners of assets (e.g., code, data, models), used in the paper, properly credited and are the license and terms of use explicitly mentioned and properly respected?

Answer: [Yes]

Justification: The usage of the COMPAS dataset is properly credited in Section 5.

- The answer NA means that the paper does not use existing assets.
- The authors should cite the original paper that produced the code package or dataset.

- The authors should state which version of the asset is used and, if possible, include a URL.
- The name of the license (e.g., CC-BY 4.0) should be included for each asset.
- For scraped data from a particular source (e.g., website), the copyright and terms of service of that source should be provided.
- If assets are released, the license, copyright information, and terms of use in the
  package should be provided. For popular datasets, paperswithcode.com/datasets
  has curated licenses for some datasets. Their licensing guide can help determine the
  license of a dataset.
- For existing datasets that are re-packaged, both the original license and the license of the derived asset (if it has changed) should be provided.
- If this information is not available online, the authors are encouraged to reach out to the asset's creators.

#### 13. New assets

Question: Are new assets introduced in the paper well documented and is the documentation provided alongside the assets?

Answer: [NA]

Justification: This paper does not release new assets.

#### Guidelines:

- The answer NA means that the paper does not release new assets.
- Researchers should communicate the details of the dataset/code/model as part of their submissions via structured templates. This includes details about training, license, limitations, etc.
- The paper should discuss whether and how consent was obtained from people whose asset is used.
- At submission time, remember to anonymize your assets (if applicable). You can either create an anonymized URL or include an anonymized zip file.

## 14. Crowdsourcing and research with human subjects

Question: For crowdsourcing experiments and research with human subjects, does the paper include the full text of instructions given to participants and screenshots, if applicable, as well as details about compensation (if any)?

Answer: [NA]

Justification: This paper does not involve crowdsourcing nor research with human subjects.

#### Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Including this information in the supplemental material is fine, but if the main contribution of the paper involves human subjects, then as much detail as possible should be included in the main paper.
- According to the NeurIPS Code of Ethics, workers involved in data collection, curation, or other labor should be paid at least the minimum wage in the country of the data collector.

## 15. Institutional review board (IRB) approvals or equivalent for research with human subjects

Question: Does the paper describe potential risks incurred by study participants, whether such risks were disclosed to the subjects, and whether Institutional Review Board (IRB) approvals (or an equivalent approval/review based on the requirements of your country or institution) were obtained?

Answer: [NA]

Justification: This paper does not involve crowdsourcing nor research with human subjects. Guidelines:

- The answer NA means that the paper does not involve crowdsourcing nor research with human subjects.
- Depending on the country in which research is conducted, IRB approval (or equivalent) may be required for any human subjects research. If you obtained IRB approval, you should clearly state this in the paper.
- We recognize that the procedures for this may vary significantly between institutions and locations, and we expect authors to adhere to the NeurIPS Code of Ethics and the guidelines for their institution.
- For initial submissions, do not include any information that would break anonymity (if applicable), such as the institution conducting the review.

#### 16. Declaration of LLM usage

Question: Does the paper describe the usage of LLMs if it is an important, original, or non-standard component of the core methods in this research? Note that if the LLM is used only for writing, editing, or formatting purposes and does not impact the core methodology, scientific rigorousness, or originality of the research, declaration is not required.

Answer: [NA]

Justification: Commercial LLM tools were used only during the writing of the manuscript and no LLM tools contributed to the analyses, methodology, or main results of the paper. As per the LLM policy, we do not disclose this use in the main body of the manuscript.

- The answer NA means that the core method development in this research does not involve LLMs as any important, original, or non-standard components.
- Please refer to our LLM policy (https://neurips.cc/Conferences/2025/LLM) for what should or should not be described.

# LEARNING WITH EQUALITY CONSTRAINTS

## ADDITIONAL MATERIAL

A	Rela	ted work	24		
В	Proc	of of Theorem 3.1	25		
	B.1	Preliminaries	25		
	B.2	Functional strong duality	31		
	B.3	Duality gap	33		
	B.4	Dual estimation error	37		
	B.5	Proof of Theorem 3.1	39		
C	Proc	of of Theorem 4.1	41		
D	Add	itional theory	45		
	D.1	Example: minimum norm interpolation	45		
	D.2	Comparing equalities with double sided approximation	49		
E	E Experiment details				
	E.1	Fairness	50		
	E.2	Boundary value problems	51		
	E.3	Interpolating classifiers	52		
F	Add	itional plots	53		
	F.1	Fairness	53		
	F.2	Boundary value problems	56		
	F.3	Interpolating classifiers	58		
Ad	lditio	nal References	60		

## A Related work

**Unconstrained learning.** Empirical risk minimization forms the cornerstone of modern machine learning applications, supported by a rich theory of generalization [1–3]. However, as machine learning becomes increasingly embedded in real world applications, there is a growing need to deal with multi-faceted problems involving multiple losses and/or requirements beyond accuracy, such as fairness [4, 5], robustness [6, 7], privacy [8], safety [9, 10], and scientific knowledge [11]. The traditional approach to handling multiple requirements is to use a weighted sum of the loss functions as the objective (e.g., [12–14, 7, 11]), choosing the weights by trial and error, cross-validation, or a problem-specific heuristic. This approach is often brittle and time consuming.

Inequality-constrained learning. Inequality-constrained learning uses constrained optimization to incorporate requirements into traditional learning problems. As in unconstrained learning, these tasks are formulated as statistical risk minimization problem, albeit with inequality constraints. Yet, this leads to non-convex programs for virtually every modern ML model, which make them challenging to solve [15, 16]. In convex settings, classical results for Sample Average Approximation (SAA) methods can be found in [17, 18]. Generalization bounds have also been derived in [19] (for linear classifiers and fairness constraints), [20] (for convex losses, convex-fractional losses, and fairness constraints), and [21] (for fairness constraints). In the general non-convex settings, certain duality properties have been shown to hold when using sufficiently expressive parametrizations, leading to a practical learning rule with generalization guarantees [22, 23]. The resulting primal-dual algorithms can be interpreted as incorporating the combination weights from unconstrained learning into the optimization process and have been used in various ML applications, such as fairness [19, 24, 22, 23, 25], invariance [26], classification [27], and robustness [28]. Our work is an extension of these works on non-convex, inequality-constrained learning problems.

**Equality constrained learning.** Equality constraints have been used in ML to express important problems, such as group fairness [29–31], invariance [32, 33], calibration [34, 35], distribution matching [36], and independence [37]. While stochastic optimization with *deterministic* equality constraints has been extensively studied in the literature [38–41], less is known about stochastic problems with *statistical* equality constraints—e.g., constraints defined via expectations over data, such as those considered in this work (Section 2).

Equality constraints are often incorporated directly into the model (see, e.g., [42, 43]) or enforced by post-processing schemes (e.g., group fairness in [30]). When the constraint function and its derivatives are only accessible by noisy oracles, solutions based on upper/lower bounds as well as Sequential Quadratic Programming (SQP) or trust region methods have been investigated [44–47].

More similar to our setup is [48] that uses a primal-dual approach similar to ours, but rely on transforming equalities into inequalities using a non-negative wrapping function (e.g., quadratic). While their empirical results are promising, this approach has severe numerical and theoretical issues as discussed in Remark 2.1. From a theoretical perspective, [49] considers the same problem we do, extending the SAA framework to equality constraints using inequality *relaxations*. They show that if the relaxation is carefully tightened as the sample size increases, it is possible to asymptotically obtain solutions of the population problem. These results are, nevertheless, (a) asymptotic, assuming access to infinitely many i.i.d. samples; (b) focused on a relaxation of the original problem; and (c) reliant on interior-point methods that are not well-suited to the large-scale, non-convex settings of ML.

#### **B** Proof of Theorem 3.1

#### **B.1** Preliminaries

#### **B.1.1** Bochner spaces

Let  $(\mathcal{X} \times \mathcal{Y}, \Sigma_{\mathcal{X} \times \mathcal{Y}}, \mathbb{P})$  be the probability space [50] corresponding to the random variables  $(\mathbf{x}, y)$ . Let  $(\mathbb{R}^K, \mathcal{B}(\mathbb{R}^K))$  be the Borel sigma algebra associated with  $\mathbb{R}^K$ , i.e. the smallest sigma algebra containing all the open sets. The Bochner spaces [51][Section 1.2.b] are a direct generalization of the  $L^p$  spaces, for measurable *vector valued* functions. Consider mappings from  $\mathcal{X}$  to  $\mathbb{R}^K$ . The Bochner space  $L^p(\mathbb{R}^K; \mathbb{P})$ , for  $p \in [1, \infty)$ , is the space of all vector valued functions  $\phi: \mathcal{X} \to \mathbb{R}^K$ , such that

$$\|\phi\|_{L^p(\mathbb{R}^K;\mathbb{P})} = \left(\int \|\phi(\mathbf{x})\|_2^p \ d\mathbb{P}\right)^{1/p}$$

is finite.  $L^{\infty}(\mathbb{R}^K; \mathbb{P})$  is similarly defined by the norm,

$$\left\|\phi\right\|_{L^{\infty}(\mathbb{P})} = \mathrm{esssup}_{\mathbf{x} \in \mathcal{X}: \mathbb{P}} \left\|\phi(\mathbf{x})\right\|_{2} = \sup \, \left\{c > 0 \mid \mathbb{P}(\left\{\left\|\phi(\mathbf{x})\right\|_{2} \geq c\right\}) = 0\right\}.$$

For  $p \in [1, \infty]$  the spaces  $L^p(\mathbb{R}^K; \mathbb{P})$  are Banach spaces. The space  $L^1(\mathbb{R}^K, \mathbb{P})$  is precisely the space of functions for which the Bochner integral, a form of vector integral, exists.

Our primary space of interest,  $L^p(\mathbb{R}^K; \mathbb{P}_+)$ , where  $\mathbb{P}_+ = \mathbb{D} + \sum_{i=1}^I \mathbb{P}_i + \sum_{j=1}^J \mathbb{Q}_j$ , is actually the intersection of the Bochner spaces corresponding to the summands. This follows from the linearity of the Lebesgue integral (for scalar functions) with respect to the measure ([50] Exercise 4.27).

**Lemma B.1.** Let  $\mathbb{B}, \mathbb{P}$  be positive measures on a measurable space, and  $f: \mathcal{X} \to \mathbb{R}$  be a non-negative measurable function. Then,

$$\int f(\mathbf{x}) \ d(\mathbb{B} + \mathbb{P}) = \int f(\mathbf{x}) \ d\mathbb{B} + \int f(\mathbf{x}) \ d\mathbb{P}.$$

and moreover, (a)  $L^1(\mathbb{B} + \mathbb{P}) = L^1(\mathbb{B}) \cap L^1(\mathbb{P})$ , and more generally (b)  $L^p(\mathbb{R}^K; \mathbb{B} + \mathbb{P}) = L^p(\mathbb{R}^K; \mathbb{B}) \cap L^p(\mathbb{R}^K; \mathbb{P})$  for  $p \in [1, \infty]$ .

*Proof.* Part 1: Suppose  $f \ge 0$ . Let  $\mathcal{X}_A$  be the characteristic function of a measurable set A. Recall that there exists a sequence of positive simple functions  $f_n(\mathbf{x}) = \sum_{i=1}^n \alpha_i \mathcal{X}_{A_i}(\mathbf{x})$  that are upper bounded by f and converge to f. The integral of  $f_n$  (with respect to any measure on the same measurable space) is an increasing convergent sequence whose limit is defined as the integral of f (see [50] Definition I.4.3(b) and Theorem I.2.17 for more details).

By applying the fact that  $(\mathbb{P} + \mathbb{B})(A) = \mathbb{P}(A) + \mathbb{B}(A)$  (by definition) for any measurable set A, we obtain that,

$$\int f_n(\mathbf{x}) \ d(\mathbb{B} + \mathbb{P}) = \int f_n(\mathbf{x}) \ d\mathbb{B} + \int f_n(\mathbf{x}) \ d\mathbb{P}.$$

If both limits on the RHS exist, then the limit on the LHS exists as well ([52] Theorem 3.4). This proves that  $f \in L^1(\mathbb{B}) \cap L^1(\mathbb{P}) \Rightarrow f \in L^1(\mathbb{B} + \mathbb{P})$ .

On the other hand, note that  $\int f_n(\mathbf{x}) d\mathbb{B} \leq \int f_n(\mathbf{x}) d(\mathbb{B} + \mathbb{P})$  since  $\int f_n(\mathbf{x}) d\mathbb{P} \geq 0$ . Therefore if  $\int f_n(\mathbf{x}) d(\mathbb{B} + \mathbb{P})$  is convergent, since  $\int f_n(\mathbf{x}) d(\mathbb{B} + \mathbb{P}) \leq \int f(\mathbf{x}) d(\mathbb{B} + \mathbb{P})$ , therefore,

$$\int f_n(\mathbf{x}) d\mathbb{B} \le \int f(\mathbf{x}) d(\mathbb{B} + \mathbb{P}).$$

Therefore  $\int f_n(\mathbf{x}) d\mathbb{B}$  is a convergent sequence since it is increasing and bounded (and by symmetry so is  $\int f_n(\mathbf{x}) d\mathbb{P}$ ). This proves that  $f \in L^1(\mathbb{B} + \mathbb{P}) \Rightarrow f \in L^1(\mathbb{B}) \cap L^1(\mathbb{P})$ . Therefore for a non-negative function  $f \in L^1(\mathbb{B} + \mathbb{P}) \iff f \in L^1(\mathbb{B}) \cap L^1(\mathbb{P})$ .

Part 2: Now, consider a general f. It is known [53][Theorem 1.33] that  $f \in L^1(\mathbb{P} + \mathbb{B})$  iff,

$$||f||_{L^1(\mathbb{P})} = \int |f(\mathbf{x})| \ (d\mathbb{P} + d\mathbb{B}) < \infty.$$

Therefore,

$$f \in L^1(\mathbb{B} + \mathbb{P}) \iff |f| \in L^1(\mathbb{B} + \mathbb{P})$$
 (11)

$$\iff |f| \in L^1(\mathbb{B}) \cap L^1(\mathbb{P}) \tag{12}$$

$$\iff f \in L^1(\mathbb{B}) \cap L^1(\mathbb{P}).$$
 (13)

Equation (12) follows from the first part, and (11) and (13) are applications of [53][Theorem 1.33]. This proves (a).

Part 3: The proof for (b) follows similarly. For  $p \in [1, \infty)$ ,

$$\phi \in L^p(\mathbb{R}^k; \mathbb{B} + \mathbb{P}) \iff \|\phi\|_2^p \in L^1(\mathbb{B} + \mathbb{P}) \tag{14}$$

$$\iff \|\phi\|_2^p \in L^1(\mathbb{B}) \cap L^1(\mathbb{P}) \tag{15}$$

$$\iff \phi \in L^p(\mathbb{R}^k; \mathbb{B}) \cap L^p(\mathbb{R}^k; \mathbb{P}).$$
 (16)

Equation (15) follows from the first part, and (14) and (16) are applications of the definition of the Bochner norm.

Part 4: Finally we consider the case where  $p = \infty$ . Let  $O(c) = \{ \mathbf{x} \in \mathcal{X} \mid \|\phi(\mathbf{x})\|_2 \ge c \}$ . Then, consider,

$$\begin{split} \|\phi\|_{L^{\infty}(\mathbb{B}+\mathbb{P})} &= \inf \ \{c > 0 \mid (\mathbb{B}+\mathbb{P})(O(c)) = 0\} \\ &= \inf \ \{c > 0 \mid \mathbb{B}(O(c)) = 0, \mathbb{P}(O(c)) = 0\} \\ &= \max \ \Big\{ \inf \ \{c > 0 \mid \mathbb{B}(O(c)) = 0\}, \inf \ \{c > 0 \mid \mathbb{P}(O(c)) = 0\} \Big\} \\ &= \max \ \Big\{ \|\phi\|_{L^{\infty}(\mathbb{B})}, \|\phi\|_{L^{\infty}(\mathbb{P})} \Big\}. \end{split} \tag{17}$$

The steps are trivial. It is obvious from Equation (17) that  $L^{\infty}(\mathbb{R}^k; \mathbb{B} + \mathbb{P}) = L^{\infty}(\mathbb{R}^k; \mathbb{B}) \cap L^{\infty}(\mathbb{R}^k; \mathbb{P})$ .

## **B.1.2** Atomless vector measures

A vector measure [54] over  $(\mathcal{X} \times \mathcal{Y}, \Sigma_{\mathcal{X} \times \mathcal{Y}})$  is a set function that takes values in a Banach space (instead of  $\mathbb{R}$ ).

**Definition B.1.** [54, pp. 1] Let  $(\mathcal{X} \times \mathcal{Y}, \Sigma_{\mathcal{X} \times \mathcal{Y}})$  be a measurable space and let  $(\mathbb{V}, \|\cdot\|)$  be a Banach space. Then  $G: \Sigma_{\mathcal{X} \times \mathcal{Y}} \to \mathbb{V}$  is a countably additive vector measure, if for all sequences of disjoint sets  $\{E_i\}_{i=1}^{\infty}$ ,  $G(\bigcup_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \sum_{i=1}^{n} G(E_i)$ .

We are interested in atomless vector measures, which appear as an intermediate object in the proof of Proposition B.1.

**Definition B.2.** Let  $(\mathcal{X}, \Sigma_{\mathcal{X}})$  be a measurable space and  $(\mathbb{V}, \|\cdot\|)$  be a Banach space. A vector measure  $G: \Sigma_{\mathcal{X}} \to \mathbb{V}$  is called non-atomic or atomless iff for any  $A \in \Sigma_{\mathcal{X}}$ , such that  $G(A) \neq 0$ , there exists  $B \in \Sigma_{\mathcal{X}}$  such that  $B \subseteq A$  and  $G(B) \notin \{0, G(A)\}$ .

We are interested in the following "closure" properties of atomless measures.

**Lemma B.2.** If  $\mathcal{L}: \mathcal{X} \to \mathbb{R}$  is integrable with respect to a measure space  $(\mathcal{X}, \Sigma_{\mathcal{X}}, \mathbb{P})$ , where  $\mathbb{P}$  is an atomless measure, then the measure  $\nu: \Sigma_{\mathcal{X}} \to \mathbb{R}$ ,

$$\forall A \in \Sigma_{\mathcal{X}}, \quad \nu(A) = \int_{A} \mathcal{L}(\mathbf{x}) \ d\mathbb{P},$$

is an atomless scalar measure.

**Lemma B.3.** Let  $N \in \mathbb{N}$ . If  $\nu_1, \ldots, \nu_N : \Sigma_{\mathcal{X}} \to \mathbb{R}$  are atomless scalar measures on a measurable space  $(\mathcal{X}, \Sigma_{\mathcal{X}})$ , then the vector measure  $G : \Sigma_{\mathcal{X}} \to \mathbb{R}^N$  defined by,

$$A \in \Sigma_{\mathcal{X}}, \quad G(A) = \left[\nu_1(A), \dots, \nu_N(A)\right]^{\top}$$

is a countably additive atomless vector measure.

Before we prove Lemma B.2 we note a few properties of the indefinite integral  $\nu(A) = \int_A \mathcal{L}(\mathbf{x}) \ d\mathbb{P}$ .

**Remark B.1.** Note that  $\nu$  is indeed a measure over the measure space  $(\mathcal{X}, \Sigma_{\mathcal{X}})$  [50, Section I.5, Indefinite integrals]. Moreover,  $\nu$  is absolutely continuous with respect to  $\mathbb{P}$ , i.e. for any  $A \in \Sigma_{\mathcal{X}}$ ,

$$\mathbb{P}(A) = 0 \Rightarrow \nu(A) = 0. \tag{18}$$

This can be inferred from the fact that if A is a  $\mathbb{P}$ -negligible set (i.e.  $\mathbb{P}(A) = 0$ ) then the indefinite integral  $\int_A \mathcal{L}(\mathbf{x}) d\mathbb{P}$  is 0 for every function  $\mathcal{L}$  [50, Proposition I.4.13].

**Remark B.2.** If  $\nu$  is a finite signed measure on  $\Sigma_{\mathcal{X}}$ , then it admits a so-called Jordan Decomposition [53, Section 6.6],

$$\forall A \in \Sigma_{\mathcal{X}}, \nu(A) = \nu_{+}(A) - \nu_{-}(A), \tag{19}$$

where  $\nu_+, \nu_-$  are non-negative measures. The Hahn decomposition theorem [53, Theorem 6.14] further states that there exists disjoint measurable sets  $\mathcal{X}_+$  and  $\mathcal{X}_-$  such that  $\mathcal{X}_+ \cup \mathcal{X}_- = \mathcal{X}$ , and for all  $A \in \Sigma_{\mathcal{X}}$ ,

$$\nu_{+}(A) = \nu(A \cap \mathcal{X}_{+}) \quad \text{and } \nu_{-}(A) = -\nu(A \cap \mathcal{X}_{-}). \tag{20}$$

In particular, Equations (19) and (20) imply that the measure of any subset of  $\mathcal{X}_+$  (resp.  $\mathcal{X}_-$ ) with respect to  $\nu$  is non-negative (resp. non-positive). We will also need the following set,

$$\mathcal{X}_0 = \{ \mathbf{x} \in \mathcal{X} \mid \mathcal{L}(\mathbf{x}) = 0 \}$$
.

 $\mathcal{X}_0$  is a measurable set since  $\mathcal{X}_0 = \mathcal{L}^{-1}(0)$ , and  $0 \in \mathcal{B}(\mathbb{R})$ , the Borel sigma algebra on  $\mathbb{R}$ , as it can be represented as a countable intersection of open sets (e.g. as  $\bigcap_{i=1}^{\infty}(-\frac{1}{n},\frac{1}{n})$ ). It is obvious that  $\nu(E) = 0$  for any  $E \subseteq \mathcal{X}_0$ .

We will now prove Lemma B.2.

#### Proof of Lemma B.2.

*Proof.* As noted in Remark B.1,  $\nu$  is indeed a (signed) measure. Suppose that for an arbitrary  $A \in \Sigma_{\mathcal{X}}$ ,  $\nu(A) \neq 0$ . Without loss of generality, assume that  $\nu(A) > 0$ , the negative case follows by symmetry. To prove that  $\nu$  is atomless we need to prove the existence of a measurable set  $B \subseteq A$  such that  $\nu(B) \notin \{\nu(A), 0\}$ .

Consider the Jordan decomposition of  $\nu$ ,

$$\nu(A) = \nu_{+}(A) - \nu_{-}(A).$$

Since  $\nu_-$  is a non-negative measure, and  $\nu(A) > 0$ , therefore  $\nu_+(A) = \nu(A) + \nu_-(A) > 0$ . Remark B.2 states that  $\nu_+(A) = \nu(A \cap \mathcal{X}_+)$ . Therefore  $\nu(A \cap \mathcal{X}_+) > 0$ . Clearly,

$$\nu(A \cap \mathcal{X}_+) = \nu(A \cap \mathcal{X}_+ \cap \mathcal{X}_0) + \nu(A \cap \mathcal{X}_+ \cap \mathcal{X}_0^c).$$

Clearly  $\nu(A \cap \mathcal{X}_+ \cap \mathcal{X}_0) = 0$ , therefore  $\nu(A \cap \mathcal{X}_+ \cap \mathcal{X}_0^c) = \nu(A \cap \mathcal{X}_+) > 0$ . Let  $A_+ = A \cap \mathcal{X}_+ \cap \mathcal{X}_0^c$ .

The absolute continuity of  $\nu$  with respect to  $\mathbb{P}$  implies that  $\mathbb{P}(A_+) > 0$  since  $\nu(A_+) > 0$ . Since  $\mathbb{P}$  is atomless, therefore there exists a set  $B \subseteq A_+$  such that  $\mathbb{P}(B) \notin \{\mathbb{P}(A_+), 0\}$ . Clearly,

$$\mathbb{P}(A_+) = \mathbb{P}(B) + \mathbb{P}(B^c \cap A_+). \tag{21}$$

Since  $\mathbb{P}$  is a non-negative measure therefore  $\mathbb{P}(B) \geq 0$  and  $\mathbb{P}(B^c \cap A_+) \geq 0$ . Now since  $\mathbb{P}(B) \notin \{\mathbb{P}(A_+), 0\}$ , (21) implies that

$$0 < \mathbb{P}(B) < \mathbb{P}(A_+) \quad \text{and } 0 < \mathbb{P}(B^c \cap A_+) < \mathbb{P}(A_+). \tag{22}$$

Now, it is also true that,

$$\nu(A_{+}) = \nu(B) + \nu(B^{c} \cap A_{+}). \tag{23}$$

Let  $\mathbbm{1}_B(\mathbf{x})$  be the indicator function of the set B, taking the value 1 when  $\mathbf{x} \in B$  and 0 otherwise. Since  $B \subseteq \mathcal{X}_+$ , therefore  $\nu(B) \ge 0$ . Now suppose, if possible, that  $\nu(B) = \int_B \mathcal{L}(\mathbf{x}) \ d\mathbb{P} = 0$ . Then according to [50, Proposition I.4.13],  $\mathbbm{1}_B(\mathbf{x})\mathcal{L}(\mathbf{x})$  is 0 almost everywhere with respect to  $\mathbb{P}$ . Since  $\mathbbm{1}_B(\mathbf{x}) = 1$  everywhere on B, this implies that  $\mathcal{L}(\mathbf{x})$  must be 0 almost everywhere on B, i.e. except for a null subset, say N. Explicitly,  $\mathcal{L}(\mathbf{x})$  is 0 on  $B \setminus N$  and since  $\mathbb{P}(B) > 0$ , therefore  $N \ne B$  and  $B \setminus N \ne \emptyset$ . However, since  $\mathcal{X}_0$  is a superset of B, therefore  $\mathcal{L}(\mathbf{x}) \ne 0$  everywhere on B, including  $B \setminus N$ , which is a contradiction. Therefore  $\nu(B) > 0$ .

Similarly, we can argue that  $\nu(B^c \cap A_+) > 0$  since  $\mathbb{P}(B^c \cap A_+) > 0$  and  $B^c \cap A_+ \subseteq \mathcal{X}_+ \cap \mathcal{X}_0$ . Therefore,

$$0 < \nu(B) < \nu(A_+).$$

Recall that we showed that  $\nu(A_+) = \nu(A \cap \mathcal{X}_+) = \nu_+(A)$ . Since  $\nu_+(A) = \nu(A) + \nu_-(A)$  and  $\nu_-$  is non-negative, therefore  $\nu(A_+) = \nu_+(A) \geq \nu(A)$ . If  $\nu(B) \neq \nu(A)$ , we are done since  $B \subseteq A$  and  $\nu(B) > 0$ . Now suppose that  $\nu(B) = \nu(A)$ . Then since  $\mathbb{P}(B) > 0$ , there exists  $B' \subseteq B$  such that  $0 < \mathbb{P}(B') < \mathbb{P}(B)$ . As before we can prove that  $0 < \nu(B') < \nu(B)$  by utilising the fact that  $\mathbb{P}(B') > 0$  and  $B' \subseteq \mathcal{X}_+ \cap \mathcal{X}_0$ . But this time,  $\nu(B') < \nu(B) = \nu(A)$  by construction. Since  $B' \subseteq A$  and  $\nu(B') \notin \{0, \nu(A)\}$  therefore  $\nu$  is atomless.

The proof for Lemma B.3 is almost trivial.

#### Proof of Lemma B.3.

*Proof.* It is easy to verify that countable additivity is preserved by concatenating scalar measures, so we will verify that atomlessness is also preserved. Consider  $A \in \Sigma_{\mathcal{X}}$  such that  $G(A) \neq 0$ . Therefore there exists an index i such that  $\nu_i(A) \neq 0$ . Since  $\nu_i$  is atomless, therefore there exists  $B \subset A$  such that  $\nu_i(B) \notin \{\nu_i(A), 0\}$ . It follows that  $G(B) \notin \{G(A), 0\}$  which proves the atomlessness of G.

## **B.1.3** Constrained optimization problems

In this section we will define and discuss the main objects involved in Lagrangian duality in a unified manner with respect to a generic optimization problem (P<sub>0</sub>). For *this section*, let  $l, g_i, h_j$  be functions from  $\mathcal{X}$  to  $\mathbb{R}$ , then consider the following optimization problem,

$$P_0^{\star} = \inf_{x \in \mathcal{X}} \quad \ell(x)$$
subject to  $g_i(x) \le 0$ , for  $i = 1, \dots, I$   $P_0$ 

$$h_j(x) = 0, \text{ for } j = 1, \dots, J.$$

Let Feas( $P_0$ ) refer to the subset of  $\mathcal X$  that satisfies all the constraints, i.e. the set of feasible solutions. If ( $P_0$ ) is infeasible, we set  $P_0^{\star}$  to  $+\infty$ . If  $P_0^{\star}=-\infty$  then we say that ( $P_0$ ) is unbounded. If  $\ell$  is bounded, then ( $P_0$ ) is also bounded (from below). Let  $Opt(P_0)$  refer to the subset of Feas( $P_0$ ) that achieves  $P_0^{\star}$ , i.e. the set of optimal solutions. Closely related to ( $P_0$ ) is its Lagrangian function,

$$L_0(x, \lambda, \mu) = \ell(x) + \sum_{i=1}^{I} \lambda_i g_i(x) + \sum_{j=1}^{J} \mu_j h_j(x),$$

and its dual function is  $q_0(\lambda, \mu) = \inf_{x \in \mathcal{X}} L_0(x, \lambda, \mu)$ . The dual function defines the dual problem,

$$D_0^{\star} = \sup_{\lambda_i \ge 0, \, \mu_j \in \mathbb{R}} q_0(\lambda, \mu) = \sup_{\lambda_i \ge 0, \, \mu_j \in \mathbb{R}} \min_{x \in \mathcal{X}} L_0(x, \lambda, \mu). \tag{D_0}$$

We call  $\lambda \in \mathbb{R}^I_{\geq 0}$ ,  $\mu \in \mathbb{R}^J$  the dual variables and denote optimal dual variables, when they exist, with a  $\star$ , such as  $(\lambda^\star, \mu^\star)$ . We denote the set of all optimizers of  $(D_0)$  as  $Opt(D_0)$ , which are also often called Lagrange multipliers. Let  $F_0: \mathcal{X} \to \mathbb{R}^{1+I+J}$  be the vector function obtained by stacking the objective and the constraint functions, i.e.,

$$\forall x \in \mathcal{X}$$
  $F_0(x) = [\ell(x), g_1(x), \dots g_I(x), h_1(x), \dots h_J(x)]^{\top}$ .

We call  $F_0$  the cost constraint vector of  $(P_0)$ , and the following set will be called the cost constraint epigraph of  $(P_0)$ ,

$$\mathcal{M}_0 = F_0(\mathcal{X}) + \mathbb{R}^{1+I}_{>0} \times \{0\}^J.$$

Another important set is the projection of  $\mathcal{M}_0$  on the constraint axes,

$$C_0 = \left\{ (g, h) \in \mathbb{R}^{I+J} \mid (\ell, g, h) \in \mathcal{M}_0 \right\}$$
  
=  $C_0(\mathcal{X}) + \mathbb{R}^I_{\geq 0} \times \left\{ 0 \right\}^J$ . (24)

where  $C_0(x) = [g_1(x), \dots g_I(x), h_1(x), \dots h_J(x)]^{\top}$  is the vector function formed by stacking only the I + J constraint functions. We call  $C_0$  the constraint vector and  $C_0$  the constraint epigraph. The relative interior of  $C_0$  is defined as,

relint 
$$(\mathcal{C}_0) = \{ y \in \mathcal{C}_0 \mid \exists \epsilon > 0, \text{ s.t. }, B(y, \epsilon) \cap \text{aff } (\mathcal{C}_0) \subseteq \mathcal{C}_0 \}$$

where aff  $(C_0)$  is the affine hull of  $C_0$  and  $B(y,\epsilon) = \{z \mid ||y-z|| \le \epsilon\}$  is an  $\epsilon$  ball centered at  $y \in \mathbb{R}^{I+J}$ .

#### **B.1.4** Geometric conditions for strong duality

 $(D_0)$  is a relaxation of  $(P_0)$ , i.e.  $D_0^\star \leq P_0^\star$ . This fact is called *weak duality* in convex optimization. When the equality holds, this fact is called strong duality and  $(P_0)$  is said to be strongly dual. Classical convex optimization provides us the following result tying the convexity of the cost constraint epigraph  $\mathcal{M}_0$  and the strong duality of  $(P_0)$ .

**Theorem B.1.** [15, Proposition 4.4.1 (variation)] Let  $P_0^* > -\infty$ . If  $0^{I+J} \in relint(C_0)$  then  $Opt(D_0) \neq \emptyset$ . Moreover, if  $\mathcal{M}_0$  is convex then  $P_0^* = D_0^*$ .

Theorem B.1 underlies our functional strong duality result (Proposition B.1). It can be seen as a consequence of the fact that  $\mathcal{M}_0$  provides a geometric embedding of  $(P_0)$ , namely,

$$P_0^{\star} = \inf_{(\ell, 0^J, 0^J) \in \mathcal{M}_0} \ell. \tag{25}$$

Similarly, the convex closure of  $\mathcal{M}_0$ , denoted  $\overline{\text{conv}}(\mathcal{M}_0)$ , provides a geometric embedding of the *dual* problem  $(D_0)$  when  $(P_0)$  is feasible ([15, Proposition 4.3.2]). Explicitly,

$$D_0^{\star} = \inf_{(\ell, 0^I, 0^J) \in \overline{\text{conv}}(\mathcal{M}_0)} \ell. \tag{26}$$

Equations (25) and (26) suggest that the convexity and closure of  $\mathcal{M}_0$  is closely related to strong duality. In fact, convexity and closure are together sufficient for strong duality [15, Proposition 4.3.2]. However, the requirement for closure can also be dropped when the target constraint values  $(0^{I+J})$  are in the *relative interior* of the constraint set  $\mathcal{C}_0$ , as seen in Theorem B.1. Theorem B.1 also provides sufficient conditions (i.e. the non-extremality of the constraint levels) for the existence of Lagrange multipliers. Note that this is non-trivial, since strong duality is not always sufficient for the existence of Lagrange multipliers, for example, see [15, Example 5.3.3] where  $\mathcal{M}_0$  is both convex and closed, but  $\mathrm{Opt}(D_0) = \emptyset$ .

If the dimension of  $C_0$  is I+J, then the set of Lagrange multipliers is bounded, and the converse is also true [15, Proposition 4.4.2]. The linear independence of the constraint functions is sufficient for the dimension of  $C_0$  to be I+J (as we implicitly assume in Assumption 1). For problems with only inequalities it is sufficient for  $C_0(\mathcal{X})$  to be non-empty for  $C_0$  to be of dimension I+J. In the next section we will analyse another perturbation of  $(P_0)$ , one where we relax all the constraints by  $\epsilon$ .

#### **B.1.5** Constraint perturbations

In this section we investigate the following relaxation of  $(P_0)$ ,

$$\begin{split} P_{\epsilon}^{\star} &= \inf_{x \in \mathcal{X}} \quad \ell(x) \\ \text{subject to} \quad g_i(x) \leq \epsilon, \qquad \text{for } i = 1, \dots, I, \\ &- \epsilon \leq h_j(x) \leq \epsilon, \quad \text{for } j = 1, \dots, J. \end{split} \tag{$\mathsf{P}_{\epsilon}$}$$

Note that  $(P_{\epsilon})$  has I + 2J inequality constraints. Its dual problem is as follows,

$$D_{\epsilon}^{\star} = \sup_{\lambda_{i}, \mu_{+,j}, \mu_{-,j} \ge 0} \min_{x \in \mathcal{X}} L_{\epsilon}(x, \lambda, \mu_{+,j}, \mu_{-,j}), \tag{D_{\epsilon}}$$

where  $L_{\epsilon}$  is the Lagrangian function of  $(P_{\epsilon})$ . While we know that  $P_0^* \geq P_{\epsilon}^*$ , the relationship is difficult to characterize further without additional assumptions. However, the relationship between the dual problems follows from the relationship between the Lagrangians.

**Lemma B.4.** Assume  $Opt(D_0), Opt(D_\epsilon) \neq \emptyset$ . If  $\gamma_0^\star = (\lambda_0^\star, \mu_0^\star) \in Opt(D_0)$  and  $\gamma_\epsilon^\star = (\lambda_\epsilon^\star, \mu_{\epsilon,+}^\star, \mu_{\epsilon,-}^\star) \in Opt(D_\epsilon)$ , then it holds that,

$$\epsilon \|\gamma_{\epsilon}^{\star}\|_{1} \le D_{0}^{\star} - D_{\epsilon}^{\star} \le \epsilon \|\gamma_{0}^{\star}\|_{1}. \tag{27}$$

Moreover, (27) implies that  $D_0^{\star} \geq D_{\epsilon}^{\star}$  and  $\|\gamma_0^{\star}\|_1 \geq \|\gamma_{\epsilon}^{\star}\|_1$ .

*Proof.* Upper bound . Let  $\lambda \in \mathbb{R}^I_{\geq 0}$  and  $\mu \in \mathbb{R}^J$  be arbitrarily chosen. Let  $\mu_+$  and  $\mu_-$  be defined pointwise as  $\mu_{+,j} = \max\{0,\mu_j\}$  and  $\mu_{-,j} = \max\{0,-\mu_j\}$ . Clearly  $\mu = \mu_+ - \mu_-$  and  $\|\mu_+\|_1 + \|\mu_-\|_1 = \|\mu\|_1$ . Applying these facts, we can verify that  $L_0$  can be rewritten as follows,

$$L_0(x,\lambda,\mu) = \left\{ \ell(x) + \sum_{i=1}^{I} \lambda_i (g_i(x) - \epsilon) + \sum_{j=1}^{J} \mu_{+,j} (h_j(x) - \epsilon) + \sum_{j=1}^{J} \mu_{-,j} (-h_j(x) - \epsilon) \right\} + \epsilon (\|\lambda\|_1 + \|\mu\|_1).$$

It is clear that the first term is  $L_{\epsilon}(\lambda, \mu_+, \mu_-)$ , therefore,

$$L_0(x, \lambda, \mu) = L_{\epsilon}(x, \lambda, \mu_+, \mu_-) + \epsilon(\|\lambda\|_1 + \|\mu\|_1). \tag{28}$$

Taking the infimum with respect to x on both sides we obtain,

$$q_0(\lambda, \mu) = q_{\epsilon}(\lambda, \mu_+, \mu_-) + \epsilon(\|\lambda\|_1 + \|\mu\|_1). \tag{29}$$

Let  $\mu_{0+}^{\star}$  and  $\mu_{0-}^{\star}$  be defined pointwise as  $\mu_{0+,j}^{\star} = \max\left\{0, \mu_{0,j}^{\star}\right\}$  and  $\mu_{0-,j}^{\star} = \max\left\{0, -\mu_{0,j}^{\star}\right\}$ . Then, we have,

$$D_0^{\star} - D_{\epsilon}^{\star} = q_0(\lambda_0^{\star}, \mu_0^{\star}) - q_{\epsilon}(\lambda_{\epsilon}^{\star}, \mu_{\epsilon,+}^{\star}, \mu_{\epsilon,-}^{\star})$$
  
$$\leq q_0(\lambda_0^{\star}, \mu_0^{\star}) - q_{\epsilon}(\lambda_0^{\star}, \mu_{0,+}^{\star}, \mu_{0,-}^{\star}).$$

since  $(\lambda_0^{\star}, \mu_{0,+}^{\star}, \mu_{0,-}^{\star})$  is not necessarily optimal with respect to  $(D_{\epsilon})$ . Applying (29) to the above inequality, we obtain,

$$D_0^{\star} - D_{\epsilon}^{\star} \le \epsilon (\|\lambda_0^{\star}\|_1 + \|\mu_0^{\star}\|_1) \le \epsilon \|\gamma_0^{\star}\|_1$$

which proves the upper bound.

**Lower bound .** Now let us prove the lower bound. Let  $\lambda \in \mathbb{R}^I_{\geq 0}$  and  $\mu_+, \mu_- \in \mathbb{R}^J_{\geq 0}$  be arbitrarily chosen. Then we can rewrite  $L_{\epsilon}$  as follows,

$$L_{\epsilon}(x,\lambda,\mu_{+},\mu_{-}) = \ell(x) + \sum_{i=1}^{I} \lambda_{i}(g_{i}(x) - \epsilon) + \sum_{j=1}^{J} \mu_{+,j}(h_{j}(x) - \epsilon) + \sum_{j=1}^{J} \mu_{-,j}(-h_{j}(x) - \epsilon)$$

$$= L_{0}(x,\lambda,\mu_{+} - \mu_{-}) - \epsilon(\|\lambda\|_{1} + \|\mu_{+}\|_{1} + \|\mu_{-}\|_{1}). \tag{30}$$

Again, taking the infimum with respect to x gives us

$$q_{\epsilon}(\lambda, \mu_{+}, \mu_{-}) = q_{0}(\lambda, \mu_{+} - \mu_{-}) - \epsilon(\|\lambda\|_{1} + \|\mu_{+}\|_{1} + \|\mu_{-}\|_{1}). \tag{31}$$

Next, we have,

$$D_{\epsilon}^{\star} - D_{0}^{\star} = q_{\epsilon}(\lambda_{\epsilon}^{\star}, \mu_{\epsilon,+}^{\star}, \mu_{\epsilon,-}^{\star}) - q_{0}(\lambda_{0}^{\star}, \mu_{0}^{\star})$$

$$\leq q_{\epsilon}(\lambda_{\epsilon}^{\star}, \mu_{\epsilon,+}^{\star}, \mu_{\epsilon,-}^{\star}) - q_{0}(\lambda_{\epsilon}^{\star}, \mu_{\epsilon,+}^{\star} - \mu_{\epsilon,-}^{\star}),$$

since  $(\lambda_{\epsilon}^{\star}, \mu_{\epsilon,+}^{\star} - \mu_{\epsilon,-}^{\star})$  is suboptimal for  $q_0$ . Applying (31) to the last inequality, we obtain,

$$D_{\epsilon}^{\star} - D_{0}^{\star} \le -\epsilon (\|\lambda_{\epsilon}^{\star}\|_{1} + \|\mu_{\epsilon,+}^{\star}\|_{1} + \|\mu_{\epsilon,-}^{\star}\|_{1}). \tag{32}$$

Since  $\|\gamma_{\epsilon}^{\star}\|_{1} = (\|\lambda_{\epsilon}^{\star}\|_{1} + \|\mu_{\epsilon,+}^{\star}\|_{1} + \|\mu_{\epsilon,-}^{\star}\|_{1})$ , multiplying (32) by -1 completes the proof.  $\square$ 

#### **B.2** Functional strong duality

#### **B.2.1** Convexity of $\mathcal{M}_{\phi}$

In this section we are concerned with the range of the functional problem.

$$\begin{split} P_{\phi}^{\star} &= \inf_{\phi \in \Phi} \quad \mathbb{E}_{\mathbb{D}} \big[ \ell(\phi(\mathbf{x}), y) \big] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_{i}} \big[ g_{i}(\phi(\mathbf{x}), y) \big] \leq 0, \quad \text{for } i = 1, \dots, I, \\ \mathbb{E}_{\mathbb{Q}_{i}} \big[ h_{j}(\phi(\mathbf{x}), y) \big] &= 0, \quad \text{for } j = 1, \dots, J. \end{split} \tag{$\mathbf{P}_{\phi}$}$$

The Lagrangian associated with  $(P_{\phi})$  is the same as for (P) (see (1)) and  $(D_{\phi})$  was defined in Section 3. Of particular interest in this section is the cost constraint function,

$$F_{\phi}(\phi) = \left[ \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right], \mathbb{E}_{\mathbb{P}_{1}} \left[ g_{1}(\phi(\mathbf{x}), y) \right], \dots, \mathbb{E}_{\mathbb{P}_{I}} \left[ g_{I}(\phi(\mathbf{x}), y) \right], \\ \mathbb{E}_{\mathbb{Q}_{1}} \left[ h_{1}(\phi(\mathbf{x}), y) \right], \dots, \mathbb{E}_{\mathbb{Q}_{I}} \left[ h_{J}(\phi(\mathbf{x}), y) \right] \right]^{\top}.$$

We will also need the constraint epigraph and the cost-constraint epigraphs of  $P_{\phi}$ , namely,

$$\mathcal{C}_{\phi} = \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{I} \times \mathbb{R}^{J} \middle| \begin{array}{l} \exists \phi \in \Phi \quad \text{s.t.} \quad \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(\phi(\mathbf{x}), y) \right] \leq u_{i}, \quad \text{for } i = 1, \dots, I, \\ \text{and} \quad \mathbb{E}_{\mathbb{Q}_{j}} \left[ h_{j}(\phi(\mathbf{x}), y) \right] = v_{j} \quad \text{for } j = 1, \dots, J, \end{array} \right\}$$
 and 
$$\mathcal{M}_{\phi} = \left\{ (f, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^{I} \times \mathbb{R}^{J} \middle| \begin{array}{l} \exists \phi \in \Phi \quad \text{s.t.} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right] = f, \\ \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(\phi(\mathbf{x}), y) \right] \leq u_{i}, \quad \text{for } i = 1, \dots, I, \\ \text{and} \quad \mathbb{E}_{\mathbb{Q}_{j}} \left[ h_{j}(\phi(\mathbf{x}), y) \right] = v_{j} \quad \text{for } j = 1, \dots, J. \end{array} \right\}$$

**Remark B.3** (Well definedness of  $P_{\phi}$ ). The expectations must exist for all  $\phi \in \Phi$  for  $(P_{\phi})$  to be well defined, i.e. we require  $\Phi \subseteq \text{dom } F_{\phi}$ . This is true, for example, when  $\ell$ ,  $g_i$ ,  $h_j$  are bounded functions. We derive another condition later in Lemma B.7, but for now we assume  $\Phi \subseteq \text{dom } F_{\phi}$  for our Lemmas.

As noted in Section B.1.3, the convexity of  $\mathcal{M}_{\phi}$  is sufficient for strong duality when 0 is in the interior of the augmented constraint set. We will prove that  $\mathcal{M}_{\phi}$  is convex under certain conditions when  $\Phi$  is a decomposable set. Let us now define decomposability.

**Definition B.3.** A function class  $\Phi = \{\phi : \mathcal{X} \to \mathbb{R}\}$  is said to be decomposable with respect to a measurable space  $(\mathcal{X}, \Sigma_{\mathcal{X}})$  iff for every  $\phi_1, \phi_2 \in \Phi$  and  $Z \in \Sigma_{\mathcal{X}}$ , there exists  $\phi_3 \in \Phi$  such that,

$$\phi_3(x) = \begin{cases} \phi_1(x) & x \in Z \\ \phi_2(x) & x \in Z^c. \end{cases}$$

We use Lyapunov's convexity theorem as a tool for the proof.

**Theorem B.2.** ([54] Chapter IX Corollary 5) Let  $(\mathcal{X}, \Sigma_{\mathcal{X}})$  be a measurable space, and let  $(\mathbb{V}, \|\cdot\|)$  be a finite dimensional Banach space. If  $G : \Sigma_{\mathcal{X}} \to \mathbb{V}$  is a countably additive vector measure, then the range of G is a compact and convex subset of  $\mathbb{V}$ .

The range of G is the set  $\{G(A) \mid A \in \Sigma_{\mathcal{X}}\}$ . We will now prove that  $\mathcal{M}_{\phi}$  is convex.

**Lemma B.5.** Let  $\Phi \subseteq dom\ F_{\phi}$ . If the distributions  $\mathbb{D}, \mathbb{P}_i, \mathbb{Q}_j$  are atomless and  $\Phi$  is a decomposable set of functions, then it follows that  $\mathcal{M}_{\phi}$  is convex.

*Proof.* Consider the following set function formed by integrating  $\ell(\phi(\mathbf{x}), y)$ ,  $g_i(\phi(\mathbf{x}), y)$ ,  $h_j(\phi(\mathbf{x}), y)$  with respect to  $\mathbb{D}$ ,  $\mathbb{P}_i$ ,  $\mathbb{Q}_j$  respectively, over sets of the form  $A \times \mathcal{Y}$ , where  $A \in \Sigma_{\mathcal{X}}$  ( $\Sigma_{\mathcal{X}}$  is the marginal sigma algebra over  $\mathcal{X}$ ):

$$\forall A \in \Sigma_{\mathcal{X}}, G(A; \phi) = \left[ \int_{A} \int_{\mathcal{Y}} \ell(\phi(\mathbf{x}), y) \ d\mathbb{D}, \int_{A} \int_{\mathcal{Y}} g_{1}(\phi(\mathbf{x}), y) \ d\mathbb{P}_{1}, \dots, \int_{A} \int_{\mathcal{Y}} h_{J}(\phi(\mathbf{x}), y) \ d\mathbb{Q}_{J} \right]^{\top}.$$

Lemma B.2 implies that each entry of  $G(A; \phi)$  is an atomless scalar measure, since  $\mathbb{D}, \mathbb{P}_i, \mathbb{Q}_j$  are atomless. Suppose that  $\phi_1, \phi_2 \in \Phi$ . Define another set function,  $\mathfrak{p}(A) : \Sigma_{\mathcal{X}} \to \mathbb{R}^{2(1+I+J)}$  given by

$$\forall A \in \Sigma_{\mathcal{X}}, \quad \mathfrak{p}(A) = \begin{bmatrix} G(A; \phi_1) \\ G(A; \phi_2) \end{bmatrix}.$$

As before, each of the entries of  $\mathfrak{p}(A)$  are atomless scalar measures, therefore Lemma B.3 implies that  $\mathfrak{p}(A)$  is an atomless *vector* measure. Therefore Lyapunov's convexity theorem (Theorem B.2) implies that the range of  $\mathfrak{p}$ , denoted  $\mathfrak{p}(\Sigma_{\mathcal{X}})$ , is convex. Therefore for any  $\lambda \in [0,1]$ , there exists a set  $\mathcal{T}_{\lambda} \subseteq \mathcal{X}$  such that,

$$\mathfrak{p}(\mathcal{T}_{\lambda}) = \lambda \mathfrak{p}(\emptyset) + (1 - \lambda)\mathfrak{p}(\mathcal{X}) = (1 - \lambda)\mathfrak{p}(\mathcal{X}), \tag{33}$$

where we applied the fact that  $\mathfrak{p}(\emptyset) = 0$ . Since for any finite measure it holds that  $\mathfrak{p}(A) = \mathfrak{p}(\mathcal{X}) - \mathfrak{p}(A^c)$ , therefore it follows that,

$$\mathfrak{p}(\mathcal{T}_{\lambda}^{c}) = \mathfrak{p}(\mathcal{X}) - \mathfrak{p}(\mathcal{T}_{\lambda}) = (1 - (1 - \lambda))\mathfrak{p}(\mathcal{X}) = \lambda \mathfrak{p}(\mathcal{X}). \tag{34}$$

We will use these relations to prove that the range of  $F_{\phi}$ , denoted  $F_{\phi}(\Phi)$ , is also convex. We do so by constructing an input  $\phi_3$  such that  $F_{\phi}(\phi_3)$  is a convex combination of  $F_{\phi}(\phi_1)$  and  $F_{\phi}(\phi_2)$ . The required function  $\phi_3$  is defined as follows,

$$\forall \mathbf{x} \in \mathcal{X}, \quad \phi_3(\mathbf{x}) = \begin{cases} \phi_1(\mathbf{x}) & \mathbf{x} \in \mathcal{T}_{\lambda}, \\ \phi_2(\mathbf{x}) & \mathbf{x} \in \mathcal{T}_{\lambda}^{\lambda}. \end{cases}$$

By the definition of decomposability,  $\phi_3$  is also an element of  $\Phi$ . Next we note that,

$$G(\mathcal{X}; \phi_3) = G(\mathcal{T}_{\lambda}; \phi_3) + G(\mathcal{T}_{\lambda}^c; \phi_3),$$

since  $\mathcal{T}_{\lambda}$ ,  $\mathcal{T}_{\lambda}^{c}$  form a mutually exclusive cover of  $\mathcal{X}$ . Therefore,

$$G(\mathcal{X};\phi_3) = G(\mathcal{T}_{\lambda};\phi_1) + G(\mathcal{T}_{\lambda}^c;\phi_2)$$

by applying the definition of  $\phi_3$ . Equations (33) and (34) applied to the above relation implies that,

$$G(\mathcal{X};\phi_3) = \lambda G(\mathcal{X};\phi_1) + (1-\lambda)G(\mathcal{X};\phi_2). \tag{35}$$

But clearly,  $F_{\phi}(\phi) = G(\mathcal{X}; \phi)$  for any  $\phi$ . Therefore,

$$F_{\phi}(\phi_3) = \lambda F_{\phi}(\phi_1) + (1 - \lambda)F_{\phi}(\phi_2).$$

This proves that the range  $F_{\phi}(\Phi)$  is convex, because  $\phi_1, \phi_2$  were picked arbitrarily.

Recall that  $\mathcal{M}_{\phi} = F_{\phi}(\Phi) + \mathbb{R}^{1+I}_{\geq 0} \times \mathbf{0}^{J}$ . Clearly  $\mathbb{R}^{1+I}_{\geq 0} \times \mathbf{0}^{J}$  is a convex set, and since the Minkowski sum of two convex sets is convex ([55] Theorem 3.1), therefore  $\mathcal{M}_{\phi}$  is also convex, which concludes the proof.

#### **B.2.2** Strong duality of $(P_{\phi})$

In this section we will utilise Theorem B.1 and Lemma B.5 along with the assumptions we made in Section 3 to prove that  $(P_{\phi})$  is strongly dual.

**Proposition B.1.** Let (a)  $\Phi \subseteq dom \ F_{\phi}$ , (b)  $\Phi$  be decomposable and (c)  $P_{\phi}^{\star} > -\infty$ . Under Assumptions 1 and 2.1,  $(P_{\phi})$  is strongly dual and  $Opt(D_{\phi})$  is non-empty.

*Proof.* Assumption 2 The premise of the proposition, along with atomlessness from Assumption 2 allows us to invoke Lemma B.5 to conclude that  $\mathcal{M}_{\phi}$  is a convex set.

From Assumption 1, we know that for some  $\xi > 0$ ,  $B(0^{I+J}, \xi) \subseteq \operatorname{int}(\mathcal{C})$  and  $\mathcal{C} \subseteq \mathcal{C}_{\phi}$ , therefore  $B(0^{I+J}, \xi) \subseteq \operatorname{int}(\mathcal{C}) \subseteq \operatorname{int}(\mathcal{C}_{\phi})$ . In particular,  $0 \in \operatorname{int}(\mathcal{C}_{\phi})$ , therefore, we can invoke Theorem B.1 to conclude that  $\operatorname{Opt}(\mathbf{D}_{\phi}) \neq \emptyset$  and  $P_{\phi}^{\star} = D_{\phi}^{\star}$ .

#### **B.3** Duality gap

In this section we will bound the duality gap of (P). For this, we define the following relaxation,

$$\begin{split} P_{\theta\nu}^{\star} &= \inf_{\theta \in \Theta} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(f_{\theta}(\mathbf{x}), y) \right] \leq L\nu, \quad \text{for } i = 1, \dots, I, \\ -L\nu \leq \mathbb{E}_{\mathbb{Q}_{i}} \left[ h_{j}(f_{\theta}(\mathbf{x}), y) \right] \leq L\nu, \quad \text{for } j = 1, \dots, J. \end{split} \tag{$\mathsf{P}_{\theta\nu}$}$$

Note that the set  $\Theta_{\nu}$  (defined in Section 3) is nothing but the feasibility set of  $(P_{\theta\nu})$ , i.e.  $Feas(P_{\theta\nu})$ . Let its dual value be  $D_{\theta\nu}^{\star}$ , defined similarly to  $D^{\star}$  (Section 3). We define its functional version,

$$P_{\phi\nu}^{\star} = \inf_{\phi \in \Phi} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right]$$
subject to 
$$\mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(\phi(\mathbf{x}), y) \right] \leq L\nu, \quad \text{for } i = 1, \dots, I,$$

$$-L\nu \leq \mathbb{E}_{\mathbb{Q}_{j}} \left[ h_{j}(\phi(\mathbf{x}), y) \right] \leq L\nu, \quad \text{for } j = 1, \dots, J.$$

$$(P_{\phi\nu})$$

Let its dual value be  $D_{\phi\nu}^{\star}$ . We will bound the duality gap based on (5) which uses the functional strong duality relation  $P_{\phi}^{\star}=D_{\phi}^{\star}$ , and which we further decompose as,

$$\begin{split} P^{\star} - D^{\star} &= P^{\star} - P_{\phi}^{\star} + D_{\phi}^{\star} - D^{\star} \\ &= (P^{\star} - P_{\theta\nu}^{\star}) + (P_{\theta\nu}^{\star} - P_{\phi}^{\star}) + (P_{\theta\nu}^{\star} - P_{\phi}^{\star}) + (D_{\phi}^{\star} - D_{\phi\nu}^{\star}) + (D_{\phi\nu}^{\star} - D_{\theta\nu}^{\star}) \\ &+ (D_{\theta\nu}^{\star} - D^{\star}). \end{split}$$

Note that this decomposition contains the duality gap of  $(P_{\theta\nu})$ , i.e. we can alternatively decompose  $P^{\star} - D^{\star}$  as,

$$P^{\star} - D^{\star} = (P^{\star} - P_{\theta\nu}^{\star}) + (P_{\theta\nu}^{\star} - D_{\theta\nu}^{\star}) + (D_{\theta\nu}^{\star} - D^{\star}).$$

First, we will prove a few helper lemmas (Sections B.3.1,B.3.2,B.3.3). Then we will bound the duality gap of  $(P_{\theta\nu})$  in Section B.3.4, and we will bound the remaining terms, i.e.,  $P^{\star} - P_{\theta\nu}^{\star}$  and  $D_{\theta\nu}^{\star} - D^{\star}$  in Section ?? and summarise everything with a bound on the duality gap  $P^{\star} - D^{\star}$  (Proposition B.2).

## **B.3.1** Lipschitz continuity of $F_{\phi}$

First, we will prove that the mapping  $F_{\phi}: L^p(\mathbb{R}^K; \mathbb{P}_+) \to \mathbb{R}^{1+I+J}$  is a Lipschitz continuous mapping.

**Lemma B.6.** Let the loss functions  $\ell, g_i, h_j$  be L-Lipschitz continuous. Then the components of the range function  $F_{\phi}: L^p(\mathbb{R}^K; \mathbb{P}_+) \to \mathbb{R}^{1+I+J}$ , e.g.  $\phi \mapsto \mathbb{E}_{\mathbb{D}}\left[\ell(\phi(\mathbf{x}), y)\right]$ , are L-Lipschitz continuous and  $F_{\phi}$  is  $(\sqrt{1+I+J})$ L-Lipschitz continuous, for  $p \in [1, \infty)$ .

*Proof.* We will prove that  $\phi \mapsto \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right]$  is Lipschitz continuous - the proof is the same for the other components. Let  $\phi_1, \phi_2 \in L^p(\mathbb{R}^K; \mathbb{P}_+)$ , then,

$$|\mathbb{E}_{\mathbb{D}}\left[\ell(\phi_{1}(\mathbf{x}), y)\right] - \mathbb{E}_{\mathbb{D}}\left[\ell(\phi_{2}(\mathbf{x}), y)\right]|^{p} \leq \mathbb{E}_{\mathbb{D}}\left[\left|\ell(\phi_{1}(\mathbf{x}), y) - \ell(\phi_{2}(\mathbf{x}), y)\right|^{p}\right]$$

$$\leq \mathbb{E}_{\mathbb{D}}\left[L^{p} \|\phi_{1}(\mathbf{x}) - \phi_{2}(\mathbf{x})\|_{2}^{p}\right]$$

$$= L^{p} \mathbb{E}_{\mathbb{D}}\left[\|\phi_{1}(\mathbf{x}) - \phi_{2}(\mathbf{x})\|_{2}^{p}\right]. \tag{36}$$

The first inequality is an application of Jensen's inequality  $(|\cdot|^p)$  is convex) and the second inequality applies the Lipschitz condition on  $\ell(\cdot,\cdot)$ . Equation (36) proves the continuity of  $\phi\mapsto \mathbb{E}_{\mathbb{D}}\left[\ell(\phi(\mathbf{x}),y)\right]$  with respect to the norm on  $L^p(\mathbb{R}^K;\mathbb{D})$ , but we need to prove it for the norm on  $L^p(\mathbb{R}^K;\mathbb{P}_+)$ . Linearity of the integral with respect to the integral (Lemma B.1) implies that,

$$\mathbb{E}_{\mathbb{P}_{+}}\left[\left\|\phi_{1}(\mathbf{x})-\phi_{2}(\mathbf{x})\right\|_{2}^{p}\right]=\mathbb{E}_{\mathbb{D}}\left[\left\|\phi_{1}(\mathbf{x})-\phi_{2}(\mathbf{x})\right\|_{2}^{p}\right]+\mathbb{E}_{\sum_{i=1}^{I}\mathbb{P}_{i}+\sum_{j=1}^{J}\mathbb{Q}_{j}}\left[\left\|\phi_{1}(\mathbf{x})-\phi_{2}(\mathbf{x})\right\|_{2}^{p}\right].$$

Clearly  $\mathbb{E}_{\sum_{i=1}^{I} \mathbb{P}_{i} + \sum_{j=1}^{J} \mathbb{Q}_{j}} [\|\phi_{1}(\mathbf{x}) - \phi_{2}(\mathbf{x})\|_{2}^{p}] \geq 0$  since both measure and integrand are nonnegative. This implies that,

$$\mathbb{E}_{\mathbb{D}} \left[ \|\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})\|_2^p \right] \le \mathbb{E}_{\mathbb{P}_+} \left[ \|\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})\|_2^p \right]. \tag{37}$$

Equation (36) and (37) together imply that,

$$\left|\mathbb{E}_{\mathbb{D}}\left[\ell(\phi_{1}(\mathbf{x}), y)\right] - \mathbb{E}_{\mathbb{D}}\left[\ell(\phi_{2}(\mathbf{x}), y)\right]\right|^{p} \leq L^{p} \mathbb{E}_{\mathbb{P}_{+}}\left[\left\|\phi_{1}(\mathbf{x}) - \phi_{2}(\mathbf{x})\right\|_{2}^{p}\right] = L^{p} \left\|\phi_{1} - \phi_{2}\right\|_{L^{p}(\mathbb{R}^{K}; \mathbb{P}_{+})}^{p},$$

which proves the first part (after taking the p<sup>th</sup> root on both sides). The second part is proved as follows.

$$||F_{\phi}(\phi_{1}) - F_{\phi}(\phi_{2})||_{2}^{2} = \sum_{k=1}^{1+I+J} (F_{\phi,k}(\phi_{1})_{k} - F_{\phi,k}(\phi_{2})_{k})^{2}$$

$$\leq (1+I+J)L^{2} ||\phi_{1} - \phi_{2}||_{L^{p}(\mathbb{R}^{K}:\mathbb{P}_{+})}^{2}.$$
(38)

Here we used the L-Lipschitz continuity of the components of  $F_{\phi}(\phi)$  that was proved in the first part. Taking the square root on both sides proves that  $F_{\phi}$  is  $(\sqrt{1+I+J})L$ -Lipschitz continuous with respect to the domain  $L^p(\mathbb{R}^K; \mathbb{P}_+)$ .

#### **B.3.2** Well definedness of $(P_{\phi})$

Continuity of the loss functions almost solves the problem of well-definedness of  $(P_{\phi})$ .

**Lemma B.7.** Let the loss functions  $\ell$ ,  $g_i$ ,  $h_j$  be L-Lipschitz continuous and  $\Phi \subseteq L^p(\mathbb{R}^K; \mathbb{P}_+)$ . If  $\Phi \cap dom\ F_\phi \neq \emptyset$ , then  $\Phi \subseteq dom\ F_\phi$ .

*Proof.* Let  $\phi_0 \in \Phi \cap \text{dom } F_{\phi}$  and  $\phi \in \Phi$ . By the triangle inequality,

$$||F_{\phi}(\phi)||_{2} \leq ||F_{\phi}(\phi) - F_{\phi}(\phi_{0})||_{2} + ||F_{\phi}(\phi_{0})||_{2}.$$

Since  $F_{\phi}: L^p(\mathbb{R}^K; \mathbb{P}_+) \to \mathbb{R}^K$  is  $(\sqrt{1+I+J})L$ -Lipschitz continuous (Lemma B.6), therefore,

$$\|F_{\phi}(\phi)\|_{2} \leq (\sqrt{1+I+J})L \|\phi_{0} - \phi\|_{L^{p}(\mathbb{R}^{K};\mathbb{P}_{+})} + \|F_{\phi}(\phi_{0})\|_{2}.$$

Since  $\phi_0 \in \text{dom } F_{\phi}$ , therefore  $\|F_{\phi}(\phi_0)\|_2 < \infty$ . Since  $\phi, \phi_0 \in L^p(\mathbb{R}^K; \mathbb{P}_+)$ , therefore the first term is also bounded. Therefore  $\|F_{\phi}(\phi)\|_2 < \infty$  and  $\Phi \subseteq \text{dom } F_{\phi}$ .

#### **B.3.3** Lipschitz continuity of parametrization

We will now show that pointwise Lipschitz continuity of the map  $f_{\theta}(\mathbf{x})$  at each  $\mathbf{x}$  implies the Lipschitz continuity of the map  $\theta \mapsto f_{\theta} \in L^p(\mathbb{R}^K; \mathbb{P}_+)$ .

**Lemma B.8.** Let  $f_{\theta}(\mathbf{x})$  be  $L_{\theta}$ -Lipschitz continuous for every  $\mathbf{x} \in \mathcal{X}$ . Then, the map  $f^{\Theta} : \Theta \ni \theta \mapsto f_{\theta} \in L^{p}(\mathbb{R}^{K}; \mathbb{P}_{+})$  is also  $L_{\theta}$ -Lipschitz continuous (for  $p \in [1, \infty)$ ).

*Proof.* Consider  $\theta_1, \theta_2 \in \Theta$ . Then,

$$\|f_{\theta_{1}} - f_{\theta_{2}}\|_{L^{p}(\mathbb{R}^{K};\mathbb{P}_{+})}^{p} = \int |f_{\theta_{1}}(\mathbf{x}) - f_{\theta_{2}}(\mathbf{x})|^{p} d\mathbb{P}_{+}$$

$$\leq \int L_{\theta}^{p} \|\theta_{1} - \theta_{2}\|_{2}^{p} d\mathbb{P}_{+}$$

$$\Rightarrow \|f_{\theta_{1}} - f_{\theta_{2}}\|_{L^{p}(\mathbb{R}^{K};\mathbb{P}_{+})}^{p} \leq L_{\theta}^{p} \|\theta_{1} - \theta_{2}\|_{2}^{p}.$$

$$(39)$$

Equation (39) is the application of the pointwise Lipschitz continuity. Taking the  $p^{th}$  root on both sides of the last inequality completes the proof.

## **B.3.4 Duality gap of** $(P_{\theta\nu})$

Lemma B.4 provides an upper bound for  $D_{\phi}^{\star} - D_{\phi\nu}^{\star}$ . Next we show that  $D_{\theta\nu}^{\star} \geq D_{\phi\nu}^{\star}$ , and thereafter bound  $P_{\theta\nu}^{\star} - P_{\phi}^{\star}$ .

**Lemma B.9.** If 
$$\mathcal{H} \subseteq \Phi$$
 then  $D_{\theta\nu}^{\star} \geq D_{\phi\nu}^{\star}$ .

*Proof.* Let  $\gamma \in \mathbb{R}^{I+2J}_{\geq 0}$  denote the dual variable for  $(P_{\theta\nu})$  or  $(P_{\phi\nu})$ . Clearly  $L_{\theta\nu}(\theta,\gamma) = L_{\phi\nu}(f_{\theta},\gamma)$ . Therefore,

$$q_{\theta\nu}(\gamma) = \inf_{\theta \in \Theta} L_{\theta\nu}(\theta, \gamma) = \inf_{f_{\theta} \in \mathcal{H}} L_{\phi\nu}(f_{\theta}, \gamma) \ge \inf_{\phi \in \Phi} L_{\phi\nu}(\phi, \gamma) = q_{\phi\nu}(\gamma).$$

The supremum preserves the inequality, therefore,

$$D_{\theta\nu}^{\star} = \sup_{\gamma \in \mathbb{R}^{I+2J}_{>0}} q_{\theta\nu}(\gamma) \ge \sup_{\gamma \in \mathbb{R}^{I+2J}_{>0}} q_{\phi\nu}(\gamma) = D_{\phi\nu}^{\star}$$

which was to be proved.

**Lemma B.10.** Under Assumption 2,  $Feas(P_{\theta\nu})$  is non-empty and  $P_{\theta\nu}^{\star} - P_{\phi}^{\star} \leq L\nu$ .

*Proof.* Let  $\phi^* \in \text{Opt}(P_\phi) \neq \emptyset$  be the solution defined by Assumption 2.2. Since  $\phi^*$  is feasible for  $(P_\phi)$ , therefore it follows that,

$$\mathbb{E}_{\mathbb{P}_i}\left[g_i(\phi^{\star}(\mathbf{x}), y)\right] \leq 0$$
 and  $\mathbb{E}_{\mathbb{Q}_i}\left[h_j(\phi^{\star}(\mathbf{x}), y)\right] = 0.$ 

By Assumption 2.2, there exists  $f_{\theta} \in \mathcal{H}$  such that  $\|f_{\theta} - \phi^{\star}\|_{L^{p}(\mathbb{R}^{K};\mathbb{P}_{+})} \leq \nu$ . Since the range function  $F_{\phi}(\phi)$  is component-wise L-Lipschitz continuous (Lemma B.6), therefore we can upper bound the inequality constraint functions as follows,

$$\mathbb{E}_{\mathbb{P}_i}\left[g_i(f_{\theta}(\mathbf{x}), y)\right] \le \mathbb{E}_{\mathbb{P}_i}\left[g_i(\phi^{\star}(\mathbf{x}), y)\right] + L \left\|f_{\theta} - \phi^{\star}\right\|_{L^p(\mathbb{R}^{K \cdot \mathbb{P}_+})} \le 0 + L\nu. \tag{40}$$

Similarly the equality constraint functions can be bounded symmetrically as,

$$\left| \mathbb{E}_{\mathbb{Q}_j} \left[ h_i(f_{\theta}(\mathbf{x}), y) \right] - \mathbb{E}_{\mathbb{Q}_j} \left[ h_i(\phi^{\star}(\mathbf{x}), y) \right] \right| \le L \left\| f_{\theta} - \phi^{\star} \right\|_{L^p(\mathbb{R}^K; \mathbb{P}_+)}$$

$$\Rightarrow \left| \mathbb{E}_{\mathbb{Q}_j} \left[ h_i(f_{\theta}(\mathbf{x}), y) \right] \right| \le L \nu. \tag{41}$$

Equations (40) and (41) imply that  $\theta$  is feasible for  $(P_{\theta\nu})$ . Therefore, Feas $(P_{\theta\nu}) \neq \emptyset$  and,

$$P_{\theta\nu}^{\star} \le \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right]. \tag{42}$$

But the L-Lipschitzness of the objective (Assumption 3, Lemma B.6) implies that,

$$\mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta}(\mathbf{x}), y)\right] \leq \mathbb{E}_{\mathbb{D}}\left[\ell(\phi^{\star}(\mathbf{x}), y)\right] + L \|f_{\theta} - \phi^{\star}\|_{L^{p}(\mathbb{R}^{K}; \mathbb{P}_{+})}$$

$$\leq P_{\phi}^{\star} + L\nu. \tag{43}$$

Therefore (42) and (43) implies,

$$P_{\theta\nu}^{\star} \leq P_{\phi}^{\star} + L\nu$$

which was to be proved.

We will now summarize our arguments and bound the duality gap of  $(P_{\theta\nu})$  in the next lemma.

**Lemma B.11.** Under Assumptions 1,2 and 3, if  $P_{\phi}^{\star} > -\infty$ ,  $Opt(D_{\phi})$  is non-empty and for any  $\gamma_{\phi}^{\star} \in Opt(D_{\phi})$ ,  $P_{\theta\nu}^{\star} - D_{\theta\nu}^{\star} \leq (1 + \left\|\gamma_{\phi}^{\star}\right\|_{1})L\nu$ .

*Proof.* We can decompose the duality gap as follows,

$$P_{\theta\nu}^{\star} - D_{\theta\nu}^{\star} = (P_{\theta\nu}^{\star} - P_{\phi}^{\star}) + (P_{\phi}^{\star} - D_{\phi}^{\star}) + (D_{\phi}^{\star} - D_{\phi\nu}^{\star}) + (D_{\phi\nu}^{\star} - D_{\theta\nu}^{\star}).$$

From Lemma B.10 (which requires Assumption 2),  $P_{\theta\nu}^{\star} - P_{\phi}^{\star} \leq L\nu$ .

Next, if  $0 \in \text{int } (\mathcal{C})$  (Assumption 1) then  $0 \in \text{int } (\mathcal{C}_{\phi})$  since  $\mathcal{C} \subseteq \mathcal{C}_{\phi}$ . Then, by the construction of  $\mathcal{C}_{\phi}$ , Feas $(P_{\phi}) \neq \emptyset$ . Clearly Feas $(P_{\phi}) \subseteq \text{dom } F_{\phi}$ , therefore  $\Phi \cap \text{dom } F_{\phi} \neq \emptyset$ . Since we also have Assumption 3 (Lipschitz losses), we can apply Lemma B.7 to conclude  $\Phi \subseteq \text{dom } F_{\phi}$ . The premise stipulates that  $P_{\phi}^{\star} > -\infty$ . Assumption 2 gives us atomlessness of the measures and decomposability of  $\Phi$ , which provides the remaining requirements to invoke Proposition B.1, giving us that  $P_{\phi}^{\star} - D_{\phi}^{\star} = 0$  and  $\text{Opt}(D_{\phi}) \neq \emptyset$ .

Since  $\operatorname{Opt}(D_{\phi}) \neq \emptyset$ , we can apply Lemma B.4 to conclude that  $D_{\phi}^{\star} - D_{\phi\nu}^{\star} \leq L\nu \left\| \gamma_{\phi}^{\star} \right\|$  for some  $\gamma_{\phi}^{\star} \in \operatorname{Opt}(D_{\phi})$ . Lemma B.9, which only requires that  $\mathcal{H} \subseteq \Phi$ , proves that  $D_{\phi\nu}^{\star} - D_{\theta\nu}^{\star} \leq 0$ . Putting these bounds together gives us the required bound on the duality gap of  $(P_{\theta\nu})$ .

**Remark B.4.** This result is the same upper bound on the parametrized duality gap presented in [23]. This proof technique, specifically Lemma B.10, however, does not work with equality constrained problems such as (P). This is because replacing  $\phi^*$  with  $\theta$  leads to a perturbation that localises the constraint function values upto an interval of size  $2L\nu$ . Therefore, no functional problem (or solution) generates a  $\theta$  that satisfies an equality constraint, e.g.  $\mathbb{E}_{\mathbb{Q}_j}\left[h_j(f_{\theta}(\mathbf{x}),y)\right]=0$ , since the constraint function value has to be exactly 0.

The limitation of this proof technique also applies to inequality constrained problems. Consider the following family of problems,

$$\begin{split} P_I^{\star} &= \inf_{\theta \in \Theta} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_i} \left[ g_i(f_{\theta}(\mathbf{x}), y) \right] \leq c_i, \quad \textit{for } i = 1, \dots, I. \end{split} \tag{P_I}$$

[23] used a tighter functional problem to upper bound  $P_I^{\star}$ , namely,

$$\begin{split} P_{\phi I}^{\star} &= \inf_{\phi \in \Phi} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(\phi(\mathbf{x}), y) \right] \\ \text{subject to} \quad \mathbb{E}_{\mathbb{P}_{i}} \left[ g_{i}(\phi(\mathbf{x}), y) \right] \leq c_{i} - L\nu, \quad \textit{for } i = 1, \dots, I. \end{split}$$

Using Lemma B.10 with J=0, we can prove that  $P_I^\star - P_{\phi I}^\star \leq L\nu$  only if  $(P_{\phi I})$  is feasible. However, this may not be the case, if  $c_i \leq \inf_{\phi \in \Phi} \mathbb{E}_{\mathbb{P}_i} \left[ g_i(\phi(\mathbf{x}),y) \right] + L\nu$ , i.e.  $c_i$  is in the neighbourhood of the minimal achievable value. In this case the analysis in the next section still applies.

#### **B.3.5 Duality gap of** (P)

In this section we will bound the duality gap of (P). Since  $D^* \geq D^*_{\theta\nu}$  (see Lemma B.4), therefore,

$$P^{\star} - D^{\star} = (P^{\star} - P_{\theta\nu}^{\star}) + (P_{\theta\nu}^{\star} - D_{\theta\nu}^{\star}) + (D_{\theta\nu}^{\star} - D^{\star}) \le (P^{\star} - P_{\theta\nu}^{\star}) + (P_{\theta\nu}^{\star} - D_{\theta\nu}^{\star}). \tag{44}$$

 $P_{\theta\nu}^{\star}-D_{\theta\nu}^{\star}$  is upper bounded by Lemma B.11. We now upper bound  $P^{\star}-P_{\theta\nu}^{\star}$ .

**Lemma B.12.** Let Feas(P) 
$$\neq \emptyset$$
. Under Assumption 3, it follows that  $P^{\star} - P_{\theta\nu}^{\star} \leq LL_{\theta}R(\nu)$ .

*Proof.* Let  $\epsilon > 0$  be chosen arbitrarily and let  $\theta_{\nu}^{\star}$  be a *feasible* solution for  $(P_{\theta\nu})$  that is  $\epsilon$ -optimal, i.e.,

$$P_{\theta\nu}^{\star} \le \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta\nu}(\mathbf{x}), y) \right] \le P_{\theta\nu}^{\star} + \epsilon. \tag{45}$$

Such a solution always exists if  $(P_{\theta\nu})$  is feasible, since the optimal value is the infimum of the objective  $\mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta^*_{\nu}}(\mathbf{x}), y)\right]$  over the feasible set. Note that since (P) is assumed feasible, so is  $(P_{\theta\nu})$ .

Next for arbitrary  $\delta > 0$ , let  $\theta_0 \in \text{Feas}(P)$  be such that  $\|\theta_{\nu}^{\star} - \theta_0\| = \inf_{\theta \in \text{Feas}(P)} \|\theta_{\nu}^{\star} - \theta\| + \delta$ , i.e.  $\theta_0$ 

is  $\delta$ -close to the projection of  $\theta_{\nu}^{\star}$  onto the feasible set of (P) (which may not be closed). Note that  $\Theta_{\nu}$  is nothing but Feas(P<sub>\theta\nu</sub>). Therefore, by the definition of  $R(\nu)$ ,

$$\|\theta_{\nu}^{\star} - \theta_0\| \le R(\nu) + \delta. \tag{46}$$

Next we apply the L-Lipschitzness of the map  $\phi \mapsto \mathbb{E}_{\mathbb{D}}[\ell(f_{\theta}(\mathbf{x}), y)]$  (Lemma B.6), and the  $L_{\theta}$ -Lipschitzness of the map  $\theta \mapsto f_{\theta}$  (Lemma B.8) to conclude that,

$$\mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta_0}(\mathbf{x}), y)\right] - \mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta_{\nu}^{\star}}(\mathbf{x}), y)\right] \le L \left\|f_{\theta_0} - f_{\theta_{\nu}^{\star}}\right\|_{L^p(\mathbb{R}^K; \mathbb{P}_+)} \le LL_{\theta} \left\|\theta_0 - \theta_{\nu}^{\star}\right\|. \tag{47}$$

Since  $\theta_0$  is feasible with respect to (P), therefore  $\mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta_0}(\mathbf{x}),y)\right] \geq P^*$ . This and (45) implies that,

$$P^{\star} - P_{\theta\nu}^{\star} \leq \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta_0}(\mathbf{x}), y) \right] - \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta_n^{\star}}(\mathbf{x}), y) \right] + \epsilon.$$

Thereafter we can apply (46) and (47) to conclude that,

$$P^* - P^*_{\theta\nu} \le LL_{\theta}R(\nu) + LL_{\theta}\delta + \epsilon.$$

Since the bound holds for every  $\epsilon > 0$  and  $\delta > 0$ , therefore it also holds for  $\epsilon = 0$  and  $\delta = 0$  (simply take the infimum on both sides). This completes the proof.

We will now state the bound on the duality gap for (P).

**Proposition B.2.** Let Assumptions 1, 2 and 3 hold and let  $P_{\phi}^{\star} > -\infty$ . Then  $Opt(D_{\phi}) \neq \emptyset$  and for any  $\gamma_{\phi}^{\star} \in Opt(D_{\phi})$ ,  $P^{\star} - D^{\star} \leq (1 + \left\|\gamma_{\phi}^{\star}\right\|_{1})L\nu + LL_{\theta}R(\nu)$ .

*Proof.* We can invoke Lemma B.11 with the premise to obtain that  $\operatorname{Opt}(D_\phi) \neq \emptyset$  and  $P_{\theta\nu}^\star - D_{\theta\nu}^\star \leq (1 + \left\|\gamma_\phi^\star\right\|_1)L\nu$  for any  $\gamma_\phi^\star \in \operatorname{Opt}(D_\phi)$ . Note that Assumption 1 implies that (P) is feasible due to the construction of  $\mathcal C$ , and this, along with Assumption 3, allows us to invoke Lemma B.12 which states that  $P^\star - P_{\theta\nu}^\star \leq LL_\theta R(\nu)$ . Applying these bounds to (44) completes the proof.

#### **B.4** Dual estimation error

In this section we will upper bound the absolute difference between the optimal values of the dual problem (D) and the empirical dual problem ( $\hat{D}$ ), i.e.  $|D^{\star} - \hat{D}^{\star}|$ , beginning with a technical lemma that helps take the conjunction of probabilistic statements.

**Lemma B.13.** Let  $(\mathcal{X}, \Sigma_{\mathcal{X}}, \mathbb{P})$  be a measure space and let  $\{a_1, \ldots, a_K\}$  be random statements. If  $a_k$  holds with probability at least  $1 - \delta$ , for  $k = 1, \ldots, K$ , then  $\bigwedge_{k=1}^K a_k$ , that is the conjuction of  $a_k$ , holds with probability at least  $1 - K\delta$ .

*Proof.* Let  $\neg a_k$  denote the negation of  $a_k$ . Then,

$$\mathbb{P}(\bigwedge_{k=1}^{K} a_k) = 1 - \mathbb{P}(\bigvee_{k=1}^{K} \neg a_k) \ge 1 - \sum_{k=1}^{K} \mathbb{P}(\neg a_k). \tag{48}$$

The last inequality is the union bound ([56] Lemma 2.2). Since  $\mathbb{P}(a_k) \geq 1 - \delta$ , therefore  $\mathbb{P}(\neg a_k) \leq \delta$ . Applying this to (48) gives us the bound,

$$\mathbb{P}(\bigwedge_{k=1}^{K} a_k) \ge 1 - K\delta,$$

which completes the proof.

**Proposition B.3.** Let  $N_{min} = \min \{M_0, M_1, \dots, M_I, N_1, \dots, N_J\}$ . If Assumptions 1 and 4 hold, then Opt(D),  $Opt(\hat{D}) \neq \emptyset$  and, for all  $\delta \in (0, 1)$ ,

$$|D^{\star} - \hat{D}^{\star}| \le (1 + \max\{\|\gamma^{\star}\|_{1}, \|\hat{\gamma}^{\star}\|_{1}) \zeta^{UC}(N_{min}, \delta),$$

holds with probability at least  $1 - (1 + I + J)\delta$ .

*Proof.* Assumption 1 states that  $B(0,\xi)\subseteq \operatorname{int}\,\mathcal{C}\cap\operatorname{int}\,\hat{\mathcal{C}}$  and therefore  $0\in\operatorname{int}\,\mathcal{C}$  and  $0\in\operatorname{int}\,\hat{\mathcal{C}}$ . Therefore Theorem B.1 implies that  $\operatorname{Opt}(D)\neq\emptyset$  and  $\operatorname{Opt}(\hat{D})\neq\emptyset$ .

Let  $\gamma^* = (\lambda^*, \mu^*)$  and  $\hat{\gamma}^* = (\hat{\lambda}^*, \hat{\mu}^*)$  be members of Opt(D) and  $Opt(\hat{D})$  respectively. Therefore,

$$D^{\star} - \hat{D}^{\star} = q(\gamma^{\star}) - \hat{q}(\hat{\gamma}^{\star})$$
  

$$\leq q(\gamma^{\star}) - \hat{q}(\gamma^{\star}), \tag{49}$$

where the inequality follows from the suboptimality of  $\gamma^*$  with respect to  $(\hat{D})$ , i.e. because  $\hat{q}(\gamma^*) \leq \hat{q}(\hat{\gamma}^*) = \hat{D}^*$ . Let  $\epsilon > 0$  be arbitrary and let  $\theta$  be  $\epsilon$ -optimal with respect to  $\hat{q}(\gamma^*)$ , i.e,

$$\hat{q}(\gamma^*) \le \hat{L}(f_\theta, \gamma^*) \le \hat{q}(\gamma^*) + \epsilon.$$
 (50)

Clearly,  $q(\gamma^*) \leq L(f_\theta, \gamma^*)$  since  $q(\gamma^*) = \inf_{\theta} L(f_\theta, \gamma^*)$ . This fact and (50), applied to (49), implies that.

$$D^{\star} - \hat{D}^{\star} \le q(\gamma^{\star}) - \hat{q}(\gamma^{\star}) \le L(f_{\theta}, \gamma^{\star}) - \hat{L}(f_{\theta}, \gamma^{\star}) + \epsilon. \tag{51}$$

Now let us expand the difference of Lagrangians in the RHS of (51), also splitting  $\gamma^*$  into its components  $(\lambda^*, \mu^*)$ ,

$$L(f_{\theta}, \lambda^{\star}, \mu^{\star}) - \hat{L}(f_{\theta}, \lambda^{\star}, \mu^{\star}) = \left(\mathbb{E}_{\mathbb{D}}\left[\ell(f_{\theta}(\mathbf{x}), y)\right] - \frac{1}{M_{0}} \sum_{m_{0}=1}^{M_{0}} \ell(f_{\theta}(\mathbf{x}_{m_{0}}), y_{m_{0}})\right)$$

$$+ \sum_{i=1}^{I} \lambda_{i}^{\star} \left(\mathbb{E}_{\mathbb{P}_{i}}\left[g_{i}(f_{\theta}(\mathbf{x}), y)\right] - \frac{1}{M_{i}} \sum_{m_{i}=1}^{M_{i}} g_{i}(f_{\theta}(\mathbf{x}_{m_{i}}), y_{m_{i}})\right)$$

$$+ \sum_{j=1}^{J} \mu_{j}^{\star} \left(\mathbb{E}_{\mathbb{Q}_{j}}\left[h_{j}(f_{\theta}(\mathbf{x}), y)\right] - \frac{1}{N_{j}} \sum_{n_{j}=1}^{N_{j}} h_{j}(f_{\theta}(\mathbf{x}_{n_{j}}), y_{n_{j}})\right). \tag{52}$$

Now, since Assumption 4 holds, therefore the following holds,

$$\left| \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right] - \frac{1}{M_0} \sum_{m_0 = 1}^{M_0} \ell(f_{\theta}(\mathbf{x}_{m_0}), y_{m_0}) \right| \le \zeta^{\text{UC}}(M_0, \delta), \tag{53}$$

for every  $\theta \in \Theta$  with probability at least  $1-\delta$ , in particular the  $\theta$  chosen. We apply the uniform convergence bound on the remaining samples corresponding to  $\mathbb{P}_i$ ,  $\mathbb{Q}_j$  to obtain inequalities analogous to (53). All of the bounds hold simultaneously with probability  $1-(1+I+J)\delta$  (see Lemma B.13). Applying all the bounds to (52), we obtain,

$$L(f_{\theta}, \lambda^{\star}, \mu^{\star}) - \hat{L}(f_{\theta}, \lambda^{\star}, \mu^{\star}) \leq \zeta^{\mathrm{UC}}(M_{0}, \delta) + \sum_{i=1}^{I} \lambda_{i}^{\star} \zeta^{\mathrm{UC}}(M_{i}, \delta) + \sum_{j=1}^{J} \mu_{j}^{\star} \zeta^{\mathrm{UC}}(N_{j}, \delta).$$

Since  $\zeta^{UC}(N, \delta)$  is decreasing with respect to N, therefore,

$$L(f_{\theta}, \lambda^{\star}, \mu^{\star}) - \hat{L}(f_{\theta}, \lambda^{\star}, \mu^{\star}) \le (1 + \|\lambda^{\star}\|_{1} + \|\mu^{\star}\|_{1})\zeta^{\text{UC}}(N_{\min}, \delta).$$
 (54)

Chaining (51) and (54) and noticing that  $\|\gamma^\star\|_1 = \|\lambda^\star\|_1 + \|\mu^\star\|_1$ , we obtain that  $D^\star - \hat{D}^\star \leq (1 + \|\gamma^\star\|_1)\zeta^{\mathrm{UC}}(N_{\min}, \delta) + \epsilon$ . Notice that we can eliminate  $\epsilon$  since none of the other terms depend on  $\epsilon$ , and so we can take the infimum on both sides. Therefore,

$$D^* - \hat{D}^* < (1 + \|\gamma^*\|_1) \zeta^{\text{UC}}(N, \delta),$$
 (55)

and this holds with probability greater than  $1 - (1 + I + J)\delta$ , since it is a consequence of (53) and its analogues for  $\mathbb{P}_i$ ,  $\mathbb{Q}_j$ .

We can make a symmetric argument, switching the roles of  $\gamma^*$  and  $\hat{\gamma}^*$ , to prove that

$$\hat{D}^{\star} - D^{\star} \le (1 + \|\hat{\gamma}^{\star}\|_{1}) \zeta^{\text{UC}}(N_{\min}, \delta). \tag{56}$$

This time we pick some  $\theta$  that is  $\epsilon$ -optimal with respect to  $q(\hat{\gamma}^\star)$  (instead of  $\hat{q}(\gamma^\star)$ ). Since (53) holds for any  $\theta$  (with probability greater than  $1-(1+I+J)\delta$ ) therefore Equations (55) and (56) hold simultaneously with probability greater than the same bound,  $1-(1+I+J)\delta$ . Therefore, they together imply that  $|D^\star - \hat{D}^\star| \leq (1+\max\{\|\gamma^\star\|_1,\|\hat{\gamma}^\star\|_1\})\zeta^{\mathrm{UC}}(N_{\min},\delta)$ , which completes the proof.

#### B.5 Proof of Theorem 3.1

**Theorem 3.1.** Let  $N_{min} = \min \{M_0, M_1, \dots, M_I, N_1, \dots, N_J\}$  and assume that  $R(\nu) < \infty$  and  $P_{\phi}^{\star} > -\infty$ . Under Assumptions 1-3 there exist  $\gamma_{\phi}^{\star} \in Opt(D_{\phi}), \gamma^{\star} \in Opt(D)$ , and  $\hat{\gamma}^{\star} \in Opt(\hat{D})$ . Moreover, for any  $\delta \in (0,1)$ , it holds with probability at least  $1 - (1 + I + J)\delta$ , that

$$|P^{\star} - \hat{D}^{\star}| \le \left(1 + \|\gamma_{\phi}^{\star}\|_{1}\right) L\nu + LL_{\theta}R(\nu) + \left(1 + \max\left\{\|\gamma^{\star}\|_{1}, \|\hat{\gamma}^{\star}\|\right\}_{1}\right) \zeta^{UC}(N_{min}, \delta).$$
 (8)

*Proof.* Note that the requirements for Proposition B.2 and B.3 are included in the premise. The triangle inequality implies that,

$$|P^* - \hat{D}^*| \le |P^* - D^*| + |D^* - \hat{D}^*|.$$

Proposition B.2 asserts that  $\operatorname{Opt}(D_{\phi}) \neq \emptyset$ , and provides an upper bound for  $|P^{\star} - D^{\star}| = P^{\star} - D^{\star}$ , which always holds. Proposition B.3 asserts that  $\operatorname{Opt}(D) \neq \emptyset$ ,  $\operatorname{Opt}(\hat{D}) \neq \emptyset$ , and provides an upper bound for  $|D^{\star} - \hat{D}^{\star}|$  that holds with probability greater than  $1 - (1 + I + J)\delta$ .

Adding both bounds gives us the necessary bound on  $|P^* - \hat{D}^*|$ , which holds in probability greater than  $1 - (1 + I + J)\delta$ .

We next discuss technical distinctions between Theorem 3.1 and [23, Theorem 1] (which pertains to problems where J=0) that were omitted in the main text (see also Remark 3.1).

**Remark B.5.** Firstly, we lift the assumption that  $\Theta$  is compact. Secondly, we prove the case for regression problems ( $\mathcal{Y}$  continuous) without additional assumptions (see [23, Assumption 6]). This is due to a (small) modification of the argument for Lemma B.5, which yields both results together. Note that [57] also provided a unified proof for functional strong duality, however, their proof takes a slightly different approach using the weak Lyapunov theorem [54, Chapter IX Theorem 10].

Finally, note that while both bounds feature the norm of the Lagrange multiplier of a functional problem, the Lagrange multiplier in [23] depends on  $\nu$ , while ours does not. This is because we have chosen to treat equalities and inequalities symmetrically (see Remark B.4 for a justification). Therefore, both  $\|\gamma_{\phi}^{\star}\|_1$  and  $R(\nu)$  depend on both inequalities and equalities. However, the proof can be repeated with a slightly different definition of  $R(\nu)$  to obtain two separate "sensitivity" terms for inequalities and equalities, with the former term depending on a Lagrange multiplier like [23], and the latter term depending on the modified  $R(\nu)$ .

In the next remark we expand upon the sensitivity interpretation of  $\|\gamma^*\|_1$  when (P) is non-convex and not strongly dual, as is generally the case in our setting.

**Remark B.6.** If (P) is strongly dual, then the perturbation function  $P^*(\mathbf{c}, \mathbf{d})$  is convex (see [58, Section 5.6.1]). In fact, its epigraph is nothing but the set we called the "cost-constraint epigraph", corresponding to (P), i.e.,

$$\mathcal{M} = \left\{ (f, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^I \times \mathbb{R}^J \middle| \begin{array}{l} \exists \theta \in \Theta \quad \textit{s.t.} \quad \mathbb{E}_{\mathbb{D}} \left[ \ell(f_{\theta}(\mathbf{x}), y) \right] = f, \\ \qquad \qquad \mathbb{E}_{\mathbb{P}_i} \left[ g_i(f_{\theta}(\mathbf{x}), y) \right] \leq u_i, \quad \textit{for } i = 1, \dots, I, \\ \textit{and} \quad \mathbb{E}_{\mathbb{Q}_j} \left[ h_j(f_{\theta}(\mathbf{x}), y) \right] = v_j \quad \textit{for } j = 1, \dots, J \end{array} \right\}.$$

If the perturbation function were additionally differentiable, then (9) holds. More generally, without differentiability,  $-(\lambda^*, \mu^*)$  is a subgradient of  $P^*(\mathbf{c}, \mathbf{d})$  at  $(0^I, 0^J)$  (see [15, Example 5.4.2]), i.e,

$$P^{\star}(\mathbf{c}, \mathbf{d}) - P^{\star} \ge -\lambda^{\star \top}(\mathbf{c} - 0^{I}) - \mu^{\star \top}(\mathbf{d} - 0^{J}). \tag{57}$$

We now derive an approximate version of (57) from Proposition B.2 as follows:

$$P^{\star}(\mathbf{c}, \mathbf{d}) - P^{\star} \ge D^{\star}(\mathbf{c}, \mathbf{d}) - P^{\star} \tag{58}$$

$$\geq D^{\star}(\mathbf{c}, \mathbf{d}) - D^{\star} - \left[ (1 + \left\| \gamma_{\phi}^{\star} \right\|_{1}) L \nu + L L_{\theta} R(\nu) \right]$$
 (59)

Here we have used the weak duality relation for  $P^*(\mathbf{c}, \mathbf{d})$  in (58) and Proposition (B.2) in (59). We can bound the gap  $D^*(\mathbf{c}, \mathbf{d}) - D^*$  in an identical manner as Lemma (B.4) to obtain,

$$D^{\star}(\mathbf{c}, \mathbf{d}) - D^{\star} \ge -\lambda^{\star \top} (\mathbf{c} - 0^{I}) - \mu^{\star \top} (\mathbf{d} - 0^{J}). \tag{60}$$

Combining (59) and (60), we obtain an approximate subgradient relation similar to (57). Explicitly,

$$P^{\star}(\mathbf{c}, \mathbf{d}) - P^{\star} \ge -\lambda^{\star \top} \mathbf{c} - \mu^{\star \top} \mathbf{d} + G(\nu), \tag{61}$$

where  $G(\nu) = -\left[ (1 + \left\| \gamma_{\phi}^{\star} \right\|_{1}) L \nu + L L_{\theta} R(\nu) \right]$  is the remainder term that disappears as  $\nu \to 0$ . Equation 61 thus enables us to interpret  $\lambda^{\star}$  as constraint sensitivity even in the non-convex case, albeit approximately.

## C Proof of Theorem 4.1

In this section we will provide guarantees for Algorithm 1, which approximately solves (D). We start by defining the empirical dual function in terms of the empirical Lagrangian (2),

$$\hat{q}(\lambda,\mu) = \inf_{\theta \in \Theta} \hat{L}(f_{\theta},\lambda,\mu).$$

The empirical dual value can be written as  $\hat{D}^* = \sup_{\lambda \in \mathbb{R}^I_+, \, \mu \in \mathbb{R}^J} \hat{q}(\lambda, \mu)$ . Let  $\gamma$  denote the tuple  $(\lambda, \mu)$  in the sequel, and similarly for  $\gamma^*, \hat{\gamma}^*, \gamma^{(t-1)}$  etc. We can prove that the set of supergradients for  $\hat{q}(\gamma)$  at  $\gamma$  is equal to the convex hull conv  $\left\{ \left. \hat{C}(\theta) \mid \theta \in \Theta, L(f_\theta, \gamma) = \hat{q}(\gamma) \right\} \right\}$  (e.g. with [59, Theorem 2.87]). We will now show that approximate minimizers produce approximate supergradients. Recall that  $\hat{F}$  is the cost-constraint vector of the empirical problem.

**Lemma C.1.** Let  $\theta^{(t)}$  satisfy  $\hat{q}(\gamma^{(t)}) \leq \hat{L}(f_{\theta}, \gamma^{(t)}) \leq \hat{q}(\gamma^{(t)}) + \rho$ . Then, it follows that, for all  $\gamma' \in \mathbb{R}^{I}_{\geq 0} \times \mathbb{R}^{J}$ 

$$\hat{q}(\gamma^{(t)}) \ge \hat{q}(\gamma') + \sum_{k=1}^{K} (\gamma_k^{(t)} - \gamma_k') \hat{F}_k(\theta^{(t)}) - \rho.$$

Proof. From the premise of the lemma we have that,

$$\hat{q}(\gamma^{(t)}) \ge \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) - \rho. \tag{62}$$

By the definition of the dual function, we have for any  $\gamma' \in \mathbb{R}^{I}_{\geq 0} \times \mathbb{R}^{J}$ ,

$$\hat{q}(\gamma') \le \hat{L}(f_{\theta^{(t)}}, \gamma'),\tag{63}$$

since  $\hat{q}(\gamma') = \inf_{\theta \in \Theta} \hat{L}(f_{\theta}, \gamma')$ . Then,

$$(62) - (63) \Rightarrow \hat{q}(\gamma^{(t)}) - \hat{q}(\gamma') \ge \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) - \hat{L}(f_{\theta^{(t)}}, \gamma') - \rho.$$

If we expand  $\hat{L}(f_{\theta^{(t)}}, \gamma^{(t)})$  and  $\hat{L}(f_{\theta^{(t)}}, \gamma')$  in terms of the cost-constraint vector  $\hat{F}$  and the dual variables  $\gamma^{(t)}, \gamma'$ , the terms correspond to the objective vanish, and the remainder is  $\sum_{k=1}^K (\gamma_k^{(t)} - \gamma_k') \hat{F}_k(\theta^{(t)})$ . Bringing  $\hat{q}(\gamma')$  to the RHS completes the proof.

Subgradient ascent algorithms that use *any* subderivative are not guaranteed to descend monotonically in the objective, in this case the dual function  $\hat{q}$ . This is also true of approximate subgradient descent. Using Lemma C.1 we can prove a recurrence which helps prove descent with respect to the *mean* squared error in the dual variable, which we define as ,

$$U_t = \inf_{\hat{\gamma}^{\star} \in \text{Opt}(\hat{\Omega})} \left\| \gamma^{(t)} - \hat{\gamma}^{\star} \right\|_2^2, \tag{64}$$

where  $\operatorname{Opt}(\hat{\mathbf{D}})$  is the set of optimal dual variables for  $(\hat{\mathbf{D}})$ . We will also need a technical lemma first. **Lemma C.2.** Define  $[\cdot]_+: \mathbb{R}^m \to \mathbb{R}^m_{\geq 0}$  such that  $[\mathbf{x}]_+ = \max(\mathbf{x}, \mathbf{0})$ , where the maximum is taken componentwise. Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ , then it follows that,

$$\|[\mathbf{x}]_{+} - [\mathbf{y}]_{+}\|_{2} \le \|\mathbf{x} - \mathbf{y}\|_{2}$$
.

*Proof.* Since  $\|[\mathbf{x}]_+ - [\mathbf{y}]_+\|_2^2 = \sum_{i=1}^m (\max(x_i, 0) - \max(y_i, 0))^2$ , therefore we only need to prove,

$$|\max(x_i, 0) - \max(y_i, 0)| \le |x_i - y_i|.$$
 (65)

If  $x_i > 0$ ,  $y_i > 0$ , both sides are equal. If  $x_i < 0$ ,  $y_i < 0$ , the LHS evaluates to 0 so (65) holds true again. If  $x_i < 0$ ,  $y_i > 0$ , the LHS is equal to  $|y_i|$  and the RHS is equal to  $|x_i| + |y_i|$  (trivial) so (65) holds true again. The last case follows by symmetry. This proves (65), which proves the lemma.  $\Box$ 

**Lemma C.3.** Let  $\theta^{(t)}, \gamma^{(t)} = (\lambda^{(t)}, \mu^{(t)})$  be the  $t^{th}$  iterate of Algorithm 1. If Assumptions 1 and 5 hold, and the loss functions  $\ell, g_i, h_j$  are B-bounded, it follows that,

$$U_t \le U_{t-1} + 2\eta \left\{ \hat{q}(\gamma^{(t-1)}) - \hat{D}^* + \rho \right\} + \eta^2 (I+J)B^2.$$

*Proof.* Let  $\hat{C}_I(\theta) = \left[\hat{C}_1(\theta), \dots, \hat{C}_I(\theta)\right]^{\top}$  and  $\hat{C}_J(\theta) = \left[\hat{C}_{I+1}(\theta), \dots, \hat{C}_{I+J}(\theta)\right]^{\top}$  denote the concatenated empirical constraint functions for the inequalities and equalities respectively. Clearly the constraint vector  $\hat{C}(\theta)$  is the concatenation of both. Since Assumption 1 holds and the loss functions are bounded, therefore  $\operatorname{Opt}(D_{\phi}) \neq \emptyset$  (Theorem B.1).

Decompose  $\gamma^{(t)} = (\lambda^{(t)}, \mu^{(t)})$ , then it holds that,

$$\|\gamma^{(t)} - \hat{\gamma}^{\star}\|_{2}^{2} = \|\lambda^{(t)} - \hat{\lambda}^{\star}\|_{2}^{2} + \|\mu^{(t)} - \hat{\mu}^{\star}\|_{2}^{2}. \tag{66}$$

Using the update rules for the dual variables, we obtain,

$$(66) \Rightarrow \left\| \gamma^{(t)} - \hat{\gamma}^{\star} \right\|_{2}^{2} = \left\| \left[ \lambda^{(t-1)} + \eta \hat{C}_{I}(\theta^{(t-1)}) \right]_{+} - \hat{\lambda}^{\star} \right\|_{2}^{2} + \left\| \mu^{(t-1)} + \eta \hat{C}_{J}(\theta^{(t-1)}) - \hat{\mu}^{\star} \right\|_{2}^{2}.$$

$$(67)$$

where  $[\mathbf{x}]_+$  applies  $\max(\cdot, 0)$  to each component of  $\mathbf{x}$ . We can write  $\hat{\lambda}^* = \left[\hat{\lambda}^*\right]_+$  since  $\hat{\lambda}^*$  is non-negative by definition. By Lemma C.2 we know that,

$$\left\| \left[ \lambda^{(t-1)} + \eta \hat{C}_I(\theta^{(t-1)}) \right]_+ - \left[ \hat{\lambda}^* \right]_+ \right\|_2 \le \left\| \lambda^{(t-1)} + \eta \hat{C}_I(\theta^{(t-1)}) - \hat{\lambda}^* \right\|_2. \tag{68}$$

Applying (68) to (67), we obtain,

$$\left\| \gamma^{(t)} - \hat{\gamma}^{\star} \right\|_{2}^{2} \leq \left\| \lambda^{(t-1)} + \eta \hat{C}_{I}(\theta^{(t-1)}) - \hat{\lambda}^{\star} \right\|_{2}^{2} + \left\| \mu^{(t-1)} + \eta \hat{C}_{J}(\theta^{(t-1)}) - \hat{\mu}^{\star} \right\|_{2}^{2}$$

$$= \left\| \gamma^{(t-1)} + \eta \hat{C}(\theta^{(t-1)}) - \hat{\gamma}^{\star} \right\|_{2}^{2}, \tag{69}$$

where in the last step we have recombined inequalities and equalities, since the rest of the proof is not affected by the difference. Expanding the squared norm on the RHS of (69), we obtain,

$$\left\| \gamma^{(t)} - \hat{\gamma}^{\star} \right\|_{2}^{2} \le \left\| \gamma^{(t-1)} - \hat{\gamma}^{\star} \right\|_{2}^{2} + 2\eta (\gamma^{(t-1)} - \hat{\gamma}^{\star})^{\top} \hat{C}(\theta^{(t-1)}) + \eta^{2} \left\| \hat{C}(\theta^{(t-1)}) \right\|_{2}^{2}. \tag{70}$$

Since Assumption 5 holds, therefore we can apply Lemma C.1, which implies that

$$(\gamma^{(t-1)} - \hat{\gamma}^{\star})^{\top} \hat{C}(\theta^{(t-1)}) \le \hat{q}(\gamma^{(t-1)}) - \hat{q}(\hat{\gamma}^{\star}) + \rho = \hat{q}(\gamma^{(t-1)}) - \hat{D}^{\star} + \rho.$$
 (71)

And from the boundedness of the loss functions we have that,

$$\left\| \hat{C}(\theta^{(t-1)}) \right\|_{2}^{2} \le (I+J)B^{2}.$$
 (72)

Applying (71) and (72) to (70) we obtain,

$$\left\| \gamma^{(t)} - \hat{\gamma}^{\star} \right\|_{2}^{2} \leq \left\| \gamma^{(t-1)} - \hat{\gamma}^{\star} \right\|_{2}^{2} + 2\eta (\hat{q}(\gamma^{(t-1)}) - \hat{D}^{\star} + \rho) + \eta^{2}(I + J)B^{2}.$$

Applying the operation  $\inf_{\hat{\gamma}^\star \in \operatorname{Opt}(\hat{\mathbb{D}})}$  to both sides preserves the inequality. Notice that only the LHS and the *first term* in the RHS depends on  $\hat{\gamma}^\star$ . Taking the other terms out of the infimum and applying the definition of  $U_t$  gives us the required bound.

Although we cannot directly show that the objective  $\hat{q}(\gamma^{(t)})$  monotonically increases, we can show that by picking the learning rate appropriately, the initial phase of the algorithm monotonically reduces the mean squared error  $(U_t)$  of the dual iterates, upto the point where  $\hat{q}(\gamma^{(t)})$  is some  $\alpha\rho$  close to  $\hat{D}^{\star}$ , for some  $\alpha > 1$ .

**Lemma C.4.** Let  $\theta^{(t)}, \gamma^{(t)} = (\lambda^{(t)}, \mu^{(t)})$  be the  $t^{th}$  iterate of Algorithm 1. Let Assumptions 1 and 5 hold, and let the loss functions  $\ell, g_i, h_j$  be B-bounded. Define  $T_0 = \min\left\{t \in \mathbb{N} \mid \hat{D}^\star - \hat{q}(\gamma^{(t)}) < \alpha\rho\right\} > 0$  for some  $\alpha > 1$  and let  $\eta \leq \frac{\alpha\rho}{(I+J)B^2}$ . Then it holds for all  $t \leq T_0$  that

$$U_t < U_{t-1}$$
.

Moreover,  $T_0 \leq T_{max} = \frac{U_0}{2\eta\rho}$ .

*Proof.* Recall from Lemma C.3 that the following holds for arbitrary t.

$$U_t \le U_{t-1} + 2\eta \left\{ \hat{q}(\gamma^{(t-1)}) - \hat{D}^* + \rho \right\} + \eta^2 (I+J)B^2.$$
 (73)

Assume  $t \leq T_0$ , then the premise states that  $\hat{D}^\star - \hat{q}(\gamma^{(t-1)}) > \alpha \rho$ , i.e.  $\hat{q}(\gamma^{(t-1)}) - \hat{D}^\star + \rho < (1-\alpha)\rho$ . Moreover  $\eta \leq \frac{\alpha \rho}{(I+J)B^2} \Rightarrow \eta(I+J)B^2 \leq \alpha \rho$ . Applying these bounds to (73) gives us,

$$U_t < U_{t-1} - 2(\alpha - 1)\eta\rho + 2\eta\alpha\rho$$
  

$$\Rightarrow U_t < U_{t-1} - 2\eta\rho.$$
(74)

(74) proves that  $U_t < U_{t-1}$  since  $-2\eta \rho < 0$ .

Now, to achieve an upper bound for  $T_0$ , apply (74) recursively to obtain,

$$U_{T_0} < U_0 - 2\eta \rho T_0. (75)$$

Since  $U_{T_0} \ge 0$ , therefore (75) implies that,

$$U_0 - 2\eta \beta T_0 > 0 \Rightarrow T_0 < \frac{U_0}{2\eta \rho}.$$

Note that the proof requires the extra margin  $(\alpha-1)\rho$  to show descent. Once  $\hat{q}(\gamma^{(t-1)}) - \hat{D}^* \leq \rho$ , (73) cannot prove descent because both terms are non-negative. Note the dependence of  $T_{\max}$  on  $\eta$  and  $\rho$ . While interpreting  $T_0$  and  $T_{\max}$ , it is important to note that we cannot detect  $T_0$  since we have neither  $\hat{D}^*$  nor  $\hat{q}$  (since our oracle is approximate). However, we can compute  $T_{\max}$ , which is simply an upper bound on  $T_0$ . Proposition C.4 does not tell us what happens between  $T_0$  and  $T_{\max}$ , in particular it does not tell us if our iterates become worse, either with respect to the mean squared error or dual function. While giving a guarantee for the *last iterate* (at  $T=T_{\max}$ ) is challenging, we can give a guarantee for the average of the Lagrangian iterates. The following can also be written as a randomized result, such as in [23].

**Theorem 4.1.** Suppose Assumptions 1 and 5 hold and let the loss functions  $\ell$ ,  $g_i$ ,  $h_j$  be B-bounded. Let  $U_0 = \inf_{\gamma^* \in Opt(\hat{D})} \|\gamma^{(0)} - \gamma^*\|$ . Then, for any  $T \in \mathbb{N}$ , it holds that,

$$\left| \hat{D}^* - \frac{1}{T} \sum_{t=0}^{T-1} \left( \hat{L}(\theta^{(t)}, \gamma^{(t)}) \right) \right| \le \rho + \frac{U_0}{2\eta T} + \frac{1}{2} (I+J)\eta B^2.$$
 (10)

If  $\eta \leq \frac{\rho}{(I+J)B^2}$  and  $T \geq \frac{U_0}{\eta \rho}$ , then the bound is equal to  $2\rho$ .

*Proof.* We know, by the definition of the oracle and the order of updates that, for any t,

$$\hat{q}(\gamma^{(t)}) \le \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) \le \hat{q}(\gamma^{(t)}) + \rho. \tag{76}$$

Since  $\hat{q}(\gamma^{(t)}) \leq \hat{D}^*$ , therefore (76) implies that  $\hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) - \hat{D}^* \leq \rho$  for all t, therefore

$$\frac{1}{T} \sum_{t=0}^{T-1} \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) - \hat{D}^* \le \rho, \tag{77}$$

which proves one part of the bound, since  $\left(\frac{U_0}{2\eta T} + \frac{1}{2}(I+J)\eta B^2\right)$  is non-negative. On the other hand, since  $\hat{q}(\gamma^{(t)}) \leq \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)})$ , therefore,

$$\frac{1}{T} \sum_{t=0}^{T-1} \left( \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) - \hat{D}^* \right) \ge \frac{1}{T} \sum_{t=0}^{T-1} \left( \hat{q}(\gamma^{(t)}) - \hat{D}^* \right). \tag{78}$$

We will now derive an upper bound for  $\sum_{t=0}^{T-1} \left( \hat{q}(\gamma^{(t)}) - \hat{D}^{\star} \right)$  by applying Lemma C.3 recursively T times to obtain,

$$U_T \le U_0 + 2\eta \sum_{t=0}^{T-1} \left( \hat{q}(\gamma^{(t)}) - \hat{D}^* \right) + T(2\eta\rho + \eta^2(I+J)B^2). \tag{79}$$

Now, since  $U_T \ge 0$ , dividing by  $2\eta T$  and rearranging the terms, we get the following inequality,

$$\frac{1}{T} \sum_{t=0}^{T-1} \left( \hat{q}(\gamma^{(t)}) - \hat{D}^* \right) \ge -\frac{U_0}{2\eta T} - \rho - \frac{1}{2} (I+J)\eta B^2.$$

Multiply the above equation by -1 and applying (78) we obtain the required upper bound,

$$\hat{D}^{\star} - \frac{1}{T} \sum_{t=0}^{T-1} \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) \le \rho + \frac{U_0}{2\eta T} + \frac{1}{2}(I+J)\eta B^2.$$
 (80)

Now since  $\eta \leq \frac{2\alpha_{\eta}\rho}{(I+J)B^2} \Rightarrow \eta(I+J)B^2 \leq 2\alpha_{\eta}\rho$ , and  $T \geq \frac{U_0\beta_T}{2\eta\rho} \Rightarrow \frac{U_0}{2\eta T} \leq \frac{\rho}{\beta_T}$ , therefore, (80) reduces to,

$$\hat{D}^* - \frac{1}{T} \sum_{t=0}^{T-1} \hat{L}(f_{\theta^{(t)}}, \gamma^{(t)}) \le \rho (1 + \frac{1}{\beta_T} + \alpha_\eta),$$

which proves the remainder.

# **D** Additional theory

### D.1 Example: minimum norm interpolation

Let  $X \in \mathbb{R}^{n \times n}$  and  $w, y \in \mathbb{R}^n$ . Consider the following problem, that finds the minimum norm solution to a system of linear equalities,

$$P_e^{\star} = \inf_{w \in \mathbb{R}^n} \quad \frac{1}{2} \|w\|_2^2$$
subject to  $Xw = y$ . (P<sub>e</sub>)

The problem  $(P_e)$  can be transformed to a problem with a single inequality constraint, by constraining the aggregated violation of the equalities, namely,

$$\begin{split} P_i^{\star} &= \inf_{w \in \mathbb{R}^n} \quad \frac{1}{2} \|w\|_2^2 \\ \text{subject to} \quad \frac{1}{2} \|Xw - y\|_2^2 &\leq \epsilon. \end{split} \tag{$\mathbf{P}_i$}$$

The feasibility and optimality sets of  $(P_i)$  coincide with  $(P_e)$  when  $\epsilon=0$ . However, the dual problems of both problems are different. In particular, the dual solution of  $(P_i)$  diverges as  $\epsilon\to 0$  and the dual problem of  $(P_i)$  becomes ill-posed. However, the dual problem for  $(P_e)$  does not depend on  $\epsilon$ , and is well posed. Note that this also implies that recovering a good approximation of the solution to  $(P_e)$  using an unconstrained objective  $\|w\|_2^2 + \lambda \|Xw - y\|_2^2$  would need a very large weight  $\lambda$ .

We will now state and prove these facts. We need the following standard identities concerning the derivatives of matrix forms (which can be found e.g. in [60] Equations 69 and 81).

**Lemma D.1.** Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix and  $x, b \in \mathbb{R}^n$ . Then, the following identities hold true.

1. 
$$\nabla_x(x^{\top}Ax) = 2Ax$$
.

2. 
$$\nabla_x(x^\top b) = \nabla_x(b^\top x) = b$$
.

We make the following simplifying assumptions.

**Assumption 6.** Assume that  $XX^{\top}$  is a full rank matrix and there exists  $w_0 \in \mathbb{R}^n$  such that  $Xw_0 = y$  and  $w_0 \neq 0$ .

We will now characterize the dual solution of  $(P_i)$ .

**Proposition D.1.** Let Assumption 6 hold. Then there always exists a dual solution of  $(P_i)$ , say  $\lambda^* \geq 0$ . Moreover, as  $\epsilon \to 0$ ,  $\lambda^* \to \infty$ .

*Proof.* Consider the constraint function,

$$g(w) = \frac{1}{2} \|Xw - y\|_2^2$$
  
=  $\frac{1}{2} [Xw - y]^\top [Xw - y]$   
=  $\frac{1}{2} (w^\top X^\top Xw - 2w^\top X^\top y + \|y\|_2^2).$ 

Existence of dual solution and strong duality. It is clear that g(w) is quadratic in w. Since its second derivative is the gram matrix  $XX^{\top}$  (Lemma D.1) which is positive semi-definite [61, Theorem 4.6.6.], therefore g(w) is convex [55, Theorem 4.5]. The existence of  $w_0$  such that  $Xw_0 = y$  further implies that a strictly feasible point  $(w_0)$  exists. Since the objective is also convex, therefore the set of dual solutions is non-empty and strong duality holds [15, Proposition 5.3.1].

**Existence of primal solution**. Since Feas(P<sub>i</sub>) is non-empty, and the objective is bounded from below by 0, therefore  $-\infty < P_i^* < \infty$ . Therefore the following modification of (P<sub>i</sub>) is well defined, for any  $\gamma > 0$ ,

$$P_{i2}^{\star} = \inf_{w \in \mathbb{R}^n} \quad \frac{1}{2} \|w\|_2^2$$
subject to 
$$\frac{1}{2} \|Xw - y\|_2^2 \le \epsilon$$

$$\frac{1}{2} \|w\|_2^2 \le P_i^{\star} + \gamma.$$
(P<sub>i2</sub>)

Clearly Feas(P<sub>i2</sub>)  $\subseteq$  Feas(P<sub>i</sub>), therefore  $P_i^{\star} \leq P_{i2}^{\star}$ . Note that since  $P_i^{\star} = \inf_{w \in \text{Feas}(P_i)} \frac{1}{2} \|w\|_2^2$ , therefore for any  $\delta > 0$ , there exists  $w \in \text{Feas}(P_i)$  such that,

$$\frac{1}{2} \|w\|_2^2 \le P_i^* + \delta. \tag{81}$$

Let  $\{\delta_j\}_{j=1}^{\infty} \subseteq \mathbb{R}$  be the sequence  $\delta_j = \frac{\gamma}{j}$ . Using (81) by setting  $\delta = \delta_j$ , we can construct a sequence  $\{w_i\}$  that satisfies,

$$w_j \in \operatorname{Feas}(\mathsf{P}_i) \text{ and } \frac{1}{2} \left\| w_j \right\|_2^2 \leq P_i^\star + \delta_j.$$

Clearly  $\{w_j\} \subseteq \operatorname{Feas}(\mathsf{P}_{i2})$  since  $\delta_1 = \gamma$  and  $\delta_j$  is strictly decreasing. Since  $\delta_j \to 0$ , therefore  $\frac{1}{2} \|w_j\|_2^2 \to P_i^{\star}$ . Therefore  $P_{i2}^{\star} = P_i^{\star}$ .

Now, since g(w) is convex, and clearly continuous, therefore its sub-level set  $g(w) \leq \epsilon$  is closed [55, Theorem 7.1]. Clearly, the set  $\left\{w\mid \frac{1}{2} \|w\|_2^2 \leq P_i^\star + \delta\right\}$  is nothing but the norm ball  $B(0,\sqrt{2P_i^\star + \delta})$ , and is closed and compact. Therefore Feas $(P_{i2})$  is compact. Setting  $\delta = \gamma$  in (81) proves that Feas $(P_{i2})$  is non-empty as well. Since the objective,  $\frac{1}{2} \|w\|_2^2$  is continuous, the compact non-emptiness of the feasibility set implies that  $\mathrm{Opt}(P_{i2}) \neq \emptyset$  (see [52, Theorem 4.1.6]).

Clearly, since  $Opt(P_{i2}) \subseteq Feas(P_i)$  and  $P_{i2}^{\star} = P_i^{\star}$ , therefore  $Opt(P_{i2}) \subseteq Opt(P_i)$ . Therefore  $Opt(P_i) \neq \emptyset$ .

**Diverging dual solution.** Now that we have established that the primal and dual solutions exist for all  $\epsilon > 0$ , let  $\lambda_e^{\star}, w_e^{\star}$  be a primal dual solution pair. Since strong duality holds, therefore they must satisfy the KKT conditions (see [58][Section 5.5.3]).

The Lagrangian function associated with  $(P_i)$  is as follows,

$$\begin{split} L(w^{\star}, \lambda) &= \frac{1}{2} \left\| w^{\star} \right\|_{2}^{2} + \lambda^{\star} \left( \frac{1}{2} \left\| X w^{\star} - y \right\|_{2}^{2} - \epsilon \right) \\ &= \frac{1}{2} w^{\star \top} w^{\star} + \frac{\lambda^{\star}}{2} \left( w^{\star \top} X^{\top} X w^{\star} - 2 w^{\star \top} X^{\top} y + \left\| y \right\|_{2}^{2} - 2\epsilon \right) \\ &= \frac{1}{2} w^{\star \top} \left[ I + \lambda^{\star} X^{\top} X \right] w^{\star} - \lambda^{\star} w^{\star \top} X^{\top} y + \frac{\lambda^{\star}}{2} \left( \left\| y \right\|_{2}^{2} - 2\epsilon \right). \end{split}$$

The KKT condition for the stationarity of the Lagrangian implies that,

$$\nabla_{w^{\star}} L(w^{\star}, \lambda^{\star}) = 0$$

$$\Rightarrow \nabla_{w^{\star}} \left( \frac{1}{2} w^{\star \top} \left[ I + \lambda^{\star} X^{\top} X \right] w^{\star} - \lambda^{\star} w^{\star \top} X^{\top} y \right) = 0$$

$$\Rightarrow \left[ I + \lambda^{\star} X^{\top} X \right] w^{\star} - \lambda^{\star} X^{\top} y = 0$$

$$\Rightarrow w^{\star} = \lambda^{\star} \left[ I + \lambda^{\star} X^{\top} X \right]^{-1} X^{\top} y$$

$$= \left[ cI + X^{\top} X \right]^{-1} X^{\top} y$$

where we have defined  $c = \frac{1}{\lambda^*}$ .

Since  $X^{\top}X$  is symmetric, we can assume the following eigen value decomposition,

$$X^{\top}X = V\Lambda V^{\top}.$$

where  $V \in \mathbb{R}^{n \times n}$  is an orthonormal matrix and  $\Lambda \in \mathbb{R}^{n \times n}$  is a diagonal matrix with *positive* entries (since  $XX^{\top}$  is positive semi definite and full rank). Now we will rewrite the constraint function evaluated at  $w^{\star}$ .

$$\begin{split} \|Xw^{\star} - y\|_{2}^{2} &= \left\|X\left[cI + X^{\top}X\right]^{-1}X^{\top}y - y\right\|_{2}^{2} \\ &= \left\|X\left[cI + X^{\top}X\right]^{-1}X^{\top}Xw_{0} - Xw_{0}\right\|_{2}^{2} \\ &= \left\|X\left[\left[cI + X^{\top}X\right]^{-1}X^{\top}X - I\right]w_{0}\right\|_{2}^{2} \\ &= \left\|X\left[\left[cVV^{\top} + V\Lambda V^{\top}\right]^{-1}V\Lambda V^{\top} - I\right]w_{0}\right\|_{2}^{2} \\ &= \left\|X\left[V^{-\top}\left[cI + \Lambda\right]^{-1}V^{-1}V\Lambda V^{\top} - I\right]w_{0}\right\|_{2}^{2} \\ &= \left\|X\left[V\left[cI + \Lambda\right]^{-1}\Lambda V^{\top} - I\right]w_{0}\right\|_{2}^{2} \\ &= \left\|X\left[V\left[cI + \Lambda\right]^{-1}\Lambda V^{\top} - I\right]^{\top}X^{\top}X\left[V\left[cI + \Lambda\right]^{-1}\Lambda V^{\top} - I\right]w_{0} \\ &= w_{0}^{\top}\left[V\left[cI + \Lambda\right]^{-1}\Lambda V^{\top} - VV^{\top}\right]^{\top}V\Lambda V^{\top}\left[V\left[cI + \Lambda\right]^{-1}\Lambda V^{\top} - VV^{\top}\right]w_{0} \\ &= w_{0}^{\top}V\left[\left[cI + \Lambda\right]^{-1}\Lambda - I\right]^{\top}V^{\top}V\Lambda V^{\top}V\left[\left[cI + \Lambda\right]^{-1}\Lambda - I\right]V^{\top}w_{0} \\ &= w_{0}^{\top}V\left[\left[cI + \Lambda\right]^{-1}\Lambda - I\right]^{\top}\Lambda\left[\left[cI + \Lambda\right]^{-1}\Lambda - I\right]V^{\top}w_{0} \\ &= q^{\top}Bq, \end{split}$$

where,

$$B = \left[ \left[ cI + \Lambda \right]^{-1} \Lambda - I \right]^{\top} \Lambda \left[ \left[ cI + \Lambda \right]^{-1} \Lambda - I \right],$$

which is a diagonal matrix, and  $q = V^{\top} w_0$ . Clearly,

$$q = V^{\top} w_0 \Rightarrow q \neq 0,$$

since  $w_0 \neq 0$ , and  $V^{\top}$  has full rank. Now,

$$B_{ii} = \sigma_i \left( \frac{\sigma_i}{c + \sigma_i} - 1 \right)^2.$$

Therefore,

$$||Xw^* - y||_2^2 = q^\top B q = \sum_{i=1}^n B_{ii} q_i^2$$
$$= \sum_{i=1}^n \left( \frac{\sigma_i}{c + \sigma_i} - 1 \right)^2 \sigma_i q_i^2.$$

Now, the feasibility of the primal solution implies,

$$\sum_{i=1}^{n} \left( \frac{\sigma_i}{c + \sigma_i} - 1 \right)^2 \sigma_i q_i^2 \le \epsilon.$$

Without loss of generality, assume that  $q_i \neq 0$ . Since the LHS is a sum of non-negative terms, therefore.

$$\left(\frac{\sigma_i}{c + \sigma_i} - 1\right)^2 \sigma_i q_i^2 \le \epsilon$$

$$\Rightarrow \frac{c^2}{(c + \sigma_i)^2} \sigma_i q_i^2 \le \epsilon$$

$$\Rightarrow \frac{1}{(1 + \lambda^* \sigma_i)^2} \sigma_i q_i^2 \le \epsilon.$$

Clearly, as  $\epsilon \to 0$ , the LHS must also vanish, which is only possible if  $\lambda^* \to \infty$  since the remaining terms don't depend on  $\epsilon$ .

Next, we will characterize the dual solution of  $(P_e)$ .

**Proposition D.2.** Let Assumption 6 hold. Then,  $\mu = -(XX^{\top})^{-1}y$  is a dual solution of  $(P_e)$ .

*Proof.* We will proceed by explicitly computing the dual function  $q(\mu)$  for  $(P_e)$ . Note that the Lagrangian

$$L(w, \mu) = \frac{1}{2} \|w\|_{2}^{2} + \mu^{\star \top} (Xw - y)$$

is a convex function of w. Therefore, the first order condition is necessary and sufficient to find a global minimiser of  $L(w, \mu)$  with respect to w ([58][Section 3.1.3.]). Proceeding,

$$\nabla_w L(w, \mu) = 0$$

$$\Rightarrow \nabla_w \left\{ \frac{1}{2} \|w\|_2^2 + \mu^\top (Xw - y) \right\} = 0$$

$$\Rightarrow w + X^\top \mu = 0$$

$$\Rightarrow w = -X^\top \mu.$$

Thus,

$$\begin{split} q(\mu) &= \inf_{w \in \mathbb{R}^n} \, L(w, \mu) = L(-X^\top \mu, \mu) \\ &= \frac{1}{2} \, \big\| - X^\top \mu \big\|_2^2 + \mu^\top \left( X(-X^\top \mu) - y \right) \\ &= \frac{1}{2} \, \big\| X^\top \mu \big\|_2^2 - \big\| X^\top \mu \big\|_2^2 - \mu^\top y \\ &= -\frac{1}{2} \, \big\| X^\top \mu \big\|_2^2 - \mu^\top y. \end{split}$$

The first order condition for  $q(\mu)$  must be satisfied by  $\mu^*$ , therefore,

$$\begin{split} & \nabla_{\mu}q(\mu)|_{\mu=\mu^{\star}} = 0 \\ \Rightarrow & \nabla_{\mu} \left\{ \frac{-1}{2} \mu^{\top} X X^{\top} \mu - \mu^{\top} y \right\} \bigg|_{\mu=\mu^{\star}} = 0 \\ \Rightarrow & - X X^{\top} \mu^{\star} - y = 0 \\ \Rightarrow & - X X^{\top} \mu^{\star} - y = 0 \\ \Rightarrow & \mu^{\star} = -(X X^{\top})^{-1} y, \end{split}$$

which completes the proof.

### D.2 Comparing equalities with double sided approximation

Consider the following equality constrained problem,

$$P_0^{\star} = \inf_{x \in \mathcal{X}} \quad \ell(x)$$
subject to  $h_j(x) = 0$ , for  $j = 1, \dots, J$ ,
$$(P_0)$$

and its symmetric relaxation,

$$P_{\epsilon}^{\star} = \inf_{x \in \mathcal{X}} \quad \ell(x)$$
subject to  $-\epsilon \le h_j(x) \le \epsilon$ , for  $j = 1, \dots, J$ .
$$(P_{\epsilon})$$

We analysed the relationship between a pair of similar problems in Appendix B.1.5 and Lemma B.4. The only difference is that we have now dropped the inequality constraints  $g_i(x) \leq 0$  from  $(P_0)$  for the sake of a clearer exposition.

The difference between  $(P_0)$  and  $(P_\epsilon)$  is in the constraints— $(P_\epsilon)$  has twice the number of constraints as  $(P_0)$ , and they are all inequality constraints, as opposed to the equality constraints in  $(P_0)$ . This implies that both problems have different dual variables. Let  $\mu \in \mathbb{R}^J$  denote the dual variable for  $(P_0)$  and let  $\mu_+, \mu_- \in \mathbb{R}^J_{\geq 0}$  denote the dual variables for  $(P_\epsilon)$ , with + corresponding to the upper bound and - corresponding to the lower bound.

Consider the Lagrangian for  $(P_{\epsilon})$ :

$$L_{\epsilon}(x,\mu_{+},\mu_{-}) = \ell(x) + \sum_{j=1}^{J} \mu_{+,j}(h_{j}(x) - \epsilon) + \sum_{j=1}^{J} \mu_{-,j}(-h_{j}(x) - \epsilon)$$

$$= \ell(x) + \sum_{j=1}^{J} (\mu_{+,j} - \mu_{-,j}) h_{j}(x) - \epsilon (\|\mu_{+}\|_{1} + \|\mu_{-}\|_{1}). \tag{82}$$

It is clear from (82) that at  $\epsilon=0$ ,  $L_{\epsilon}(x,\mu_{+},\mu_{-})=L_{0}(x,\mu_{+}-\mu_{-})$ . Moreover, Lemma B.4 proves that when  $\epsilon=0$ ,  $D_{0}^{\star}=D_{\epsilon}^{\star}$ , i.e. the value of the dual problems are equal. Taking the infimum with respect to x in Equation 82, we can obtain,

$$q_{\epsilon}(\mu_{+}, \mu_{-}) = q_{0}(\mu_{+} - \mu_{-}) - \epsilon(\|\mu_{+}\|_{1} + \|\mu_{-}\|_{1}). \tag{83}$$

Clearly  $Opt(D_{\epsilon})$  is not directly equal to  $Opt(D_0)$ , however from (83) it is clear that any solution of  $(D_{\epsilon})$  can be transformed to a solution of  $(D_0)$ , when  $\epsilon = 0$ , by the following mapping,

$$\mu := \mu_+ - \mu_-.$$

Similarly, we can verify (as in the proof of Lemma B.4), that any solution of  $(D_0)$  can be transformed to a solution of  $(D_{\epsilon})$  (when  $\epsilon = 0$ ) by the following mapping,

$$\mu_{+,j} := \max(\mu_j, 0) \text{ and } \mu_{-,j} := \max(-\mu_j, 0).$$

It is easy to verify that if the constraint functions are B-bounded, and the dual variables for  $(P_{\epsilon})$  are initialised with some large positive value,

$$\mu_{+,j}, \mu_{-,j} := B' \gg \eta BT,$$

then the trajectories for the dual variables for both problems under Algorithm 1 are virtually the same. More precisely, if  $\mu$  is the dual variable for  $(P_0)$ , and  $\mu_e = \mu_+ - \mu_-$  is the *effective* dual variable of  $(P_\epsilon)$ , then their trajectories are identical. Theoretically, if  $\mu_+, \mu_-$  are not initialised to a large enough values, the projection by  $\max{(\cdot,0)}$  on line 6 may cause differences in the trajectories. Figure 7 compares the dual trajectories for (P-DP) and its double sided relaxation for a small value of  $\epsilon$ , although all the dual variables were initialised to 0. See also Figure 1(b), where we see that "effective" dual variables of the relaxed problem converge to the dual variables of the equality constrained problem as  $\epsilon \to 0$ .

Note that if the "effective" dual trajectories are identical, then (82) implies that the primal trajectories are identical too, meaning that both methods are virtually the same. Finally, observe that when  $\epsilon > 0$ , (83) suggests that relaxing the constraints by  $\epsilon$  is equivalent to regularizing the dual solution.

# **E** Experiment details

Our implementation was made with pytorch and other standard ML libraries, and our codebase can be found at https://github.com/abarthakur/equality-constrained-learning. All experiments were run on an internal computing resources.

#### E.1 Fairness

**Dataset.** All our experiments on fairness applications were conducted on the COMPAS dataset downloaded from https://github.com/propublica/compas-analysis. Our starting point is the compas-scores-two-years.csv file, which was processed by the following pipeline following [22].

- 1. Remove rows where the attribute days\_b\_screening\_arrest is not in the range [-30, 30].
- Remove all columns except sex, age, race, juv\_misd\_count, juv\_other\_count, priors\_count, c\_charge\_degree and two\_year\_recid.
- Recode the race attribute by clubbing values Asian, Native-American, Other to Other.
- 4. Split the data into train (70%) and test (30%) sets.
- 5. Encode the categorical attribute race with one-hot encoding.
- 6. Encode the binary attributes sex, c\_charge\_degree as (single) binary columns.
- 7. Quantize the numerical attributes juv\_misd\_count, juv\_other\_count, priors\_count using the following bins, and then use one-hot encoding for the resultant categorical variable.
  - (a)  $priors\_count [(0, 0.99), (0.99, 1), (1, 2), (2, 3), (3, 4), (4, 1000)]$
  - (b)  $juv_misd_count [(0, 0.99), (0.99, 1), (1, 1000)]$
  - (c)  $juv_other_count_{-}[(0, 0.99), (0.99, 1), (1, 1000)]$
- 8. Bin the numerical attribute age by quantiles using 5 bins. Use the quantiles of the training set to quantize both training and test sets. Encode the resultant categorical variable using one-hot encoding.
- 9. Remove the binary attribute two\_year\_recid from the features and set it as the target variable y.
- 10. Copy the categorical attribute race from the features, and set it as the protected attribute defining  $G_i$  in (P-DP) and (P-F).

This process was repeated 10 times to produce 10 different train-test splits. After preprocessing and filtering, the full dataset (train and test together) consists of 23 features and 6,172 samples.

**Sigmoidal relaxation.** We replace the indicator functions in (P-DP) and (P-F) with a sigmoid function in order to make the problem tractable (and aligned with our previous results). Explicitly, we use

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{I}\left[f_{\theta}(\mathbf{x}) > 0.5\right] | \mathbf{x} \in \mathcal{G}_{j}\right] \approx \mathbb{E}_{\mathbb{P}}\left[\sigma\left(\alpha\left(f_{\theta}(\mathbf{x}) - 0.5\right)\right) | \mathbf{x} \in \mathcal{G}_{j}\right],$$

and similarly for the overall rate  $\mathbb{E}_{\mathbb{P}}\left[\mathbb{I}\left[f_{\theta}(\mathbf{x})>0.5\right]\right]$ , where  $\sigma$  denotes the sigmoid function. A similar approach was taken, e.g., in [24, 23].

**Model and hyperparameters.** The following are the details of the hyperparameters.

- 1. The Adam optimizer was used for both primal (step 5 of Alg. 1) and dual updates (step 6–7 of Alg. 1) with learning rates of 0.2 (primal) and 0.001 (dual). The other hyperparameters for Adam were set to their default values ( $\epsilon = 10^{-8}, \beta_1 = 0.9, \beta_2 = 0.999$ ).
- 2. A logistic regression classifier was used as the model. The cross-entropy loss was used as the objective.

- 3. The full dataset was used to compute the objective and constraint functions at each step.
- 4. The temperature parameter  $\alpha$  in the sigmoidal approximation of the rate constraints (see Section 5), was set to 8.0.
- 5. Training was terminated when the average of the last 100 Lagrangian iterates, changed by less than  $10^{-5}$  in a window of 100 steps. In case the termination condition was not met, the algorithm ran till 16,000 epochs.

**Computing resources.** The fairness experiments were run on CPU-only nodes (AMD EPYC 7713) with 32 threads, with about 10-20 runs in parallel. Each run took at most 20m (when the termination condition was not met).

### E.2 Boundary value problems

Let  $\Omega\subseteq\mathbb{R}^d$  be a bounded connected region with boundary  $\partial\Omega$ , and define the domain of a BVP as  $\mathcal{D}=\Omega\times(0,T]$  (where  $T\in\mathbb{R}_{\geq 0}$ ) and let  $\mathcal{H}=\{f_\theta\mid\theta\in\Theta\}$  be a set of functions defined on  $\mathcal{D}$ . A BVP is typically posed as the following problem: We call the three constraints the partial differential equation (PDE), boundary condition (BC) and initial condition (IC) respectively. As discussed in Section 2, the PDE constraint can be transformed to the constraint  $\mathbb{E}_{\mathbb{P}_p}[(D[f_\theta](\mathbf{x},t)-\tau(\mathbf{x},t))^2]=0$  for some distribution  $\mathbb{P}_p$  that has support over the entire domain  $\mathcal{D}$ . The difference between this and the original formulation is essentially that the latter is with respect to the  $L^2$  norm, while the former is a pointwise constraint, or equivalently over the  $L^\infty$  norm with respect to any distribution whose support contains  $\mathcal{D}$ . The BC and IC constraints can similarly be transformed into their counterparts in (P-BVP) by picking  $\mathbb{P}_b$  and  $\mathbb{P}_i$  as distributions over  $\partial\Omega\times(0,T)$  and  $\Omega$  respectively.

**Dataset.** In our experiments we solve the convection BVP, with sinusoidal initial condition and periodic boundary conditions:

find 
$$f_{\theta} \in \mathcal{H}$$
  
subject to  $\frac{\partial f_{\theta}}{\partial t} + \beta \frac{\partial f_{\theta}}{\partial x} = 0$  for  $x \in [0, 2\pi], t \in (0, 1],$  (BVP-C)  
 $f_{\theta}(0, t) = f_{\theta}(2\pi, t)$  for  $t \in [0, 1],$   
 $f_{\theta}(x, 0) = \sin(x)$  for  $x \in [0, 2\pi].$ 

We express this as the following instance of (P-BVP):

Here  $\mathbb{P}_p$ ,  $\mathbb{P}_b$  and  $\mathbb{P}_i$  are taken to be uniform distributions over  $[0,2\pi] \times [0,1]$ , [0,1] and  $[0,2\pi]$  respectively. We followed a training and evaluation setup (including model and hyperparameters) similar to [62]. For the training dataset, 1000 collocation points (x,t) were sampled uniformly, and dynamically at each iteration/epoch, from the domain  $\mathcal{X} \times \mathcal{T} = [0,2\pi] \times [0,1]$  which were used to compute the PDE, IC and BC constraints. For evaluation, a uniform grid over (x,t) with (512,251) divisions was used as the test set. We used the implementation by [62] to generate the ground truth. Evaluation was done with respect to the relative L2 error on the test set, which can be written as,

$$\text{Relative } L^2 \text{ Error} = \sqrt{\frac{\sum_{n=1}^{N} \left(f_{\theta}(x_n,t_n) - f^{\dagger}(x_n,t_n)\right)^2}{\left(f^{\dagger}(x_n,t_n)\right)^2}},$$

where  $f^{\dagger}$  is the ground truth solution and  $(x_n,t_n)_{n=1}^N$  is the test set.

**Model and hyperparameters.** The following are the details of the hyperparameters.

- 1. All models were trained for 300k iterations or epochs.
- 2. Both methods (PINN/ (P-BVP)) used a 4 layered MLP with 50 hidden neuron layers and Tanh activation to represent the primal model  $f_{\theta}$ .
- 3. Adam was used to optimize the primal model, with an initial learning rate of 1e-3, and a learning rate scheduler was used that multiplies the learning rate by a factor 0.9 every 5000 steps/epochs (available as the class StepLR from pytorch). The other hyperparameters for Adam were left at their default values  $((\beta_1, \beta_2, \epsilon) = (0.9, 0.999, 1e-8))$ .
- 4. A batch size of 1000 was used, and each batch was sampled dynamically at each step.
- 5. For the PINN formulation, the multipliers for the PDE, BC and IC losses were picked to be (1, 100, 100) respectively.
- 6. No weight decay was applied to either method. In particular,  $\alpha$  was set to 0 in (P-C).
- 7. The dual variables were optimized using Adam with learning rate 1e-4. The other hyperparameters were left at their default values.

**Computing resources.** The BVP experiments were run (one run at a time) on an accelerated node (AMD EPYC 7713, Nvidia A40 GPU) with 16 CPU threads and one GPU. Both PINN and (P-C) took approximately 1h40m for each run (300k epochs), with negligible overhead for the constrained method.

## E.3 Interpolating classifiers

**Dataset.** We used the CIFAR-10 and CIFAR-100 datasets from the torchvision library. We applied dynamic data augmentation, i.e. we transformed samples at train time using the following pipeline:

- 1. random cropping of the image after padding with 4 pixels,
- 2. random horizontal flip,
- 3. random rotation between -15 to 15 degrees,
- 4. channel-wise normalization by the (pixel) mean and standard deviation from the training set.

# Model and hyperparameters.

- 1. The cross entropy loss function was used for  $\ell_0$ . All runs were for 200 epochs.
- 2. Both methods (ERM/(P-CI)) were trained with a ResNet18 model, with a standard modification to the first layer using smaller filter sizes to adapt it to smaller images (compared to ImageNet). The reader is referred to our codebase (or the open source repository https://github.com/kuangliu/pytorch-cifar.git) for the exact implementation.
- 3. SGD was used to optimize the primal model for both datasets, with a batch size of 128, and an initial learning rate of 1e-3, and a momentum hyperparameter of 0.9. For CIFAR-10, a cosine annealing learning rate scheduler (CosineAnnealingLR) was used with  $T_{\rm max}=200$ . For CIFAR-100, the initial learning rate was decayed by a factor of 0.2 at the 60th, 120th and 160th epoch (i.e. these were the milestones passed to MultiStepLR).
- 4. A weight decay value of 5e-4 was used for all runs.
- 5. Adam was used to optimize the dual variables, with a learning rate of 1e-4 for CIFAR-10, and 1e-5 for CIFAR-100. All other hyperparameters were held to their default values.

**Computing resources.** The interpolation experiments were run (one run at a time) on a workstation (AMD Ryzen 5 7600X 6-Core Processor, NVIDIA GeForce RTX 5090) with 12 CPU threads and one GPU. On CIFAR-10, ERM took about 16m while (P-CI) took about 18m. On CIFAR-100, the overhead was larger, with ERM still taking about 16m while the constrained method took out 31m. This may be mitigated by a vectorised implementation of the Lagrangian, but is left for future work.

# F Additional plots

## F.1 Fairness

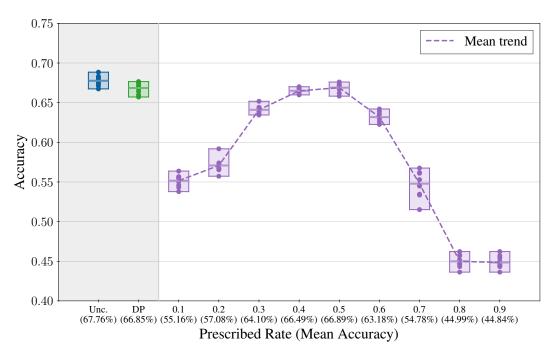


Figure 5: Prescribed rates. We see that at  $r_j = 0.5$ , the mean accuracy of the model is slightly better than the Exact DP solution. At the same time, for  $r_j = 0.5$ , the model achieves a group disparity that is comparable to the Exact DP solution (see Figure 2(b)) but at a different rate. Thus (P-F) enables new tradeoffs between group disparity and accuracy, that cannot be found by using (P-DP).

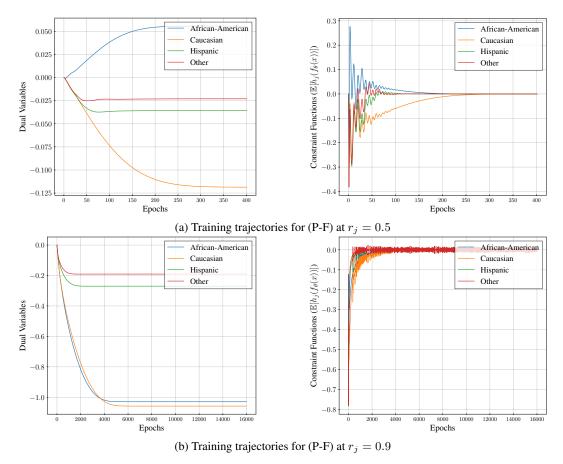


Figure 6: Prescribed rates. We see that while training converges quickly for  $r_j=0.5$ , at about 400 epochs, it takes more than 10 times as many epochs for  $r_j=0.9$  to converge (and it still does not reach the termination condition). This is understandable, since the population prevalence of the binary label, i.e. P(y=1), is about 0.46. As such, prescribing a very high rate, e.g.  $r_j=0.9$ , is a hard constraint to satisfy while maximising the predictive performance. And in fact, it is so restrictive that the model yields the constant classifier after thresholding.

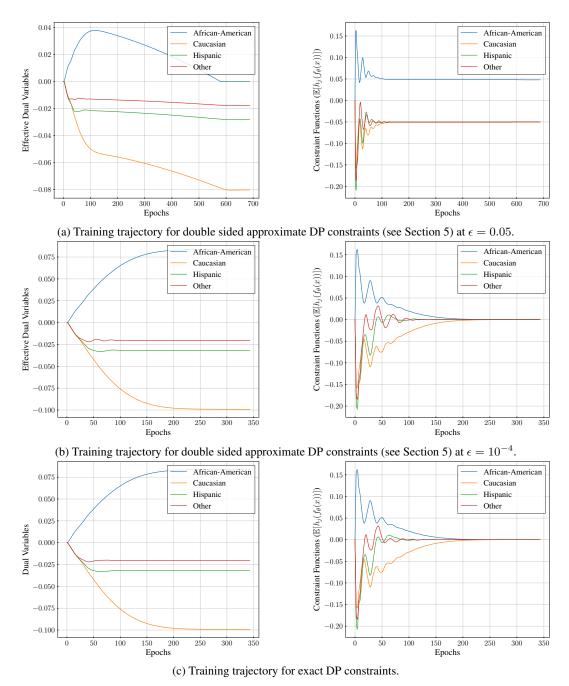


Figure 7: Exact vs approximate fairness. The trajectories of the effective dual variables (i.e. the difference of pairs) and constraint functions of the approximately constrained problem and the exact constrained problem are very similar at  $\epsilon = 10^{-4}$ , as compared to at  $\epsilon = 0.05$ . Section D.2 discusses theoretical reasons for why this is the case.

# F.2 Boundary value problems

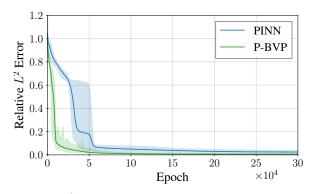


Figure 8: Evolution of relative  $L^2$  error for Convection BVP ( $\beta=30$ ). We see that besides a smaller final error, P-BVP also converges much quicker to a smaller error.

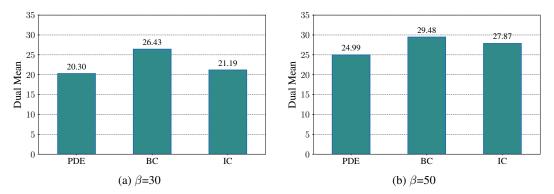


Figure 9: This figure shows the final dual variable per constraint, averaged across 5 runs. The dual values for the boundary condition are the largest for both values of  $\beta$ , which can possibly be explained by the simplicity of the solution to the convection equation with a sinusoidal initial condition - which makes the *propagation* of the information through the boundary condition the challenging part of the problem.

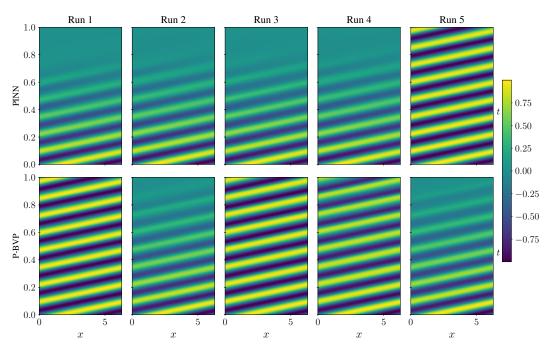


Figure 10: Predicted solutions for  $\beta=50$ . We can visually see the difference between the solutions of PINN and (P-BVP), where the former struggles to solve the problem for most runs.

## F.3 Interpolating classifiers

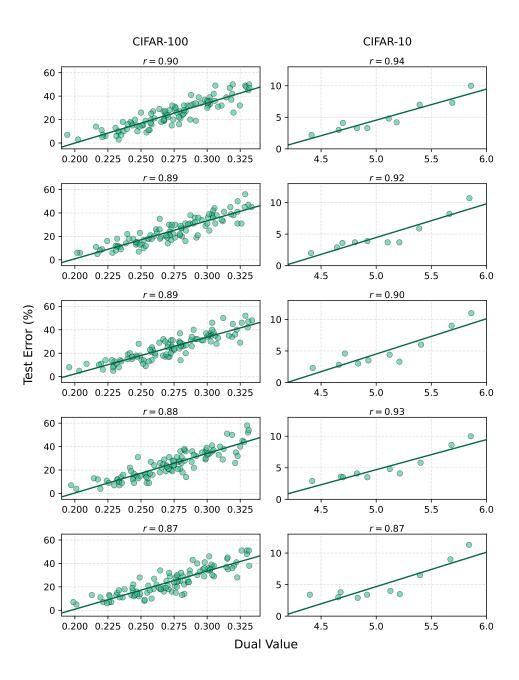


Figure 11: *Dual value vs test error (for all seeds)*. We see a strong linear relationship between the dual value and the test error of each class across runs, for both CIFAR-10 and CIFAR-100, with correlation consistently greater than 0.87.

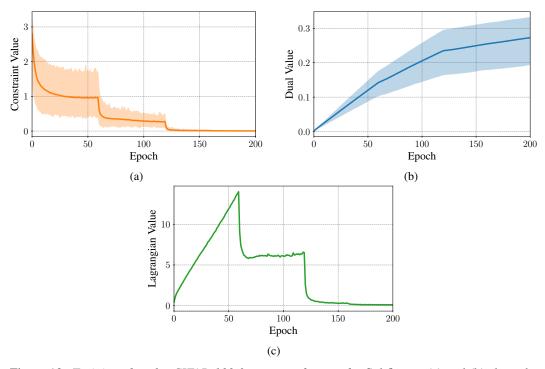


Figure 12: *Training plots for CIFAR-100 for one random seed.* Subfigures (a) and (b) show the constraint and dual values during training, with the mean across classes shown by the solid line, while the maximum and minimum (across classes) is denoted by the extents of the shaded region. Subfigure (c) shows the evolution of the Lagrangian.

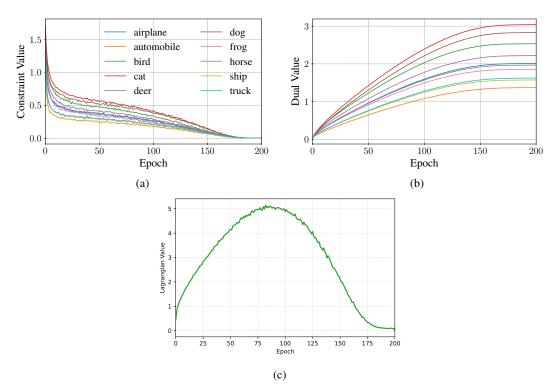


Figure 13: *Training plots for CIFAR-10 for one random seed*. Each class is denoted by a different color in subfigures (a) and (b).

# **Appendix references**

## References

- [1] Ingo Steinwart and Andreas Christmann. *Support Vector Machines*. Springer, 2008. Information Science and Statistics.
- [2] Vladimir N. Vapnik. The Nature of Statistical Learning Theory. Springer, 2000.
- [3] Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. *Foundations of Machine Learning*. MIT Press, 2012.
- [4] Solon Barocas, Moritz Hardt, and Arvind Narayanan. Fairness and Machine Learning: Limitations and Opportunities. MIT Press, 2023.
- [5] Sam Corbett-Davies, Emma Pierson, Avi Feller, Sharad Goel, and Aziz Huq. Algorithmic Decision Making and the Cost of Fairness. In *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 797–806, 2017.
- [6] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards Deep Learning Models Resistant to Adversarial Attacks. In *International Conference on Learning Representations*, 2018.
- [7] Hongyang Zhang, Yaodong Yu, Jiantao Jiao, Eric P. Xing, Laurent El Ghaoui, and Michael I. Jordan. Theoretically Principled Trade-off between Robustness and Accuracy. In *International Conference on Machine Learning*, pages 7472–7482, 2019.
- [8] Cynthia Dwork and Aaron Roth. The Algorithmic Foundations of Differential Privacy. *Foundations and Trends*® *in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [9] Javier García and Fernández. A Comprehensive Survey on Safe Reinforcement Learning. *Journal of Machine Learning Research*, 16(42):1437–1480, 2015.
- [10] Santiago Paternain, Miguel Calvo-Fullana, Luiz F. O. Chamon, and Alejandro Ribeiro. Safe Policies for Reinforcement Learning via Primal-Dual Methods. *IEEE Transactions on Automatic Control*, 68(3):1321–1336, 2023.
- [11] M. Raissi, P. Perdikaris, and G.E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2019.
- [12] Ian J. Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and Harnessing Adversarial Examples. In *International Conference on Learning Representations*, 2015.
- [13] Richard Berk, Hoda Heidari, Shahin Jabbari, Matthew Joseph, Michael J. Kearns, Jamie Morgenstern, Seth Neel, and Aaron Roth. A Convex Framework for Fair Regression. CoRR, 2017.
- [14] Irina Higgins, Loic Matthey, Arka Pal, Christopher Burgess, Xavier Glorot, Matthew Botvinick, Shakir Mohamed, and Alexander Lerchner. beta-VAE: Learning Basic Visual Concepts with a Constrained Variational Framework. In *International Conference on Learning Representations*, 2017.
- [15] Dimitri Bertsekas. Convex Optimization Theory. Athena Scientific, 2009.
- [16] J. F. Bonnans and A. Shapiro. Perturbation Analysis of Optimization Problems. Springer Series in Operations Research and Financial Engineering. Springer New York, 2000.
- [17] B. K. Pagnoncelli, S. Ahmed, and A. Shapiro. Sample Average Approximation Method for Chance Constrained Programming: Theory and Applications. *Journal of Optimization Theory and Applications*, 142(2):399–416, March 2009.
- [18] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczynski. Lectures on Stochastic Programming: Modeling and Theory, Third Edition. Society for Industrial and Applied Mathematics, 2021.

- [19] Andrew Cotter, Michael P. Friedlander, Gabriel Goh, and Maya R. Gupta. Satisfying Real-World Goals with Dataset Constraints. arXiv:1606.07558, 2017.
- [20] Harikrishna Narasimhan. Learning with Complex Loss Functions and Constraints. In *International Conference on Artificial Intelligence and Statistics*, volume 84, pages 1646–1654, 2018.
- [21] Alekh Agarwal, Alina Beygelzimer, Miroslav Dudík, John Langford, and Hanna Wallach. A Reductions Approach to Fair Classification, 2018.
- [22] Luiz F. O. Chamon and Alejandro Ribeiro. Probably Approximately Correct Constrained Learning. In Advances in Neural Information Processing Systems, 2020.
- [23] Luiz F. O. Chamon, Santiago Paternain, Miguel Calvo-Fullana, and Alejandro Ribeiro. Constrained Learning With Non-Convex Losses. *IEEE Transactions on Information Theory*, 69(3):1739–1760, 2023.
- [24] Andrew Cotter, Heinrich Jiang, Maya R. Gupta, Serena Lutong Wang, Taman Narayan, Seungil You, and Karthik Sridharan. Optimization with Non-Differentiable Constraints with Applications to Fairness, Recall, Churn, and Other Goals. *Journal of Machine Learning Research*, 20:172:1–172:59, 2019.
- [25] Michael Kearns, Seth Neel, Aaron Roth, and Zhiwei Steven Wu. Preventing Fairness Gerrymandering: Auditing and Learning for Subgroup Fairness. In *International Conference on Machine Learning*, volume 80, pages 2564–2572, 2018.
- [26] I. Hounie, L. F. O. Chamon, and A. Ribeiro. Automatic Data Augmentation via Invariance-Constrained Learning. In *International Conference on Machine Learning*, pages 13410–13433, 2023.
- [27] Gilles Gasso, Aristidis Pappaioannou, Marina Spivak, and Léon Bottou. Batch and online learning algorithms for nonconvex Neyman-Pearson classification. *ACM Transactions on Intelligent Systems and Technology*, 2:1–19, 2011.
- [28] A. Robey, L. F. O. Chamon, G. J. Pappas, H. Hassani, and A. Ribeiro. Adversarial Robustness with Semi-Infinite Constrained Learning. In *Conference on Neural Information Processing Systems*, pages 6198–6215, 2021. \* equal contribution.
- [29] Muhammad Bilal Zafar, Isabel Valera, Manuel Gomez-Rodriguez, and Krishna P. Gummadi. Fairness Constraints: Mechanisms for Fair Classification. In *International Conference on Artificial Intelligence and Statistics*, pages 962–970, 2017.
- [30] Moritz Hardt, Eric Price, and Nati Srebro. Equality of Opportunity in Supervised Learning. In Advances in Neural Information Processing Systems, pages 3315–3323, 2016.
- [31] Matt J. Kusner, Joshua R. Loftus, Chris Russell, and Ricardo Silva. Counterfactual Fairness. In *Advances in Neural Information Processing Systems*, pages 4066–4076, 2017.
- [32] Taco Cohen and Max Welling. Group Equivariant Convolutional Networks. In *International Conference on Machine Learning*, pages 2990–2999, 2016.
- [33] Risi Kondor and Shubhendu Trivedi. On the Generalization of Equivariance and Convolution in Neural Networks to the Action of Compact Groups. In *International Conference on Machine Learning*, pages 2752–2760, 2018.
- [34] Geoff Pleiss, Manish Raghavan, Felix Wu, Jon M. Kleinberg, and Kilian Q. Weinberger. On Fairness and Calibration. In *Advances in Neural Information Processing Systems*, pages 5680–5689, 2017.
- [35] Chuan Guo, Geoff Pleiss, Yu Sun, and Kilian Q. Weinberger. On Calibration of Modern Neural Networks. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70, pages 1321–1330. Proceedings of Machine Learning Research, 2017.
- [36] Shengjia Zhao, Jiaming Song, and Stefano Ermon. The Information Autoencoding Family: A Lagrangian Perspective on Latent Variable Generative Models, 2018.

- [37] Xi Chen, Yan Duan, Rein Houthooft, John Schulman, Ilya Sutskever, and Pieter Abbeel. InfoGAN: Interpretable Representation Learning by Information Maximizing Generative Adversarial Nets. In *Advances in Neural Information Processing Systems*, 2016.
- [38] Richard H. Byrd, Frank E. Curtis, and Jorge Nocedal. An Inexact SQP Method for Equality Constrained Optimization. SIAM Journal on Optimization, 19(1):351–369, 2008.
- [39] Albert S. Berahas, Frank E. Curtis, Daniel Robinson, and Baoyu Zhou. Sequential Quadratic Optimization for Nonlinear Equality Constrained Stochastic Optimization. *SIAM Journal on Optimization*, 31(2):1352–1379, 2021.
- [40] Frank E. Curtis, Daniel P. Robinson, and Baoyu Zhou. Inexact Sequential Quadratic Optimization for Minimizing a Stochastic Objective Function Subject to Deterministic Nonlinear Equality Constraints, 2021.
- [41] Michael J. O'Neill. A Two Stepsize SQP Method for Nonlinear Equality Constrained Stochastic Optimization, 2024.
- [42] Hao Chen, Gonzalo E. Constante Flores, and Can Li. Physics-informed neural networks with hard linear equality constraints. *Computers & Chemical Engineering*, 189:108764, 2024.
- [43] Linlin Cao, Ran He, and Bao-Gang Hu. Locally imposing function for Generalized Constraint Neural Networks A study on equality constraints. *International Joint Conference on Neural Networks*, pages 4795–4802, 2016.
- [44] M. Hintermüller. Solving Nonlinear Programming Problems with Noisy Function Values and Noisy Gradients. *Journal of Optimization Theory and Applications*, 114(1):133–169, 2002.
- [45] Figen Oztoprak, Richard Byrd, and Jorge Nocedal. Constrained Optimization in the Presence of Noise. SIAM Journal on Optimization, 33:2118–2136, 2023.
- [46] Shigeng Sun and Jorge Nocedal. A Trust-Region Algorithm for Noisy Equality Constrained Optimization. arXiv:2411.02665.
- [47] Haoming Shen, Yang Zeng, and Baoyu Zhou. Sequential Quadratic Optimization for Solving Expectation Equality Constrained Stochastic Optimization Problems, 2025.
- [48] Ferdinando Fioretto, Pascal Van Hentenryck, Terrence W.K. Mak, Cuong Tran, Federico Baldo, and Michele Lombardi. Lagrangian Duality for Constrained Deep Learning, 2020.
- [49] Thomas Lew, Riccardo Bonalli, and Marco Pavone. Sample average approximation for stochastic programming with equality constraints. *SIAM Journal on Optimization*, 34(4):3506–3533, 2024.
- [50] E. Cinlar. Probability and Stochastics. Graduate Texts in Mathematics. Springer New York, 2011.
- [51] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces*. Springer, 2016.
- [52] W. Rudin. Principles of Mathematical Analysis. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.
- [53] Walter Rudin. Real and Complex Analysis, Third Edition. McGraw-Hill, Inc., 1987.
- [54] J. Diestel and J. J. Uhl. Vector Measures. Mathematical Surveys and Monographs. American Mathematical Society, 1977.
- [55] R.T. Rockafellar. Convex Analysis. Princeton University Press, 1997.
- [56] Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: From Theory to Algorithms*. Cambridge University Press, 2014.
- [57] Dionysis Kalogerias and Spyridon Pougkakiotis. Strong Duality in Risk-Constrained Nonconvex Functional Programming, 2023.

- [58] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
- [59] Andrzej Ruszczyński. Nonlinear Optimization. 2006.
- [60] Kaare Brandt Petersen, Michael Syskind Pedersen, et al. The matrix cookbook. 2008.
- [61] Lokenath Debnath and Piotr Mikusinski. *Introduction to Hilbert Spaces with Applications, Third Edition*. Academic Press, 2005.
- [62] Arka Daw, Jie Bu, Sifan Wang, Paris Perdikaris, and Anuj Karpatne. Mitigating Propagation Failures in Physics-informed Neural Networks using Retain-Resample-Release (R3) Sampling. In *International Conference on Artificial Intelligence and Statistics*, volume 202, pages 7264–7302, 2023.