Adaptation of Kruskal's Uniqueness Conditions to multiview CP

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Abstract

Several uniqueness conditions have been formulated for the Candecomp/Parafac(CP) model. Though a single condition which is both sufficient and necessary is yet to be discovered, there exist several necessary conditions and a few sufficient ones as well. Here we observe the adaptation of the most general known necessary as well as sufficient conditions of CP uniqueness to the multiview CP case.

1 Introduction

CP Decomposition and its Uniqueness

$$\mathcal{X} = [\mathbf{A}, \mathbf{B}, \mathbf{C}] \tag{1}$$

$$\mathcal{X} = \sum_{r=1:R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r \tag{2}$$

Explain the main uniqueness concept by Harshman [1970] and Kruskal [1977]. This is the most important concept in Parafac and was the reason for creating Parafac.

$$\mathcal{X} = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}] = [\mathbf{A}\Pi\Lambda_a, \mathbf{B}\Pi\Lambda_b, \mathbf{C}\Pi\Lambda_c] \tag{3}$$

where Π is a permutation matrix and $\Lambda_{a,b,c}$ are non-singular diagonal scale matrices such that $\Lambda_a\Lambda_b\Lambda_c = I$.

1.1 Neccessary Conditions

Several attempts have been made at identifying neccessary conditions of CP uniqueness. The three required conditions are as follows:

No Zero Column by Stegeman2007

Condition: None of the loading matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ should have a zero column, for the reason that if a column is zero - then the corresponding column in other two matrices can take arbitrary values.

Minimum k-rank by Stegeman2007

Condition:

$$k_A \geq 2, \qquad k_B \geq 2, \qquad k_C \geq 2 \tag{4}$$

is a necessary condition for essential uniqueness.

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Full Rank Khatri-Rao by Sidiropoulos2001

CP can be re-written as:

$$\mathcal{X} = (\mathbf{A} \odot \mathbf{B})\mathbf{C} = (\mathbf{A} \odot \mathbf{C})\mathbf{B} = (\mathbf{B} \odot \mathbf{C})\mathbf{A} \tag{5}$$

Condition:

$$(\mathbf{A} \odot \mathbf{B}), \qquad (\mathbf{A} \odot \mathbf{C}), \qquad (\mathbf{B} \odot \mathbf{C}), \qquad \text{have full column rank} \tag{6}$$

As otherwise there exists a vector n such that $(\mathbf{A} \odot \mathbf{B})n = 0$. Adding n to any column of $\mathbf{C}$ preserves the model $\mathcal{X} = (\mathbf{A} \odot \mathbf{B})\mathbf{C}$, but changes $\mathbf{C}$. Hence not unique.

1.2 Sufficient Condition

The most general sufficient condition of CP uniqueness is by Kruskal [1977]. We will consider only this one, as it encompasses all other sufficient conditions. It involves the use of a variant of matrix rank concept(k-rank or the kruskal-rank) he introduced. The k-rank of a matrix is defined as the largest value k such that every k subset of columns of the matrix is linearly independent. Kruskal showed that a CP solution is essentially unique if

$$2R + 2 \leq k_A + k_B + k_C \tag{7}$$

2 Two-view CP and Uniqueness

2.1 Two-view CP

Two-view CP decomposition should be seen as two seperate CP's, one for each view:

$$\mathcal{X}^1 = [\mathbf{Z}^1, \mathbf{U}^1, \mathbf{W}^1] \tag{8}$$

$$\mathcal{X}^2 = [\mathbf{Z}^2, \mathbf{U}^2, \mathbf{W}^2] \tag{9}$$

The multiview model assumes $\mathbf{Z}$ and $\mathbf{U}$ have two parts each, $\mathbf{Z}_S$, $\mathbf{U}_S$ and $\mathbf{Z}_p$, $\mathbf{U}_p$ which capture the shared and specific components, such that $\mathbf{Z}^1$ is a concatenation of $\mathbf{Z}_S$ and $\mathbf{Z}_p^1$, leading to the form:

$$\mathcal{X}^1 = [\mathbf{Z}_s\mathbf{Z}_p^1, \mathbf{U}_s\mathbf{U}_p^1, \mathbf{W}^1] \tag{10}$$

$$\mathcal{X}^2 = [\mathbf{Z}_s\mathbf{Z}_p^2, \mathbf{U}_s\mathbf{U}_p^2, \mathbf{W}^2] \tag{11}$$

Equations 11 and 12 represent two instances of CP decomposition with the additional $\mathbf{Z}_S$, $\mathbf{U}_S$ shared constraint. Allowing the following adjustments of the CP decomposition to its multi-view case.

2.2 Uniqess

2.2.1 Necessary Conditions

No Zero Column

For similar reasons as in CP, the projection matrices in two-view CP should not have any zero column.

Minimum k-rank

CP conditions can now be represented as:

$$k_{\mathbf{Z}_s\mathbf{Z}_p^1} \geq 2,\; k_{\mathbf{Z}_s\mathbf{Z}_p^2} \geq 2, \quad k_{\mathbf{U}_s\mathbf{U}_p^1} \geq 2,\; k_{\mathbf{U}_s\mathbf{U}_p^2} \geq 2, \quad k_{\mathbf{W}^1} \geq 2,\; k_{\mathbf{W}^2} \geq 2 \tag{12}$$

Full Rank Khatri-Rao

Two-view CP can be re-written as:

$$\mathcal{X}^1 = (\mathbf{Z}_s\mathbf{Z}_p^1 \odot \mathbf{U}_s\mathbf{U}_p^1)\mathbf{W}^1 \tag{13}$$

$$\mathcal{X}^2 = (\mathbf{Z}_s\mathbf{Z}_p^2 \odot \mathbf{U}_s\mathbf{U}_p^2)\mathbf{W}^2 \tag{14}$$

Condition:

$$(\mathbf{Z}_s\mathbf{Z}_p^1 \odot \mathbf{U}_s\mathbf{U}_p^1), \qquad (\mathbf{Z}_s\mathbf{Z}_p^1 \odot \mathbf{W}^1), \qquad (\mathbf{U}_s\mathbf{U}_p^1 \odot \mathbf{W}^1), \qquad \text{have full column rank} \tag{15}$$

Where $(\mathbf{Z}_s\mathbf{Z}_p^1 \odot \mathbf{U}_s\mathbf{U}_p^1)$ is full rank iff $(\mathbf{Z}_s \odot \mathbf{U}_s)$ and $(\mathbf{Z}_p^1 \odot \mathbf{U}_p^1)$ are full rank.

And similarly for m=2.

2.2.2 Sufficient Condition

Applying Kruskul's uniqueness condition we get:

$$2(R_s + R_p^1) + 2 \le k_{\mathbf{Z}_s\mathbf{Z}_p^1} + k_{\mathbf{U}_s\mathbf{U}_p^1} + k_{\mathbf{W}^1} \tag{16}$$

$$2(R_s + R_p^2) + 2 \le k_{\mathbf{Z}_s\mathbf{Z}_p^2} + k_{\mathbf{U}_s\mathbf{U}_p^2} + k_{\mathbf{W}^2} \tag{17}$$

where R_s is the matrix rank due to shared components and R_p due to view specific components. Combining the total rank R of the two-view CP decomposition and some algebra we get

$$2(2R_s + R_p^1 + R_p^2 + 2) \le k_{\mathbf{Z}_s\mathbf{Z}_p^1} + k_{\mathbf{Z}_s\mathbf{Z}_p^2} + k_{\mathbf{U}_s\mathbf{U}_p^1} + k_{\mathbf{U}_s\mathbf{U}_p^2} + k_{\mathbf{W}^1} + k_{\mathbf{W}^2} \tag{18}$$

$$2(R_s + R + 2) \le \text{as above} \tag{19}$$

$$(2R + 2) + (2R_s + 2) \le k_{\mathbf{Z}_s\mathbf{Z}_p^1} + k_{\mathbf{Z}_s\mathbf{Z}_p^2} + k_{\mathbf{U}_s\mathbf{U}_p^1} + k_{\mathbf{U}_s\mathbf{U}_p^2} + k_{\mathbf{W}^1} + k_{\mathbf{W}^2} \tag{20}$$

Specific Cases:

- All Specific Components: $\mathbf{Z}_s$ and $\mathbf{U}_s$ vanish, resulting in 2 independent CP decompositions.

$$2R + 2 \le k_{\mathbf{Z}_p^1} + k_{\mathbf{Z}_p^2} + k_{\mathbf{U}_p^1} + k_{\mathbf{U}_p^2} + k_{\mathbf{W}^1} + k_{\mathbf{W}^2} \tag{21}$$

- All Shared Components: $\mathbf{Z}_p$ and $\mathbf{U}_p$ vanish, resulting in 2 independent CP decompositions.

$$2R + 2 = 2R_s + 2 \le k_{\mathbf{Z}_s} + k_{\mathbf{U}_s} + (k_{\mathbf{W}^1} + k_{\mathbf{W}^2})/2 \tag{22}$$

2.2.3 Empirical Validation

Empirical Validation with test data shows that an $8 \times 8 \times 8$ a single view CP can identify unique solutions upto $R \le 11$, while for $M = 2$ datasets of $8 \times 8 \times 8$ each, multiview CP can uniquely identify 5 shared and 6 specific components in each view, i.e. a total of R=17.

3 Conclusion

Necessary conditions of CP uniqueness extend to two-view case also, while the sufficient CP condition by Kruskal:

$$2R + 2 \le k_A + k_B + k_C \tag{23}$$

becomes

$$(2R + 2) + (2R_s + 2) \le k_{\mathbf{Z}_s\mathbf{Z}_p^1} + k_{\mathbf{Z}_s\mathbf{Z}_p^2} + k_{\mathbf{U}_s\mathbf{U}_p^1} + k_{\mathbf{U}_s\mathbf{U}_p^2} + k_{\mathbf{W}^1} + k_{\mathbf{W}^2} \tag{24}$$

for two-view CP decomposition, accomodating View-Shared and Specific Components. Extension to multiview CP will be notationally complicated as components shared between a subset of views will make the equation cluttered.

References

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