Participation Incentives in Online Cooperative Games

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ABSTRACT

This paper studies cooperative games where coalitions are formed online and the value generated by the grand coalition must be irrevocably distributed among the players at each time step. We investigate the fundamental issue of strategic participation incentives and address these concerns by formalizing natural participation incentive axioms. Our analysis reveals that existing value-sharing mechanisms fail to meet these criteria. Consequently, we propose a family of equal sharing rules that fulfill these desirable participation incentive axioms. Additionally, we refine our mechanisms under superadditive valuations to ensure individual rationality while preserving the previously established axioms.

KEYWORDS

Participation Incentives, Cooperative Games, Online Mechanism Design

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1 INTRODUCTION

Cooperative game theory studies the behavior of self-interested players in strategic settings when binding agreements among groups are feasible. A central objective is to design allocation rules that fairly distribute the collective benefits among players, thereby ensuring the cooperation incentives. Canonical cooperative games typically assume the static grand coalition containing all the involved players and the focus lies in determining how the total surplus should be allocated. Prominent solution concepts in this setting include the Shapley Value [33], the Banzhaf Index [6], and the Nucleolus [32], among others.

In many real-world contexts, however, coalitions do not emerge instantaneously. Instead, players may join sequentially, with the value generated at each stage requiring irrevocable allocation before the eventual grand coalition can be realized. Consider, for example, the formation of a startup: the venture may begin with a handful of founders, while additional contributors arrive over time, each bringing distinct skills and resources. It is typically infeasible for participants to defer all compensation until the final composition of the coalition is known, and in practice, it may be unclear whether the set of contributors has even stabilized. Such scenarios naturally give rise to **online cooperative games**, where each newly arriving player augments the coalition's value, which must then be distributed among current members. This dynamic setting introduces new strategic considerations, as players may strategically manipulate the timing of their entry into the coalition.

The formal online cooperative game model that considers strategic arriving behavior was first proposed by Ge et al. [22]. They study axioms including incentivizing players for early arrival, i.e., joining the coalition as early as possible, staying in the coalition as long as possible, and ex-ante fairness termed *Shapley-Fairness*, which requires the expected reward achieved by each player under uniform-distributed permutations equals to the offline Shapley Value. The authors first study the classic Shapley Value rule and the simple *Distributing Marginal Contribution (DMC)* rule, under which newly arriving player receives the whole marginal value. Unfortunately, the former rule faces the fundamental flaw that players might have incentive to leave the coalition as the total reward could decrease over time while the later one fails to incentivize players joining the coalition as early as possible as some players may delay their arrival for obtaining higher marginal contribution. In view of this, they proposed the *Rewarding First Critical Player* (RFC) rule, which allocates rewards to the first arriving player who is essential in generating value. While the RFC rule does not generally satisfy the incentive for early arrival, it has been shown that in 0–1 games, the incentive for early arrival, the incentive for staying, and Shapley-fairness cannot all be satisfied simultaneously, and the RFC rule fulfills these axioms whenever feasible.

In this paper, we primarily focus on participation incentive axioms, which not only include the *incentive for early arrival* and the *incentive for staying*, but also introduce new natural incentive principles. The first axiom we examine is the *Strong Incentive to Stay (S-STAY)*, which refines the incentive for staying studied by Ge et al. [22]. It not only requires that each player's allocated share be non-decreasing over time, but also ensures that any player being essential in generating marginal value with a newly arriving player receives a strictly positive reward. Intuitively, it provides a strong incentive for players to remain in the coalition, as continued participation can lead to additional benefits through collaboration with future arrivals. The second axiom we introduce is the Incentive for Participation (PART). This axiom requires that whenever a newly joining player contributes a positive marginal value to the existing coalition, she must receive an immediate positive reward, which can be interpreted as an *instant* incentive for players to participate in the coalition. The third aspect is a classic participation incentive axiom known as *Individual Rationality* (IR). According to IR, if a new player joins the coalition but cannot secure a share of the value greater than what she could achieve on her own, she may be discouraged from participating in the coalition. In summary, this paper aims to address the following central questions:

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For the online cooperative game setting, what are the key participation incentive properties? How do the existing rules fare with respect to these properties? Can we design new rules that perform even better with respect to participation properties?

1.1 Our Contribution

In this paper, we focus on participation incentive axioms in online cooperative games with strategic arrivals. Beyond the STAY and EA axioms studied by Ge et al. [22], we introduce three new axioms, S-STAY, PART, and IR, which capture natural and practical concerns in online value-sharing design. We first show that existing sharing rules, including the DMC, SV, and RFC rules, fail to satisfy these incentive axioms, motivating the development of new rules that fulfill all participation incentives. To this end, we propose a class of "Equal Sharing" rules, where at each stage, a subset of players equally share the new marginal contribution. We establish sufficient conditions under which these rules satisfy S-Stay, Part, and EA, and introduce two representative rules, *Marginal Equal Share* (MES) and *Non-Dummy Marginal Equal Share* (NDMES) rules. We further explore a greedy-based variant, termed *Upward Lexicographic Marginal Equal Share* (ULMES) rule, and extend it to eULMES rule via the game decomposition framework from Ge et al. [22]. Finally, we investigate the IR axiom under superadditive valuation functions and refine our proposed rules to ensure IR satisfaction while preserving all previously held axiomatic properties. Table 1 summarizes both existing and newly proposed rules with respect to their satisfaction of the studied axioms. All omitted proofs are provided in the Appendix due to space constraints.

Rules	**IR***	**PART**	EA	STAY	**S-STAY**	**OD**	SF	Poly-time
DMC	✓	✓	–	✓	–	✓	✓	✓
SV	✓	✓	✓	–	–	✓	✓	✓
eRFC	–	–	–	✓	–	✓	✓	–†
MES	–	✓	✓	✓	✓	–	–	✓
NDMES	–	✓	✓	✓	✓	✓	–	–‡
ULMES	–	✓	–	✓	✓	✓	–	✓
eULMES	–	✓	✓	✓	✓	✓	–	–†
IR-eULMES	✓	✓	✓	✓	✓	✓	–	–†

Table 1: Summary of results: axioms and rules in bold are newly presented in this paper. †: Poly-time in 0-1 online cooperative games; ‡: Poly-time with subadditive valuation. *: IR axiom is considered in superadditive valuation.

1.2 Related Work

Cooperative Games. Cooperative game theory, originating from the last century [33, 35], is a significant branch of game theory that studies scenarios where players can benefit by forming coalitions and making collective decisions. One of the key problems in this area is how to distribute the value created by coalitions among players, considering axiomatic characterizations (e.g., stability, consistency, etc.). The Shapley Value [33] initiated the research, laying the foundation for a series of subsequent works. von Neumann et al. [35] first proposed the core concept for cooperative games. In the context of transferable utility cooperative games, Shubik [34] studied market games, while Aumann and Maschler [3] investigated cooperative bargaining scenarios. Schmeidler [32] first introduced the concept of the nucleolus, and Roth and Sotomayor [31] bridged cooperative game theory with practical matching markets. There is also a line of research focusing on cooperative games with hedonic preferences [1, 4, 5, 12, 16]. Further details about classic cooperative game theory can be found in several books (see, e.g., [11, 15]).

Online cooperative games study the games in an online manner where players arrive in a random order and the coalition formation decision should be made without any knowledge regarding the players arriving in the future. Our paper is closely related to the work by Ge et al. [22], which was the first to study online cooperative games with consideration of strategic arrivals. Recently, Zhang et al. [36] explores the cost sharing game in the context of online strategic arrivals and propose the Shapley-fair shuffle cost sharing mechanisms. Zhang et al. [37] consider the online cooperative game model where each arriving player can choose to create new coalition or join an existing coalition and design value-sharing policies to optimize the competitive ratio with respect to social welfare. Another branch studying cooperative game in an online manner, mainly concerning on hedonic games, focuses on addressing approximation to the social welfare and stability [13, 19]. The biggest difference from the aforementioned online cooperative game is that it typically assume that players reveal their preferences truthfully without incentive to misreport. Moreover, Flammini et al. [19] studied the online coalition structure generation problem, while Bullinger and Romen [13] investigated online coalition formation with random arrival. An online or dynamic perspective has also been applied to matching and hedonic games (see, e.g., [10, 14, 17]).

Online Mechanism Design. In the online cooperative game model where players can strategically manipulate their arrival time, the arrival time is private information and the problem can be viewed as a dynamic mechanism design problem. Mechanism design in dynamic environments focuses on problems involving multiple players with private information, where the goal is to elicit this private information while making decisions without knowledge of future events. There is a vast body of work considering mechanism design in the online manner [2, 7, 8, 27]. Lavi and Nisan [26] initiated the study of truthful online auctions in dynamic environments. Later, Friedman and Parkes [20] coined the concept of online mechanism design. Some works [28, 29] discussed the state-of-the-art VCG mechanism in dynamic online settings. Online matching has also been a hot topic in dynamic algorithm design [18, 21, 24, 25]. Moreover, there is a wide literature on solutions for different sequential mechanism design problems, including scheduling [30], online combinatorial optimization [9, 23].

2 PRELIMINARY

2.1 Model

An online cooperative game (OCG) G is a triple (N, v, π), where $N := \{1, 2, \ldots, n\}$ is the set of n players, $v : 2^N \to \mathbb{R}_+$ is the valuation function mapping a subset of players to a non-negative real number, and $\pi \in \Pi(N)$ is a permutation of N representing the arrival order of all the players over all possible permutations $\Pi(N)$. Given any subset $S \subseteq N$, S creates a coalition with value $v(S)$. In this paper, we focus on *normalized* and *monotone* general valuation function: (1). *Normalized*: $v(\emptyset) = 0$; (2). *Monotone*: $S \subseteq T \subseteq N$, $v(T) \geq v(S)$. We then introduce several types of functions regarding the valuation set function. A valuation function $v : 2^N \to \mathbb{R}$ is *submodular* if for any S, T s.t. $S \subseteq T \subseteq N$, we have $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$; A valuation function $v : 2^N \to \mathbb{R}$ is *subadditive* if for any S, T s.t. $S \subseteq T \subseteq N$, $v(S) + v(T) \geq v(S \cup T)$; A valuation function $v : 2^N \to \mathbb{R}$ is *superadditive* if for any S, T s.t. $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$.

Given a permutation π, for any pair of players $i, j \in N$, let $i \prec_\pi j$ denote that player i arrives earlier than player j, according to the permutation π. An *online value-sharing rule* ϕ maps the game $G = (N, v, \pi)$ to an n-tuple $\phi(G) = (\phi(G, 1), \ldots, \phi(G, n))$, where $\phi(G, i)$ denotes the value assigned to player i. For any player $i \in N$, we assume $\phi(G, i) \geq 0$ and $\sum_{i \in N} \phi(G, i) = v(N)$.

We now introduce two significant definitions that will serve as the foundation for the axioms and sharing rules. The first is the notion of a **prefix subgame**. A subset $S \subseteq N$ is called a *prefix* of the arriving order π if S consists of the first $|S|$ players to arrive. In this case, we write $S \sqsubseteq \pi$. Given an OCG $G = (N, v, \pi)$ and a prefix $S \sqsubseteq \pi$, the prefix subgame is defined as $G^S = (S, v_{|S}, \pi_{|S})$, where $v_{|S}$ is the valuation function restricted to coalitions $C \subseteq S$, and $\pi_{|S}$ denotes the arrival order of players in S. For any player i in the arriving order π, we denote by $N_{\pi|i}$ the prefix consisting of all players who arrive no later than i (including i). The corresponding prefix subgame is then $G^{N_{\pi|i}} = (N_{\pi|i}, v_{|N_{\pi|i}}, \pi_{|N_{\pi|i}})$. When the context is clear, we simplify this notation by writing $G^i = (N_{|i}, v_{|i}, \pi_{|i})$.

We next define the **Simple Online Cooperative Game** (SOCG) with a restrictive 0-1 valuation function. In an SOCG, there is a *pivotal* player such that, upon her arrival, the value of the coalition jumps from 0 to 1. Given that the valuation function is monotone, the grand coalition value remains at 1 after all players have arrived, i.e., $v(N) = 1$. Formally, an OCG $G = (N, v, \pi)$ is an SOCG if the valuation function $v(\cdot)$ satisfies: $\forall S \subseteq N, v(S) \in \{0, 1\}$ and $v(\emptyset) = 0, v(N) = 1$.

2.2 Incentive Axioms from Ge et al. [22]

We revisit some existing axioms introduced by Ge et al. [22]. The first property is termed *Incentive to Stay* [1], which guarantees that each player's shared value is non-decreasing as more players arrive. This encourages the arrived players staying in the grand coalition for more potential rewards.

Definition 2.1 (Incentive to Stay (STAY)). An online value-sharing rule ϕ_G satisfies *incentive to stay (STAY)* if given an OCG $G = (N, v, \pi)$, for any two prefix subgames $G^S = (S, v_{|S}, \pi_{|S})$ and $G^T = (T, v_{|T}, \pi_{|T})$, where $T \subseteq S$, and $T, S \sqsubseteq \pi$, every player $q \in T$ satisfies $\phi(G^T, q) \leq \phi(G^S, q)$.

STAY is the first natural participation incentive axiom studied by Ge et al. [22]. Next, we revisit another participation incentive axiom called *Incentive for Early Arrival*. Recall that in the online cooperative game model, the arrival time of each agent is treated as private information. So players might strategically choose to delay their arrival for extra benefits. The axiom of EA requires that for every player i, when fixing the arrival order of all other players, arriving as early as possible is the dominant strategy for player i.

Definition 2.2 (Incentive for Early Arrival (EA)). An online value-sharing rule $\phi(G)$ incentivizes early arrival (EA) if, for any two OCGs $G = (N, v, \pi)$ and $G' = (N, v, \pi')$, for each player i, it always holds $\phi(G, i) \geq \phi(G', i)$ whenever $\pi_{|N \setminus \{i\}} = \pi'_{|N \setminus \{i\}}$ and $N_{\pi|i} \subset N_{\pi'|i}$.

Before introducing the next axiom called *Shapley-Fairness*, we first introduce the concept of marginal contribution and the classic Shapley Value. Given an OCG $G = (N, v, \pi)$, for each player $i \in N$, we define the *marginal contribution* of i to a coalition S in G as $\mathrm{MC}(G, S, i) = v(S \cup \{i\}) - v(S)$. Based on the definition of marginal contribution, we introduce the Shapley Value.

Definition 2.3 (Shapley Value [33] (SV)). Given an OCG $G = (N, v, \pi)$, each player i's Shapley Value is

$$\mathrm{SV}(G, i) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|! \cdot (|N| - |S| - 1)! \cdot \mathrm{MC}(G, S, i).$$

The Shapley value assigns to each player their average marginal contribution across all possible coalitions. It ensures that players are rewarded fairly based on how much they add to the value of any coalition they join. A follow-up definition, termed *Shapley-Fairness* extends the Shapley Value in online cooperative games.

Definition 2.4 (Shapley-Fairness (SF)). Given an OCG $G = (N, v, \pi)$, an online value-sharing rule $\phi(G)$ is said to satisfy *Shapley-Fairness (SF)* if, for each player $i \in N$,

$$\frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \phi(G, i) \; = \; \mathrm{SV}(G, i),$$

Intuitively, *SF* requires that, for any online coalition game (OCG) G, if all arrival orders of the players are equally likely, then the expected payoff of each player coincides with her Shapley value. However, Shapley-Fairness only guarantees fairness in expectation across permutations; for a fixed arrival order, the payoffs assigned by an SF rule need not match the Shapley values exactly. In online settings, by contrast, only one arrival order is realized, and the total value must be allocated sequentially as players appear. This makes SF overly restrictive and motivates relaxing the requirement for two key reasons. First, incentivizing early arrivals necessarily reduces the shares available to later players, since the total value of the grand coalition is fixed; Shapley-Fairness, being insensitive to arrival order, fails to capture this trade-off and therefore cannot serve as a participation-incentive axiom. Second, beyond its incompatibility with STAY and EA (as shown by Ge et al. [22]), *SF* is conceptually misaligned with the sequential nature of online value sharing.

2.3 New Incentive Axioms

In the previous section, we revisited two participation incentive axioms, STAY and EA, originally introduced by Ge et al. [22]. We now turn to additional natural axioms that further refine our understanding of participation incentives in online cooperative games. To formulate these axioms, we first introduce an auxiliary notion

[1] In the original paper [22], this property is referred to as Online Individual Rationality (OIR). However, it differs from the classic notion of individual rationality. As discussed in Section 2.3, we revisit the axiom of individual rationality (IR) that aligns with the classic notion.

for players whose arrival generates a strictly positive marginal contribution, which we refer to as *contributional players*.

Definition 2.5 (Contributional Player). Given an OCG $G = (N, v, \pi)$, for each player i in arriving order π, if $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{i\})$, then player i is called a *contributional* player under permutation π in G.

Building on this notion, we propose new participation incentive axioms. The first, termed *Strong Incentive to Stay* (S-STAY), strengthens the Stay axiom by imposing a more stringent requirement.

Definition 2.6 (Strong Incentive to Stay (S-STAY)). An online value-sharing rule $\phi(G)$ satisfies *Strong Incentive to Stay* (S-STAY) if, for any OCG $G = (N, v, \pi)$, it satisfies the STAY axiom, and, for every contributional player i, and every player $j \prec_\pi i$ such that $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, it holds that for player j, $\phi(G^i, j) > \phi(G^j, j)$.

Intuitively, S-STAY requires not only that each player's cumulative allocation be non-decreasing, but also that if an already-arrived player j is *essential* for creating the positive marginal value generated by a newly-arrived contributional player i, then j must receive a share of this newly created value. In this way, S-STAY provides stronger incentives for players to remain in the grand coalition, as it ensures that they can benefit from potential future cooperation with later arrivals. We refer to the second axiom as *Incentive for Participation* (PART), which ensures that each arriving player receives an immediate share of value if they contribute to the coalition. We consider this a minimal and natural requirement in the online setting.

Definition 2.7 (Incentive for Participation (PART)). An online value-sharing rule $\phi(G)$ satisfies *Incentive for Participation* (PART) if, for any OCG $G = (N, v, \pi)$ and every player i, whenever i is a contributional player, it holds that $\phi(G^i, i) > 0$.

Intuitively, PART requires that in any OCG G, every contributional player i receives a strictly positive share immediately upon joining the coalition. S-STAY and PART are two natural axioms which give players strong incentive to participate to join the coalition. To give a more intuitive feeling, consider a simple example, an SOCG with two players, player 1 joins the coalition first. When player 2 subsequently joins, the coalition value increases to 1, whereas neither player alone can generate this value. In this case, S-STAY provides a strong guarantee for player 1 to join and remain in the coalition as she will benefit once player 2 arrives. PART incentivizes player 2 joining since she is a contributional player and she will receive immediate reward. However, in the next section, we will show that prior existing sharing rules fail these simple and natural axioms.

Apart from these two new axioms, we also consider the classical notion of *Individual Rationality* (IR), which aligns with its definition in standard cooperative games and can likewise be viewed as a participation incentive axiom.

Definition 2.8 (Individual Rationality (IR)). Given an OCG $G = (N, v, \pi)$, an online value-sharing rule ϕ is individual rational (IR) if for each player $i \in N$, $\phi(G, i) \geq v(\{i\})$.

Although we relax the Shapley-Fairness axiom in this paper, we consider an alternative fairness criterion by introducing the concept of *Online-Dummy* (OD) to evaluate value-sharing rules. The Online-Dummy concept is defined within each prefix subgame G^i, requiring that any *dummy player* in G^i receives no share of the marginal value $\mathrm{MC}(G^i, N_{\pi|i} \setminus i, i)$. Consequently, the Online-Dummy property generalizes the Dummy axiom from classical cooperative games. We begin by formally defining a *dummy player*.

Definition 2.9 (Dummy Player). Given an OCG $G = (N, v, \pi)$, for player i, if $v(S \cup \{i\}) = v(S)$ for any $S \subseteq N$, then player i is a dummy player in G.

In classical cooperative games, the *Dummy* axiom requires that every dummy player receives no positive share of the value. We now extend this axiom to the online setting by introducing the *Online Dummy* axiom.

Definition 2.10 (Online Dummy (OD)). An online value-sharing rule ϕ satisfies Online-Dummy if for each prefix subgame $G^i = (N_{|i}, v_{|i}, \pi_{|i})$, for any dummy player j in game G^i, $\phi(G^i, j) = 0$.

OD axiom requires that dummy players receive zero payoff not only in the final allocation but also at every subgames. In particular, any dummy player in a certain prefix subgame receives no share at that stage, even though they may later contribute positively through interactions with subsequently arriving players.

3 LIMITATION OF EXISTING RULES

In this section, we first revisit three existing online value-sharing rules: the Distribute Marginal Contribution (DMC) rule, the Shapley Value (SV) rule, and the extended Reward First Critical Player (eRFC) rule [22] and highlight their limitations with respect to the participation incentive axioms introduced above.

DMC rule simply distributes the whole new marginal value to the newly arrived player, i.e., for each arriving player i, $\phi(G, i) = \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$. Ge et al. [22] prove that it satisfies STAY and SF, but fails EA in general[2]. SV rule simply computes the Shapley value for each prefix subgame and distributes these values among the players. When the grand coalition is eventually formed, each player's allocation coincides with its Shapley value. Ge et al. [22] prove that it satisfies both EA and SF, but fails STAY. Furthermore, Ge et al. [22] proposed a novel rule, termed the **R**eward the **F**irst **C**ritical Player (RFC) rule, which was initially developed for SOCGs and later extended to general OCGs through game decomposition.

Reward First Critical Player (RFC) Rule

Input: $G = (N, v, \pi)$
$\forall\, i \in N$, initialize $\phi(G, i) \leftarrow 0$;
For each player i in arriving order π,
  Let $S^i \leftarrow \{j \mid j \in N_{\pi|i}, v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})\}$;
  $j^* \leftarrow$ first arrived player in S^i;
  $\phi(G, j^*) \leftarrow \phi(G, j^*) + \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$;
Output: $\phi(G)$.

Given any SOCG G and any prefix subgame G^i with arriving player i, the RFC rule first identifies the set of players S^i who are essential for generating the positive marginal contribution, namely, those players whose absence would cause the coalition value to

[2]It satisfies EA if and only if $v(\cdot)$ is submodular

drop from 1 to 0. The entire marginal contribution is then allocated to the first-arriving player in S^i according to the arrival order π.

For general OCGs, Ge et al. [22] further proposed a greedy monotone (GM) decomposition method (see Appendix A for algorithm details and examples) and extended the RFC rule to the eRFC rule by decomposing an OCG into multiple SOCGs and aggregating the results obtained from applying the RFC rule to each decomposed SOCG. The RFC rule is shown to satisfy Stay and SF, but not EA, even within SOCGs. Moreover, Ge et al. [22] demonstrated the inherent incompatibility among Stay, EA, and SF by proving the existence of a class of unsolvable games, in which no value-sharing rule can simultaneously satisfy all three properties. The RFC rule thus satisfies all three properties except for these unsolvable cases.

In light of the aforementioned impossibility result and our focus on participation incentive axioms, we relax the SF axiom and investigate the extent to which all participation incentive axioms, including Stay, EA, S-Stay, and Part, can be simultaneously satisfied by value-sharing rules. Our first observation is that, unfortunately, the three existing value-sharing rules not only violate Stay and EA, but also fail to satisfy our newly introduced S-Stay and Part axioms. We summarize these shortcomings below.

Proposition 3.1. *SV and DMC satisfy PART while eRFC does not.*

Proof. Consider the following SOCG $G = (N, v, \pi)$, where $N = \{1, 2\}, \pi = (1, 2)$ and $v(\{1\}) = v(\{2\}) = 0, v(\{1, 2\}) = 1$. Firstly, player 1 arrives with no value, then player 2 arrives and the grand coalition value is 1. eRFC rule allocates the entire value 1 to player 1, who is the first arrived player and contributes to the grand coalition. However, player 2, as a contributional player, receives zero shared value, thereby violating the PART axiom. □

Proposition 3.2. *DMC, SV, and eRFC rules do not satisfy S-STAY.*

Proof. Since SV rule fails to satisfy STAY, it does not satisfy S-STAY. For DMC and eRFC rules, consider an SOCG $G = (N, v, \pi)$ where $N = \{1, 2, 3\}$, $\pi = (1, 2, 3)$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0, v(\{1, 2, 3\}) = 1$. DMC allocates the value 1 to player 3 because 3's arrival creates the new marginal value 1, i.e., $\phi(G, 1) = \phi(G, 2) = 0, \phi(G, 3) = 1$. It does not satisfy S-STAY for player 1 and 2 as they both contribute to the grand coalition after player 3's arrival, in which S-STAY requires player 1 and 2 should receive some positive value. With regard to eRFC rule, value 1 will be wholely allocated to player 1, i.e., $\phi(G, 1) = 1, \phi(G, 2) = \phi(G, 3) = 0$. Note that player 2, who is essential for creating the value 1, however, gets 0 in G. This violates S-STAY which requires that $\phi(G, 2) > 0$. □

Remark 1. *The DMC, and eRFC rules fail not only the EA axiom but also our newly introduced incentive axioms, S-STAY and PART, in general. Although the SV rule satisfies EA, this is rather trivial, as its allocation is entirely independent of the arrival order. These observations motivate us to investigate whether there exist sharing rules satisfying all three main incentive axioms simultaneously. In the next section, we address this question by introducing a family of sharing rules, termed the "Equal Sharing Rule".*

4 NEW DESIRABLE RULES

In the previous section, we highlighted the failure of existing rules to satisfy the existing and newly proposed incentive axioms, including EA, S-STAY, PART, etc. Motivated by these concerns, we study a family of value-sharing rules, termed **E**qual **S**haring rules, which based on the simple and natural idea that once a new contributional player joins the coalition, the value is equally by a set of players where the set varies differently by designing different types of rules.

> Equal Sharing Scheme
>
> **Input**: $G = (N, v, \pi)$
> **For** each player i in arriving order π,
>   Initialize $\phi(G, i) \leftarrow 0$.
>   Decide the sharing set S^i.
>   **For** each player $j \in S^i$,
> $\phi(G, j) \leftarrow \phi(G, j) + \frac{1}{|S^i|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$.
> **Output**: $\phi(G)$.

4.1 Initial Attempts: MES and NDMES

Based on the selection of the sharing set S^i in each subgame, we consider three different types of equal sharing rules. We begin by considering a simple sharing rule, termed **Marginal Equal Share (MES)** rule, which distributes the marginal contribution among all the existing players, i.e., for each prefix subgame G^i with arriving player i, $S^i = N_{\pi|i}$. Interestingly, this simple MES rule satisfies not only PART and S-STAY, but also EA (Proposition 4.4).

Although the simple MES rule satisfies all of these three participation incentive axioms, it is easy to verify that the value sharing may be perceived unfair as some of the players could be the "free-riders", i.e., those dummy players who never contribute to the grand coalition also receive positive rewards. That is, MES rule fails the Online Dummy (OD) axiom. This directly rises a immediate question, can we exclude these dummy players while persevering these incentive axioms? This leads to our second value-sharing rule, **Non-Dummy Marginal Equal Share (NDMES)** Rule, which allocates the marginal contribution to all the existing non-dummy players. Specifically, for each prefix subgame G^i, let D_i denote the dummy player set under G^i. Define $S^i = N_{\pi|i} \setminus D_i$ as the sharing player set. For each player $j \in S^i$, j receives an equal share of $\frac{1}{|S^i|} \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$. It is straightforward that NDMES rule satisfies OD as in each prefix subgame, by the definition of NDMES, dummy players will never be included in the value sharing set. Surprisingly, this rule not only satisfies OD, but also maintains the satisfaction of S-STAY, EA, and PART (Proposition 4.4).

Before formally proving that the MES and NDMES rules satisfy the S-Stay, Part, and EA axioms, we first introduce two key structural properties, *sharing consistency* and *order independence*.

Definition 4.1 (Sharing Consistency). Given any OCG $G = (N, v, \pi)$, for any Equal Sharing rule ϕ, we say that ϕ is *sharing consistent* if, for any prefix subgames G^i and G^j with arriving players i and j, where $i \prec_\pi j$, it holds that $S^i \subseteq S^j$.

Recall that for any prefix subgame G^i with arriving player i, S^i denotes the sharing set of players in G^i under the Equal Sharing scheme. The property of *sharing consistency* requires that once an player $k \in S^i$, this player remains in the sharing set for all subsequent prefix subgames G^j where $j \succ_\pi i$.

Definition 4.2 (Order Independence). Given two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$, for any Equal Sharing rule ϕ, we say that ϕ is *order independent* if, for any $i, j \in N$ such that $N_{\pi_1|i} = N_{\pi_2|j}$, it holds that $S_1^i = S_2^j$.

Intuitively, *order independence* means that for any two prefix subgames G_1^i and G_2^j with the arriving players i and j, if the set of existing arrived players (including i and j) is identical, i.e., $N_{\pi_1|i} = N_{\pi_2|j}$, then the resulting sharing sets S_1^i and S_2^j are the same. In other words, the sharing sets of subgame G_1^i and G_2^j are independent of the arrival order among the players in $N_{\pi_1|i}$ and $N_{\pi_2|j}$.

Building on these two properties, we establish a key lemma stating that if an Equal Sharing rule satisfies S-STAY, PART, sharing consistency, and order independence, then it also satisfies the EA axiom.

Lemma 4.3. *For any Equal Sharing rule ϕ that satisfies S-STAY and PART, if ϕ satisfies Sharing Consistency and Order Independence, then it satisfies EA.*

Proof. Let ϕ denote an Equal Sharing rule that satisfies S-Stay and Part. Assume further that ϕ satisfies *Sharing Consistency* and *Order Independence.* Consider two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$ that differ only in the arrival order of some player $i \in N$, where i delays her arrival in π_2 relative to π_1:

$$\pi_1 : (1, 2, \ldots, i-1, \mathbf{i}, i+1, \ldots, j-1, j, j+1, \ldots, n);$$
$$\pi_2 : (1, 2, \ldots, i-1, i+1, \ldots, j-1, j, \mathbf{i}, j+1, \ldots, n).$$

We prove that ϕ satisfies EA axiom by discussing two cases based on whether i is a contributional player in G_1. For any player i, let S_1^i (resp. S_2^i) denote the sharing set in subgame G_1^i (resp. G_2^i).

Case 1. i is a contributional player in G_1. Since ϕ satisfies Part, we have $i \in S_1^i$. By sharing consistency, the total reward received by i in G_1 can be expressed as

$$\phi(G_1, i) = \sum_{k=i}^{n} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k).$$

On the other hand, in G_2, player i's total share can be upper-bounded by

$$\phi(G_2, i) \leq \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + \sum_{k=j+1}^{n} \frac{1}{|S_2^k|} \mathrm{MC}(G_2, N_{\pi_2|k} \setminus \{k\}, k).$$

Since ϕ satisfies order independence, the corresponding sharing sets in the two games satisfy $S_1^j = S_2^i$, and for all $k \in \{j+1, \ldots, n\}$, $S_1^k = S_2^k$. Thus, the value shared with i from players arriving after j is identical in both games and cancels out in the comparison. We therefore focus on the difference of shared values for player j and derive that

$$\begin{aligned}
&\phi(G_1, i) - \phi(G_2, i) \\
&\geq \sum_{k=i}^{j} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k) - \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) \\
&\geq \frac{1}{|S_1^j|} \sum_{k=i}^{j} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k) - \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) \\
&\hfill \text{(Sharing consistency)} \\
&\geq \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j}) - v(N_{\pi_1|i} \setminus \{i\}) \big) - \frac{1}{|S_2^i|} \big(v(N_{\pi_2|i}) - v(N_{\pi_2|j}) \big) \\
&\hfill \text{(MC definition)} \\
&\geq \frac{1}{|S_1^j|} \big(v(N_{\pi_2|j} - v(N_{\pi_1|i} \setminus \{i\})) \big) \qquad (S_1^j = S_2^i, N_{\pi_1|j} = N_{\pi_2|i}) \\
&\geq 0. \qquad \text{(Valuation monotonicity)}
\end{aligned}$$

This implies that when player i is a contributional player, i has no incentive to delay her arrival.

Case 2. If i is not a contributional player in G_1. The first sub-case is that i is also not a contributional player in G_2, then it follows that $\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) = 0$. Since ϕ satisfies order consistency, for every $k \in \{j+1, \ldots, n\}$ we have $S_1^k = S_2^k$, implying that after player $j+1$'s arrival, the value shared with i is identical in G_1 and G_2. However, because ϕ satisfies S-Stay, player i may obtain extra benefit in G_1 if she is essential in generating the marginal contribution of some player q who arrives between i and j in π_1. Thus, i again has no incentive to delay her arrival. Now consider the remaining sub-case where i is not a contributional player in G_1 but becomes one in G_2 after delaying. This implies that i is essential for generating the marginal contribution of some player arriving between i and j in π_1 (Otherwise i cannot be a contributional player after delay the arrival). Let q be the **first** such player, i.e., $v(N_{\pi_1|q}) > v(N_{\pi_1|q} \setminus i)$ with $i \prec_{\pi_1} q \prec_{\pi_1} j$ (or $q = j$). Recall that ϕ satisfies S-Stay and sharing consistency. i will be included in the sharing set since q's arrival. Hence, i's total share in G_1 is

$$\phi(G_1, i) = \sum_{k=q}^{n} \frac{1}{|S_1^k|} \mathrm{MC}(G_1, N_{\pi_1|k} \setminus \{k\}, k).$$

For G_2, by Part and sharing consistency, we have

$$\phi(G_2, i) = \frac{1}{|S_2^i|} \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + \sum_{k=j+1}^{n} \frac{1}{|S_2^k|} \mathrm{MC}(G_2, N_{\pi_2|k} \setminus \{k\}, k).$$

Applying the same reasoning as in Case 1, we obtain the difference between $\phi(G_1, i)$ and $\phi(G_2, i)$

$$\begin{aligned}
\phi(G_1, i) - \phi(G_2, i) &= \frac{1}{|S_1^j|} \big(v(N_{\pi_2|j}) - v(N_{\pi_1|(q-1)}) \big) \\
&= \frac{1}{|S_1^j|} \big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)}) \big).
\end{aligned}$$

We claim that $v(N_{\pi_1|(q-1)}) = v(N_{\pi_1|(q-1)} \setminus \{i\})$. To see this, suppose not, by monotonicity, it must be the case that $v(N_{\pi_1|(q-1)}) > v(N_{\pi_1|(q-1)} \setminus \{i\})$, implying that player i is essential to create the marginal value when player $q-1$ joins the coalition, contradicting our assumption that player q is the first player such that i is essential to create the marginal contribution. Therefore, we have $v(N_{\pi_1|(q-1)}) = v(N_{\pi_1|(q-1)} \setminus \{i\})$. Consequently,

$$\phi(G_1, i) - \phi(G_2, i) = \frac{1}{|S_1^j|}\big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)})\big)$$
$$= \frac{1}{|S_1^j|}\big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|(q-1)} \setminus \{i\})\big) \geq 0.$$
(Valuation monotonicity)

Combining both cases, we conclude that player i never benefits from delaying her arrival. Therefore, any Equal Sharing rule ϕ that satisfies S-Stay, Part, sharing consistency, and order independence also satisfies EA. □

Proposition 4.4. *MES and NDMES satisfy S-STAY, PART, and EA.*

Proof. We first show that both the MES and NDMES rules satisfy S-Stay and Part. For the MES and NDMES rules, the Stay axiom is immediately satisfied since every player's shared value is non-decreasing throughout the process. Now consider any online cooperative game $G = (N, v, \pi)$ and any prefix subgame G^i corresponding to the arrival of player i. Take any player j such that $j \prec_\pi i$ and $v(N_{\pi|i}) > v(N_{\pi|i} \setminus j)$. For the MES rule, the sharing set in G^i is $N_{\pi|i}$, which includes j. Therefore, player j receives an additional positive share in G^i, given by $\frac{1}{|N_{\pi|i}|}\,\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > 0$, which implies that $\phi(G^i, j) > \phi(G^j, j)$. The same reasoning applies to the NDMES rule: since every player j with $v(N_{\pi|i}) > v(N_{\pi|i} \setminus j)$ is a non-dummy player, j is included in the sharing set S^i of G^i, and thus also satisfies S-Stay. Next, we verify Part. Under the MES rule, any newly arriving player who makes a positive marginal contribution immediately receives a positive share, $\frac{1}{|N_{\pi|i}|}\,\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > 0$, which establishes Part. For the NDMES rule, any such contributional player is non-dummy and thus included in the sharing set S^i, again satisfying Part. Having established that both the MES and NDMES rules satisfy S-Stay and Part, by lemma 4.3, it remains to show that they satisfy sharing consistency and order independence.

The MES rule trivially satisfies both properties, as its sharing set in each subgame is simply the set of all existing players. For the NDMES rule, observe that for any two subgames G^i and G^j with $i \prec_\pi j$, every non-dummy player in G^i remains non-dummy in G^j. Moreover, the set of non-dummy players in each subgame depends solely on the player set rather than the arrival order. Hence, NDMES also satisfies sharing consistency and order independence. Applying lemma 4.3, we conclude that both MES and NDMES satisfy the EA axiom. □

Remark 2. *Designing rules under the Equal Sharing scheme that satisfy S-Stay and Part is relatively straightforward. In contrast, ensuring the EA axiom is considerably more challenging and non-trivial. Our lemma 4.3 provides a characterization, in fact, a sufficient condition, showing that there exists a family of Equal Sharing rules satisfying sharing consistency and order independence that also guarantee the EA axiom. Notably, this family includes simple rules such as MES and NDMES. It remains open to provide a complete characterization for Equal Sharing rules to satisfy EA axiom.*

4.2 Alternative Rules: ULMES and eULMES

In the previous subsection, we characterized a class of Equal Sharing rules that satisfy three key participation incentive axioms, S-Stay, Part, and EA, and introduced two representative instances: MES and NDMES. However, the MES rule fails to satisfy the fairness notion of Online Dummy, while the NDMES rule faces computational challenges, as identifying all dummy players in each subgame requires exponential time due to the need to verify all coalitions[3].

In this subsection, we move beyond the scope of the previous characterization and focus on designing a polynomial-time computable Equal Sharing rule. The proposed rule follows a greedy approach, which we term the Upward Lexicographic Marginal Equal Share (ULMES) rule.

Upward Lexicographic Marginal Equal Share (ULMES)

Input: $G = (N, v, \pi)$
For each player i in arriving order $\pi = (1, 2, \ldots, n)$:
  Initialize $\phi(G, i) \leftarrow 0$, $S^i \leftarrow N_{\pi|i}$, $\ell \leftarrow i$.
  While $\ell > 0$
    If $v(S^i \setminus \{\ell\}) = v(N_{\pi|i})$:
      Update $S^i \leftarrow S^i \setminus \{\ell\}$
    $\ell \leftarrow \ell - 1$
  For $j \in S^i$,
    $\phi(G, j) \leftarrow \phi(G, j) + \frac{1}{|S^i|}\,\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i)$.
Output: $\phi(G)$.

The ULMES rule follows a greedy procedure. For any prefix subgame G^i, it first initializes the sharing set as $S^i = N_{\pi|i}$ and then determines which players to retain in S^i in an upward lexicographic order. Specifically, for each player ℓ (starting from i), the rule checks whether removing ℓ decreases the grand coalition value in G^i. If the coalition $S^i \setminus \ell$ yields the same value as $v(N_{\pi|i})$, then player ℓ is considered dispensable for generating the marginal contribution and is removed from S^i. This procedure proceeds iteratively with $\ell \leftarrow \ell - 1$ until all players have been examined. The remaining players in S^i equally share the marginal contribution.

Intuitively, when a new player i arrives, there may exist multiple coalitions capable of generating the same marginal contribution. Among all such coalitions, ULMES selects the player set S^i for which the last arriving player is the earliest, that is, it favors the coalition that is **lexicographically minimal in terms of arrival order**. The ULMES rule runs in $O(n^2)$ time as computing the sharing set S^i requires $O(n)$ time for verifying whether each existing player should be removed from S^i.

We show that ULMES satisfies the S-Stay, Part, and OD axioms. However, it unfortunately fails to satisfy the EA axiom. Due to space limitations, we defer the formal proofs of these properties to Appendix B.2, but provide here a counterexample illustrating why ULMES fails to meet the EA condition.

Theorem 4.5. *ULMES satisfies S-STAY, PART, and OD, but fails EA.*

[3]We show that NDMES runs in $O(n)$ time when the valuation function is subadditive; see proposition B.1 in Appendix B.1.

Example 4.6. Consider two OCGs $G_1 = (N, v, \pi_1)$, $N = \{1, 2, 3, 4\}$, $\pi_1 = (1, 2, 3, 4)$ and $G_2 = (N, v, \pi_2)$ where $\pi_2 = (1, 2, 4, 3)$, that is, the instance where player 3 delays her arrival in G_1, which leads to G_2. For the valuation function, we have

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0;$$
$$v(\{1,2\}) = v(\{1,4\}) = v(\{2,4\}) = v(\{1,2,4\}) = 0;$$
$$0 \le v(\{1,3\}) \le v(\{2,3\}) < v(\{1,2,3\}) = x \ (x > 0);$$
$$v(\{3,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,3,4\}) = y > x.$$

We now focus on player 3. For G_1, when player 3 arrives, the value jumps from 0 to x and ULMES selects the sharing set $S^3 = \{1, 2, 3\}$, so we have $\phi(G_1^3, 3) = \frac{x}{3}$, when player 4 comes, only $\{3, 4\}$ survive in ULMES and the marginal contribution $(y - x)$ is equally shared by 3 and 4. Hence we have $\phi(G_1, 3) = \frac{x}{3} + \frac{y-x}{2}$. In contrast, when 3 delays her arrival, i.e., in G_2, 1, 2, 4 share no value as $v(\{1, 2, 4\}) = 0$. When 3 arrives in π_2, the marginal contribution will be y and it is shared by $S_2^3 = \{3, 4\}$ according to ULMES rule. Then we have $\phi(G_2, 3) = \frac{y}{2}$, which is greater than $\phi(G_1, 3) = \frac{x}{3} + \frac{y-x}{2}$. This implies that ULMES fails EA axiom.

Intuitively, ULMES fails to satisfy EA in general because a player might choose to delay their arrival to become a contributional player who shares a larger marginal value with fewer players. Although this player might forfeit some shared value from previous timesteps, the potential gain from the new marginal value can outweigh the losses, thereby undermining the EA property. Although ULMES fails EA in general, we observe that it adheres to the EA axiom for every simple online cooperative game (SOCG). The proof proceeds via a case analysis, considering whether a player i is pivotal and how her delayed arrival position affects the sharing set. Due to space constraints, the detailed proof is deferred to Appendix B.3.

Lemma 4.7. *For any SOCG, ULMES satisfies EA.*

Recall the Greedy Monotone (GM) decomposition algorithm proposed by Ge et al. [22], which generalizes the RFC rule for SOCGs to the eRFC rule for general OCGs without compromising any axiomatic guarantees. Following a similar idea, we extend our ULMES rule to the *extended Upward Lexicographic Marginal Equal Share* (eULMES) rule by leveraging the GM decomposition framework. We briefly outline the main steps of the GM decomposition and refer readers to Appendix A for detailed explanations and examples.

In general, the GM decomposition algorithm takes any OCG G as input and outputs a linear combination of component games, denoted as $D(G)$. Each component is represented as a pair $(c, \bar{G})$, where c is the coefficient and $\bar{G}$ is an SOCG. Given any OCG $G = (N, v, \pi)$, the eULMES rule proceeds as follows:

(1) apply the GM decomposition to decompose G into multiple component games;
(2) execute the ULMES rule within each component game;
(3) aggregate the weighted results across all components to obtain each player's total share.

Theorem 4.8. *eULMES satisfies S-STAY, EA, PART, and OD.*

Due to space limitations, we defer the formal proof of theorem 4.8 to Appendix B.4. To further illustrate the behavior of the eULMES rule and its distinction from ULMES, we provide a specific example in Appendix B.5, in which ULMES fails the EA axiom, while eULMES successfully satisfies it.

5 INDIVIDUAL RATIONALITY UNDER SUPERADDITIVITY VALUATION

In the previous sections, we explored participation incentives through the lens of the EA, S-Stay, and Part axioms. We now shift our focus to the individual rationality (IR) axiom, which reflects a fundamental participation incentive, requiring that a player chooses to join the grand coalition only if the value they receive from it is at least as great as their standalone (singleton) valuation.

We begin by presenting an impossibility result regarding the satisfaction of the IR axiom in online cooperative games.

Proposition 5.1 (Impossibility). *There is no value sharing rule satisfying IR for OCG under general valuation.*

Proof. Consider an OCG $G = (N, v, \pi)$, where $N = 1, 2, v(\{1\}) = 2$, $v(\{2\}) = 3$, $v(\{1, 2\}) = 4$, and $\pi = (1, 2)$. To satisfy the IR axiom, player 1 must receive at least $v(\{1\}) = 2$ upon arrival. When player 2 subsequently joins, the total additional value created by their arrival is $v(\{1, 2\}) - v(\{1\}) = 2$. Hence, the maximum value that can be allocated to player 2 is $2 < v(\{2\}) = 3$, violating the IR requirement. □

In light of the impossibility result, we restrict our attention to the *superadditive* valuation. Specifically, for an OCG $G = (N, v, \pi)$, we say that G is a *superadditive OCG* if its valuation function v is superadditive. In what follows, we first verify that both the DMC and SV rules satisfy the IR axiom, whereas the eRFC rule does not. We then introduce an IR-refinement paradigm that modifies all three rules to ensure IR satisfaction while preserving their other desirable axiomatic properties.

Proposition 5.2. *In OCGs with supe-radditive valuations, DMC and SV satisfy IR while eRFC does not.*

Unfortunately, MES, NDMES, and eULMES also fail IR axiom. To address this, we propose a simple IR refinement to modify these rules such that IR axiom is satisfied. The refinement follows a simple yet effective idea. For each subgame G^i, we first allocate value $v(\{i\})$ to player i to ensure satisfaction of the IR axiom. After that, we apply the Equal Sharing rule to determine the sharing set S^i, and then equally distribute the remaining marginal contribution, $\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})$, among S^i.

IR Refinement Framework

Input: $G = (N, v, \pi)$, value-sharing rule ϕ
$\forall\, i \in N$, initialize $\hat{\phi}(G, i) \leftarrow 0$.
For i in arriving order π:
  $\hat{\phi}(G, i) \leftarrow v(\{i\})$.
  Compute sharing set S^i by the rule ϕ.
  For each player $j \in S^i$:
    $\hat{v}_i \leftarrow \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})$.
    $\hat{\phi}(G, i) \leftarrow \hat{\phi}(G, i) + \frac{1}{|S^i|}\hat{v}_i$.
Output: $\hat{\phi}(G)$.

For each Equal Sharing rule, we denote its refined counterpart by adding the "IR-" prefix, for example, eULMES becomes the IR-eULMES rule. This refinement guarantees compliance with the IR axiom while preserving all other desired properties.

Theorem 5.3. *For any superadditive OCG, the IR-MES, IR-NDMES, and IR-eULMES rules satisfy IR while preserving all other axioms that were satisfied prior to the refinement.*

6 DISCUSSION

In this paper, we study participation incentive axioms in online cooperative games with strategic arrivals, including the existing Stay and EA axioms, and our newly studied S-Stay, Part, and IR. We first identify that existing sharing rules fail the axioms. To address this, we design a class of "Equal Sharing" rules and provide sufficient conditions for satisfying S-Stay, Part, and EA. An intriguing open question that remains is: "What are the necessary and sufficient conditions for an Equal Sharing rule to satisfy these axioms?" Moreover, studying online hedonic games with strategic arrivals presents a promising direction for future research.

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Below is the Technical Appendix for our submission (#249) "Participation Incentives in Online Cooperative Games".

A GREEDY MONOTONE (GM) DECOMPOSITION

We revisit the greedy monotone (GM) decomposition algorithm proposed by Ge et al. [22]. GM decomposition maps a general OCG $G = (N, v, \pi)$ into a linear combination of multiple SOCGs. The detailed procedures of GM decomposition is provided in Algorithm 1.

Algorithm 1 Greedy Monotone (GM) Decomposition

Input: : An OCG $G = (N, v, \pi)$.
Output: : A decomposition $D(G)$.

Initialize $D(G) \leftarrow \emptyset$, and $k \leftarrow 1$;
while $\max_{T \subseteq N}\{v(T)\} > 0$ **do**
    $S \leftarrow \underset{T \subseteq N, v(T)>0}{\arg\min}\ v(T)$;
    Coefficient $c_k \leftarrow v(S)$;
    Initialize component SOCG $\bar{G}_k = (N, \bar{v}_k, \pi)$;
    for $T \subseteq N$ **do**
        if $v(T) > 0$ **then**
            $\bar{v}_k(T) \leftarrow 1$;
        else
            $\bar{v}_k(T) \leftarrow 0$;
        end if
        Update $v(T) \leftarrow v(T) - c_k \bar{v}_k(T)$;
    end for
    Add component SOCG $(c_k, \bar{G}_k)$ into $D(G)$;
    Update $k \leftarrow k + 1$;
end while

Algorithm 1 first initializes $D(G) = \emptyset$ and create an iteration counter $k = 1$. While there exists a subset of N with a positive value, it identifies the subset S with the minimum value among all subsets with positive value. Let $c_k = v(S)$ denote the coefficient of the k-th component game. To determine the valuation function $\bar{v}_k(\cdot)$ for the k-th component game, for any subset $T \subseteq N$ such that $v(T) > 0$, it assigns $\bar{v}_k(T) = 1$. For all other coalitions with zero value, their values remain zero in $\bar{v}_k(\cdot)$. Accordingly, it updates the original valuation function $v(\cdot)$ for each subset T with positive value by setting $v(T) = v(T) - c_k$. This completes the creation of the k-th component game $\bar{G}_k = (N, \bar{v}_k, \pi)$ with coefficient c_k. To better illustrate the algorithm, we provide the following example showing how GM decomposition runs.

Example A.1. Consider an OCG $G = (N, v, \pi)$, where $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 1, v(\{1, 2\}) = 5$, and $\pi = (1, 2)$. We first initialize $D(G) = \emptyset$ and $k = 1$. For the first component game $\bar{G}_1 = (N, \bar{v}_1, \pi)$, we first identify the minimum positive coalition value is 1 and let the coefficient $c_1 = 1$. For the valuation, since $v(\{1\}) = v(\{2\}) = 1, v(\{1, 2\}) = 5$, we have $\bar{v}_1(\{1\}) = 1, \bar{v}_1(\{2\}) = 1, \bar{v}_1(\{1, 2\}) = 1$ and updates $v(\{1\}) = v(\{2\}) = 0, v(\{1, 2\}) = 4$. Next we update k to 2, and for the second component game $\bar{G}_2 = (N, \bar{v}_2, \pi)$, we identify the minimum positive coalition value is 4 (i.e., $v(\{1, 2\}) = 4$), and let $c_2 = 4$. For the valuation function, note that only $v(\{1, 2\}) = 4$ is positive, we then have $\bar{v}_2(\{1\}) = 0, \bar{v}_2(\{2\}) = 0, \bar{v}_2(\{1, 2\}) = 1$. After that, we update the original valuation function $v(\{1\}) = v(\{2\}) = 0, v(\{1, 2\}) = 4 - c_2 = 0$. Since there is no coalition with positive value, the algorithm terminates and returns $D(G) = \{(c_1 = 1, \bar{G}_1), (c_2 = 4, \bar{G}_2)\}$, where

$$c_1 = 1, \bar{G}_1 : N = \{1, 2\}, \pi = (1, 2), \bar{v}_1(\{1\}) = 1, \bar{v}_1(\{2\}) = 1, \bar{v}_1(\{1, 2\}) = 1$$

$$c_2 = 4, \bar{G}_2 : N = \{1, 2\}, \pi = (1, 2), \bar{v}_2(\{1\}) = 0, \bar{v}_2(\{2\}) = 0, \bar{v}_2(\{1, 2\}) = 1.$$

for $\bar{G}_1$, we have $\bar{v}_1(\{1\}) = 1, \bar{v}_1(\{2\}) = 1, \bar{v}_1(\{1, 2\}) = 1$ while for $\bar{G}_2$, we have $\bar{v}_2(\{1\}) = 0, \bar{v}_2(\{2\}) = 0, \bar{v}_2(\{1, 2\}) = 1$.

Ge et al. [22] proved the following properties of the GM decomposition method,

- GM decomposition outputs the $D(G)$ satisfying for each $T \subseteq N$, $v(T) = \sum_k c_k \bar{v}_k(T)$ and $\bar{v}_k(\cdot)$ is 0-1 valued monotone functions.
- Given a decomposition $D(G)$, for any player i in π with subgame G^i, the decomposition of $D(G^i)$ is consistent with $D(G)$ within players in $N_{|i}$ (consistency between the global game and prefix subgames).

B OMITTED PROOFS IN SECTION 4

B.1 Proof of Proposition B.1

Proposition B.1. *Given an OCG $G = (N, v, \pi)$, if the valuation function $v(\cdot)$ is subadditive, NDMES rule runs in linear time $O(n)$.*

Proof. To identify all the dummy players in each prefix subgame G^i, we need to verify for every player $j \in N_{\pi|i}$ whether $v(S \cup \{j\}) = v(S)$ holds for all subsets $S \subseteq N_{\pi|i}$. However, when the valuation function $v(\cdot)$ is subadditive, we show that player j is a dummy player in G^i *if and only if* $v(\{j\}) = 0$. Note that $v(\cdot)$ is subadditive, for any subset S, $v(S \cup \{j\}) \leq v(S) + v(\{j\}) = v(S)$. On the other hand, we assume the valuation function $v(\cdot)$ is monotone, i.e., $v(S \cup \{j\}) \geq v(S)$. Combining the two inequalities yields $v(S \cup \{j\}) = v(S)$ for all S, implying that j is a dummy player. Conversely, if j is a dummy player, then taking $S = \emptyset$ gives $v(\{j\}) = v(\emptyset) = 0$. Hence, under subadditivity, j is a dummy in G^i if and only if $v(\{j\}) = 0$. Therefore, determining all dummy players in each subgame G^i can be done in linear time $O(n)$, as it suffices to check whether $v(\{j\}) = 0$ for every $j \in N_{\pi|i}$. □

B.2 Proof of Theorem 4.5

Proof. **(S-STAY)** ULMES rule satisfies STAY as each player's shared value is non-decreasing. To show ULMES rule satisfies S-STAY, consider an OCG $G = (N, v, \pi)$, for any player $j \in N$ and any prefix subgame G^i where $j \prec_\pi i$, assume i is a contributional player, if j satisfies $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, then j must be in the final S^i after the elimination process of ULMES rule. The proof is as follows. Denote $S^i(j)$ as the tentative S^i when checking whether player j should be eliminated or not. Recall the valuation function is monotone and $S^i(j) \subseteq N_{\pi|i}$. It means $v(N_{\pi|i} \setminus \{j\}) \geq v(S^i(j) \setminus \{j\})$. Since $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, then we have $v(N_{\pi|i}) > v(S^i(j) \setminus \{j\})$, which implies j must be kept in $S^i(j)$. Then, j will share $\frac{1}{|S^i|}\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > 0$. Therefore, ULMES rule satisfies S-STAY.
(PART) Consider an OCG $G = (N, v, \pi)$, for any prefix subgame G^i with arriving player i, if i is a contributional player, then i is kept in S^i as $\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) = v(N_{\pi|i}) - v(N_{\pi|i} \setminus \{i\}) > 0$.

(OD) Consider an OCG $G = (N, v, \pi)$, for any prefix subgame G^i with an arriving contributional player i, we show that ULMES rule never assigns positive value to dummy players in G^i. Prove by contradiction. Assume that there exists some dummy player j in G^i who gets assigned positive value, then j must survive in the elimination of S^i, however, by the definition of dummy player, for any subset $T \subseteq N_{\pi|i}$, $v(T) = v(T \cup \{j\})$. Denote $S^i(j)$ as the tentative $S^i(j)$ for the time-step when j is checked whether she should be eliminated from S^i. According to ULMES rule, $j \in S^i(j)$ and $v(S^i(j)) = v(N_{\pi|i})$, let $T = S^i(j) \setminus \{j\}$, we have $v(S^i(j) \setminus \{j\}) = v(S^i(j)) = v(N_{\pi|i})$, meaning j must be eliminated from $S^i(j)$. Hence, j will not survive in the final sharing set S^i to share the value, contradicting to the assumption that j gets some positive sharing value in G^i. □

B.3 Omitted Proof of Lemma 4.7

Proof. Consider an SOCG $G_1 = (N, v, \pi_1)$ where $\pi_1 = (1, 2, \dots, i-1, \mathbf{i}, i+1, \dots, n)$. Let q denote the pivotal player in G_1, that is, in subgame G^q, $v(N_{\pi_1|q} \setminus \{q\}) = 0$ and $v(N_{\pi_1|q}) = 1$. Let $G_2 = (N, v, \pi_2)$, where $\pi_2 = (1, 2, \dots, i-1, i+1, \dots, j, \mathbf{i}, \dots, n)$ represents the arriving order in which all other players' arriving orders are fixed and i delays her arrival. For i, there are three cases: $q \prec_{\pi_1} i$, $q = i$ and $i \prec_{\pi_1} q$.

Case 1: $q \prec_{\pi_1} i$. If a coalition with value 1 has formed before i's arrival in π_1, i shares no value in both π_1 and π_2 and $\phi(G_1, i) = \phi(G_2, i) = 0$.

Case 2: $q = i$. If i is the pivotal player in G_1, there are two possible cases if i delays her arrival.

(a). i loses her pivotal role in π_2, i.e., some player in $\{i+1, i+2, \dots, j\}$ becomes the pivotal player. Then, $\phi(G_1, i) > \phi(G_2, i) = 0$. Player i has no incentive to delay.

(b). i remains to be pivotal in π_2. Let S_1^i be the player set sharing value 1 in G_1. When i delays in π_2, there will be no change of $S_1^i \setminus \{i\}$ which creates the new marginal value along with i. Thus, ULMES will eliminate all the players in $\{i+1, \dots, j\}$ (because of the existence of $S_1^i \setminus \{i\}$) in G_2. Hence, players in S_1^i still share the value 1 in G_2. Player i receives the same value in G_1 and G_2. Therefore, i has no incentive to delay in case 2.

Case 3: $i \prec_{\pi_1} q$ Denote S_q^1 as the set of players among whom the value 1 is shared. There are two possible cases.

(a). $i \notin S_1^q$, i.e., $\phi(G_1, i) = 0$. $i \notin S_1^q$ is *either* because there exists no coalition including i along with q creating value 1 *or* because i is in some coalition creating value 1 with q, however, eliminated by ULMES rule. In the former case, **i)** i delays between $i+1$ and q, there is still no coalition including i can creating value 1 with q; **ii)** i delays after q's arrival, which makes no change for the value sharing as i's delay does not influence S_1^q; For the latter case, it implies that among players $\{1, 2, \dots, i, \dots, q-1\}$, there are multiple coalitions with q creating value 1 and i is in one of these coalitions, denoting it by $\bar{S}_1^q$. However, $\bar{S}_1^q$ is eliminated because of the existence of S_1^q. Since $i \notin S_1^q$, i's any delay strategy in π_2 has no effect on S_1^q, keeping $\phi(G_2, i) = \phi(G_1, i) = 0$.

(b). $i \in S_1^q$, i.e., $\phi(G_1, i) = \frac{1}{|S_1^q|}$. There are two different situations: **i)** S_1^q is the unique coalition creating value 1 in G^q. If i delays between $i+1$ and q, it does not change value sharing in π_2 and $\phi(G_2, i) = \phi(G_1, i) = \frac{1}{|S_1^q|}$; If i delays the arrival after q, one case is some players in π_2 between q and i, along with some players in $\{1, 2, \dots, i-1, i+1, \dots, q\}$ construct a coalition with value 1, then $\phi(G_2, i) = 0$; the other case is that after i's delay, i becomes the pivotal player. However, i will still share the value in S_1^q as all the other such coalitions creating value 1 will be eliminated by ULMES because of the existence of S_1^q. Therefore, $\phi(G_2, i) = \phi(G_1, i) = \frac{1}{|S_1^q|}$. **ii)** there are multiple coalitions including some players in $\{1, 2, \dots, q-1\}$ creating value 1 with q and S_1^q is the coalition survives in ULMES rule. If i delays her arrival between $i+1$ and q, it could be either S_1^q is still the coalition to share the value ($\phi(G_2, i) = \phi(G_1, i) = \frac{1}{|S_1^q|}$) or because of i's delay, S_1^q get eliminated in ULMES rule, making some other coalition survives ($\phi(G_1, i) > \phi(G_2, i) = 0$); If i delays her arrival after q, $\phi(G_2, i) = 0$ as there exists some other coalition creating and sharing the value 1. Combining the aforementioned three cases, we conclude that ULMES satisfies the EA axiom in any SOCG. □

B.4 Omitted Proof of Theorem 4.8

Proof. **(S-STAY)** Consider an OCG $G = (N, v, \pi)$, for any prefix subgame G^i with an arriving contributional player i, for every player j satisfying $j \prec_\pi i$ and $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$. According to the GM decomposition $D(G)$, the valuation is decomposed into linear combinations. Since $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, there exists at least one component in $D(G)$ such that $\bar{v}(N_{\pi|i}) > \bar{v}(N_{\pi|i} \setminus \{j\})$ and i is a contributional player in $\bar{G}^i$. For this component $(c, \bar{G})$, according to Theorem 4.5, ULMES rule satisfies S-STAY, meaning in $\bar{G}^i$, $\phi(\bar{G}^i, j) > 0$. Due to the consistency of $D(G)$ and $D(G^i)$, we have $\phi(G^i, j) = \sum_k c_k \phi(\bar{G}_k^i, j) > 0$. eULMES rule satisfies S-STAY.

(PART) Consider an OCG $G = (N, v, \pi)$, for any prefix subgame G^i with an arriving contributional player i. In the decomposition $D(G)$ of G, there exists at least one component $(c, \bar{G})$ such that i is still a contributional player in $\bar{G}$. Prove by contradiction, assume there is no component such that i is a contributional player. By the linearity and consistency of $D(G)$ regarding the valuation function, i is not a contributional player in G, contradicting i is the contributional player in G^i. For $\bar{G}$, as ULMES rule is PART, meaning i get positive value in $\bar{G}^i$: $\phi(\bar{G}^i, i) > 0$. So i has positive shared value in G^i as the value $c \cdot \phi(\bar{G}^i, i)$ will be added into $\phi(G^i, i)$. Then eULMES rule satisfies PART.

(OD) Notice that ULMES rule satisfies OD, assigning no value to dummy players for each component $(c, \bar{G})$ of the decomposition $D(G)$. Also, consider any OCG $G = (N, v, \pi)$, for any player i, if i is dummy in G, then for SOCG $\bar{G}$, in each component of $D(G)$, i is still a dummy player. Thus, for eULMES rule which outputs the weighted sum of ULMES rule's output over each component, it never assigns a positive value to dummy players in G. So, eULMES rule satisfies the OD axiom.

(EA) Consider an OCG $G = (N, v, \pi)$, according to lemma 4.7, ULMES rule satisfies EA for every SOCG. It means ULMES rule satisfies EA for every SOCG in each component $(c, \bar{G})$ of $D(G)$. For any player i in arriving order π, assume i delays her arrival and changes the order into π' w.r.t. the OCG $G' = (N, v, \pi')$. Notice that the GM decomposition only depends on the valuation function and is unrelated with the arriving order. Thus, for player i, for each component $(c, \bar{G} = (N, \bar{v}, \pi))$ and $(c, \bar{G}' = (N, \bar{v}, \pi'))$, ULMES rule satisfies the EA axiom implies $\phi(\bar{G}, i) \geq \phi(\bar{G}', i)$. Furthermore, $\phi(G, i) = \sum_k c_k \cdot \phi(\bar{G}_k, i) \geq \sum_k c_k \cdot \phi(\bar{G}'_k, i) = \phi(G', i)$. So i has no incentive to delay her arrival, meaning eULMES rule satisfies the EA axiom. □

B.5 Omitted Example for ULMES and eULMES

Example B.2 (ULMES and eULMES rules). Consider an OCG $G = (N, v, \pi)$ where $N = \{1, 2, 3, 4\}$, $\pi = (1, 2, 3, 4)$. For the valuation function $v(\cdot)$, all the coalition valuation are enumerated in the first row of Table 2.

Under the ULMES rule, during the first two timesteps, players 1 and 2 arrive but generate no value. When player 3 joins, the coalition $\{1, 2, 3\}$ generates a value of 2, and no other sub-coalition achieves this value. Consequently, the three players share the total value equally, each receiving $\frac{3}{2}$. Upon the arrival of player 4, the grand coalition achieves a total value of 3, with a marginal contribution of 1. To determine S_4, we start from player 4. Removing player 4 reduces the value to 2, indicating that player 4 must be included in S_4 Similarly, player 3 must also be in S_4 since $v(\{1, 2, 4\}) = 0$. Conversely, player 1 and 2 are excluded from S_4 since $v(\{1, 3, 4\}) = v(\{3, 4\}) = 3$. Hence, the marginal contributional value of 1 is evenly distributed between player 3 and 4, each receiving $\frac{1}{2}$. The final value distribution is $\phi(G) = (\frac{2}{3}, \frac{2}{3}, \frac{7}{6}, \frac{1}{2})$. Now consider the OCG G', where player 3 delays her arrival, changing the arriving order from $\pi = (1, 2, 3, 4)$ to $\pi' = (1, 2, 4, 3)$. In this scenario, player 3 strategically delays her arrival to ensure that she shares the entire value of 3 solely with player 4, resulting in each receiving $\frac{3}{2}$. Hence, the final value distribution under π' is $\phi(G') = (0, 0, \frac{3}{2}, \frac{3}{2})$. Table 3 compares the outcomes for ULMES rule in G and G'. This shows that ULMES rule fails to satisfy the EA axiom as player 3 can gain extra benefit from delaying her arrival.

Regarding eULMES rule, firstly, since the GM decomposition is independent with the arriving order, OCG G and G' share the same game decomposition components shown in Table 2, i.e., decomposing G into three components $D(G) = \{(1, \bar{G}_1), (1, \bar{G}_2), (1, \bar{G}_3)\}$ with three distinct valuation functions $\bar{v}_1(\cdot), \bar{v}_2(\cdot)$, and $\bar{v}_3(\cdot)$. After the decomposition, we run ULMES rule in each decomposed component in G and G', respectively. Taking G as the example, under the arriving order $\pi = (1, 2, 3, 4)$, for $\bar{G}_1$, player 1 and 2 arrive with no value sharing, when player 3 comes, we compute $S_3 = \{1, 3\}$ since $\bar{v}_1(\{1, 3\}) = 1$, then player 1 and 3 each shares $\frac{1}{2}$. Similarly, we can compute the value distribution in $\bar{G}_2$ and $\bar{G}_3$. The results are demonstrated in the left part of Table 4 while the outcome of eULMES rule in G' is shown in the right part of Table 4.

For the three components with $\bar{G}_1, \bar{G}_2$, and $\bar{G}_3$, we omit the coefficient for each SOCG as their coefficients are all the same 1 in this example. It is not hard to see that player 3 has no incentive to delay her arrival when we apply eULMES rule.

C OMITTED PROOF IN SECTION 5

C.1 Omitted Proof of Proposition 5.2

Proof. Consider an OCG $G = (N, v, \pi)$, DMC rule satisfies the IR axiom since for each player $i \in N$, $\phi(G, i) = \mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) \geq v(\{i\})$ with superadditive valuation function. For SV rule, it degenerates to the classic cooperative game in which the Shapley Value satisfies IR in games with superadditive valuation functions. Next, we show that eRFC rule does not satisfy the IR property with superadditive valuation function by the following example. Consider a game $G = (N, v, \pi)$ where $N = \{1, 2\}$, $\pi = (1, 2)$, and $v(\{1\}) = 1, v(\{2\}) = 2, v(\{1, 2\}) = 5$. According to eRFC rule in [22], G will be decomposed into $D(G) = \{(1, \bar{G}_1), (1, \bar{G}_2), (3, \bar{G}_3)\}$. The decomposed valuation function is provided in Table 5.

Finally, for the original game G, $\phi(G, 1) = 4$ and $\phi(G, 2) = 1$. Note that $v(\{2\}) = 2 > \phi(G, 2) = 1$. Thus, eRFC rule does not satisfy the IR property. □

C.2 Omitted Proof of Theorem 5.3

Proof. For the IR-MES rule, IR-NDMES rule, and IR-eULMES rule, all three rules satisfy the IR axiom. For any OCG $G = (N, v, \pi)$, each rule first assigns the value $v(\{i\})$ to any arriving player i in the order π. The superadditive valuation function ensures the validity of this allocation.

Next, we show each rule maintains to satisfy all the axioms before the refinement.

We first clarify S-STAY is incompatible with IR if there exists some player i, $\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) = v(\{i\}) > 0$. Consider an OCG $G = (N, v, \pi)$ where $N = \{1, 2\}$, $\pi = (1, 2)$, and $v(\{1\}) = v(\{2\}) = 1$, $v(\{1, 2\}) = 2$. According to IR, it must be $\phi(G, 1) = \phi(G, 2) = 1$. However, this value distribution does not satisfy S-STAY axiom. Therefore, we next mainly focus on the superadditive valuation where for any OCG $G = (N, v, \pi)$, $\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) > v(\{i\})$ for each arriving player i when discussing the S-STAY axiom.

We first show that **IR-MES** satisfies S-STAY, PART, EA axioms as follows.

(S-STAY) Consider an OCG $G = (N, v, \pi)$ with arriving order $\pi = (1, 2, \ldots, n)$. For the prefix sub-game G^i with arriving player i, $\phi(G^i, i) = \frac{1}{|S^i|}\big(\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})\big)$. For any other prefix subgame G^j where $i \prec_\pi j$. $\phi(G^j, i) = \frac{1}{|S^i|}\big(\mathrm{MC}(N_{\pi|i}, i) - v(\{i\})\big) + \sum_{k=i+1}^{j}\big(\mathrm{MC}(N_{\pi|k}, k) - v(\{k\})\big) \geq \phi(G^i, i)$. IR-MES rule satisfies STAY. Furthermore, if in G^j, $\mathrm{MC}(N_{\pi|j}, j) > v(\{j\})$, then $\phi(G^j, i) > \phi(G^i, i)$, satisfying S-STAY.

(PART) Consider an OCG $G = (N, v, \pi)$, for any arriving player i, the value shared immediately by i is $\phi(G^i, i) = v(\{i\}) + \frac{1}{|S^i|}\big(\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})\big) \geq 0$. Hence, IR-MES rule satisfies PART.

(EA) Consider any player $i \in N$ and two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$ with two different arriving orders π_1 and π_2.

$$\pi_1 : (1, 2, \ldots, i-1, \mathbf{i}, i+1, \ldots, j-1, j, j+1, \ldots, n),$$
$$\pi_2 : (1, 2, \ldots, i-1, i+1, \ldots, j-1, j, \mathbf{i}, j+1, \ldots, n).$$

For each prefix subgame $G_1^i(G_2^i)$ within $G_1(G_2)$, denote the player set sharing the marginal value by $S_1^i(S_2^i)$ in IR-MES rule. Then, the value shared by i in G_1 can be represented as

$$\phi(G_1, i) = v(\{i\}) + \sum_{k=i}^{n} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi|k} \setminus \{k\}, k) - v(\{k\})\Big). \quad (1)$$

The value shared by player i in G_2 can be written as

$$\begin{aligned}\phi(G_2, i) =& v(\{i\}) + \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi|j}, i) - v(\{i\})\Big) \\ &+ \sum_{k=j+1}^{n} \frac{1}{|S_2^k|}\Big(\mathrm{MC}(G_2^k, N_{\pi|k} \setminus \{k\}, k\Big) - v(\{k\})\Big).\end{aligned} \quad (2)$$

Coalition	Coef	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
Valuation		0	0	0	0	0	1	0	1	0	3	2	0	3	3	3
$\bar{v}_1(\cdot)$	$c_1 = 1$	0	0	0	0	0	1	0	1	0	1	1	0	1	1	1
$\bar{v}_2(\cdot)$	$c_2 = 1$	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1
$\bar{v}_3(\cdot)$	$c_3 = 1$	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1

Table 2: Coalition valuation and decomposition of the OCG G with 3 components $(1, \bar{G}_1), (1, \bar{G}_2), (1, \bar{G}_3)$.

$\phi(G, i)$	1	2	3	4
$\pi = (1, 2, 3, 4)$	2/3	2/3	**7/6**	1/2
$\pi' = (1, 2, 4, 3)$	0	0	**3/2**	3/2

Table 3: ULMES rule outcome in G and G'

For every player k who arrives after player j, we have $N_{\pi_1|k} = N_{\pi_2|k}$. Hence,

$$\begin{aligned}
&\phi(G_1, i) - \phi(G_2, i) \\
=&\sum_{k=i}^{j} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big) \\
&\quad - \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|j} \setminus \{i\}, i) - v(\{i\})\Big) \\
\geq&\frac{1}{|S_2^i|}\sum_{k=i}^{j} \Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big) \\
&\quad - \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|j} \setminus \{i\}, i) - v(\{i\})\Big) \\
\geq&\frac{1}{|S_2^i|}\Big(v(N_{\pi_1|j}) - v(N_{\pi_1|j} \setminus \{i\}) - \sum_{k=i}^{j} v(\{k\})\Big) \\
&\quad - \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|j} \setminus \{i\}, i) - v(\{i\})\Big).
\end{aligned} \tag{3}$$

Note that $N_{\pi_2|i} \setminus \{i\} = \{1, 2, \ldots, i-1, i+1, \ldots, j\} = N_{\pi_1|j} \setminus \{i\} \cup \{i+1, \ldots, j\}$. Therefore we have

$$\begin{aligned}
&\frac{1}{|S_2^i|}\Big(v(N_{\pi_1|j}) - v(N_{\pi_1|j} \setminus \{i\}) - \sum_{k=i}^{j} v(\{k\})\Big) \\
&\quad - \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|j} \setminus \{i\}, i) - v(\{i\})\Big) \\
=&\frac{1}{|S_2^i|}\Big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|i} \setminus \{i\}) - \sum_{k=i+1}^{j} v(\{k\})\Big) \\
\geq&\frac{1}{|S_2^i|}\left(v(\{i+1, \ldots, j\}) - \sum_{k=i+1}^{j} v(\{k\})\right) \geq 0.
\end{aligned} \tag{4}$$

The last two steps are from superadditivity, we have $v(N_{\pi_1|j} \setminus \{i\}) \geq v(N_{\pi_1|i} \setminus \{i\}) + v(\{i+1, \ldots, j\})$ and $v(\{i+1, \ldots, j\}) \geq \sum_{k=i+1}^{j} v(\{k\})$.

Next, we show that **IR-NDMES** satisfies S-STAY, PART, OD, and EA axioms as follows.

(S-STAY) Obviously, IR-NDMES rule satisfies STAY as every player's shared value is non-decreasing. Next, we show the satisfication of S-STAY. Consider an OCG $G = (N, v, \pi)$, for any player j, consider any prefix subgames G^i, $(j \prec_\pi i)$ and i is a contributional player, if j satisfies $v(N_{\pi|i}) > v(N_{\pi|i} \setminus \{j\})$, then j is non-dummy in G^i. So, j is included in S^i and share the value $\frac{1}{|S^i|}\big(\mathrm{MC}(G^i, N_{\pi|i} \setminus \{i\}, i) - v(\{i\})\big) > 0$. Hence, for player j, $\phi(G^i, j) > \phi(G^j, j)$, satisfying S-STAY axiom.

(PART) For each arriving player i, if i is a contributional player, i.e., $\mathrm{MC}(N_{\pi|i}, i) > 0$, i is not a dummy player and will be included in S^i. Then, i immediately gets $\frac{1}{|S^i|}\big(\mathrm{MC}(N_{\pi|i} \setminus \{i\}, i) - v(\{i\})\big) > 0$, satisfying the PART axiom.

(OD) Consider an OCG $G = (N, v, \pi)$. For every prefix subgame G^i, the IR-NDMES rule eliminates all the dummy players in G^i, just like the NDMES rule, aligning directly with the definition of Online-Dummy.

(EA) Consider two OCGs $G_1 = (N, v, \pi_1)$ and $G_2 = (N, v, \pi_2)$ with two different orders π_1 and π_2 as follows

$$\begin{aligned}
\pi_1 &: (1, 2, \ldots, i-1, \mathbf{i}, i+1, \ldots, j-1, j, j+1, \ldots, n) \\
\pi_2 &: (1, 2, \ldots, i-1, i+1, \ldots, j-1, j, \mathbf{i}, j+1, \ldots, n)
\end{aligned} \tag{5}$$

For any player $i \in N$, we discuss the satisfication of the EA axiom in the following four cases.

Case 1. $\mathrm{MC}(G_1^i, N_{\pi_1|i} \setminus \{i\}, i) = \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) = 0$. The shared value after player $j+1$'s arrival remains the same for these two orders. The only difference is that player i might gain some shared value when i is not a dummy player in subgames from G^{i+1} to G^j. Therefore, the value shared by player i in π_1 will always be at least as much as in π_2

Case 2. $\mathrm{MC}(G_1^i, N_{\pi_1|i} \setminus \{i\}, i) > 0, \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) = 0$. For π_1, i will never be a dummy player in all subgames after her arrival, thus i shares all the marginal value from i to n. However, in π_2, player i only shares marginal value from player $j+1$ to player n.

Case 3. $\mathrm{MC}(G_1^i, N_{\pi_1|i} \setminus \{i\}, i) = 0, \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) > 0$. In thise case, we discuss cases concerning whether i is dummy or not. The first situtation is that i keeps to be dummy (No shared value from i to $j-1$) until j's arrival (This is because i has some positive marginal value in π_2).

$$\phi(G_1, i) = v(\{i\}) + \sum_{k=j}^{n} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^i, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big). \tag{6}$$

$$\phi(G_2, i) = v(\{i\}) + \sum_{k=j}^{n} \frac{1}{|S_2^k|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|k} \setminus \{k\}, k) - v(\{k\})\Big). \tag{7}$$

Let S_1^j be the set of players who share the value from j's participation in π_1. $S_1^j = S_2^i$ because they both represent the set of non-dummy players in players $\{1, 2, \ldots, j\}$. Another observation is that

	G				G'			
$\phi(\cdot, i)$	1	2	3	4	$\phi(\cdot, i)$	1	2	4	3
$\bar{G}_1$	1/2	0	1/2	0	$\bar{G}'_1$	1/2	0	0	1/2
$\bar{G}_2$	1/3	1/3	1/3	0	$\bar{G}'_2$	1/3	1/3	0	1/3
$\bar{G}_3$	0	0	1/2	1/2	$\bar{G}'_3$	0	0	1/2	1/2
G	5/6	1/3	4/3	1/2	G'	5/6	1/3	1/2	4/3

Table 4: eULMES rule outcomes in G and G'. The left part shows results in G, and the right part shows results in G' where player 3 delays her arrival.

Coalition	Coef	{1}	{2}	{1, 2}
Valuation		1	2	5
$\bar{v}_1(\cdot)$	$c_1 = 1$	1	1	1
$\bar{v}_2(\cdot)$	$c_2 = 1$	0	1	1
$\bar{v}_3(\cdot)$	$c_3 = 3$	0	0	1

Table 5: Coalition valuation and game decomposition of G

starting from player $j+1$'s participation, the value shared by player i in π_1 and π_2 will be the same. Hence,

$$\begin{aligned}&\phi(G_1, i) - \phi(G_2, i)\\ =&\frac{1}{|S_1^j|}\Big[\big(\mathrm{MC}(G_1^j, N_{\pi_1|j} \setminus \{j\}, j)\\ &- v(\{j\})\big) - \big(\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) - v(\{i\})\big)\Big].\end{aligned} \tag{8}$$

Note that player i creates no new marginal valution with $N_{\pi_1|i}$ and she is a dummy player for all subgames before j's arrival in this situation, thus we have $v(\{i\}) = 0$ and $v(N_{\pi_1|j} \setminus \{j\}) = v(N_{\pi_1|j} \setminus \{i, j\})$. Therefore,

$$\begin{aligned}&\phi(G_1, i) - \phi(G_2, i)\\ =&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|j} \setminus \{j\}) - v(\{j\})\Big)\\ \geq&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|j} \setminus \{i, j\}) + v(j) - v(N_{\pi_1|j} \setminus \{i, j\}) - v(j)\Big)\\ =&0.\end{aligned} \tag{9}$$

The other situation is that player i is not dummy after some player q's arrival where $i \prec_{\pi_1} q \prec_{\pi_1} j$, in this case, $\phi(G_1, i)$ can be written as

$$\phi(G_1, i) = v(\{i\}) + \sum_{k=q}^{n} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big). \tag{10}$$

Note that $v(\{i\}) = 0$ because of the superadditivity. Also, denote $\sum_{k=j+1}^{n} \frac{1}{|S_1^k|}\big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\big)$ by Δ for simplicity. Then, we have

$$\begin{aligned}\phi(G_1, i) =&\sum_{k=q}^{j} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big) + \Delta\\ \geq&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|j} - v(N_{\pi_1|q} \setminus \{q\}) - \sum_{k=q}^{j} v(k)\Big) + \Delta.\end{aligned} \tag{11}$$

For the game G_2, $\phi(G_2, i)$ can be written as

$$\begin{aligned}\phi(G_2, i) =&v(\{i\}) + \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) - v(i)\Big) + \Delta\\ =&\frac{1}{|S_2^i|}\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + \Delta.\end{aligned} \tag{12}$$

where $\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) = v(N_{\pi_1|j}) - v(N_{\pi_1|j} \setminus \{i\})$. Further,

$$\begin{aligned}&\phi(G_1, i) - \phi(G_2, i)\\ =&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|j}) - v(N_{\pi_1|q} \setminus \{q\}) - \sum_{k=q}^{j} v(\{k\})\Big)\\ &- \frac{1}{|S_2^i|}\Big(v(N_{\pi_1|j}) - v(N_{\pi_1|j} \setminus \{i\})\Big).\end{aligned} \tag{13}$$

For every player k who arrives after player j, we have $N_{\pi_1|k}$ and $N_{\pi_2|k}$. That is $|S_1^j| = |S_2^i|$,

$$\begin{aligned}&\phi(G_1, i) - \phi(G_2, i)\\ =&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|q} \setminus \{q\}) - \sum_{k=q}^{j} v(\{k\})\Big)\\ \geq&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|q} \setminus \{i, q\}) + \sum_{k=q}^{j} v(k) - v(N_{\pi_1|q} \setminus \{q\}) - \sum_{k=q}^{j} v(\{k\})\Big)\\ =&\frac{1}{|S_1^j|}\Big(v(N_{\pi_1|q} \setminus \{i, q\}) - v(N_{\pi_1|q} \setminus \{i, q\})\Big) = 0.\end{aligned} \tag{14}$$

The penultimate inequality follows from superadditivity and the last step is because i is a dummy player of prefix $N_{\pi_1|q} = \{1, 2, \ldots, i, \ldots, q\}$.

Case 4. $\mathrm{MC}(G_1^i, N_{\pi_1|i} \setminus \{i\}, i) = 0, \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) > 0$ For G_1, $\phi(G_1, i)$ can be represented as

$$\begin{aligned}\phi(G_1, i) = v(\{i\}) +& \sum_{k=i}^{j} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big)\\ &+ \sum_{k=j+1}^{n} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big).\end{aligned} \tag{15}$$

For G_2, $\phi(G_2, i)$ can be represented as

$$\phi(G_2, i) = v(\{i\}) + \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) - v(\{i\})\Big) + \sum_{k=j+1}^{n} \frac{1}{|S_k^2|}\Big(\mathrm{MC}(G_2^k, N_{\pi_2|k} \setminus \{k\}, k) - v(\{k\})\Big). \quad (16)$$

Still, we have $S_1^j = S_2^i$, and from the arrival of player $j+1$ to the last player, the value shared by player i in G_1 and G_2 remains the same. Thus,

$$\begin{aligned} &\phi(G_1, i) - \phi(G_2, i) \\ =&\sum_{k=i}^{j} \frac{1}{|S_1^k|}\Big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\Big) \\ &- \frac{1}{|S_2^i|}\Big(\mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) - v(\{i\})\Big) \quad (17) \\ \geq& \frac{1}{|S_1^j|}\Big(\sum_{k=i}^{j}\big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\big) \\ &- \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + v(\{i\})\Big). \end{aligned}$$

To show $\phi(G_1, i) - \phi(G_2, i) \geq 0$, we next show $\sum_{k=i}^{j}\big(\mathrm{MC}(G_1^k, N_{\pi_1|k} \setminus \{k\}, k) - v(\{k\})\big) - \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) + v(\{i\}) \geq 0$. We first rewrite the equation as

$$v(N_{\pi_1|j}) - v(N_{\pi_1|i} \setminus \{i\}) - \sum_{k=i+1}^{j} v(\{k\}) - \mathrm{MC}(G_2^i, N_{\pi_1|i} \setminus \{i\}, i). \quad (18)$$

Notably, $N_{\pi_2|i} \setminus \{i\} = \{1, 2, \ldots, i-1, i+1, \ldots, j\}$ and the valuation function v is superadditive, then we have,

$$\begin{aligned} &v(N_{\pi_1|j}) - v(N_{\pi_1|i} \setminus \{i\}) - \sum_{k=i+1}^{j} v(\{k\}) - \mathrm{MC}(G_2^i, N_{\pi_2|i} \setminus \{i\}, i) \\ &= v(N_{\pi_1|j} \setminus \{i\}) - v(N_{\pi_1|i} \setminus \{i\}) - \sum_{k=i+1}^{j} v(\{k\}) \\ &\geq v(N_{\pi_1|i} \setminus \{i\}) + \sum_{k=i+1}^{j} v(\{k\}) - v(N_{\pi_1|i} \setminus \{i\}) - \sum_{k=i+1}^{j} v(\{k\}) \\ &= 0. \end{aligned}$$

From Case 1 to Case 4, for any player i, i has no incentive to delay her arrival under the IR-NDMES sharing scheme. Thus, the IR-NDMES rule satisfies the EA axiom.

Finally, we show that **IR-eULMES** satisfies S-STAY, PART, OD, and EA. Recall IR-eULMES rule runs as follows. For an OCG $G = (N, v, \pi)$, we first decompose G into $D(G)$ by Algorithm 1. For each SOCG component $(c, \bar{G})$, we run the IR refinement paradigm, that is, first assign $v(\{i\})$ for each arriving player i; then compute S^i in $\bar{G}$ and equally share the value $\mathrm{MC}(\bar{G}^i, N_{\pi|i}) - v(\{i\})$.

(S-STAY) It is obviously that IR-eULMES rule satisfies the STAY axiom as each player shared value is the linear combination (positive coefficient) of the outputs of SOCGs, in which the shared value is non-decreasing. Next we show for each component SOCG $\bar{G}$, the rule satisfies S-STAY. Consider an SOCG $\bar{G} = (N, \bar{v}, \pi)$. For any prefix subgame $\bar{G}^i$, since we assume that $\mathrm{MC}(\bar{G}^i, N_{\pi|i} \setminus \{i\}, i) - \bar{v}(\{i\}) > 0$, then for any player j who satisfies $\bar{v}(N_{\pi|i}) > \bar{v}(N_{\pi|i} \setminus \{j\})$, j must be included in S^i in $\bar{G}$ (The same proof procedures of the S-STAY axiom for ULMES rule in Theorem 4.5). Then j must get a positve fraction $\frac{1}{|S^i|}$ of $(\mathrm{MC}(\bar{G}^i, N_{\pi|i} \setminus \{i\}, i) - \bar{v}(\{i\})) > 0$, satisfying S-STAY.

(PART) Since IR-eULMES rule outputs the linear combination of IR-ULMES rule in each decomposed component SOCG $\bar{G}$, we show that the PART axiom is satisfied in each $\bar{G}$. Consider an SOCG $\bar{G} = (N, \bar{v}, \pi)$, for any prefix subgame $\bar{G}^i$ with arriving player i, if i is a contributional player, $\bar{v}(N_{\pi|i}) > \bar{v}(N_{\pi|i} \setminus \{i\})$, then i is kept in S^i as $\bar{v}(N_{\pi|i}) - \bar{v}(N_{\pi|i} \setminus \{i\}) > 0$. Futher, we have $\phi(\bar{G}, i) = \bar{v}(\{i\}) + \frac{1}{|S^i|}\big(\mathrm{MC}(\bar{G}^i, N_{\pi|i} \setminus \{i\}, i) - \bar{v}(\{i\})\big)$. If $\bar{v}(\{i\}) > 0$, then $\phi(\bar{G}, i) > 0$ as $\mathrm{MC}(\bar{G}^i, N_{\pi|i} \setminus \{i\}, i) - \bar{v}(\{i\}) \geq 0$. When $\bar{v}(\{i\}) = 0$, $\phi(\bar{G}, i) = \frac{1}{|S^i|}\big(\mathrm{MC}(\bar{G}^i, N_{\pi|i} \setminus \{i\}, i) - \bar{v}(\{i\})\big) > 0$. Therefore, IR-eULMES rule satisfies PART.

(OD) The proof of satisfication of the Online-Dummy axiom directly holds as the IR refinement does not affect the selection of S^i for any OCG G. Thus the proof of the OD axiom directly follows the proof in Theorem 4.8.

(EA) To show IR-eULMES rule satisfies EA, it is sufficient to show that IR-ULMES rule satisfies EA in each decomposed component SOCG $\bar{G}$. The reason is that if player i has no incentive to delay her arrival in every decomposed SOCG $\bar{G}$, then i can not change the shared value by delaying her arrival because the GM decomposition does not depend on the arrival order π. Now we focus on the EA property of IR-ULMES rule in SOCG $\bar{G} = (N, \bar{v}, \pi)$. Notice that in SOCG with superadditive valuation function, there could be at most one player with singleton value 1. We discuss the following two cases: (1). There is no player with singleton value 1 in $\bar{G}$. In this case, IR-ULMES rule degenerates to ULMES rule. This means IR-ULMES rule satisfies EA as well as we have proved that ULMES rule satisfies EA in SOCGs in Theorem 4.7. (2). The other situation is that there exists some player q satisfying $\bar{v}(\{q\}) = 1$. In this case, as we know IR-ULMES rule satisfies IR, it means that in such an SOCG with $\bar{v}(\{q\}) = 1$, the value 1 is wholely allocated to player q. Then, firstly q has no incentive to delay her arrival as it does not change the outcome. Secondly, for any other player $i \neq q$, i cannot delay her arrival to share some part of the value 1 because IR property guarantees IR-ULMES rule distributes the value 1 to player q. This completes the proof of the EA satisfication of IR-ULMES rule. Therefore, we have IR-eULMES rule satisfies the EA axiom.

□