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Multi-player evolutionary game of federated learning incentive mechanism based on system dynamics

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ABSTRACT

Federated learning has emerged as a new way of data sharing. The participants in the federation tend to choose different strategies based on their benefits, which is formalized into an evolutionary game model. Existing techniques can limit the malicious behavior of participants by detecting betrayers or weakening their influence. The problem that whether there is an incentive mechanism which makes participants spontaneously choose to cooperate honestly and maintains the stability of the federated learning system is urgent. In this paper, we develop a multi-player evolutionary game model in federated learning. We model the federated learning process by evaluating the payoffs of the central server, internal clients, and external clients. The stability of the federated learning system in the long-term dynamics process is assessed by seeking the evolutionarily stable equilibrium solutions. In this paper, mathematical reasoning and computer simulation are combined to analyze the impact of reward and punishment strategies in incentive mechanisms on the game process and game equilibrium. An incentive mechanism is designed to achieve evolutionarily stable equilibrium while make most clients join the federation spontaneously and cooperate honestly. Finally, the effectiveness, stability, and generalizability of this incentive mechanism are verified by sensitivity analysis and Lyapunov stability theory.

1. Introduction

Federated learning has been developed as a new distributed machine learning technique to solve the data island problem [1]. Specifically, the process of federated learning is that clients train the local data and then upload the parameters to the central server, which aggregates and obtains the global model parameters and distributes them to each client. The learning is completed through multiple communications and interactions [2]. Federated learning is received increasing attention due to its significant advantages in privacy protection. However, there are still many challenges, for example, participant challenges [3]. As the principal members of federated learning, local clients are the foundation of federated learning. Since clients will upload and download parameters through the central server, data pollution caused by a betrayal participant may spread to the whole federated learning cluster. Therefore, how to deal with malicious clients and attract more honest clients to participate in federated learning is a necessary issue [4].

Previous studies that only focused on technical aspects are not enough to solve this problem. To some extent, the research on cooperative strategies for federated learning in the context of specific application scenarios is more critical than technical research [5]. The operation of the federated learning system is determined by the interaction of the strategy of multiple players, which consist of the central server (CS), internal clients (IC), and external clients (EC). Non-cooperation, dependence on strategy, and pursuit of profit maximization among the players are the essential characteristics of game theory. Therefore, some scholars applied game theory to solve the design problem of federated learning mechanisms and achieved good results [5–10]. For example, Wang et al. [11] modeled the dynamics of regret minimization in large agent populations, helped us deepen our understanding of the multi-agent learning process. This study shows the importance of evolution and stability in multi-agent system, and inspires us to use game theory to model federated learning.

Most of the current studies are under the framework of complete information and complete rationality of the players, which differs from the reality and ignores the fluctuations of the changing strategies of the players in the system. This leads to the risk of instability of the federated learning system in the dynamic process. In reality, the participants in federated learning form a complex dynamic system, and the players are always in a state of limited rationality. In this scenario, it is difficult to

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achieve a stable equilibrium in traditional games, and evolutionary game theory is needed for analysis and research.

Evolutionary game that combines game analysis and dynamic evolution process analysis has been widely used in economics, operations research, and management, etc. [12]. Evolutionary game theory focuses on the dynamics of strategy changes and which strategies can persist in the above settings. The success of a player's strategy is directly related to the strategy chosen by other players. Once an evolutionarily stable strategy is adopted by a group, no player intends to deviate from it, and it can still survive when there is a small external incursion. In the field of federated learning, some scholars also use evolutionary game to model the interaction between the clients, where the individual utility of clients has been successfully improved by adjusting the client training scheme or cluster selection [13–15]. To the best of our knowledge, no researcher embodied all the complex strategies influences among the CS, honest and betrayed IC, and EC not joining the federation in the modeling process of evolutionary games, and obtained evolutionarily stable equilibrium strategies.

This paper aims to develop a multi-player evolutionary game model by analyzing the operation process of the federated learning system. We analyze each player's strategy choices and benefit needs in the dynamic process based on current techniques and a variety of specific application scenarios in federated learning. In this paper, we use the system dynamics (SD) method [16], which is widely used in the numerical simulation, to simulate the strategy changes and interaction of each player, and to study the implementation effects of each strategy and the evolutionarily stable equilibrium. In this paper, through the mathematical analysis based on Lyapunov stability theory and the computer simulation of SD, we study the participant's challenge in the dynamic process of federated learning from qualitative and quantitative aspects. We obtain the incentive mechanism that can achieve the evolutionarily stable equilibrium. Finally, the corresponding recommendations for the design of the federated learning mechanism are given, aiming to maintain the healthy operation of the federated learning system stably way in the long term. The main contributions are as follows:

- In this paper, the model combines evolutionary game theory and system dynamics to analyze the strategy choice and game processes of the CS, IC, and EC in federated learning. An incentive mechanism is designed, which can achieve evolutionary stability equilibrium, and made most of the EC spontaneously choose to join the federation, and most of the IC choose to cooperate honestly.
- Different from the previous studies, this paper's model is based on the assumption of incomplete information and limited rationality. We analyzes the CS's reward and punishment behavior, the IC's attack behavior, and the EC's impact on the federation as a whole, which is more in line with the actual operation process of a federated learning system.
- In this paper, the effectiveness, stability, and generalizability of this incentive are verified by sensitivity analysis and Lyapunov stability theory. The combination of mathematical reasoning and computer simulation provides ideas and recommendations for the design of incentive mechanisms in federated learning.

The remainder of this paper is organized as follows. In Section 2, literature review is presented. In Section 3, the analysis of the multi-player evolutionary game in federated learning is presented. In Section 4, the stability and robustness of the mechanism are analyzed. Finally, the conclusions are described in Section 5.

2. Literature review

In 2016, McMahan et al. [17] proposed the concept of federated learning, a distributed machine learning, which provides a feasible solution to the problem of data islands. It integrates multi-player models by not directly sharing data, so that the AI system can use data more accurately, efficiently [18], and safely [19,20]. Most existing studies [21,22] assume that the client is willing to participate in constructing federated learning models, which can only be realized on the premise that there is an effective incentive mechanism. Otherwise, clients seeking for maximize benefits may not be willing to participate in federated learning, or even attacks spontaneously [23]. Therefore, the design of the federated learning mechanism needs to consider the game process of the players with conflicting benefits in the model [24].

In recent years, scholars have made some achievements in the design of federated learning mechanism by using game theory. Hu et al. [25] and Xiao et al. [7] designed incentive mechanisms by reducing the client privacy loss and optimizing the server utility. They constructed a Stackelberg game model based on federated learning, derived the optimal equilibrium solution, and demonstrated the significant performance of the proposed mechanism. Ding et al. [5] and Wu et al. [6] studied the problem of designing incentives for the CS by using a multidimensional contract theory approach in game theory, which reveal the impact of the level of information asymmetry on the strategy, cost, and utility of the server. Ng et al. [8] proposed a multi-player game model of federated learning to study how each participant makes choices among different incentives. Cong et al. [9] split the incentive mechanism design problem in the game-theoretic framework into two sub-problems: supply-side and demand-side. It is noteworthy that these studies are based on the assumption of complete information or complete rationality. However, the behavior of each participant in federated learning is limited rational in actuality. And if the limitation of incomplete information is ignored, the modeling and analysis of the federated learning mechanism will deviate from the actual situation, which reduce the effectiveness and application value of the design method of the mechanisms. To solve the above problems, we investigated the literatures that have successfully addressed them in other fields. For example, in the field of climate change mitigation, when face collective-risk social dilemmas, Wang et al. [26] successfully safeguarded the group optimal outcome while balancing selfish interests and common good. Wang et al. [27] proposed and proved through social dilemma experiments that reducing anonymity can effectively promote the cooperation between individuals. These studies inspire us to consider: in the field of federated learning, there is no pressure of group loss from social dilemmas, and strict privacy requirements make it difficult to achieve onymous communication between individuals. In such a case, how can the game promote egoists to cooperate actively? Can we promote the cooperative behavior between individuals through incentive mechanism?

We wonder if introduce the evolutionary game into federated learning can find an answer to the above idea. The evolutionary game model can analyze the game behavior between limited rational players in an incomplete information model. Because the strategies of the CS, IC, and EC are constantly evolving in the operation of a real federated learning system, evolutionary game theory is precisely applicable to studying this interaction of strategies. By applying evolutionary game theory, Zou et al. [13] explored a method to maximize the utility of mobile devices in federated learning. Lim et al. [14] proposed a hierarchical game framework to study edge association and resource allocation in hierarchical federated learning networks by combining the Stackelberg game with an evolutionary game. In addition, Lim et al. [15] simulated the dynamic process of cluster selection to solve the problem of secondary resource allocation and incentive design. The articles above have all achieved excellent results. To our knowledge, no researcher has embodied all the complex strategies influences among CS, honest and betrayed IC, and EC not joining the federation in the modeling process of evolutionary game. There is still a lack of discussion on motivating more IC to choose honesty while more EC choose to join in the federated learning. The above research work provides a reference for us to analyze the strategic choices of all players in the federated learning system. The comparison between state of art related techniques and ours is listed in Table 1.

Table 1
Comparison with related works in federated learning (FL).

Schemes	Game model	Rational limitation	Players	Research objectives	Verification method	Sensitivity analysis
[25]	Stackelberg	×	CS, IC	The incentive design for compensating privacy leakage cost of providing reliable data users for general FL systems	✓	×
[28]	Stackelberg	×	CS, IC, market	Build market-oriented model to analyze and solve optimal behavior in general FL systems	✓	✓
[29]	Mixed-strategy	×	CS, IC	Algorithm optimization of detect and discard bad updates provided by the clients in robust FL System	×	✓
[30]	Reverse	✓	CS, IC	Employ federal learning as the proof of work for blockchain	✓	×
[6]	Contract theory	✓	CS, IC	Jointly considering the task expenditure and privacy issue of FL incentive design	✓	×
[14]	Stackelberg, Evolutionary	✓	CS, IC, edge servers	Capture the dynamics of edge association and resource allocation in hierarchical FL networks	✓	✓
[31]	Evolutionary	✓	CS, fog providers	Optimize algorithms that improve system availability by solving instability problems within the fog federation	×	×
[13]	Evolutionary	✓	CS, IC	Optimization strategies for enhancing individual utility of mobile devices in general FL systems	✓	×
Ours	Evolutionary	✓	CS, IC, EC	Analyze and solve the problem of stability and incentive mechanism design for dynamic processes of general FL systems	✓	✓

Verification method: whether mathematical reasoning is used to interpret and verify the computer simulation results.
Rational limitation: whether to restrict to 'limited rationality'.

3. Multi-player evolutionary game analysis of federated learning

In the operation of a federated learning system [17], because all clients are in the same social market, their payoffs are mutually influenced. IC are managed by CS, so the payoffs between these two are also mutually influential and interdependent. The above-mentioned individuals are limited rational, their behavior are not static. They always calculate, compare their payoffs and change their choices accordingly, thus the behavior of the game is formed. The players with lower payoffs keep replicating the dynamics of the "successful ones" in the process of the game. The strategies of each player need to experience a process of adaptive change rather than being obtained by an immediate optimization calculation. So the whole federated learning system evolves over time according to some regular pattern. A system with the above characteristics can be called a dynamic system [32]. Evolutionary game theory differs from traditional game theory, because it applies to analyzing dynamic game processes with limited rational players [33]. Therefore, evolutionary game theory is more suitable for studying the dynamic process of federated learning, and making the study more realistic.

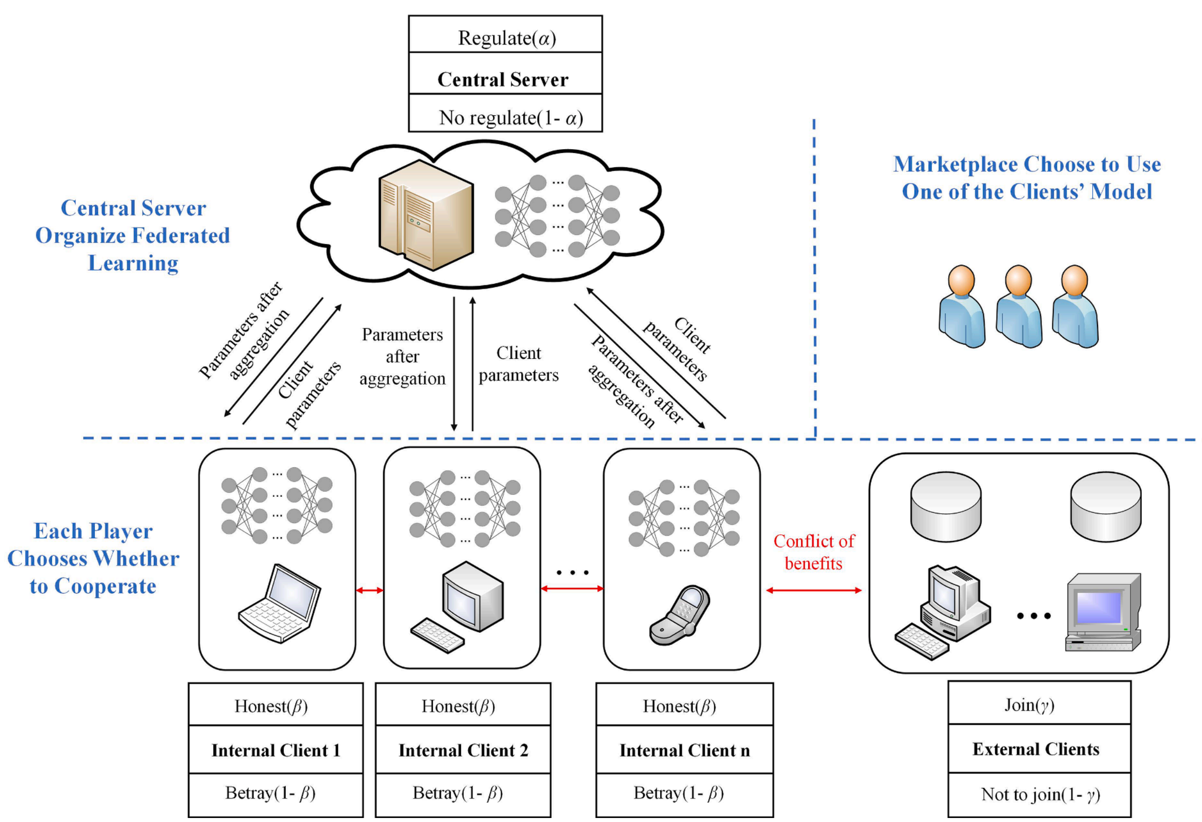

Fig. 1. Federated Learning System Framework.

3.1. Game design and description

Based on the theory of federated learning, the central servers and clients participating in the federation will make different decisions, which result in different payoffs. As time goes on, players continuously improve their cooperation strategies, and then form new federated cooperation scenarios. We further set the payoffs for each player by analyzing the federated learning process.

Assumption1. *In the game, each player is limited rational and pursues payoff maximization. Each player can choose one of two mutually exclusive strategies and continuously check their own payoffs to decide whether to change their strategies.*

The multi-player game among the CS, IC, and EC is shown in Fig. 1.

Central server: In our setting, the CS may be a cloud platform providing coordination services and is responsible for organizing federation collaboration and aggregating local models. The goal of CS is to enable participating clients to cooperate honestly and recruit more clients to join the federation at a low cost. CS coordinates more honest IC can get better global model and thus higher market returns so as to improve its net income. The strategy space of CS is {Regulate, No regulate}. 'Regulate' means to pay extra cost to deploy defenses. For example, the schemes proposed in the literature such as [34] and [35] can be used to detect attacks from malicious IC. At this time, there is a probability to capture malicious IC and implementing a penalty. 'No regulate' does not require the above costs to detect malicious attacks. The specific interactions and corresponding symbols related to CS in the multi-player game process are described as follows:

The CS chooses $\alpha(0\leqslant\alpha\leqslant1)$ as its strategy, where α represents the regulation rate. The bigger the value of α, the bigger the regulation investment of the CS. The CS charges the client a model usage fee $w_1\cdot(client^{'}smarketreturns)$ and gives the honest client a reward of $w_3\cdot(modelusagefee)$. The CS needs to pay the additional regulation cost $c(c>1)$ when strengthening regulation. At this time, it is possible to detect the betrayal behavior of the IC and implement punishment. When the CS chooses 'no regulate', it is vulnerable to attack by malicious IC, and the quality of the global model is reduced. Then the market returns of other IC is lower, and the model usage fee is low. This leads to low payoffs for the CS and a low-quality global model, which makes EC tend to be unwilling to join. Therefore, the CS considers changing its strategy to 'regulate', which also leads to changes in other players' strategies. In reality, multiple entities (clients) in a federated learning system collaborate to solve machine learning problems under the coordination of a CS. The original data of each client is stored locally and will not be transmitted directly. The clients are independent decision-making and use local updates to achieve learning goals. Therefore, the system operation of federated learning is a long-term process. Each player will observe and compare their different payoffs and change their strategies dynamically.

Assumption2. *The CS probabilistically catches the betrayed IC with a probability density of* $\frac{1-e^{-c}}{T}$, *then the probability of catching the betrayal is* $\int_0^T \frac{1-e^{-ct}}{T}dt = 1+\frac{1}{TC}e^{-ct}+\frac{1}{TC}$.

Internal clients: The clients may be private cell phones, computers, and other terminals or small enterprises that own the data. We call the clients who have participated in federated learning as IC, they are responsible for uploading the parameters of local models to CS for aggregation, and then downloading the aggregated models and putting them into the market for payoffs. The goal of IC is to maximize their net income, their strategy space is {Honest, Betray}. 'Honest' means that IC honestly implement procedures such as uploading or downloading parameters and reporting market revenue in the process of federated learning, CS and several honest IC can form a reliable federation to achieve federated learning. 'Betray' means that malicious IC choose to implement attack methods such as model poisoning [36] and backdoor attacks [37] on CS. This can destroy the market share of other honest IC by destroying the global model of CS, so as to indirectly increase their market returns, or to reduce the cost of participating in the federation to increase their net benefits by free-riding [38], misreporting market returns, etc. The specific interactions and corresponding symbols related to IC in the game process are described as follows:

IC choose $\beta(0\leqslant\beta\leqslant1)$ as their strategy, where β represents the honesty rate of IC. The bigger the value of β, the more IC choose honesty. Global model prediction accuracy can be improved when the IC honestly participates in federated learning (formalized in the text as improving the model quality score). IC pay a model usage fee to the CS and receives reward. When a client betrays the federation, it uploads fake parameters. In this case, the federated model quality score is decreased, and the payoffs are slightly misreported. However, it is possible to be detected by the CS and punished.

External clients: EC refer to clients that are not already joining in the aforementioned federated cooperation, and their strategy space is {Join, Not to join}. If the payoff of joining the federation is higher, EC choose 'Join', and if not, choose 'Not to join'. EC that do not join in federated learning only gain market returns through their own local models, and do not need to interact with CS. The specific interactions and corresponding symbols related to EC in the game process are described as follows:

EC choose $\gamma(0\leqslant\gamma\leqslant1)$ as their strategy, where γ represents the joining rate of EC. The bigger the value of γ, the more EC choose to join. When an external client joins the federation, it defaults to honest cooperation. It needs to pay model usage fees, while it can improve the model quality score (i.e., increases market returns) and receives a reward.

All variables in the multi-player game model are shown in Table 2. Table 3 shows the payoff matrix under eight different strategies combinations. Each part of the payoff function represents the payoff of the CS, IC, and EC.

3.2. Game solution

According to evolutionary game theory [39] and the payoff matrix in Table 3, the expected benefits U_α when the CS chooses to regulate and the expected benefits $U_{1-\alpha}$ when it does not regulate can be obtained as follows, respectively.

$$\begin{aligned} U_\alpha &= -c+(1-w_3)w_1m\beta\left[2w_2^{'}\gamma+w_2(1-\gamma)\right] \\ &+\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S(1-\beta)-\left(-\frac{1}{Tc}e^{-Tc}+\frac{1}{Tc}\right) \\ &w_3w_1m(1-p)(1-\beta)(\gamma w_2+1-\gamma) \\ &+w_1m(1-p)(1-\beta)[\gamma(2-w_3)w_2+1-\gamma] \end{aligned} \tag{1}$$

Table 2
Meanings of variables in the multi-player game.

Variables	Meaning of the variables	Notes
c	Supervision cost of central server	$c>1$
m	Model quality score when the client does not join federated learning	$m>1$
w_2	The quality enhancement coefficient for the initial client when only one player in the game honestly joins the federated learning	$1<w_2<w_2^{'}<2$
$w_2^{'}$	The quality enhancement coefficient for the initial client when two players in the game honestly join the federated learning	$1<w_2<w_2^{'}<2$
w_1	The proportion of model usage fee handed over to the central server on the client in the client's market return	$0<w_1<1$
w_3	The proportion of rewards distributed by the central server to the client in the model usage fee	$0<w_3\leqslant1$
p	Damage coefficient of the betrayed client to federated learning model	$0<p<<1$
T	Update cycle of a federated learning model	$T>0$
S	Punishment for internal clients who choose to betray	$S>0$

Table 3
Payoff matrix of the game among the central server, the internal clients and the external clients.

Three players strategies	Central server payoff	Internal clients payoff	External clients payoff
α,β,γ	$(1-w_3)2w_1w_2'm-c$	$(1+w_3w_1-w_1)w_2'm$	$(1+w_3w_1-w_1)w_2'm$
$\alpha,\beta,1-\gamma$	$(1-w_3)w_1w_2m-c$	$(1+w_3w_1-w_1)w_2m$	$m\frac{1}{w_2+1}$
$\alpha,1-\beta,\gamma$	$(2-w_3)w_1w_2m(1-p)-c-\left(-\frac{1}{Tc}e^{-Tc}+\frac{1}{Tc}\right)w_3w_1w_2m(1-p)+\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S$	$m(1+p)-w_1w_2m(1-p)+\left(-\frac{1}{Tc}e^{-Tc}+\frac{1}{Tc}\right)w_3w_1w_2m(1-p)-\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S$	$(1+w_3w_1-w_1)(1-p)w_2m$
$\alpha,1-\beta,1-\gamma$	$w_1m(1-p)-c-\left(-\frac{1}{Tc}e^{-Tc}+\frac{1}{Tc}\right)w_3w_1m(1-p)+\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S$	$m(1+p)-w_1m(1-p)+\left(-\frac{1}{Tc}e^{-Tc}+\frac{1}{Tc}\right)w_3w_1m(1-p)-\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S$	m
$1-\alpha,\beta,\gamma$	$(1-w_3)2w_1w_2'm$	$(1+w_3w_1-w_1)w_2'm$	$(1+w_3w_1-w_1)w_2'm$
$1-\alpha,\beta,1-\gamma$	$(1-w_3)w_1w_2m$	$(1+w_3w_1-w_1)w_2m$	$m\frac{1}{w_2+1}$
$1-\alpha,1-\beta,\gamma$	$2(1-w_3)w_1w_2m(1-p)$	$m(1+p)+(w_3-1)w_1w_2m(1-p)$	$(1+w_3w_1-w_1)(1-p)w_2m$
$1-\alpha,1-\beta,1-\gamma$	$(1-w_3)w_1m(1-p)$	$m(1+p)+(w_3-1)w_1m(1-p)$	m

$$U_{1-\alpha}=(1-w_3)w_1m\beta\left[2w_2'\gamma+w_2(1-\gamma)\right]+(1-w_3)w_1m(1-p)(1-\beta)(2\gamma w_2+1-\gamma) \tag{2}$$

Therefore, the average expected benefit of the central server is:

$$\overline{U}_{\alpha,1-\alpha}=\alpha U_\alpha+(1-\alpha)U_{1-\alpha} \tag{3}$$

According to replication dynamics [40], more players gradually adopt strategies with better-than-average benefits in a limited rational population. Therefore, the proportion of players using each strategy in the population will change. The change rate of α can be calculated by replication dynamic equation, which is derived as follows:

$$F(\alpha)=\frac{d\alpha}{dt}=\alpha\left(U_\alpha-\overline{U}_{\alpha,1-\alpha}\right)=\alpha(1-\alpha)(U_\alpha-U_{1-\alpha})=\alpha(1-\alpha)\left\{p-c+\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)\left(1-\beta\right)[S+w_3w_1m(1-p)(\gamma w_2+1-\gamma)]\right\} \tag{4}$$

Similarly, the expected benefits U_β when IC are honest and the expected benefits $U_{1-\beta}$ when they betray can be calculated as follows:

$$U_\beta=(1+w_3w_1-w_1)m\left[w_2'\gamma+w_2(1-\gamma)\right] \tag{5}$$

$$U_{1-\beta}=m(1+p)-\alpha\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S+(\gamma w_2+1-\gamma)w_1m(1-p)\left[w_3-1-\alpha w_3\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)\right] \tag{6}$$

The expected benefits U_γ when EC join and the expected benefits $U_{1-\gamma}$ when they do not join can be calculated as follows:

$$U_\gamma=(1+w_3w_1-w_1)m\left[w_2'\beta+w_2(1-\beta)(1-p)\right] \tag{7}$$

$$U_{1-\gamma}=m\beta\frac{1}{w_2+1}+m(1-\beta) \tag{8}$$

To sum up, the multi-player evolutionary game of federated learning can be expressed by the following replication dynamic equation group:

$$\begin{cases} F(\alpha)=\alpha(1-\alpha)\left\{-c+\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)(1-\beta)[S+w_3w_1m(1-p)(\gamma w_2+1-\gamma)]\right\} \\ F(\beta)=\beta(1-\beta)\left\{\left(1+w_3w_1-w_1\right)m\left[w_2'\gamma+w_2(1-\gamma)\right]-m(1+p)+\alpha\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)S-(\gamma w_2+1-\gamma)w_1m(1-p)\left[w_3-1-\alpha w_3\left(1+\frac{1}{Tc}e^{-Tc}-\frac{1}{Tc}\right)\right]\right\} \\ F(\gamma)=\gamma(1-\gamma)\left\{\left(1+w_3w_1-w_1\right)m\left[w_2'\beta+w_2(1-\beta)(1-p)\right]-m\beta\frac{1}{w_2+1}-m\left(1-\beta\right)\right\} \end{cases} \tag{9}$$

Set $F(\alpha)=F(\beta)=F(\gamma)=0$, the local equilibrium solutions can be calculated as: $E_1(0,0,0)^T$, $E_2(0,0,1)^T$, $E_3(0,1,0)^T$, $E_4(0,1,1)^T$, $E_5(1,0,0)^T$, $E_6(1,0,1)^T$, $E_7(1,1,0)^T$, $E_8(1,1,1)^T$, α, β_1, $\beta_2\in[0,1]$. The other three equilibrium solutions are in the appendix A.

Friedman [39] proposed that the stability of the equilibrium point of the replication dynamic equations can be judged by the determinant and eigenvalues of the Jacobian matrix of the system at the equilibrium point. According to Lyapunov stability theory, if all eigenvalues have non-positive real parts, the system is stable; otherwise, the system is unstable. The Jacobian matrix of the replication dynamic equation group (9) is:

$$J=\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \tag{10}$$

Which,

$$J_{11}=\frac{\partial F(\alpha)}{\partial\alpha}=(1-2\alpha)\left\{-c+\left(1-\frac{1}{Tc}+\frac{e^{-Tc}}{Tc}\right)(1-\beta)\left[S+w_3m(1-p)w_1(1-\gamma+w_2\gamma)\right]\right\}$$

$$J_{12}=\frac{\partial F(\alpha)}{\partial\beta}=-\left(1-\frac{1}{Tc}+\frac{e^{-Tc}}{Tc}\right)(1-\alpha)\alpha[S+w_3m(1-p)w_1(1-\gamma+w_2\gamma)]$$

$$J_{13} = \frac{\partial F(\alpha)}{\partial \gamma} = w_3 m(1-p)w_1\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\left(-1 + w_2\right)(1-\alpha)\alpha(1-\beta)$$

$$J_{21} = \frac{\partial F(\beta)}{\partial \alpha} = (1-\beta)\beta\left[S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right) + w_3 m(1-p)w_1\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\left(1 - \gamma + w_2\gamma\right)\right]$$

$$J_{22} = \frac{\partial F(\beta)}{\partial \beta} = (1-2\beta)\left\{-m(1+p) + S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\alpha + m\left(1 - w_1 + w_3 w_1\right)\left[w_2(1-\gamma) + w_2'\gamma\right] - m(1-p)w_1\left[-1 + w_3 - w_3\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\alpha\right]\left(1 - \gamma + w_2\gamma\right)\right\}$$

$$J_{23} = \frac{\partial F(\beta)}{\partial \gamma} = \left\{m\left(1 - w_1 + w_3 w_1\right)\left(w_2' - w_2\right) - m(1-p)w_1\left(-1 + w_2\right)\left[-1 + w_3 - w_3\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\alpha\right]\right\}(1-\beta)\beta$$

$$J_{31} = \frac{\partial F(\gamma)}{\partial \alpha} = 0$$

$$J_{32} = \frac{\partial F(\gamma)}{\partial \beta} = \left[m - \frac{m}{1+w_2} + m\left(1 - w_1 + w_3 w_1\right)\left(w_2' - (1-p)w_2\right)\right](1-\gamma)\gamma$$

$$J_{33} = \frac{\partial F(\gamma)}{\partial \gamma} = \left\{-m\left(1-\beta\right) - \frac{m\beta}{1+w_2} + m\left(1 - w_1 + w_3 w_1\right)\left[(1-p)w_2(1-\beta) + w_2'\beta\right]\right\}\left(1 - 2\gamma\right)$$

Take eight pure strategy solutions as an example, substituted into the Jacobian matrix (10) respectively. The eigenvalues of the Jacobian matrix corresponding to the equilibrium solutions are shown in Table 4.

If and only if $\lambda_1, \lambda_2, \lambda_3 \leqslant 0$, the equilibrium solution is in a stable state. By analyzing the calculated results in Table 4, we know that many factors jointly influence the evolutionary equilibrium state of the federated learning system. At this time, it is impossible to determine whether there is an evolutionary stable strategy (ESS) in the above equilibrium solutions, so we can use computer simulation to analyze the federated learning evolution. The primary purpose of mathematical modeling is to catch the essence of the problem, explore the specific effects of each factor, and find the optimal solution of the problem. Therefore, when the full use of theoretical analysis cannot achieve the purpose of mathematical modeling, then computer simulation can be used to conduct scientific prediction analysis by simulating the implementation effect of different strategies.

3.3. Game analysis based on system dynamics

The SD approach relies on computer simulation technology and the feedback control theory. It combines quantitative and qualitative analysis to deeply explore the information feedback behavior in complex systems, find and study relevant and important influencing factors within the system [16]. Simulation the game by using SD allows for an integrated view of the dynamic features behind the game equilibrium from a global perspective. In addition, the analysis of evolutionary game theory plays a key role in modeling and formulating the corresponding decisions [41].

Based on the above assumptions and analysis, this paper uses Vensim DSS v5.6a to develop an evolutionary game model for federated learning, which consists of three sub-systems, namely the central server, internal clients, and external clients, as shown in Fig. 2. The rectangles represent level variables showing the cumulative results. The valves represent the rate variables showing the physical flows of items feeding into or depleting; the other variables are auxiliary and exogenous variables. In this model, there are three flow level variables, three flow rate variables, nine exogenous variables, and twelve auxiliary variables. Functional relationships among all variables are set based on the above equations (1)-(8).

Table 4
Eigenvalues of Jacobian matrix.

Equilibrium solution	λ_1	λ_2	λ_3
$E_1(0,0,0)^{\mathrm{T}}$	$-c + \left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)[S + w_3 m(1-p)w_1]$	$-m(1+p) + m(1 - w_1 + w_3 w_1)w_2 - m(1-p)w_1(-1 + w_3)$	$-m + m(1 - w_1 + w_3 w_1)(1-p)w_2$
$E_2(0,0,1)^{\mathrm{T}}$	$-c + \left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)[S + w_3 m(1-p)w_1 w_2]$	$-m(1+p) + m(1 - w_1 + w_3 w_1)w_2' - m(1-p)w_1[-1 + w_3]w_2$	$m - m(1 - w_1 + w_3 w_1)(1-p)w_2$
$E_3(0,1,0)^{\mathrm{T}}$	$-c$	$m(1+p) - m(1 - w_1 + w_3 w_1)w_2 + m(1-p)w_1(-1 + w_3)$	$-\frac{m}{1+w_2} + m(1 - w_1 + w_3 w_1)w_2'$
$E_4(0,1,1)^{\mathrm{T}}$	$-c$	$m(1+p) - m(1 - w_1 + w_3 w_1)w_2' + m(1-p)w_1(-1 + w_3)w_2$	$\frac{m}{1+w_2} - m(1 - w_1 + w_3 w_1)w_2'$
$E_5(1,0,0)^{\mathrm{T}}$	$c - \left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)[S + w_3 m(1-p)w_1]$	$-m(1+p) + S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right) + m(1 - w_1 + w_3 w_1)w_2 - m(1-p)w_1\left[-1 - w_3\left(-\frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\right]$	$-m + m(1 - w_1 + w_3 w_1)(1-p)w_2$
$E_6(1,0,1)^{\mathrm{T}}$	$c - \left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)[S + w_3 m(1-p)w_1 w_2]$	$-m(1+p) + S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right) + m(1 - w_1 + w_3 w_1)w_2' - m(1-p)w_1\left[-1 - w_3\left(-\frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\right]w_2$	$m - m(1 - w_1 + w_3 w_1)(1-p)w_2$
$E_7(1,1,0)^{\mathrm{T}}$	c	$m(1+p) - S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right) - m(1 - w_1 + w_3 w_1)w_2 + m(1-p)w_1\left[-1 - w_3\left(-\frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\right]$	$-\frac{m}{1+w_2} + m(1 - w_1 + w_3 w_1)w_2'$
$E_8(1,1,1)^{\mathrm{T}}$	c	$m(1+p) - S\left(1 - \frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right) - m(1 - w_1 + w_3 w_1)w_2' + m(1-p)w_1\left[-1 - w_3\left(-\frac{1}{Tc} + \frac{e^{-Tc}}{Tc}\right)\alpha\right]w_2$	$\frac{m}{1+w_2} - m(1 - w_1 + w_3 w_1)w_2'$

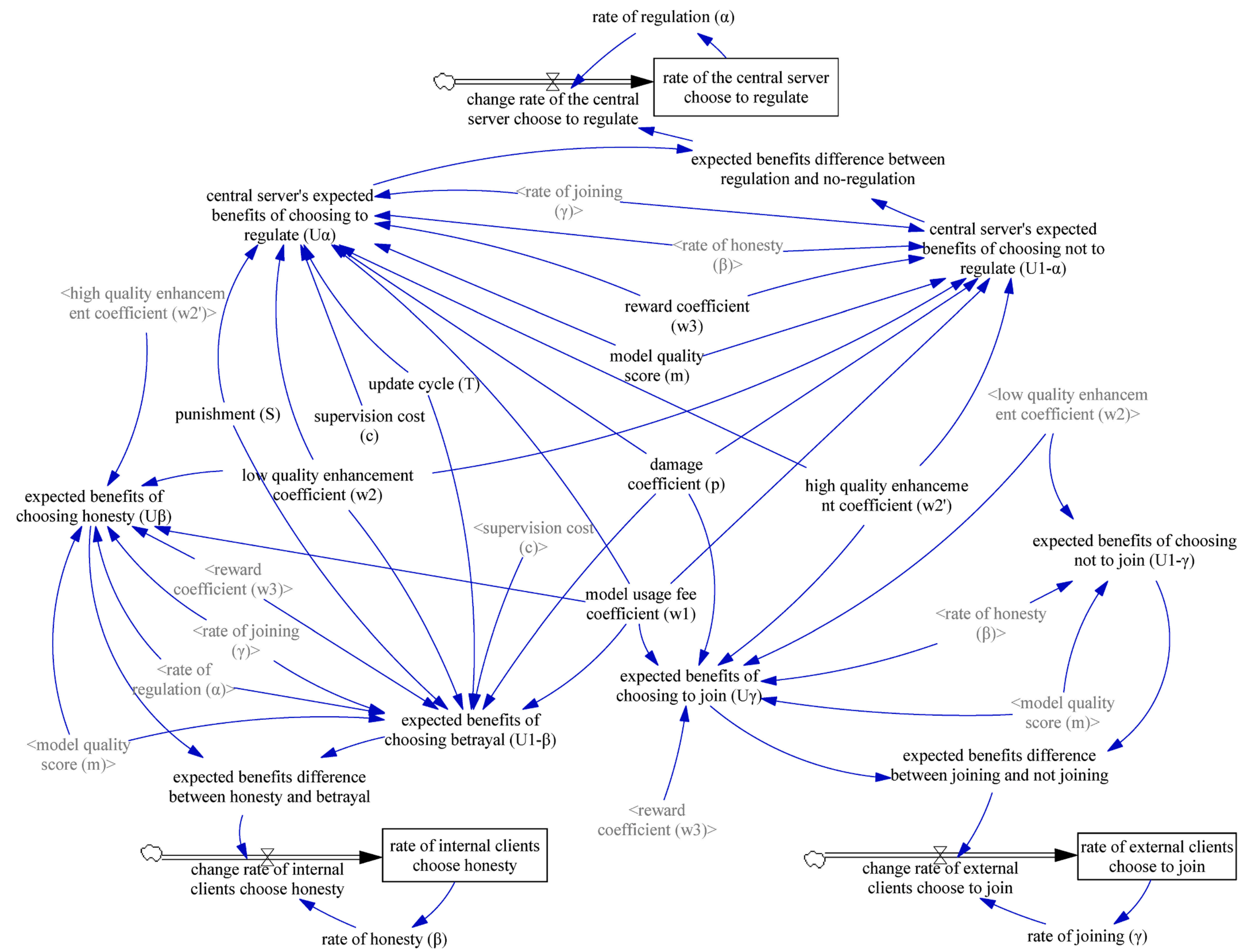

Fig. 2. Evolutionary game SD model of federated learning.

The system parameters are set as: INITIAL TIME = 0, FINAL TIME = 100, TIME STEP = 0.25, Integration Type: Euler. According to publicly available data sources (i.e., literature, institutional reports, and announcements), the initial values of the exogenous variables in the model are shown in Table 5 after pre-processing.

Table 5
The initial values of exogenous variables.

Variables	Meaning of the variables	Initial values
c	Supervision cost of central server	2
m	Model quality score when the client does not join federated learning	10
w_2	The quality enhancement coefficient for the initial client when only one player in the game honestly joins the federated learning	1.2
$w_2^{'}$	The quality enhancement coefficient for the initial client when two players in the game honestly join the federated learning	1.3
w_1	The proportion of model usage fee handed over to the central server on the client in the client's market return	0.8
w_3	The proportion of rewards distributed by the central server to the client in the model usage fee	0.8
p	Damage coefficient of the betrayed client to federated learning model	0.5
T	Update cycle of a federated learning model	2
S	Punishment for internal clients who choose to betray	4

In the replication dynamic equations group (9), set $F(\alpha) = F(\beta) = F(\gamma) = 0$, 11 equilibrium solutions can be obtained as follows: $E_1(0,0,0)^T$, $E_2(0,0,1)^T$, $E_3(0,1,0)^T$, $E_4(0,1,1)^T$, $E_5(1,0,0)^T$, $E_6(1,0,1)^T$, $E_7(1,1,0)^T$, $E_8(1,1,1)^T$, $E_9(1, 0.4376, 0.885388)^T$, $E_{10}(0.758333, 0.631877, 0)^T$, $E_{11}(0.527392, 0.661928, 1)^T$. $E_1 \sim E_8$ are pure strategy equilibrium solutions and $E_9 \sim E_{11}$ are mixed strategy equilibrium solutions.

Next, we take E_{11} as an example and substitute E_{11} into the SD model, the simulation results are shown in Fig. 3. This result shows a relatively balanced state, in which the CS, IC, and EC do not spontaneously change their strategies. However, it still needs to be tested whether E_{11} is an ESS. According to evolutionary game theory, the population who adopt ESS should be sufficient to withstand small mutations [42]. We make a small mutation in the initial strategy of the EC, i.e., change the initial value γ of the EC from 1 to 0.9, and re-simulating the model. The result is shown in Fig. 4.

As can be seen from Fig. 3, when $\gamma = 1$, i.e., when all EC choose to join, IC maintain a high honesty rate, and the CS maintain a moderate regulation rate. Further, the results in Fig. 4 show that E_{11} is not an ESS, because a mutation in the initial value of γ breaks the equilibrium of E_{11}, leading to fluctuating and unstable strategies for other players. The reason for this phenomenon is that EC have a mutation, i.e., a change in their strategy results in less benefit. The strategy of EC will continue to change, so that the CS and IC will also change their strategies based on their benefits. Similarly, we checked the equilibrium states of other ten

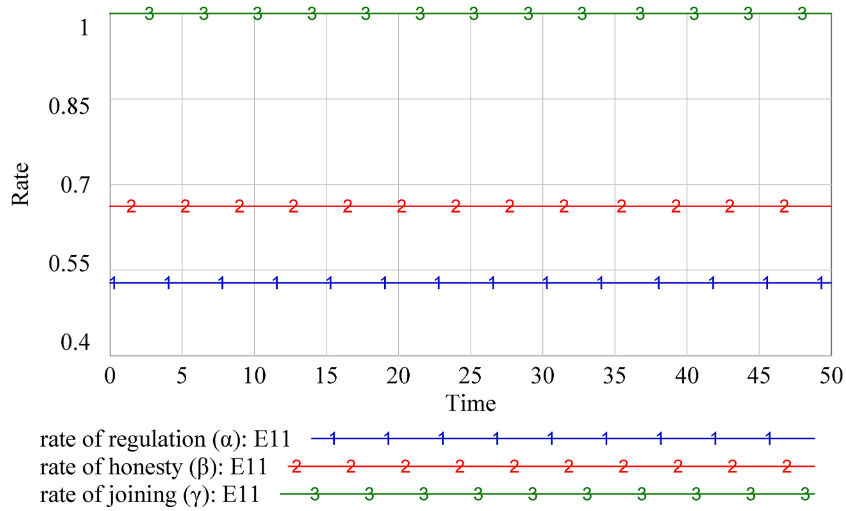

Fig. 3. Game results under initial strategy E_{11}.

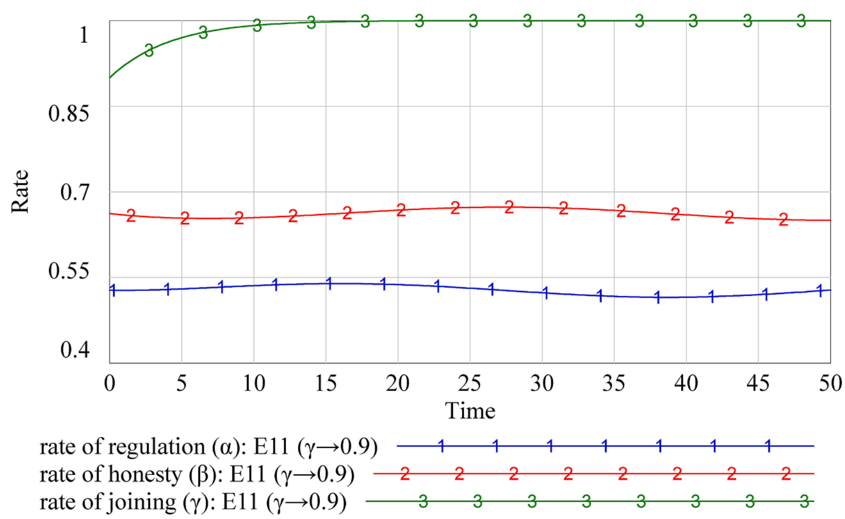

Fig. 4. Game results exists mutation $E_{11}(\gamma \rightarrow 0.9)$.

equilibrium solutions by observing the simulation results and found that $E_1 \sim E_{10}$ are also not ESS.

To sum up, a practical approach to analyzing the stability of equilibrium solutions is to use SD to simulate multi-player evolutionary game. When the CS, IC, and EC all maintain the initial strategies, the state of the system is stable and will not change with simulation time, and each player makes the best choice based on its own benefits. However, this equilibrium is unstable, once there is a mutation in one player's strategy, this steady state will be broken, which shows that all equilibrium solutions are not ESS. Therefore, there is no ESS in this game, and the behavior of multi-players will not be effectively controlled within a certain time.

4. Stability control scheme of federated learning

It is difficult to design a mature federated learning mechanism when the system is unstable. Therefore, it is necessary to study how to ensure the stability of the system to produce an effective mechanism.

4.1. Static incentive mechanism

A common idea in the design of federated learning incentive mechanisms is to increase the penalty for betrayed IC. In the above model, we adjust the strength of the CS's punishment on IC, i.e., the CS's punishment on betrayed clients is changed from $S = 4$ to $S = 6$ and to $S = 8$. The initial strategies of the three players are set as: $\alpha = 0.5$, $\beta = 0.5$, and $\gamma = 0.5$. The strategy choices for the CS and the IC under different penalty strengths are shown in Fig. 5 and Fig. 6.

According to the simulation results in Fig. 5, it can be seen that the fluctuation frequency and amplitude of the CS supervision rate will increase with punishment. Similarly, as can be seen from Fig. 6, in the same period, the honesty rate of the IC increases with penalty intensity. But the frequency and amplitude of fluctuations of the IC in the game process also increase with penalty intensity.

In conclusion, in the design of the federated learning incentive mechanism, simply increasing the penalty is not effective in restraining the fluctuations in the players' strategic choices, and there is still no ESS in the game. In addition, increasing the punishment can restrain the betrayal of IC in the short term. As the punishment rises, the honesty rate of IC will rise faster, and IC will temporarily choose not to betray. However, this strategic choice of players is not sustainable in the long term; this design approach can only obtain short-term achievements. There is still fluctuation in this game, and the magnitude and frequency of fluctuation increase. To address similar problems, some scholars proved that only increasing the punishment in a mixed strategy game cannot actually change the equilibrium position of the honesty

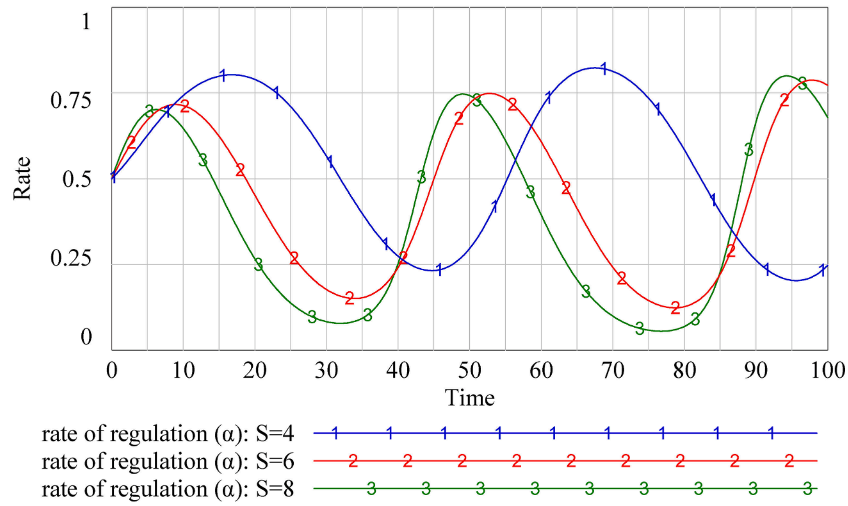

Fig. 5. Effect of different punishment on CS in static incentive mechanism.

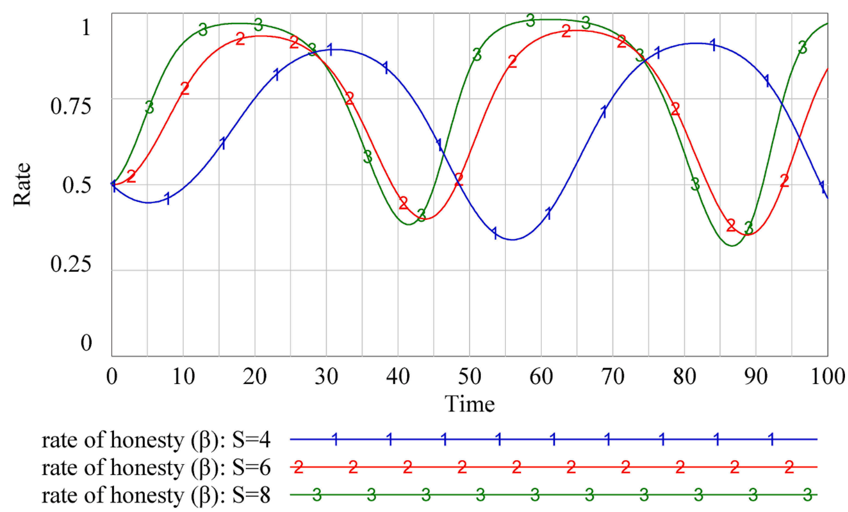

Fig. 6. Effect of different punishment on IC in static incentive mechanism.

probability of the punished[43]. In practice, increasing the punishment is widely used because it can increase the equilibrium position of the punishment in the short term[44]. However, this approach ignores that the payoff matrix of the game players is impacted by the increase of punishment, which makes the game more difficult to control.

4.2. Dynamic incentive mechanism

To restrain fluctuations in the operation of the federated learning system, several studies correlated rewards and punishments with the performance of all players [45]. Therefore, in the dynamic incentive mechanism, the CS implements dynamic punishment based on its regulation rate and dynamic reward based on the IC honesty rate and EC joining rate. The stricter the CS regulation, the heavier the punishment for IC. The lower the honesty rate of IC and the joining rate of EC, the more rewards that the CS will give to clients. So as to improve the multi-player strategy fluctuation situation, as shown in the following two formulas:

$$S^* = q_1 S\alpha, w_3^* = q_2 w_3(1-\beta)(1-\gamma)$$

Where q_1 and q_2 are the reward and punishment adjustment factors of the CS respectively, which are set to 8 and 10 here. The modified system dynamics model is shown in Fig. 7.

Set the initial strategies for the CS, IC, and EC to $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$ and $\alpha = 0.2$, $\beta = 0.6$, $\gamma = 0.9$ unplanned, the simulation results are shown in Fig. 8 and Fig. 9.

According to the simulation results in Fig. 8 and Fig. 9, it can be seen that in the dynamic incentive mechanism, even if the initial strategies are different, the three players will keep playing over time and finally stabilize at $E^*(0.4790, 0.8421, 0.7480)$. The strategy choices of CS, IC and EC in both figures need different time to reach stable state, which is influenced by the initial strategies. But they all show a trend of gradually decreasing amplitude until they no longer fluctuate. This is different from the simulation results in Fig. 5 and Fig. 6 in section 4.1. We subsequently simulated other 20 groups with randomly different initial strategies and get the same evolutionary process and stable point E^*. In addition, the time of reaching the stable state is influenced by the distance between the initial three-player strategy and E^*, the closer the two are, the faster the convergence rate. Thus, the fluctuations in the previous static incentive mechanism are eliminated and converge to a point. Then E^* is an evolutionarily stable equilibrium solution.

Since the possible values of the initial strategy are infinite, the SD simulation cannot traversal all possibilities, which is not rigorous enough in academic research. So we need to further verify the correctness of E^* according to Lyapunov stability theory , and substitute S^* and w_3^* into the equation group (9). The new replication dynamic equations

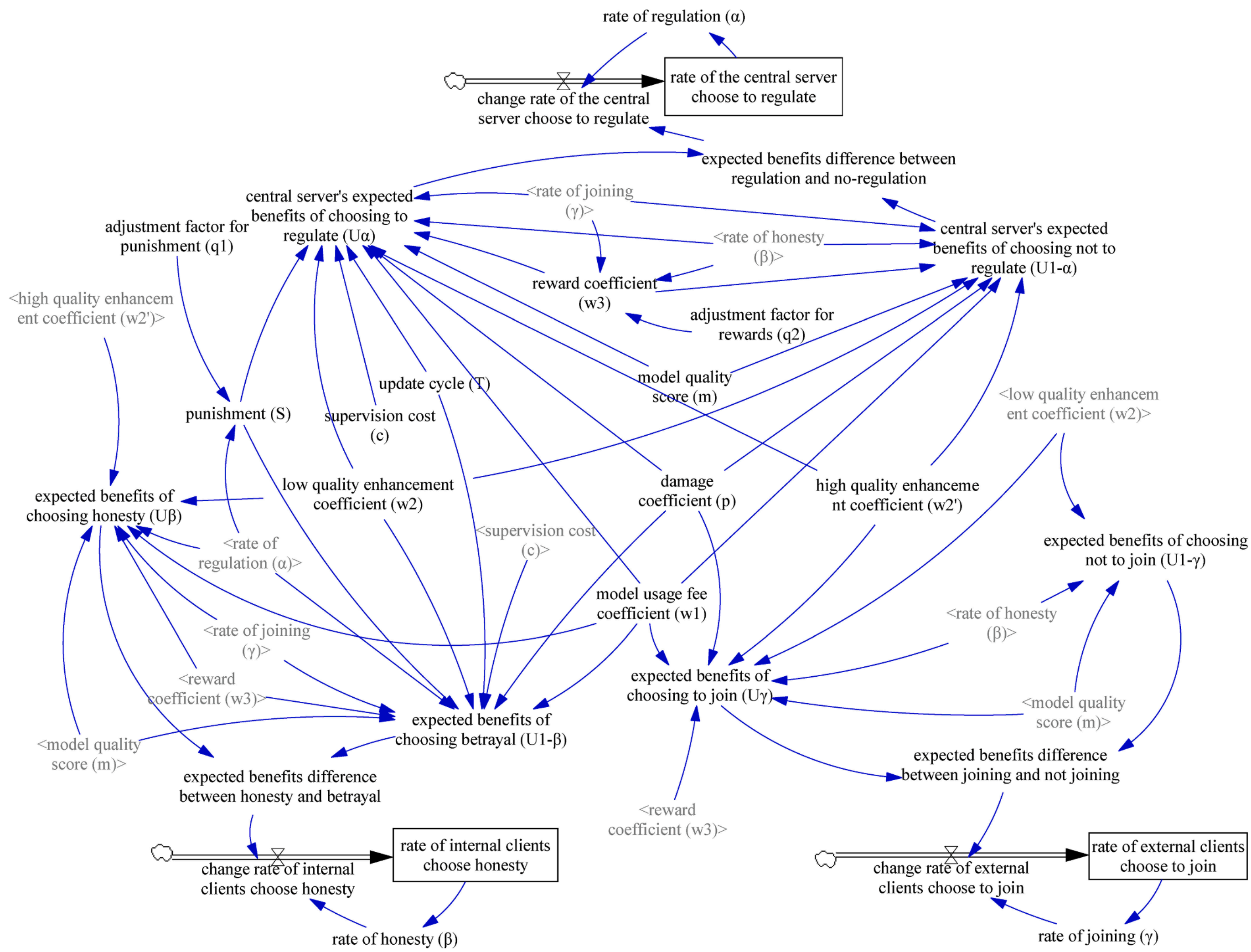

Fig. 7. Evolutionary game SD model under the dynamic incentive mechanism.

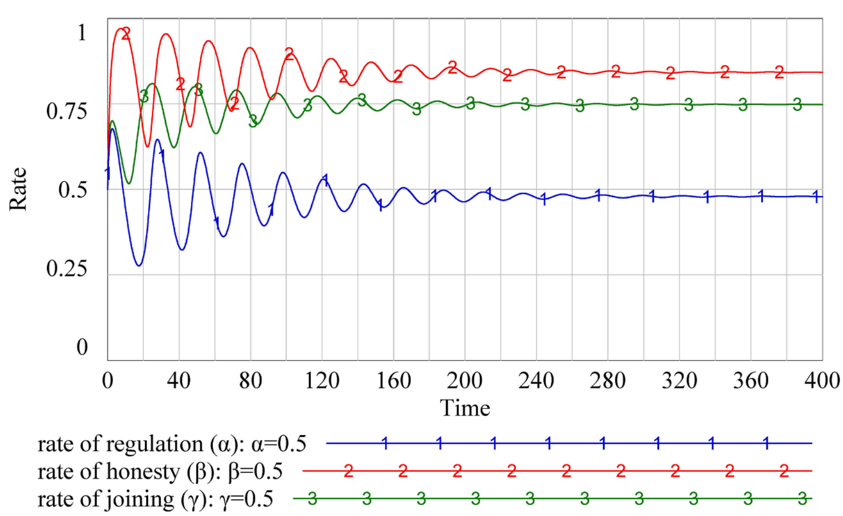

Fig. 8. Game results under dynamic incentive mechanism.

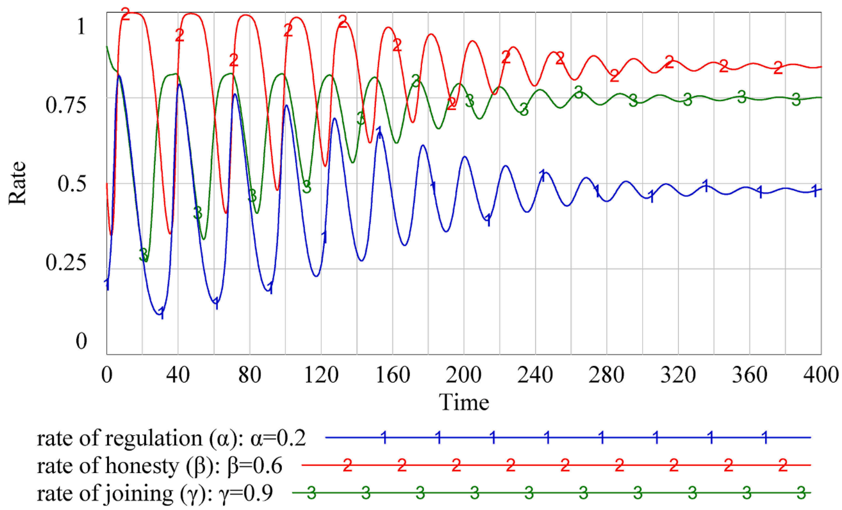

Fig. 9. Game results under dynamic incentive mechanism.

group is obtained as:

$$\begin{cases} F^{'}(\alpha) = \alpha(1-\alpha)\{ -c + \left(1 + \frac{1}{Tc}e^{-Tc} - \frac{1}{Tc}\right)(1-\beta) \\ [q_1 S\alpha + q_2 w_3(1-\beta)(1-\gamma)w_1 m(1-p)(\gamma w_2 + 1 - \gamma)]\} \\ F^{'}(\beta) = \beta(1-\beta)\{[1 + q_2 w_3(1-\beta)(1-\gamma)w_1 - w_1]m \\ [w_2^{'}\gamma + w_2(1-\gamma)] - m(1+p) + \alpha\left(1 + \frac{1}{Tc}e^{-Tc} - \frac{1}{Tc}\right) \\ q_1 S\alpha - (\gamma w_2 + 1 - \gamma)w_1 m(1-p)[q_2 w_3(1-\beta)(1-\gamma) \\ -1 - \alpha q_2 w_3(1-\beta)(1-\gamma)\left(1 + \frac{1}{Tc}e^{-Tc} - \frac{1}{Tc}\right)]\} \\ F^{'}(\gamma) = \gamma(1-\gamma)\{[1 + q_2 w_3(1-\beta)(1-\gamma)w_1 - w_1]m \\ [w_2^{'}\beta + w_2(1-\beta)(1-p)] - m\beta\frac{1}{w_2+1} - m(1-\beta)\} \end{cases} \tag{11}$$

In replication dynamic equation group (11), set $F^{'}(\alpha) = F^{'}(\beta) = F^{'}(\gamma) = 0$, the equilibrium solutions of 8 pure strategies and 7 mixed strategies can be obtained as follows: $E_1^*(0, 0, 0)^{\mathrm{T}}$, $E_2^*(0, 0, 1)^{\mathrm{T}}$, $E_3^*(0, 1, 0)^{\mathrm{T}}$, $E_4^*(0, 1, 1)^{\mathrm{T}}$, $E_5^*(1, 0, 0)^{\mathrm{T}}$, $E_6^*(1, 0, 1)^{\mathrm{T}}$, $E_7^*(1, 1, 0)^{\mathrm{T}}$, $E_8^*(1, 1, 1)^{\mathrm{T}}$, $E_9^*(0.561021, 0.852363, 1)^{\mathrm{T}}$, $E_{10}^*(0, 0, 0.770833)^{\mathrm{T}}$, $E_{11}^*(1, 0, 0.770833)^{\mathrm{T}}$, $E_{12}^*(0.0828277, 0, 1)^{\mathrm{T}}$, $E_{13}^*(0, 0.808036, 0)^{\mathrm{T}}$, $E_{14}^*(0, 0.144036, 0.796504)^{\mathrm{T}}$, $E_{15}^*(0.479061, 0.842166, 0.748053)^{\mathrm{T}}$.

The Jacobian matrix of the replicated dynamic equation group (11) is:

$$J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix} \tag{12}$$

In which,

$$\begin{gathered} J_{11}^* = \frac{\partial F^*(\alpha)}{\partial \alpha} = 8\left(3+e^{-4}\right)(1-\alpha)\alpha(1-\beta) \\ +(1-\alpha)\{ -2 + \left(3+e^{-4}\right)(1-\beta)[8\alpha + 8(1-\beta) \\ (1-\gamma)(1+0.2\gamma)]\} - \alpha\{ -2 + \left(3+e^{-4}\right)(1-\beta) \\ [8\alpha + 8(1-\beta)(1-\gamma)(1+0.2\gamma)]\} \end{gathered}$$

$$\begin{gathered} J_{12}^* = \frac{\partial F^*(\alpha)}{\partial \beta} = (1-\alpha)\alpha\{ - \left(3+e^{-4}\right)[8\alpha + 8(1-\beta) \\ (1-z)(1+0.2\gamma)] - 8\left(3+e^{-4}\right)(1-\beta)(1-\gamma)(1+0.2\gamma)\} \end{gathered}$$

$$\begin{gathered} J_{13}^* = \frac{\partial F^*(\alpha)}{\partial \gamma} = \left(3+e^{-4}\right)(1-\alpha)\alpha(1-\beta) \\ [1.6(1-\beta)(1-\beta) - 8(1-\beta)(1+0.2\gamma)] \end{gathered}$$

$$\begin{gathered} J_{21}^* = \frac{\partial F^*(\beta)}{\partial \alpha} = (1-\beta)\beta[16\left(3+e^{-4}\right)\alpha \\ +8\left(3+e^{-4}\right)(1-\beta)(1-\gamma)(1+0.2\gamma)] \end{gathered}$$

$$\begin{gathered} J_{22}^* = \frac{\partial F^*(\beta)}{\partial \beta} = (1-\beta)\beta\{ - 64(1-\gamma)(1.2+0.1\gamma) \\ -4[-8(1-\gamma) + 2\left(3+e^{-4}\right)\alpha(1-\gamma)](1+0.2\gamma)\} \\ +(1-\beta)\{ -15 + 8\left(3+e^{-4}\right)\alpha^2 + [2 + 64(1-\beta) \\ (1-\gamma)](1.2+0.1\gamma) - 4[-1 + 8(1-\beta)(1-\gamma) \\ -2\left(3+e^{-4}\right)\alpha(1-\beta)(1-\gamma)](1+0.2\gamma)\} \\ -\beta\{ -15 + 8\left(3+e^{-4}\right)\alpha^2 + [2 + 64(1-\beta) \\ (1-\gamma)](1.2+0.1\gamma) - 4[-1 + 8(1-\beta) \\ (1-\gamma) - 2\left(3+e^{-4}\right)\alpha(1-\beta)(1-\gamma)](1+0.2\gamma)\} \end{gathered}$$

$$J_{23}^{*} = \frac{\partial F^{*}(\beta)}{\partial \gamma} = (1-\beta)\beta\Big\{0.1[2+64(1-\beta)(1-\gamma)] -0.8\Big[-1+8(1-\beta)(1-\gamma)-2\left(3+e^{-4}\right)\alpha(1-\beta) (1-\gamma)\Big] - 64(1-\beta)(1.2+0.1\gamma) - 4\Big[-8(1-\beta) +2\left(3+e^{-4}\right)\alpha(1-\beta)\Big](1+0.2\gamma)\Big\}$$

$$J_{31}^{*} = \frac{\partial F^{*}(\gamma)}{\partial \alpha} = 0$$

$$J_{32}^{*} = \frac{\partial F^{*}(\gamma)}{\partial \beta} = \left\{\frac{60}{11} + 0.7[2+64(1-\beta)(1-\gamma)] -64(0.6+0.7\beta)(1-\gamma)\right\}(1-\gamma)\gamma$$

$$J_{33}^{*} = \frac{\partial F^{*}(\gamma)}{\partial \gamma} = \left\{-10+\frac{60\beta}{11}+(0.6+0.7\beta)\Big[2+64(1-\beta)(1-\gamma)\Big]\right\}(1-\gamma) - \left\{-10+\frac{60\beta}{11}+(0.6+0.7\beta)\Big[2+64(1-\beta)(1-\gamma)\Big]\right\}\gamma - 64(1-\beta)(0.6+0.7\beta)(1-\gamma)\gamma$$

Substituting $E_1^{*} \sim E_{14}^{*}$ into the Jacobian matrix (12) respectively, there are eigenvalues greater than 0 in the calculation results, so $E_1^{*} \sim E_{14}^{*}$ are not evolutionarily stable equilibrium solutions. After substituting E_{15}^{*} into the matrix, we get:

$$J^{*}\left(E_{15}^{*}\right) = \begin{bmatrix} 0.951115 & -3.43781 & -0.165013 \\ 3.22193 & -1.94567 & -1.07373 \\ 0 & -1.98731 & -2.2646 \end{bmatrix} \tag{13}$$

The eigenvalues of the matrix (13) are: λ_1 = - 0.253886 + 2.80762i, λ_2 = - 0.253886 −2.80762i, λ_3 = −2.75138. The real part of the eigenvalues are all less than 0. Therefore, $E_{15}^{*}(0.479061, 0.842166, 0.748053)^{\mathrm{T}}$ is the ESS of the system.

In summary, the mathematical derivation results are consistent with the computer simulation results. The ESS can be accurately obtained by simulating the evolutionary game through the SD model. The dynamic incentive mechanism effectively restrains the fluctuation and makes the model have a stable evolutionary equilibrium solution. In addition, under certain system parameter designs, the mechanism makes IC tend to be honest and EC tend to join while the CS maintains a moderate regulation rate. It shows a good incentive effect and enables the federated learning system to operate stably and efficiently.

4.3. Sensitivity analysis

In reality, the federated learning system may be subject to external perturbations or operate under some uncertain conditions, and the proposed mechanism should still be able to maintain its function. Therefore, sensitivity analysis is performed to further verify the robustness of the system. We find out the parameters in the model from several uncertainties one by one, which have an important influence on federated learning. We analyze the degree of influence and sensitivity of the parameters on the three players in the game, thus determining the ability of the system to resist risk. If a small change in a parameter can lead to a large change in the strategies of three players, then this parameter is called a sensitive factor, and otherwise, it is called a non-sensitive factor [46]. In this paper, we use the sensitivity module embedded in Vensim to analyze all the variables prone to fluctuate in the model. We make these variables fluctuate with a normal distribution, the fluctuation range is set to [-20%, +20%], and the initial strategies for the CS, IC, and EC are α = 0.5, β = 0.5, γ = 0.5. Taking the CS regulation cost $c = 2$ as an example, the minimum value of c is 1.6, the

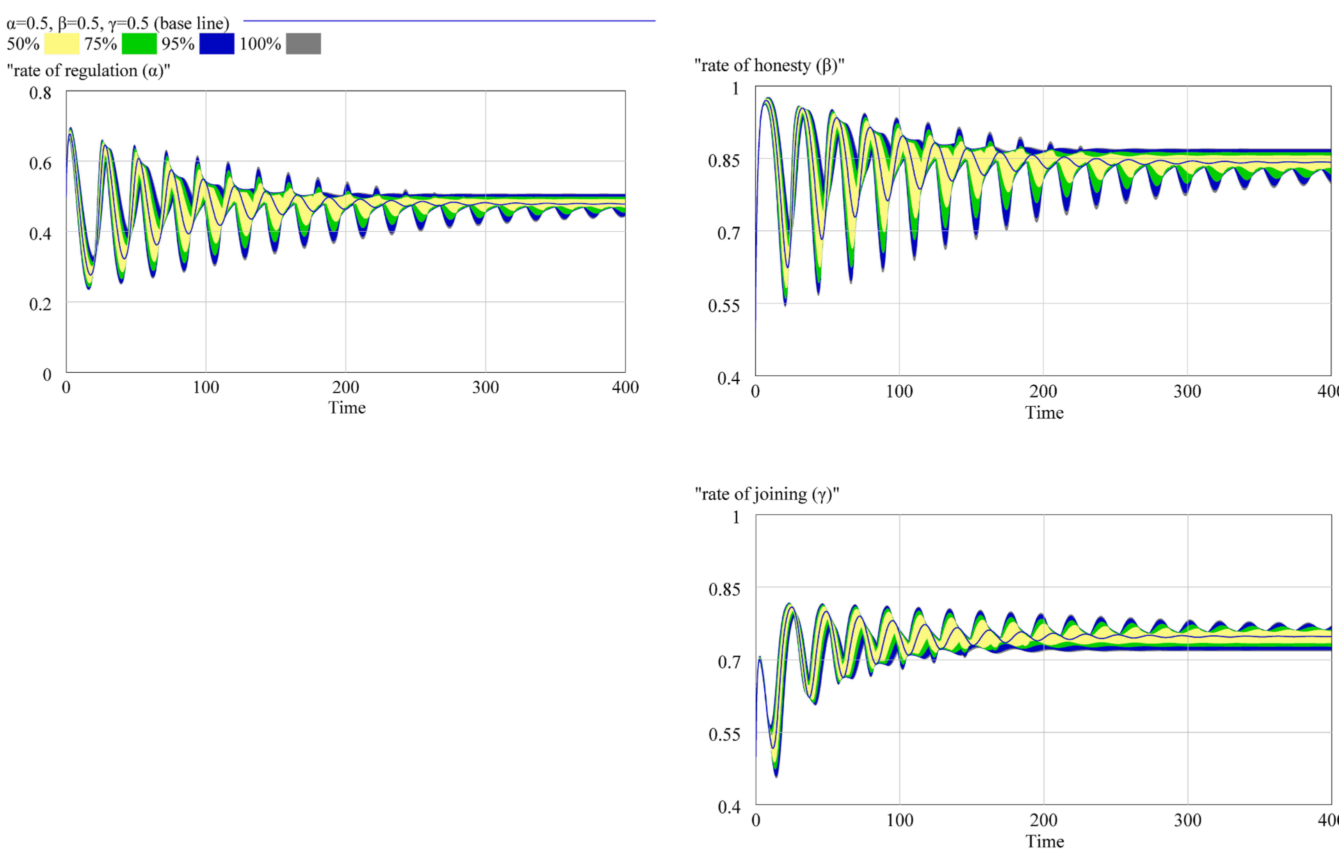

Fig. 10. Sensitivity analysis results (when the CS regulation cost c fluctuates).

maximum value is 2.4, the mean is 2, and the variance is 1, which follows the normal distribution. The simulation results are shown in Fig. 10. The colored areas in the figure are the range of possible changes of the player's strategy when the parameter changes. For example, the yellow area means that the player's strategy falling in the yellow area with 50% confidence level when the external parameter changes. Likewise, the confidence level for falling in the green area is 75%, blue is 95% and gray is 100%. As shown in the figures, the area of all colored areas is extremely small, so the change of c does not lead to a substantial change in the strategies of three players, so c is a non-sensitive factor. When c is changed, all players still achieve an evolutionary stable equilibrium, so the federated learning mechanism proposed in this paper maintains the stability. Similarly, after sensitivity analysis of other vulnerable variables, the simulation results further verify that the mechanism is robust and can contribute to the long-term stable and healthy operation of the federated learning system.

5. Conclusions

In this paper, a multi-player evolutionary game model with a CS, IC, and EC is developed to solve the problems in the operation of the federated learning system. A combination of mathematical analysis and computer simulation was used to mutually verify the correctness, and analyze the strategies of each player in different incentive mechanisms. The conclusions are as follows:

When the incentive mechanism is static, the strategy choices of the CS, IC, and EC fluctuate continuously. In other words, there does not exist an ESS in the game. In addition, the frequency and magnitude of fluctuations vary with the initial values of some variables. Simply changing the rewards and punishments will only get results in the short term and cannot restrain fluctuations. Instead, in the long term, it will increase the fluctuation of strategy choices for all players, and make the actual problem more difficult to control effectively.

In the dynamic incentive mechanism, the fluctuation of each player in the game is effectively restrained. The stable state and the equilibrium value are not affected by the change of initial values of the variables. The game has an ESS, and the mechanism shows a good incentive effect under a certain range of system parameters design. It enables the federated learning system to operate stably and efficiently.

Static incentives can quickly restrain the betrayal behavior of IC in the short term, and dynamic incentives can effectively restrain the fluctuations in the game. A good federated learning mechanism should not simply increase the punishment. However, it should increase the joining rate of EC while decreasing the betrayal rate of IC and avoid fluctuations in the game. This ensures the stability of the system while keeping all players' strategies in an ideal situation. In addition, Lyapunov stability theory and the method of SD simulation evolutionary game both can effectively analyze the system stability and determine the equilibrium solution. The sensitivity analysis shows that the SD model is generalize and can provide a reference for developing federated learning incentives.

However, there are some limitations of this study. For example, this paper does not consider the conspiracy between clients. Gaming becomes more complicated when multiple clients cooperate to betray the central server. Therefore, our future work will further explore this issue.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix. Equilibrium solutions

$$E_9 \left(\frac{ce^{cT}mT\left(-1-p+w_1-w_3w_1-pw_1+w_3pw_1+w_2-w_1w_2+w_3w_1w_2\right)}{\left(-w_3mw_1+w_3mpw_1-S\right)\left(1-e^{ct}+ce^{ct}T\right)}, \frac{-w_3mw_1+e^{cT}w_3mw_1+w_3mpw_1-e^{cT}w_3mpw_1-S+e^{cT}S+c^2e^{cT}T-ce^{cT}w_3mw_1T+ce^{cT}w_3mpw_1T-ce^{cT}ST}{\left(-w_3mw_1+w_3mpw_1-S\right)\left(1-e^{cT}+ce^{cT}T\right)}, 0 \right)^{\mathrm{T}}$$

$$E_{10} \left(\frac{ce^{cT}mT\left(-1-p+w_2^{'}-w_1w_2^{'}+w_3w_1w_2^{'}+w_1w_2-w_3w_1w_2-pw_1w_2+w_3pw_1w_2\right)}{\left(1-e^{cT}+ce^{cT}T\right)\left(-S-w_3mw_1w_2+w_3mpw_1w_2\right)}, \frac{-e^{cT}\left(S+c^2T-cST+w_3mw_1w_2-w_3mpw_1w_2-cw_3mw_1Tw_2+cw_3mpw_1Tw_2\right)-S-w_3mw_1w_2+w_3mpw_1w_2}{\left(1-e^{cT}+ce^{cT}T\right)\left(S+w_3mw_1w_2-w_3mpw_1w_2\right)}, 1 \right)^{T}$$

$$E_{11}\left(0,\frac{(1+w_2)(1-w_2+pw_2+w_1w_2-w_3w_1w_2-pw_1w_2+w_3pw_1w_2)}{w_2\left[w_1\left(1-w_3-p+w_3p-w_2^{'}+w_3w_2^{'}\right)+p+w_2^{'}\right]+w_2^2\left(-1+p+w_1-w_3w_1-pw_1+w_3pw_1\right)+w_2^{'}-w_1w_2^{'}+w_3w_1w_2^{'}},\frac{1+p-w_1+w_3w_1+pw_1-w_3pw_1-w_2+w_1w_2-w_3w_1w_2}{-w_1+w_3w_1+pw_1-w_3pw_1+w_2^{'}-w_1w_2^{'}+w_3w_1w_2^{'}-w_2+2w_1w_2-2w_3w_1w_2-pw_1w_2+w_3pw_1w_2}\right)^T$$

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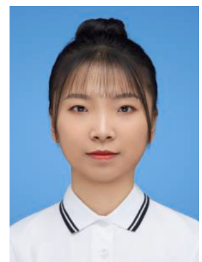

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