Rotated time-frequency lattices are sets of stable sampling for continuous wavelet systems

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***Abstract*—We provide an example for the generating matrix A of a two-dimensional lattice $\Gamma = A\mathbb{Z}^2$, such that the following holds: For any sufficiently smooth and localized mother wavelet ψ, there is a constant $\beta(A,\psi) > 0$, such that $\beta\Gamma \cap (\mathbb{R}\times\mathbb{R}^+)$ is a set of stable sampling for the wavelet system generated by ψ, for all $0 < \beta \leq \beta(A,\psi)$. The result and choice of the generating matrix are loosely inspired by the studies of low discrepancy sequences and uniform distribution modulo 1. In particular, we estimate the number of lattice points contained in any axis parallel rectangle of fixed area. This estimate is combined with a recent sampling result for continuous wavelet systems, obtained via the oscillation method of general coorbit theory.**

***Index Terms*—wavelet transforms, frames, lattice rules, oscillation method**

I. Introduction

A wavelet system [1] is a collection of functions generated from a single prototype, the *mother wavelet*, by translation and dilation. Here, we consider a complex mother wavelet $\psi \in \mathbf{L}^2(\mathbb{R})$, such that its Fourier transform vanishes for negative frequencies, i.e., $\hat{\psi}(\xi) = 0$ for $\xi \in (-\infty, 0]$. With a slight abuse of notation, we denote the space of square-integrable functions with this property as $\mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$. It is a Hilbert space with respect to the standard inner product on $\mathbf{L}^2(\mathbb{R})$. Mother wavelets $\psi \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$ are sometimes called *analytic*, although the terminology *analytic wavelet transform* is not used consistently [2], [3]. The associated continuous wavelet system is defined as

$$\Psi = (\psi_{x,s})_{x\in\mathbb{R}, s\in\mathbb{R}^+}, \text{ where} \tag{1}$$

$$\psi_{x,s} := \mathbf{T}_x \mathbf{D}_{1/s}\psi = \sqrt{s}\psi\left(s(\bullet - x)\right). \tag{2}$$

Here, $\mathbf{D}_s$ with $s \in \mathbb{R}^+$ denotes a unitary dilation by the factor s, and $\mathbf{T}_x$ denotes translation by $x \in \mathbb{R}$. We note that the above definition is slightly unusual in the sense that we associate large scales with small values of s [1]. Under the given conditions on the mother wavelet ψ, the natural extension of Ψ to negative scales $s < 0$ has no further consequences on our results and is therefore omitted. In the following, we refer to $\Lambda = \mathbb{R}\times\mathbb{R}^+$ as the *phase space* associated with Ψ.

Sets of stable sampling for the continuous wavelet system Ψ are discrete sets $\Lambda_D \subset \Lambda$, such that $(\psi_{x,s})_{(x\,;\,s)^\intercal\in\Lambda_D}$ forms a frame, cf. Section II. Such sets have been systematically studied at least since the popularization of frame theory and multiresolution analyses in the 1980s [4]–[6], although certain wavelet bases were known earlier, as discussed in [1, Chapter 4.2.1]. Most established sampling schemes for wavelets follow the same formula:

1) Select a basis $a \in (1,\infty)$ and consider all scales a^j, $j \in \mathbb{Z}$.
2) Select a relative translation step $b \in (0,\infty)$ and consider, at scale a^j, the translations $a^{-j}\cdot lb$, $l \in \mathbb{Z}$.

The density of the resulting set $\Lambda_D = \left\{\binom{a^{-j}\cdot lb}{a^j} : j,l\in\mathbb{Z}\right\} \subset \Lambda$ can be controlled by scaling a and b. Although variations on this scheme are occasionally studied, e.g., for the construction of shift-invariant wavelet systems [7], they often only differ by adding additional points to Λ_D.

A few recent works have explored low discrepancy sets and sequences, as commonly used in the quasi-Monte Carlo method [8], for selecting discrete subsets from Ψ and other time-frequency systems [9], [10]. In particular, the numerical results in the work [10] suggest that the restriction of certain sheared lattices to the upper half-plane form sets of stable sampling for *inhomogeneous* wavelet systems which are obtained by removing the scales $s < 1$ and adding an appropriate substitute.

Contribution: In this work, we consider the rotated square lattice $\Gamma = A\mathbb{Z}^2$, with $A = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1\end{pmatrix}$, for a specific choice of α. We show that isotropic dilations of Γ, restricted to the upper half-plane, form sets of stable sampling for the continuous wavelet system Ψ. The main step towards this result is achieved by proving that the number of lattice points contained in an arbitrary half-open rectangle is proportional to its area, provided that the rectangle has at least a certain minimal area. Our result relies on choosing α to be badly approximable, a property that also plays a prominent role in the construction of Kronecker sequences [11], [12]. The choice of Γ further bears some resemblance to lattice rules [13], a popular construction rule for low discrepancy sequences.

II. Preliminaries

For any $\phi \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$, the continuous wavelet transform defined for all $(x\,;\,s)^\intercal \in \Lambda$ by

$$W_\psi f(x,s) = \langle f, \psi_{x,s}\rangle_{L_2}, \quad \text{for all } f \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+)), \tag{3}$$

is a function in $\mathbf{L}^\infty(\Lambda)$. If the so-called admissibility constant $C_\psi = \|\psi/(\bullet)\|_2^2$ is finite, then the system $\Psi \subset \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$ is a *continuous tight frame* [14], [15] with frame bound C_ψ. In particular, if $C_\psi = 1$, then $\|f\|_{\mathbf{L}^2(\mathbb{R})} = \|W_\psi f\|_{\mathbf{L}^2(\Lambda)}$, for all $f \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$. The image space

$\widetilde{\mathcal{H}} = W_\psi(\mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+)))$ of the wavelet transform with mother wavelet ψ is a reproducing kernel Hilbert space of continuous, square-integrable functions [1], [15].

This work is concerned with the study of certain discrete subsets $\Lambda_D \subset \Lambda$, such that $(\psi_{x,s})_{(x\,;\,s)^\intercal \in \Lambda_D}$ is a discrete frame [15], i.e.,

$$A\|f\|_2^2 \leq \sum_{(x\,;\,s)^\intercal \in \Lambda_D} |\langle f, \psi_{x,s}\rangle|^2 \leq B\|f\|_2^2, \tag{4}$$

for all $f \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$ and some constants $0 < A \leq B < \infty$. When we say that Λ_D is a set of stable sampling for the continuous wavelet system Ψ, we mean precisely that Λ_D is a set of stable sampling for $\widetilde{\mathcal{H}}$, which, noting $C_\psi\|f\|_2^2 = \|W_\psi f\|_{\widetilde{\mathcal{H}}}^2$, is equivalent to (4).

III. The Frame Property from Oscillation Estimates

The oscillation method was introduced by Feichtinger and Gröchenig in their seminal works studying atomic decompositions in *coorbit spaces* [16]–[18]. Although the discretization results in those works can be applied to the wavelet transform on $\mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$, it is nontrivial to show the density and separation requirements therein for the specific point sets $c\Gamma$ that we consider. Hence, it is more convenient to rely on *generalized coorbit theory*, as introducted in [19] and extended in [20], [21]. Applied to continuous frames on Hilbert spaces $\mathcal{H}$, this variant of the oscillation method yields a sufficient condition for irregular sets of stable sampling that can informally be summarized as follows: Assume that $\Phi = (\phi_m)_{m\in M}$ is a localized [19], continuous tight frame for $\mathcal{H}$, and there is a countable covering $\mathcal{V} = (V_j)_{j\in J}$ of M, such that the oscillation of Φ with respect to $\mathcal{V}$ is small in the norm of some Schur-type algebra $\mathcal{A}$ [22]. Then any set $(m_j)_{j\in J}$, with $m_j \in V_j$ for all $j \in J$, is a set of stable sampling for Φ, or equivalently, $(\phi_{m_j})_{j\in J}$ is a discrete frame for $\mathcal{H}$. More precisely, given a function $\Xi\colon \Lambda\times\Lambda \to \mathbb{C}$, with $|\Xi| \equiv 1$, the Ξ-oscillation $\mathrm{osc}_{\mathcal{V}}$ of Φ with respect to $\mathcal{V}$ is given by

$$\begin{aligned} &\mathrm{osc}_{\mathcal{V}}(m_0, m_1) \\ &= \sup_{n\in\mathcal{V}_{m_1}} |\langle \phi_{m_0}, \phi_{m_1}\rangle - \Xi(m_1,n)\langle \phi_{m_0}, \phi_n\rangle|, \end{aligned} \tag{5}$$

where $\mathcal{V}_{m_1} := \bigcup_{j\in J: m_1\in V_j} V_j$. The appropriate choice of Ξ is crucial to achieve small $\mathcal{A}$-norm of $\mathrm{osc}_{\mathcal{V}}$.

In a recent paper concerned with warped time-frequency systems [23], of which the continuous wavelet systems Ψ are a special case, it was shown that there exists a family of coverings $\mathcal{V}^\delta = (V_j^\delta)_{j\in J}$ of $\Lambda = \mathbb{R}\times\mathbb{R}^+$, comprised of half-open rectangles of identical area δ^2, such that the oscillation of the wavelet system Ψ with respect to $\mathcal{V}^\delta$ converges to zero in $\mathcal{A}$-norm, for $\delta \to 0$. In particular, [23, Example 4.1, Corollary 6.9] imply the following result.

Theorem III.1. *Let $\psi \in \mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$, with $C_\psi = 1$, such that $\mathcal{F}(\psi) \in \mathcal{C}^2(\mathbb{R})$, with $\max\{(\bullet)^5, (\bullet)^{-5}\}\cdot\mathcal{F}(\psi) \in \mathcal{C}_0(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}^+)$. Define $\mathcal{V}^\delta = (V_{k,\ell}^\delta)_{k,\ell\in\mathbb{Z}}$ by*

$$V_{k,\ell}^\delta = \left[\frac{\delta^2 k}{|I_\ell^\delta|}, \frac{\delta^2(k+1)}{|I_\ell^\delta|}\right) \times I_\ell^\delta,$$

with $I_\ell^\delta = [e^{\delta\ell}, e^{\delta(\ell+1)})$. Then there exists $\delta(\Psi) > 0$, such that for any $0 < \delta \leq \delta(\Psi)$, and any discrete set $\Lambda_D \subset \Lambda$ with

$$\inf_{k,\ell\in\mathbb{Z}} |\Lambda_D \cap V_{k,\ell}^\delta| > 0 \text{ and } \sup_{k,\ell\in\mathbb{Z}} |\Lambda_D \cap V_{k,\ell}^\delta| =: N < \infty,$$

the collection $(\psi_{x,s})_{(x\,;\,s)^\intercal\in\Lambda_D}$ is a frame for $\mathcal{F}^{-1}(\mathbf{L}^2(\mathbb{R}^+))$.

Since Theorem III.1 is an application of the results referred to above, its proof is not self-contained and deferred to the Appendix. In the next section, we show that certain rotated lattices satisfy the conditions on Λ_D in Theorem III.1.

IV. Wavelet Frames by Sampling on Lattices

In the following, we set $\alpha = \varphi^{-1} = \frac{\sqrt{5}-1}{2}$, where φ is the golden ratio, and consider the matrix

$$A = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}. \tag{6}$$

With this choice of A, we define the lattice

$$\Gamma = A\mathbb{Z}^2, \tag{7}$$

i.e., $\Gamma = \{A\binom{n}{m}\colon n, m \in \mathbb{Z}\}$. As we will see, any axis parallel rectangle Δ of a given size has clear bounds on the number of points in $\Delta \cap \Gamma$ that do not depend on its position or ratio of its side lengths. Before we proceed to prove a formal version of this statement, we derive some properties of α that will be subsequently used.

Proposition IV.1. *Let $\alpha = \varphi^{-1} = \frac{\sqrt{5}-1}{2}$. Then, for all $n \in \mathbb{N}$,*

$$\alpha^n = (-1)^{n-1}(f_n\alpha - f_{n-1}), \tag{8}$$

where f_n is the n-th Fibonacci number, defined by $f_0 = 0$, $f_1 = 1$ and, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

Moreover, for any $n \in \mathbb{N}$, we have

$$\alpha^{n-1} f_{n+2} + \alpha^n f_{n+1} = 2 + \alpha = \varphi^2. \tag{9}$$

Proof. We first prove (8). For $n = 1$, the statement trivially holds, while for $n = 2$,

$$\alpha^2 = \varphi^{-2} = \frac{1}{1+\varphi} = 1 - \frac{\varphi}{1+\varphi} = 1 - \alpha \tag{10}$$

as desired. The induction step follows from

$$\begin{aligned} \alpha^{n+1} = \alpha^n\cdot\alpha &= (-1)^{n-1}(f_n\alpha - f_{n-1})\cdot\alpha \\ &\overset{(10)}{=} (-1)^{n-1}(f_n(1-\alpha) - f_{n-1}\alpha) \\ &= (-1)^{n-1}(f_n - (f_n + f_{n-1})\alpha) \\ &= (-1)^n(f_{n+1}\alpha - f_n). \end{aligned}$$

We next prove (9). Inserting $n = 1$ on the left hand side immediately yields

$$\alpha^0 f_3 + \alpha^1 f_2 = 2 + \alpha,$$

as desired. The induction step follows from

$$\begin{aligned} \alpha^n f_{n+3} + \alpha^{n+1} f_{n+2} &= \alpha^n(f_{n+2} + f_{n+1}) + \alpha^{n+1} f_{n+2} \\ &= (\alpha^2 + \alpha)\alpha^{n-1} f_{n+2} + \alpha^n f_{n+1} \\ &\overset{(10)}{=} \alpha^{n-1} f_{n+2} + \alpha^n f_{n+1} \\ &= 2 + \alpha. \end{aligned}$$

$\square$

We are now ready to show that any axis-parallel rectangle with sufficient area contains at least one lattice point.

Lemma IV.2. *Let $\Delta = [a,b) \times [c,d)$ be an axis parallel half-open rectangle of size $\mu(\Delta) \geq 2+\alpha = \varphi^2$, where μ is the standard Lebesgue measure on $\Lambda = \mathbb{R}\times\mathbb{R}^+$. Then $|\Delta\cap\Gamma| \geq 1$.*

Proof. Given $\Delta = [a,b)\times[c,d)$, with $\Delta\cap\Gamma = \emptyset$, we assume, without loss of generality, that there exist $a_0 \in [a,b)$ and $c_0 \in [c,d)$, such that

$$\begin{pmatrix} b \\ c_0 \end{pmatrix}, \begin{pmatrix} a_0 \\ d \end{pmatrix} \in \Gamma.$$

Otherwise, there is a nonzero $\left(\begin{smallmatrix} b_0 \\ d_0 \end{smallmatrix}\right) \in [0,\infty)^2$, such that $(\Delta + [0,b_0)\times[0,d_0)) \cap \Gamma = \emptyset$.

We now show that $\Delta\cap\Gamma = \emptyset$ implies that $\mu(\Delta) < 2+\alpha$. To this end, we first consider the case that both $b-a \geq 1$ and $d-c \geq 1$. Because $\left(\begin{smallmatrix} a_0+\alpha \\ d-1 \end{smallmatrix}\right) \in \Gamma$ and $d-1 \in [c,d)$, we have $a_0+\alpha \geq b$, since otherwise $\left(\begin{smallmatrix} a_0+\alpha \\ d-1 \end{smallmatrix}\right) \in \Gamma\cap\Delta$. Similarly, because $\left(\begin{smallmatrix} a_0-1 \\ d-\alpha \end{smallmatrix}\right) \in \Gamma$ and $d-\alpha \in [c,d)$, we have $a_0 - 1 < a$. Hence,

$$b \leq a_0 + \alpha < a + 1 + \alpha$$

and we conclude that $b-a < 1+\alpha$. Analogously, we obtain $d-c < 1+\alpha$. Together, this implies $\mu(\Delta) < (1+\alpha)^2$. Finally, (10) yields

$$(1+\alpha)^2 = 1 + 2\alpha + \alpha^2 = 2+\alpha,$$

as desired.

For the remainder of the proof, we assume that $d-c<1$ and remark that the proof for the case $b-a<1$ is analogous, using the rotational symmetry $\Gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Gamma$. Furthermore, since $\Gamma \subset \mathbb{R}^2$ is a lattice, it is in particular a subgroup of $\mathbb{R}^2$: The problem is invariant under shifts $\Delta - \gamma$ of Δ by lattice points $\gamma \in \Gamma$. Hence, we can assume, without loss of generality, that $\left(\begin{smallmatrix} a_0 \\ d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, implying $a \leq 0$, $b > 0$ and $c < 0$. Further, let $n \in \mathbb{N}$ be such that

$$\alpha^n \leq -c < \alpha^{n-1}.$$

In particular, this implies

$$c \leq -\alpha^n < -\alpha^{n+1} < 0. \tag{11}$$

If $n \in 2\mathbb{N}$ then, by (8) in Proposition IV.1, the chain of inequalities in (11) is equivalent to

$$c \leq f_n\alpha - f_{n-1} < f_n - f_{n+1}\alpha < 0. \tag{12}$$

For $p_0 = \left(\begin{smallmatrix} f_n \\ -f_{n-1} \end{smallmatrix}\right)$ and $p_1 = \left(\begin{smallmatrix} -f_{n+1} \\ f_n \end{smallmatrix}\right)$, we have

$$Ap_0 = \begin{pmatrix} f_n + \alpha f_{n-1} \\ f_n\alpha - f_{n-1} \end{pmatrix} \quad \text{and} \quad Ap_1 = \begin{pmatrix} -f_{n+1} - \alpha f_n \\ -f_{n+1}\alpha + f_n \end{pmatrix}.$$

In particular, (12) yields $Ap_0 \in ((0,\infty)\times[c,0))\cap\Gamma$ and $Ap_1 \in ((-\infty,0)\times[c,0))\cap\Gamma$. Thus, $\Delta\cap\Gamma = \emptyset$ implies $b \leq (Ap_0)_1$ and $a > (Ap_1)_1$. Together, we obtain

$$\begin{aligned} b - a &< (Ap_0)_1 - (Ap_1)_1 \\ &= f_n + \alpha f_{n-1} - (-f_{n+1} - \alpha f_n) \\ &= f_{n+2} + \alpha f_{n+1}. \end{aligned}$$

Similarly, if $n \in 2\mathbb{N}-1$, we set $p_0 = \left(\begin{smallmatrix} -f_n \\ f_{n-1} \end{smallmatrix}\right)$ and $p_1 = \left(\begin{smallmatrix} f_{n+1} \\ -f_n \end{smallmatrix}\right)$ and we arrive at the same bound for $b-a$.

Since $d-c = -c < \alpha^{n-1}$, we obtain

$$\mu(\Delta) < \alpha^{n-1}f_{n+2} + \alpha^n f_{n+1} = 2+\alpha,$$

by (9) in Proposition IV.1. This completes the proof. □

Furthermore, any axis-parallel rectangle that has sufficiently small area contains no more than a single element of Γ.

Lemma IV.3. *Let $\Delta = [a,b)\times[c,d)$ be an axis parallel half-open rectangle of size $\mu(\Delta) \leq 1/(3+2\alpha)$. Then $|\Delta\cap\Gamma| \leq 1$.*

Proof. Given $\Delta = [a,b)\times[c,d)$, with $\Delta\cap\Gamma \neq \emptyset$, we assume, without loss of generality, that there exist $a_0 \in [a,b)$ and $c_0 \in [c,d)$, such that

$$\begin{pmatrix} a \\ c_0 \end{pmatrix}, \begin{pmatrix} a_0 \\ c \end{pmatrix} \in \Gamma.$$

Otherwise, there is a nonzero $\left(\begin{smallmatrix} b_0 \\ d_0 \end{smallmatrix}\right) \in [0,\infty)^2$, such that $[a+b_0, b)\times[c+d_0,d)\cap\Gamma \neq \emptyset$. We now show that $|\Delta\cap\Gamma| \geq 2$ implies $\mu(\Delta) > 1/(3+2\alpha)$.

Analogous to the previous proof, we only consider the case $d-c<1$ and assume that $\left(\begin{smallmatrix} a_0 \\ c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, implying $a \leq 0$, $b>0$ and $d>0$. Since α is a so-called badly approximable number, we can bound how well it can be approximated by any rational. Specifically, by [24, Sec. 11.7], we have

$$\left|\alpha + \frac{m}{n}\right| \geq \frac{1}{(3+2\alpha)n^2} \tag{13}$$

for any $n,m \in \mathbb{Z}$, $n \neq 0$. In turn, we have $|n\alpha+m| \geq \frac{1}{(3+2\alpha)|n|}$. Hence, any lattice point in Γ with second component in $[0,d)$ must satisfy $\frac{1}{(3+2\alpha)|n|} < d$, i.e.,

$$|n| > \frac{1}{(3+2\alpha)d}. \tag{14}$$

On the other hand, $|n\alpha+m| < d < 1$, implies that the cases $n>0$ and $m>0$ as well as $n<0$ and $m<0$ can be excluded, which implies $nm \leq 0$. Thus, for the first component of any point in $\Delta\cap\Gamma$, we have $|n - m\alpha| = |n| + |m\alpha|$. Furthermore, $a \leq n - m\alpha < b$ and, hence,

$$|n| \leq |n| + |m\alpha| < b - a. \tag{15}$$

Combining (14) and (15), we obtain $\frac{1}{(3+2\alpha)} < (b-a)d = \mu(\Delta)$. □

Corollary IV.4. *Let $\Delta = [a,b)\times[c,d)$ be an axis parallel half-open rectangle of size $\mu(\Delta) = 2+\alpha = \varphi^2$. Then $1 \leq |\Delta\cap\Gamma| \leq 12$.*

In particular, for any $\delta > 0$, there is a $\beta := \beta(\delta) > 0$, such that the following holds: If $\mathcal{V}^\delta = (V^\delta_{k,\ell})_{k,\ell\in\mathbb{Z}}$ is as in Theorem III.1, then

$$1 \leq |\beta\Gamma \cap V^\delta_{k,\ell}| \leq 12,$$

for all $k,\ell \in \mathbb{Z}$. Hence, if $\delta \leq \delta(\Psi)$, we can choose $\Lambda_D = \beta\Gamma$ in Theorem III.1.

Proof. The lower bound in the first assertion follows from Lemma IV.2. For the upper bound note that $2+\alpha < \frac{12}{2\alpha+3}$, such that an axis-parallel, half-open rectangle of area $2+\alpha$ can be contained in no more than 12 axis-parallel, half-open rectangles of area $(2\alpha+3)^{-1}$. The bound now follows from Lemma IV.3.

For the second assertion, it is sufficient to note that each $V_{k,l}^{\delta}$ is an axis-parallel, half-open rectangle of area $\mu(V_{k,l}^{\delta}) = \delta^2$. Choose $\beta(\delta) = \delta^2 \cdot (2+\alpha)^{-1}$ to obtain the desired result. □

Considering the coarse estimates used in both previous proofs, it seems quite likely that the upper bound in Corollary IV.4 can be further improved.

V. Conclusion and Outlook

We have shown that a certain scale of rotated lattices provides sets of stable sampling for continuous wavelet systems, when restricted to the upper half plane. Our proof relies on prior work on discretization of wavelet systems in the context of coorbit spaces, and a property of the proposed lattices that evokes discrepancy theory. It should be noted that our result generalizes to other warped time frequency systems or, more generally, localized continuous frames that satisfy oscillation estimates with respect to a phase-space covering comprised of axis-parallel rectangles with identical area. A generalization to higher dimensional phase space seems quite feasible.

We expect analogues of Lemmas IV.2 and IV.3 for any badly approximable number in place of α. Specifically, some bound in the style of Lemma IV.3 exists for any badly approximable number, but [24, Sec. 11.7] only yields explicit estimates for algebraic numbers. We are currently working on a proof for Lemma IV.2 that does not rely on the properties of the golden ratio φ and generalizes to arbitrary badly approximable numbers. With this extended result, it will be possible to show that a more general class of time-frequency lattices generates sets of stable sampling for wavelet systems and other localized continuous frames.

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Appendix

Proof of Theorem III.1. First note that [23, Example 4.1] shows that Ψ is a warped time-frequency system with respect to the warping function $\Phi = \log$ (the natural logarithm) and the prototype $\theta = \mathcal{F}(\psi) \circ \exp$, where $\circ$ denotes composition. In particular, it is a warping function in the sense of [23, Definition 4.2], with associated weight function $w = (\Phi^{-1})' = \exp$. Clearly, $w \in \mathcal{C}^{\infty}(\mathbb{R})$ satisfies $w(t+s) = w(t)w(s)$, for all $t, s \in \mathbb{R}$, such that it is self-moderate. Finally, the k-th derivative $w^{(k)}$ of w equals w, such that $|w^{(k)}/w| = 1$, and $\Phi = \log$ satisfies all assumptions on Φ in [23, Corollary 6.9]. The covering $\mathcal{V}^{\delta}$ in the statement of Theorem III.1 is precisely the Φ-induced δ-cover, for $\Phi = \log$, considered in [23, Corollary 6.9]. Further, Equation (16) in [23] is satisfied with $Y = \mathbf{L}^2(\Lambda)$ and $m \equiv 1$, as stated in [19, Section 3]. It only remains to verify the conditions on $\theta = \mathcal{F}(\psi) \circ \exp$, before [23, Corollary 6.9] can be applied.

With this choice of m, and estimating $w(t) = \exp(t) \leq \exp(|t|)$, the conditions on θ in [23, Corollary 6.9] simplify to

1) $\theta \in \mathcal{C}^2(\mathbb{R})$ with $\theta^{(k)} \cdot \exp(|\bullet|)^3 \in \mathcal{C}_0(\mathbb{R})$, for $0 \leq k \leq 2$,
2) $\theta^{(k)} \cdot \exp(|\bullet|)^{\frac{7-2k}{2}} \in \mathbf{L}^2(\mathbb{R})$.

Since $\exp(|\log(\tau)|) = \max\{\tau, \tau^{-1}\}$, for all $\tau \in \mathbb{R}^+$, the first condition is equivalent to

$$\theta^{(k)}(\log(\bullet)) \cdot \max\{\bullet^3, \bullet^{-3}\} \in \mathcal{C}_0(\mathbb{R}^+).$$

Furthermore,

$$\theta^{(1)} \circ \log(\tau) = \tau \cdot (\theta \circ \log)^{(1)}(\tau) = \tau \cdot (\mathcal{F}(\psi))^{(1)}(\tau) \quad (16)$$

$$\text{and} \quad \theta^{(2)} \circ \log(\tau) = \tau \cdot (\mathcal{F}(\psi))^{(2)}(\tau) - (\mathcal{F}(\psi))^{(1)}(\tau), \quad (17)$$

for all $\tau \in \mathbb{R}^+$. We conclude that Item (1) is equivalent to $\max\{\bullet^4, \bullet^{-4}\} \cdot (\mathcal{F}(\psi))^{(k)} \in \mathcal{C}_0(\mathbb{R})$, for all $0 \leq k \leq 2$.

Further, for all $l \in \mathbb{N}$ and measurable $g \colon \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned}\int_{\mathbb{R}} & |g(t) \cdot \exp(|t|)^l|^2 \, dt \\ &= \int_{\mathbb{R}^+} \tau^{-1} |\max\{\tau^l, \tau^{-l}\} \cdot (g \circ \log)(\tau)|^2 \, d\tau.\end{aligned}$$

Inserting $g = \theta^{(k)}$ and $l = \frac{7-2k}{2}$, for $0 \leq k \leq 2$, and using Equations (16) and (17) once more, we see that Item (2) is implied by $\max\{(\bullet)^5, (\bullet)^{-5}\} \cdot \mathcal{F}(\psi) \in \mathbf{L}^2(\mathbb{R}^+)$. Altogether, $\theta = \mathcal{F}(\psi)$ satisfies the conditions of [23, Corollary 6.9], as desired, such that we can invoke said corollary.

Note that [23, Corollary 6.9] requires a pairwise association of the points $\lambda_{k,l} \in \Gamma$ and the elements $V_{k,l}^{\delta}$ of $\mathcal{V}^{\delta}$, for all $k, l \in \mathbb{Z}$, but the choice of point in $V_{k,l}^{\delta}$ is arbitrary. The covering $\mathcal{V}^{\delta}$ is, in fact, a tiling of Λ, such that $\lambda \in V_{k,l}^{\delta}$ implies $\lambda \notin V_{k',l'}^{\delta}$, for any $k, k', l, l' \in \mathbb{Z}$ with $\binom{k}{l} \neq \binom{k'}{l'}$. Hence, we can find $\Gamma_0 \subset \Gamma$, such that $|\Gamma_0 \cap V_{k,l}^{\delta}| = 1$, for all $k, l \in \mathbb{Z}$, and Γ_0 is a set of stable sampling for Ψ by [23, Corollary 6.9].

We can further find $\Gamma_1, \ldots, \Gamma_{N-1} \subset \Gamma$ that satisfy the following:

1') $\bigcup_{n=0}^{N-1} \Gamma_n = \Gamma$, and $\Gamma_n \cap \Gamma_{n'} = \emptyset$, for all $n, n' \in \{0, \ldots, N-1\}$ with $n \neq n'$.
2') $|\Gamma_n \cap V_{k,l}^{\delta}| \leq 1$, for all $n \in \{0, \ldots, N-1\}$ and all $k, l \in \mathbb{Z}$.

Since the points $(\lambda_{k,l})_{k,l \in \mathbb{Z}}$ with $\lambda_{k,l} \in V_{k,l}^{\delta}$, for all $k, l \in \mathbb{Z}$ in [23, Corollary 6.9] are arbitrary, it is clear that each Γ_n satisfies the implicit upper frame bound in that result. Hence, we conclude that Γ is a set of stable sampling for Ψ. Precisely, the lower bound in (4) equals the lower bound implied in [23, Corollary 6.9], whereas the upper bound in (4) is no larger than N times the upper bound implied in [23, Corollary 6.9]. □

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