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Finite-time High-probability Bounds for Polyak-Ruppert Averaged Iterates of Linear Stochastic Approximation

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This paper provides a finite-time analysis of linear stochastic approximation (LSA) algorithms with fixed step size, a core method in statistics and machine learning. LSA is used to compute approximate solutions of a d-dimensional linear system $\bar{\mathbf{A}}\theta = \bar{\mathbf{b}}$, for which $(\bar{\mathbf{A}}, \bar{\mathbf{b}})$ can only be estimated through (asymptotically) unbiased observations $\{(\mathbf{A}(Z_n), \mathbf{b}(Z_n))\}_{n\in\mathbb{N}}$. We consider here the case where $\{Z_n\}_{n\in\mathbb{N}}$ is an i.i.d. sequence or a uniformly geometrically ergodic Markov chain, and derive p-moments inequality and high probability bounds for the iterates defined by LSA and its Polyak-Ruppert averaged version. More precisely, we establish bounds of order $(p\alpha t_{\mathrm{mix}})^{1/2} d^{1/p}$ on the p-th moment of the last iterate of LSA. In this formula α is the step size of the procedure and t_{mix} is the mixing time of the underlying chain ($t_{\mathrm{mix}} = 1$ in the i.i.d. setting). We then prove finite-time instance-dependent bounds on the Polyak-Ruppert averaged sequence of iterates. These results are sharp in the sense that the leading term we obtain matches the local asymptotic minimax limit, including tight dependence on the parameters (d, t_{mix}) in the higher order terms.

Key words: linear stochastic approximation; Polyak–Ruppert averaging; stability of random matrix product
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1. Introduction This paper is concerned with the linear stochastic approximation (LSA) algorithm for solving the linear system $\bar{\mathbf{A}}\theta = \bar{\mathbf{b}}$ with unique solution $\theta^\star$, based on a sequence of observations $\{(\mathbf{A}(Z_n), \mathbf{b}(Z_n))\}_{n\in\mathbb{N}}$. Here $\mathbf{A} : \mathsf{Z} \to \mathbb{R}^{d\times d}$, $\mathbf{b} : \mathsf{Z} \to \mathbb{R}^d$ are measurable functions, and $(Z_k)_{k\in\mathbb{N}}$ is

1. either an i.i.d. sequence taking values in a state space $(\mathsf{Z}, \mathcal{Z})$ with distribution π satisfying $\mathbb{E}[\mathbf{A}(Z_1)] = \bar{\mathbf{A}}$ and $\mathbb{E}[\mathbf{b}(Z_1)] = \bar{\mathbf{b}}$;
2. or a Z-valued ergodic Markov chain with unique invariant distribution π, such that $\lim_{n\to+\infty} \mathbb{E}[\mathbf{A}(Z_n)] = \bar{\mathbf{A}}$ and $\lim_{n\to+\infty} \mathbb{E}[\mathbf{b}(Z_n)] = \bar{\mathbf{b}}$.

For a fixed step size $\alpha > 0$, burn-in period $n_0 \in \mathbb{N}$, and initialization θ_0, consider the sequences of estimates $\{\theta_n\}_{n\in\mathbb{N}}, \{\bar{\theta}_{n_0,n}\}_{n\geq n_0+1}$ given by

$$\begin{aligned} \theta_k &= \theta_{k-1} - \alpha\{\mathbf{A}(Z_k)\theta_{k-1} - \mathbf{b}(Z_k)\}\,, \quad k \geq 1, \\ \bar{\theta}_{n_0,n} &= (n-n_0)^{-1} \textstyle\sum_{k=n_0}^{n-1} \theta_k\,, \quad n \geq n_0 + 1\,. \end{aligned} \tag{1}$$

The sequence $\{\theta_k\}_{k\in\mathbb{N}}$ are the standard LSA iterates, while $\{\bar{\theta}_{n_0,n}\}_{n\geq n_0+1}$ corresponds to the Polyak-Ruppert (PR) averaged iterates; see Ruppert [35],Polyak and Juditsky [32].

LSA algorithms is a core algorithms in statistics and machine learning. It plays a central role in linear system identification Eweda and Macchi [14], Widrow and Stearns [40], Benveniste et al. [4], Kushner and Yin [23]. More recently, they have reignited interest in machine learning, particularly for high-dimensional least squares estimation and reinforcement learning (RL) problems; Bertsekas and Tsitsiklis [6], Bottou et al. [9], Sutton [37], Bertsekas [5], Watkins and Dayan [39]. LSA and LSA-PR recursions (1) have been the subject of a wealth of works and it is difficult to give a proper credit to all the contributions. Polyak and Juditsky [32], Kushner and Yin [23], Borkar [8], Benveniste et al. [4] provided asymptotic convergence guarantees (almost sure convergence, central limit theorem) under both the i.i.d. and Markovian settings. In particular, it has been established that LSA-PR can accelerate LSA and satisfies a central limit theorem with a covariance matrix which is minimax optimal asymptotically.

Although asymptotic convergence guarantees are of theoretical interest, the current trend is to obtain non-asymptotic guarantees that take into account both the limited sample size and the dimension of the parameter space. For these reasons, non-asymptotic analysis of both i.i.d. and Markovian SA procedures has recently attracted much attention.

In the i.i.d. setting, Rakhlin et al. [33], Nemirovski et al. [28], Jain et al. [20, 21] studied the mean squared error in finite time, Durmus et al. [13] provided tight high probability bounds for the LSA sequence $\{\theta_n\}_{n\in\mathbb{N}}$. For least squares regression problems, where $\mathbf{A}(Z_n)$ is a symmetric matrix almost surely, Bach and Moulines [3], Jain et al. [19] showed that for a constant step size, the mean squared error (MSE) of $\bar{\theta}_{n_0,n} - \theta^\star$ converges as $\mathcal{O}(1/n)$. For general LSA, which includes instrumental variable methods for linear system identification and temporal difference in reinforcement learning (TD), Lakshminarayanan and Szepesvari [24] showed a rate of convergence of the mean squared error $\mathcal{O}(1/n)$. Close to the present work, Mou et al. [26] provided a non-asymptotic high-probability bounds for LSA-PR with independent observations. However, the proof of their main result Mou et al. [26, Theorem 3] relies on tools from Markov chain theory which assume strong conditions on $\{(\mathbf{A}(Z_n), \mathbf{b}(Z_n)\}_{n\in\mathbb{N}}$ and it is not clear how to adapt their method to the general case. Our first contribution is to extend and improve this result using a total different approach relying on the stochastic expansion for LSA (1) introduced in Aguech et al. [1]. More precisely, we derive sharp finite-time bounds on p-th moment of $\{\|\theta_n - \theta^\star\|\}_{n\in\mathbb{N}}$ and $\{\|\bar{\theta}_{n_0,n} - \theta^\star\|\}_{n\in\mathbb{N}}$. As a corollary, we provide optimized high probability bounds for the LSA and LSA-PR iterates for a fixed tolerance parameter $\delta \in (0,1)$ and number of iterations n, choosing appropriately the stepsize α. For LSA-PR, the leading term of these bounds matches the one of the central limit theorem making appear the same asymptotic covariance matrix up to numerical constants.

Regarding the Markovian setting, the literature is scarcer. Assuming a mixing time upper-bound bound on the Markov chain, a projected variant of the linear SA was analyzed by Bhandari et al. [7], which report non-asymptotic rates for the mean squared error that are sharp in their dependence on sample size, but not on the dimension. This result was later extended by Srikant and Ying [36], which analyzed LSA without the projection step - and obtained the same convergence rate. In Chen et al. [10] the authors obtained sharp MSE bound for the last iterate of the linear SA procedure for the observations being V-uniformly ergodic Markov chain and the decreasing sequence of stepsizes $\alpha_k = 1/k$. Recently, Mou et al. [27] established p-moment bounds for the last iterates of LSA and showed that the mean-square error obtained with Polyak-Ruppert averaged LSA matches the local asymptotic minimax optimal limit. Our second contribution is to improve and extend their result. First, we derive sharper p-moments bounds on $\{\|\theta_n - \theta^\star\|\}_{n\in\mathbb{N}}$. Second, we provide the same type of results for LSA-PR which allows us to provide high-probability bounds which are natural counterpart to the i.i.d. setting. To the best of our knowledge, this results for Markovian observations are new. Moreover, our bound for mean-square error with Markovian noise improves the dependence upon the problem dimension compared to the results of Mou et al. [27, Theorem 1].

The paper is organized as follows. In Section 2 we introduce the decomposition of the error which is key to our proof (see Aguech et al. [1]) and formulate our main assumptions. In Section 3 we present our results in the independent setting. In Section 4 we extend our results when $\{Z_n\}_{n\in\mathbb{N}^*}$ is a uniformly geometrically ergodic Markov chain. The proofs are postponed to the appendix. For reader's convenience the notations and key constants appearing in the text are summarized in Section A.

2. Stochastic expansions for LSA and LSA-PR As an introduction, we present tools and some preliminary results that will be central to our analysis of LSA and LSA-PR which will be used for the case where $\{Z_n\}_{n\in\mathbb{N}^*}$ is i.i.d. but also where this sequence is a Markov chain. Set

$$\tilde{\mathbf{A}}(z) = \mathbf{A}(z) - \bar{\mathbf{A}}\,, \quad \tilde{\mathbf{b}}(z) = \mathbf{b}(z) - \bar{\mathbf{b}}\,, \quad \varepsilon(z) = \tilde{\mathbf{A}}(z)\theta^\star - \tilde{\mathbf{b}}(z)\,, \tag{2}$$

and denote by $\Gamma_{1:n}^{(\alpha)}$ the product of random matrices

$$\Gamma_{m:n}^{(\alpha)} = \textstyle\prod_{i=m}^{n}(\mathrm{I} - \alpha\mathbf{A}(Z_i))\,, \quad m,n\in\mathbb{N}^*, \quad m \le n\,. \tag{3}$$

The definition (1) implies the following decomposition $\theta_n - \theta^\star = \tilde{\theta}_n^{(\mathsf{tr})} + \tilde{\theta}_n^{(\mathsf{fl})}$ where $\tilde{\theta}_n^{(\mathsf{tr})}$ is a transient term (reflecting the forgetting of initial condition) and $\tilde{\theta}_n^{(\mathsf{fl})}$ is a fluctuation term (reflecting misadjustement noise)

$$\tilde{\theta}_n^{(\mathsf{tr})} = \Gamma_{1:n}^{(\alpha)}\{\theta_0 - \theta^\star\}\,, \quad \tilde{\theta}_n^{(\mathsf{fl})} = -\alpha\textstyle\sum_{j=1}^{n}\Gamma_{j+1:n}^{(\alpha)}\varepsilon(Z_j)\,. \tag{4}$$

We first bound the p-th moments for $\{\|\tilde{\theta}_n^{(\mathsf{tr})}\| : n\in\mathbb{N}\}$ by establishing that the sequence of random matrices $\{\mathbf{A}(Z_i)\}_{i\in\mathbb{N}^*}$ is *exponentially stable* (see Guo and Ljung [15], Ljung [25]): for $q\ge 1$, there exist $\mathsf{a}_q, \mathrm{C}_q > 0$ and $\alpha_{\infty,q} < \infty$ such that, for any step size $\alpha \le \alpha_{\infty,q}$, $m,n\in\mathbb{N}$, $m<n$,

$$\mathbb{E}[\|\Gamma_{m:n}^{(\alpha)}\|^q] \le \mathrm{C}_q \exp\left(-\mathsf{a}_q\alpha(n-m)\right)\,. \tag{5}$$

In the sequel, this result is established in Theorem 1 for the i.i.d. setting and in Theorem 3 in the Markovian setting. Intuitively, exponential stability means that the q-th moment of the product of random matrices $\Gamma_{m:n}^{(\alpha)}$ behaves similarly to that of the product of *deterministic* matrices $(\mathrm{I}-\alpha\bar{\mathbf{A}})^{n-m}$, for $m,n\in\mathbb{N}$, $m\le n$, under the assumption that $-\bar{\mathbf{A}}$ is Hurwitz, i.e., for any eigenvalue λ of $\bar{\mathbf{A}}$, we have $\mathrm{Re}(\lambda) > 0$.

PROPOSITION 1 (**[12, Proposition 1]**). *Assume that $-\bar{\mathbf{A}}$ is Hurwitz. There exists a unique symmetric positive definite matrix Q satisfying the Lyapunov equation $\bar{\mathbf{A}}^\top Q + Q\bar{\mathbf{A}} = \mathrm{I}$. In addition, setting*

$$a = \|Q\|^{-1}/2\,, \quad \text{and} \quad \alpha_\infty = (1/2)\|\bar{\mathbf{A}}\|_Q^{-2}\|Q\|^{-1} \wedge \|Q\|\,, \tag{6}$$

for any $\alpha\in[0,\alpha_\infty]$, it holds that $\|\mathrm{I}-\alpha\bar{\mathbf{A}}\|_Q^2 \le 1 - a\alpha$, and $\alpha a \le 1/2$.

Finally, note that the condition that $-\bar{\mathbf{A}}$ is Hurwitz implies the existence of a unique solution $\theta^\star$ to $\bar{\mathbf{A}}\theta = \bar{\mathbf{b}}$.

For the fluctuation term, we use the perturbation-expansion technique introduced in [1] that recursively decomposes $\tilde{\theta}_n^{(\mathsf{fl})}$. It is important to note that the expansion order is chosen to achieve the desired approximation accuracy. Indeed, by definition of $\tilde{\theta}_n^{(\mathsf{fl})}$ (see (4)), it satisfies the recurrence

$$\tilde{\theta}_n^{(\mathsf{fl})} = (\mathrm{I} - \alpha\mathbf{A}(Z_n))\tilde{\theta}_{n-1}^{(\mathsf{fl})} - \alpha\varepsilon(Z_n)\,.$$

Using the definition of $\tilde{\mathbf{A}}$ (see (2)), and an induction argument, it is easy to verify that the following decomposition holds for any $n\in\mathbb{N}$:

$$\tilde{\theta}_n^{(\mathsf{fl})} = J_n^{(0)} + H_n^{(0)}\,, \tag{7}$$

where the latter terms are defined by the following pair of recursions

$$\begin{aligned} J_n^{(0)} &= \left(\mathrm{I} - \alpha \bar{\mathbf{A}}\right) J_{n-1}^{(0)} - \alpha \varepsilon(Z_n)\,, & J_0^{(0)} &= 0\,, \\ H_n^{(0)} &= \left(\mathrm{I} - \alpha \mathbf{A}(Z_n)\right) H_{n-1}^{(0)} - \alpha \tilde{\mathbf{A}}(Z_n) J_{n-1}^{(0)}\,, & H_0^{(0)} &= 0\,. \end{aligned} \tag{8}$$

Furthermore, the same decomposition can be applied to $H_n^{(0)}$ to obtain *higher order expansions*. Fix $L \geq 1$ to be the desired expansion order. Then a double recursion in $\ell \in \{1, \ldots, L\}$ and $n \in \mathbb{N}$ shows that

$$H_n^{(0)} = \textstyle\sum_{\ell=1}^{L} J_n^{(\ell)} + H_n^{(L)}\,, \tag{9}$$

where for any $\ell \in \{1, \ldots, L\}$,

$$\begin{aligned} J_n^{(\ell)} &= \left(\mathrm{I} - \alpha \bar{\mathbf{A}}\right) J_{n-1}^{(\ell)} - \alpha \tilde{\mathbf{A}}(Z_n) J_{n-1}^{(\ell-1)}\,, & J_0^{(\ell)} &= 0\,, \\ H_n^{(L)} &= \left(\mathrm{I} - \alpha \mathbf{A}(Z_n)\right) H_{n-1}^{(L)} - \alpha \tilde{\mathbf{A}}(Z_n) J_{n-1}^{(L)}\,, & H_0^{(L)} &= 0\,. \end{aligned} \tag{10}$$

The choice of parameter L controls the desired approximation accuracy. Combining (7) and (9), we obtain the decomposition which is the cornerstone of our analysis:

$$\theta_n - \theta^\star = \tilde{\theta}_n^{(\mathsf{tr})} + \textstyle\sum_{\ell=0}^{L} J_n^{(\ell)} + H_n^{(L)}\,, \tag{11}$$

where $\{J_n^{(\ell)} : \ell \in \{1, \ldots, \ell\}\}$ and $H_n^{(L)}$ are defined in (10) and (8) respectively. Following Durmus et al. [12], this decomposition can be used to obtain sharp bounds on the p-th moment of the final LSA iterate θ_n. We refine the bounds obtained in Durmus et al. [12] with respect to the dimension d in the i.i.d. setting in Proposition 3 and then provide extensions to the Markovian setting in Proposition 7.

In addition, (11) naturally leads to a decomposition for the PR averaged sequence $\bar{\theta}_{n_0,n} - \theta^\star$ starting from the observation that by definition (1), for any $n, n_0 \in \mathbb{N}$, $n_0 \leq n$,

$$\bar{\mathbf{A}}\left(\bar{\theta}_{n_0,n} - \theta^\star\right) = \{\alpha(n-n_0)\}^{-1}(\theta_{n_0} - \theta_n) - (n-n_0)^{-1} \textstyle\sum_{t=n_0}^{n-1} e\left(\theta_t, Z_{t+1}\right)\,, \tag{12}$$

$$e(\theta, z) = \tilde{\mathbf{A}}(z)\theta - \tilde{\mathbf{b}}(z) = \varepsilon(z) + \tilde{\mathbf{A}}(z)(\theta - \theta^\star)\,. \tag{13}$$

Using (11), we may further decompose

$$\textstyle\sum_{t=n_0}^{n-1} e\left(\theta_t, Z_{t+1}\right) = E_{n_0,n}^{\mathsf{tr}} + E_{n_0,n}^{\mathsf{fl}}\,, \tag{14}$$

where we have set

$$\begin{aligned} E_{n_0,n}^{\mathsf{tr}} &= \textstyle\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}(Z_{t+1}) \Gamma_{1:t}^{(\alpha)}\{\theta_0 - \theta^\star\}\,, \\ E_{n_0,n}^{\mathsf{fl}} &= \textstyle\sum_{t=n_0}^{n-1} \varepsilon(Z_{t+1}) + \sum_{\ell=0}^{L} \sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}(Z_{t+1}) J_t^{(\ell)} + \sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}(Z_{t+1}) H_t^{(L)}\,. \end{aligned}$$

Based on the decompositions (12) and (14), our analysis of the PR recursion proceeds in bounding the three terms $\{\alpha(n-n_0)\}^{-1}(\theta_{n_0} - \theta_n)$, $E_{n_0,n}^{\mathsf{tr}}$ and $E_{n_0,n}^{\mathsf{fl}}$ separately. For the first one, we use the bounds derived on the p-th moment of $\theta_k - \theta^\star$, for $k \in \mathbb{N}$. For the second one, we use the exponential stability for $\{\Gamma_{m:n}^{(\alpha)} : m, n \in \mathbb{N}, m \leq n\}$ (5). Finally, the fluctuation term $E_{n_0,n}^{\mathsf{fl}}$ is dealt with the conditions we impose on the sequence $\{Z_n\}_{n \in \mathbb{N}^*}$.

Throughout this paper, we impose the following assumption regarding $z \mapsto \tilde{\mathbf{A}}(z)$ and $\bar{\mathbf{A}}$.

A 1. $\mathrm{C}_{\mathbf{A}} = \sup_{z \in \mathsf{Z}} \|\mathbf{A}(z)\| \vee \sup_{z \in \mathsf{Z}} \|\tilde{\mathbf{A}}(z)\| < \infty$ and the matrix $-\bar{\mathbf{A}}$ is Hurwitz.

We further require the following assumptions on the noise term $\varepsilon(z)$ and the stationary distribution π of the sequence $\{Z_n\}_{n \in \mathbb{N}^*}$.

A 2. There exists $\mathrm{C}_\varepsilon < +\infty$, such that for any $z \in \mathsf{Z}$, $\|\varepsilon(z)\| \leq \mathrm{C}_\varepsilon \sqrt{\operatorname{Tr} \Sigma_\varepsilon}$, where

$$\Sigma_\varepsilon = \textstyle\int_{\mathsf{Z}} \varepsilon(z) \varepsilon(z)^\top \mathrm{d}\pi(z)\,. \tag{15}$$

The assumption A 2 is considered in Vershynin [38, Theorem 5.6.1]. It can be generalized in some sense, leaving the basic ingredients of our proof unchanged. In particular, in Section C we provide the counterparts of the results of Section 3 under a sub-Gaussian moment assumption for $\varepsilon(Z_n)$.

Our main results are presented and discussed in the following two sections. Section 3 focuses on the i.i.d. case, while Section 4 deals with the Markovian setting.

3. Finite-time Moment and High-probability Bounds in the Independent Noise Setting Consider the following assumption:

IND 1. $\{Z_k\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with distribution π.

We begin our analysis by deriving sharp bounds for the p-th moment on $\{\|\theta_n-\theta^\star\| : n\in\mathbb{N}\}$ based on the decomposition (11). Our first step is to estimate the transient term $\tilde{\theta}_n^{(\mathsf{tr})}$. Using the notations of Proposition 1, we define under the assumption A 1 for $q\geq 2$

$$\kappa_Q=\lambda_{\mathsf{max}}(Q)/\lambda_{\mathsf{min}}(Q)\,,\quad b_Q=2\sqrt{\kappa_Q}\,\mathrm{C}_{\mathbf{A}}\,, \tag{16}$$
$$\alpha_{q,\infty}=\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}}/q\,,\quad \mathrm{c}_{\mathbf{A}}=a/\{2b_Q^2\}\,. \tag{17}$$

A key property from which we derive our bounds is an exponential stability result for the p-th moment of the product of $\Gamma_{1:n}^{(\alpha)}$.

Theorem 1 **(Durmus et al. [12, Corollary 1])**. *Assume A 1 and IND 1. Then, for any $p,q\in\mathbb{N}$, $2\leq p\leq q$, $\alpha\in(0,\alpha_{q,\infty}]$ and $n\in\mathbb{N}^*$, it holds*

$$\mathbb{E}^{1/p}\left[\|\Gamma_{1:n}^{(\alpha)}\|^p\right]\leq\sqrt{\kappa_Q}d^{1/q}(1-a\alpha+(q-1)b_Q^2\alpha^2)^{n/2}\,.$$

Theorem 1 implies that $\sup_{n\in\mathbb{N}^*}\mathbb{E}[\|\Gamma_{1:n}^{(\alpha)}\|^p]<+\infty$ for any $\alpha\in(0,\alpha_{q,\infty}]$ and $2\leq p\leq q$. This condition relating the choice of the stepsize α with p and q is unavoidable; see Durmus et al. [12, Example 1].

Decomposition (11) applied with $L=0$ is enough to obtain tight MSE bound for LSA-PR procedure. We first provide p-moment bounds for the sequence $\{J_n^{(0)} : n\in\mathbb{N}\}$.

Proposition 2. *Assume A 1, A 2, and IND 1. Then, for any $\alpha\in(0,\alpha_\infty]$, $p\geq 2$, and $n\in\mathbb{N}$, it holds*

$$\mathbb{E}^{1/p}\left[\|J_n^{(0)}\|^p\right]\leq \mathsf{D}_1\sqrt{\alpha a p\operatorname{Tr}\Sigma_\varepsilon}\,,\ \textit{where}\ \mathsf{D}_1=\sqrt{2\kappa_Q}\,\mathrm{C}_\varepsilon/a\,. \tag{18}$$

Proof. Expanding the recurrence (8), we represent

$$J_n^{(0)}=-\alpha\textstyle\sum_{k=1}^n(\mathrm{I}-\alpha\bar{\mathbf{A}})^{n-k}\varepsilon(Z_k)\,. \tag{19}$$

Now the bound (18) follows from a Hoeffding-type bound for sums of independent random vectors (see [31, Theorem 3.1]) combined with Proposition 1. Detailed argument is postponed to Section B.1.

In the statement of (18) we could simplify the term a^{-1} occurring in D_1 and the same factor in $\sqrt{\alpha a p\operatorname{Tr}\Sigma_\varepsilon}$. However, for homogeneity reasons, we have chosen to state it in this form. Indeed, if we multiply $\bar{\mathbf{A}}$ by a positive constant M, then the quantities $a,\mathrm{C}_{\mathbf{A}},\mathrm{C}_\varepsilon$ should also scale similarly, implying that D_1 remains unchanged. We adhere to this reasoning for all bounds we derive below to obtain constants that remain unchanged under a scaling factor.

We emphasise that $J_n^{(0)}$ is the leading (in terms of the step size α) term in the error decomposition (11). Indeed, (18) and the stability result (Theorem 1) are sufficient to obtain a rough bound $\mathbb{E}^{1/p}[\|H_n^{(0)}\|^p]\leq C\sqrt{\alpha}$ for a constant $C\geq 0$ (see (8) for the definition of $H_n^{(0)}$). Combining these results gives the following p-th moment bound for the LSA error $\|\theta_n-\theta^\star\|$:

Proposition 3. *Assume A 1, A 2, and IND 1. Then, for any $p,q\in\mathbb{N}$, $2\leq p\leq q$, $\alpha\in(0,\alpha_{q,\infty}]$, $n\in\mathbb{N}$, and $\theta_0\in\mathbb{R}^d$ it holds*

$$\mathbb{E}^{1/p}\left[\|\theta_n-\theta^\star\|^p\right]\leq d^{1/q}\kappa_Q^{1/2}\,(1-\alpha a/4)^n\,\|\theta_0-\theta^\star\|+d^{1/q}\mathsf{D}_2\sqrt{\alpha a p\operatorname{Tr}\Sigma_\varepsilon}\,, \tag{20}$$

where the constant D_2 is given by

$$\mathsf{D}_2=(2\kappa_Q)^{1/2}\,\mathrm{C}_\varepsilon\,a^{-1}(1+4\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,a^{-1})\,.$$

Proof. Using the decomposition (11) and Minkowski's inequality,

$$\mathbb{E}^{1/p}\left[\|\theta_n-\theta^\star\|^p\right]\le \mathbb{E}^{1/p}\left[\|\Gamma^{(\alpha)}_{1:n}(\theta_0-\theta^\star)\|^p\right]+\mathbb{E}^{1/p}\left[\|J^{(0)}_n\|^p\right]+\mathbb{E}^{1/p}\left[\|H^{(0)}_n\|^p\right]. \tag{21}$$

Applying Theorem 1, using that $\alpha a\le 1/2$ by Proposition 1, and $(1-t)^{1/2}\le 1-t/2$ for $t\in[0,1]$,

$$\mathbb{E}^{1/p}\left[\|\Gamma^{(\alpha)}_{1:n}(\theta_0-\theta^\star)\|^p\right]\le \kappa_Q^{1/2}d^{1/q}(1-\alpha a/4)^n\|\theta_0-\theta^\star\|. \tag{22}$$

With Proposition 2, we get $\mathbb{E}^{1/p}[\|J^{(0)}_n\|^p]\le \mathsf{D}_1\sqrt{\alpha a p\operatorname{Tr}\Sigma_\varepsilon}$. It remains to bound $\mathbb{E}^{1/p}[\|H^{(0)}_n\|^p]$. Expanding the recurrence (8), we represent

$$H^{(0)}_n=-\alpha\textstyle\sum_{j=1}^n\Gamma^{(\alpha)}_{j+1:n}\tilde{\mathbf{A}}(Z_j)J^{(0)}_{j-1}.$$

Using Minkowski's inequality and since $\tilde{\mathbf{A}}(Z_j)$ and $J^{(0)}_{j-1}$ are independent under IND 1, we obtain with A 1, that

$$\mathbb{E}^{1/p}\left[\|H^{(0)}_n\|^p\right]\le \alpha\,\mathrm{C}_{\mathbf{A}}\textstyle\sum_{j=1}^n\mathbb{E}^{1/p}\left[\|\Gamma^{(\alpha)}_{j+1:n}\|^p\right]\mathbb{E}^{1/p}\left[\|J^{(0)}_{j-1}\|^p\right].$$

Now (22) and Proposition 2 yield

$$\mathbb{E}^{1/p}\left[\|H^{(0)}_n\|^p\right]\le \mathrm{c}_1\,d^{1/q}\sqrt{\alpha a p\operatorname{Tr}\Sigma_\varepsilon},\ \text{ where }\ \mathrm{c}_1=4\mathsf{D}_1\,\mathrm{C}_{\mathbf{A}}\,\kappa_Q^{1/2}/a.$$

Combining the bounds above in (21) completes the proof.

The bound of Proposition 3 can be used to derive a high-probability bound for a particular choice of step size α when the number of iterations n is fixed.

Corollary 1. *Assume A 1, A 2, IND 1 and set $\delta\in(0,1)$. Then, for any $\theta_0\in\mathbb{R}^d$, sample size $n\in\mathbb{N}$ satisfying*

$$n/\log n\ge (a/4)\left\{\alpha_\infty^{-1}\vee \mathrm{c}_{\mathbf{A}}^{-1}(1+\log d)\log(2\mathrm{e}/\delta)\right\}, \tag{23}$$

and step size $\alpha=4\log n/(an)$, it holds with probability at least $1-\delta$, that

$$\|\theta_n-\theta^\star\|\le 4\mathrm{e}\mathsf{D}_2\sqrt{\frac{\{\operatorname{Tr}\Sigma_\varepsilon\}\log n\log(2\mathrm{e}/\delta)}{n}}+\frac{2\mathrm{e}\kappa_Q^{1/2}\|\theta_0-\theta^\star\|}{n}.$$

Proof. Under (23), the step size $\alpha=4\log n/(an)$ satisfies $\alpha\le\alpha_{(1+\log d)\log(2\mathrm{e}/\delta),\infty}$. Now the statement follows from Proposition 3 applied with $p=\log(2\mathrm{e}/\delta)$, $q=(1+\log d)\log(2\mathrm{e}/\delta)$, and the Markov inequality applied with the same p.

Corollary 1 is closely related to Durmus et al. [12, Theorem 1]. We choose the probability tolerance parameter $\delta>0$ and optimize the moment p, which implies a constraint on the number of iterations (23) and the step size α, while in Durmus et al. [12, Theorem 1] the authors fix p and derive high probability bounds that hold for any $\alpha\in(0,\alpha_{p,\infty}]$. Note that at the cost of the logarithmic dependence of α on dimension, we can get rid of the dependence of d in (20).

Now we have all the ingredients to state our first main result for a bound on the mean square error of n-steps LSA-PR $\bar\theta_{\lceil n/2\rceil,n}-\theta^\star$. The leading term we obtain is related to the noise covariance matrix Σ_ε defined in (15). The other terms depending on $\Delta^{(\mathrm{tr})}_{n,\alpha}$ and $\Delta^{(\mathrm{fl})}_{n,\alpha}$ correspond to the *transient* and *fluctutation* components of the LSA error and are defined as

$$\Delta^{(\mathrm{tr})}_{n,\alpha}=32\mathrm{e}\kappa_Q/(\alpha^2 n)+64\mathrm{e}\kappa_Q\,\mathrm{C}_{\mathbf{A}}^2/(7\alpha a n),\qquad \Delta^{(\mathrm{fl})}_{n,\alpha}=64\mathrm{e}\mathsf{D}_2^2/(\alpha n)+8\mathrm{e}\alpha\mathsf{D}_2^2. \tag{24}$$

For this result we only need the decomposition of the LSA- PR error (12) together with the p-th moment boundary for the LSA error given in Proposition 3. Subsequent p-th moment bounds for the LSA-PR error require further exploitation of the decomposition (14).

For $t\in\mathbb{N}$, we define the filtration $\mathcal{F}_t=\sigma(Z_s:1\le s\le t)$, $\mathcal{F}_0=\{\emptyset,\mathsf{Z}\}$, and denote by $\mathbb{E}^{\mathcal{F}_t}$ the conditional expectation with respect to $\mathcal{F}_t$.

PROPOSITION 4. *Assume A 1, A 2, and IND 1. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_\infty \wedge \mathrm{c}_{\mathbf{A}} / \{2 + 2\log d\})$, $\theta_0 \in \mathbb{R}^d$, it holds*

$$(n/2)\mathbb{E}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2 \rceil, n} - \theta^\star\right)\|^2\right] \le 4 \operatorname{Tr}\Sigma_\varepsilon + \Delta^{(\mathrm{fl})}_{n,\alpha} \operatorname{Tr}\Sigma_\varepsilon + \Delta^{(\mathrm{tr})}_{n,\alpha}\|\theta_0 - \theta^\star\|^2 (1 - \alpha a/4)^n \,.$$

Proof. Let $q \ge 2$ be a number to be fixed later, and assume that $\alpha \in (0, \alpha_{q,\infty})$. We need this additional degree of freedom to ensure that our bounds are dimension-free. Our proof is based on the decomposition (12). Under IND 1, $\mathbb{E}^{\mathcal{F}_t}[e(\theta_t, Z_{t+1})] = 0$ $\mathbb{P}$-a.s., showing that $e(\theta_t, Z_{t+1})$ is an $\mathcal{F}_t$-martingale increment. Thus, exploiting (12), we obtain

$$\mathbb{E}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{n_0,n} - \theta^\star\right)\|^2\right] \le T_1 + T_2 \,, \tag{25}$$

where we set

$$T_1 = \frac{2\sum_{t=n_0}^{n-1} \mathbb{E}\left[\|\mathrm{e}(\theta_t, Z_{t+1})\|^2\right]}{(n-n_0)^2} \,, \qquad T_2 = \frac{2\mathbb{E}[\|\theta_{n_0} - \theta_n\|^2]}{\alpha^2 (n-n_0)^2} \,.$$

Now we estimate the terms T_1 and T_2 separately. To control the remainder term T_2, we apply Proposition 3 with $p = 2$, and obtain

$$T_2 \le \frac{16 d^{2/q} \kappa_Q \|\theta_0 - \theta^\star\|^2 (1 - \alpha a/4)^{2n_0}}{\alpha^2 (n-n_0)^2} + \frac{32 d^{2/q} \mathsf{D}_2^2 a \operatorname{Tr}\Sigma_\varepsilon}{\alpha (n-n_0)^2} \,.$$

Now we bound T_1. Recall that for $\theta \in \mathbb{R}^d$, $z \in \mathsf{Z}$, $e(\theta, z) = \varepsilon(z) + \tilde{\mathbf{A}}(z)(\theta - \theta^\star)$. Hence,

$$\mathbb{E}\left[\|\mathrm{e}(\theta_t, Z_{t+1})\|^2\right] \le 2 \operatorname{Tr}\Sigma_\varepsilon + 2\mathbb{E}\left[\|\tilde{\mathbf{A}}(Z_{t+1})\{\theta_t - \theta^\star\}\|^2\right] \,,$$

where we used that $\mathbb{E}[\|\varepsilon_t\|^2] = \operatorname{Tr}\Sigma_\varepsilon$. Proposition 3, the fact that $\alpha \le \alpha_\infty$ and $\alpha a \le 1/2$ by Proposition 1, give

$$\begin{aligned}\textstyle\sum_{t=n_0}^{n-1} \mathbb{E}\left[\|\mathrm{e}(\theta_t, Z_{t+1})\|^2\right] \le 2(n-n_0)\operatorname{Tr}\Sigma_\varepsilon + 8\alpha a (n-n_0) d^{2/q} \mathsf{D}_2^2 \, \mathrm{C}_{\mathbf{A}}^2 \operatorname{Tr}\Sigma_\varepsilon \\ + \frac{32 d^{2/q} \kappa_Q \, \mathrm{C}_{\mathbf{A}}^2 \, \|\theta_0 - \theta^\star\|^2}{7\alpha a} (1 - \alpha a/4)^{2n_0} \,.\end{aligned}$$

It remains to combine the bounds above in (25), choose $q = 2(1 + \log d)$, $n_0 = \lceil n/2 \rceil$, and use the elementary inequality $d^{2/(2+2\log d)} \le \mathrm{e}$.

The bounds above can be simplified under a particular choice of α, depending on n. The fluctuation error $\Delta^{(\mathrm{fl})}$ in (24) suggests that α should scale with n as $n^{-1/2}$. Let us choose

$$\alpha(n, d) = (\alpha_\infty \wedge \mathrm{c}_{\mathbf{A}} / \{2 + 2\log d\}) \, n^{-1/2} \,.$$

Then Proposition 4 implies the MSE bound

$$\mathbb{E}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2 \rceil, n} - \theta^\star\right)\|^2\right] \lesssim \frac{\operatorname{Tr}\Sigma_\varepsilon}{n} + \frac{\operatorname{Tr}\Sigma_\varepsilon}{n^{3/2}} + \|\theta_0 - \theta^\star\|^2 \exp\left\{-\frac{(\alpha_\infty \wedge \mathrm{c}_{\mathbf{A}})\sqrt{n}}{8(1 + \log d)}\right\} \,, \tag{26}$$

where $\lesssim$ stands for inequality up to a constant, depending on κ_Q, a, $\mathrm{C}_{\mathbf{A}}$, C_ε, and polylogarithmic factors in d. Note that the bound (26) shows the same (optimal) leading term $n^{-1}\operatorname{Tr}\Sigma_\varepsilon$ as in Mou et al. [27, Theorem 1], improving the dependence upon sample size n in the remainder term.

To obtain the above MSE bound, we used the expansions (9)–(11) with $L = 0$. But this choice is not sufficient to show *scale separation* with respect to the step size α between $\{J_n^{(0)} : n \in \mathbb{N}\}$ and $\{H_n^{(0)} : n \in \mathbb{N}\}$. More precisely, we have only proved in Proposition 4 that $\sup_{n \in \mathbb{N}} \mathbb{E}^{1/p}[\|H_n^{(0)}\|^p] \le C\alpha^{1/2}$ for α small enough and some constant $C \ge 0$. In fact, the expansion (9) allows to refine this bound and obtain that $\sup_{n \in \mathbb{N}} \mathbb{E}^{1/p}[\|H_n^{(0)}\|^p] \le C\alpha$ if α is small enough, for some constant $C \ge 0$.

PROPOSITION 5. *Assume A 1, A 2, and IND 1. Then, for any $\alpha \in (0, \alpha_\infty]$, $p \geq 2$, and $n \in \mathbb{N}$, it holds*

$$\mathbb{E}^{1/p}\left[\|J_n^{(1)}\|^p\right] \leq \mathsf{D}_3 \alpha a p^{3/2}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}, \text{ where } \mathsf{D}_3 = 2\kappa_Q \operatorname{C}_{\mathbf{A}} \operatorname{C}_\varepsilon / a^2 . \tag{27}$$

Moreover, for any $2 \leq p \leq q$ and $\alpha \in (0, \alpha_{q,\infty}]$, $n \in \mathbb{N}$,

$$\mathbb{E}^{1/p}\left[\|H_n^{(1)}\|^p\right] \leq \mathsf{D}_4 \alpha a p^{3/2} d^{1/q}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}, \text{ where } \mathsf{D}_4 = 4\kappa_Q^{1/2} \operatorname{C}_{\mathbf{A}} \mathsf{D}_3 / a . \tag{28}$$

Proof. The proof is provided in Section B.2. We stress that the constants D_3 and D_4 depend only in the constants in A 1 and A 2, and do not depend on the dimension d.

We use Proposition 5 to obtain the p-th moment error bound for LSA-PR procedure. Similarly to (24), we introduce the fluctuation and transient components of the LSA-PR error

$$\begin{aligned}
\Delta_{n,p,\alpha}^{(\mathrm{fl})} &= \frac{4\mathrm{e}^{1/p}\mathsf{D}_2 p^{1/2}}{(\alpha n)^{1/2}} + \mathrm{e}^{1/p}\operatorname{C}_{\mathbf{A}}(\mathsf{D}_3+\mathsf{D}_4)\alpha a p^{5/2} + \frac{\sqrt{2}\operatorname{C}_{\mathsf{Rm},2}\operatorname{C}_\varepsilon p}{n^{1/2}} \\
&\quad + \operatorname{C}_{\mathbf{A}}\mathsf{D}_1 \alpha^{1/2} p^{3/2}, \\
\Delta_{n,p,\alpha}^{(\mathrm{tr})} &= \mathrm{e}^{1/p}\kappa_Q^{1/2}(2\sqrt{2}/(\alpha n^{1/2}) + 2^{-1/2} n^{1/2}\operatorname{C}_{\mathbf{A}}) .
\end{aligned} \tag{29}$$

THEOREM 2. *Assume A 1, A 2, IND 1. Then, for any $n \in \mathbb{N}$, $p \geq 2$, $\alpha \in (0, \alpha_\infty \wedge \mathrm{c}_{\mathbf{A}}/\{p(1+\log d)\})$, $\theta_0 \in \mathbb{R}^d$, it holds*

$$\begin{aligned}
(n/2)^{1/2}\mathbb{E}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n} - \theta^\star\right)\|^p\right] &\leq \operatorname{C}_{\mathsf{Rm},1}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}p^{1/2} + \{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\Delta_{n,p,\alpha}^{(\mathrm{fl})} \\
&\quad + \Delta_{n,p,\alpha}^{(\mathrm{tr})}(1-\alpha a/4)^{n/2}\|\theta_0 - \theta^\star\| ,
\end{aligned}$$

where $\operatorname{C}_{\mathsf{Rm,i}}$, $i = 1, 2$ are defined in Section A.

Proof. Let $q \geq 2$ be a number to be fixed later, and assume that $\alpha \in (0, \alpha_{q,\infty})$. The proof is based on exploiting the representation (12), and the LSA error decomposition (11) with $L = 1$. Below we use shorthand notations $\tilde{\mathbf{A}}_t, \mathbf{A}_t, \varepsilon_t$ for $\tilde{\mathbf{A}}(Z_t), \mathbf{A}(Z_t)$, and $\varepsilon(Z_t)$, respectively. Applying (12) and Minkowski's inequality, we get

$$\begin{aligned}
&(n-n_0)\mathbb{E}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{n_0,n} - \theta^\star\right)\|^p\right] \leq T_1 + T_2 , \\
&T_1 = \mathbb{E}^{1/p}\left[\|\textstyle\sum_{t=n_0}^{n-1}\mathrm{e}(\theta_t, Z_{t+1})\|^p\right], \quad T_2 = \alpha^{-1}\mathbb{E}^{1/p}[\|\theta_{n_0} - \theta_n\|^p] .
\end{aligned} \tag{30}$$

The term T_2 is a remainder one, which is controlled with Proposition 3 and Minkowski's inequality:

$$T_2 \leq 2\alpha^{-1}d^{1/q}\kappa_Q^{1/2}(1-\alpha a/4)^{n_0}\|\theta_0 - \theta^\star\| + 2\alpha^{-1/2}d^{1/q}\mathsf{D}_2(ap\operatorname{Tr}\Sigma_\varepsilon)^{1/2} .$$

Now we proceed with the leading term T_1. Using Minkowski's inequality, (13), and (11) with $L = 1$,

$$\begin{aligned}
T_1 &\leq \mathbb{E}^{1/p}\left[\|\textstyle\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\right] + \mathbb{E}^{1/p}[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}\tilde{\theta}_t^{(\mathsf{tr})}\|^p] \\
&\quad + \mathbb{E}^{1/p}[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\|^p] + \mathbb{E}^{1/p}[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(1)}\|^p] + \mathbb{E}^{1/p}[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}H_t^{(1)}\|^p] .
\end{aligned} \tag{31}$$

We first estimate the leading term $\mathbb{E}^{1/p}\left[\|\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\right]$. Applying Rosenthal's inequality for martingales from Pinelis [31, Theorem 4.1], and using that $\mathbb{E}[\|\varepsilon(Z)\|^2] = \operatorname{Tr}\Sigma_\varepsilon$, we obtain

$$\begin{aligned}
\mathbb{E}^{1/p}\left[\|\textstyle\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\right] &\leq \operatorname{C}_{\mathsf{Rm},1}p^{1/2}(n-n_0)^{1/2}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2} \\
&\quad + \operatorname{C}_{\mathsf{Rm},2}\,p\,\mathbb{E}^{1/p}[\max_{t\in\{n_0,\dots,n-1\}}\|\varepsilon_{t+1}\|^p] .
\end{aligned}$$

With the assumption A 2, we get from the previous bound

$$\mathbb{E}^{1/p}\left[\|\textstyle\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\right] \leq \operatorname{C}_{\mathsf{Rm},1}p^{1/2}(n-n_0)^{1/2}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2} + \operatorname{C}_{\mathsf{Rm},2}\operatorname{C}_\varepsilon\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}p .$$

Now we proceed with the other terms. The term $\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}_{t+1}\tilde{\theta}_t^{(\mathsf{tr})}$ is controlled with Minkowski's inequality and Theorem 1:

$$\mathbb{E}^{1/p}\big[\| \textstyle\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}_{t+1}\tilde{\theta}_t^{(\mathsf{tr})}\|^p\big] \le \mathrm{C}_{\mathbf{A}}(n-n_0)\kappa_Q^{1/2} d^{1/q}(1-\alpha a/4)^{n_0}\|\theta_0-\theta^\star\| \,.$$

Note that the sequences $\{\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\}_{t=n_0}^{n-1}$, $\{\tilde{\mathbf{A}}_{t+1}J_t^{(1)}\}_{t=n_0}^{n-1}$, and $\{\tilde{\mathbf{A}}_{t+1}H_t^{(1)}\}_{t=n_0}^{n-1}$ are $(\mathcal{F}_t)_{t\in\mathbb{N}}$-martingale increments. Hence, applying the Burkholder inequality Osekowski [29, Theorem 8.6] and the Minkowski inequality,

$$\begin{aligned}\mathbb{E}^{1/p}\big[\| \textstyle\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}_{t+1}H_t^{(1)}\|^p\big] &\le p\Big(\textstyle\sum_{t=n_0}^{n-1} \mathbb{E}^{2/p}[\|\tilde{\mathbf{A}}_{t+1}H_t^{(1)}\|^p]\Big)^{1/2}\\ &\le \mathrm{C}_{\mathbf{A}}\,\mathsf{D}_4(n-n_0)^{1/2}p^{5/2}\alpha a d^{1/q}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,,\end{aligned}$$

where the last inequality follows from Proposition 5. Similarly, using Proposition 2 and Proposition 5, we get

$$\begin{aligned}\mathbb{E}^{1/p}[\| \textstyle\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}_{t+1}J_t^{(0)}\|^p] &\le p\Big(\textstyle\sum_{t=n_0}^{n-1} \mathbb{E}^{2/p}[\|\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\|^p]\Big)^{1/2}\\ &\le \mathrm{C}_{\mathbf{A}}\,\mathsf{D}_1(n-n_0)^{1/2}p^{3/2}\{\alpha a\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,.\end{aligned}$$

By the same reasoning, with Proposition 5, we get

$$\mathbb{E}^{1/p}[\| \textstyle\sum_{t=n_0}^{n-1} \tilde{\mathbf{A}}_{t+1}J_t^{(1)}\|^p] \le \mathrm{C}_{\mathbf{A}}\,\mathsf{D}_3(n-n_0)^{1/2}p^{5/2}\alpha a\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,.$$

It remains to choose $q=p(1+\log d)$, $n_0=\lceil n/2\rceil$, and combine the bounds above in (30).

We again can simplify the bounds of Theorem 2 with a special choice of the step size α, proceeding as in (26). The fluctuation error term (29) suggests that α should scale with n as $n^{-1/2}$, therefore we set

$$\alpha(n,d,p) = (\alpha_\infty \wedge \mathrm{c}_{\mathbf{A}}/\{1+\log d\})\,(pn^{1/2})^{-1}\,. \tag{32}$$

Then Theorem 2 implies the following p-th moment bound:

$$\begin{aligned}\mathbb{E}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n}-\theta^\star\right)\|^p\right] &\lesssim \tfrac{\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}p^{1/2}}{n^{1/2}} + \{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\left(\tfrac{p}{n^{3/4}}+\tfrac{p^{3/2}}{n}\right)\\ &\quad + (p+\sqrt{n})\|\theta_0-\theta^\star\|\exp\left\{-\tfrac{(\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}})\sqrt{n}}{8p(1+\log d)}\right\}\,,\end{aligned} \tag{33}$$

where $\lesssim$ stands for inequality up to a constant, depending on κ_Q, a, $\mathrm{C}_{\mathbf{A}}$, C_ε, and polylogarithmic factors in d. Note that the bound (33) can be formulated as a high-probability bound using the Markov inequality with $p=\log(3\mathrm{e}/\delta)$.

Corollary 2. *Assume A 1, A 2, IND 1 and set $\delta\in(0,1)$. Then, for any $\theta_0\in\mathbb{R}^d$, $n\in\mathbb{N}$, with $\alpha=\alpha(n,d,\log(3\mathrm{e}/\delta))$ (see (32)), it holds with probability at least $1-\delta$, that*

$$(n/2)^{1/2}\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n}-\theta^\star\right)\| \le 3\mathrm{e}\,\mathrm{C}_{\mathsf{Rm},1}\sqrt{\{\operatorname{Tr}\Sigma_\varepsilon\}\log(3\mathrm{e}/\delta)} + c_2\Delta^{(\mathrm{HP})}(n,\theta_0,\delta)\,,$$

where

$$\begin{aligned}\Delta^{(\mathrm{HP})}(n,\theta_0,\delta) &= n^{-1/4}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\log^{3/2}(3\mathrm{e}/\delta)\\ &\quad + (\log(3\mathrm{e}/\delta)+\sqrt{n})\|\theta_0-\theta^\star\|\exp\left\{-\frac{(\alpha_\infty\wedge\mathrm{c}_{\mathbf{A}})\sqrt{n}}{8(1+\log d)\log(3\mathrm{e}/\delta)}\right\}\,,\end{aligned}$$

and the constant c_2, defined in (52), depends only on κ_Q, a, $\mathrm{C}_{\mathbf{A}}$, C_ε, and polylogarithmic factors in d.

For completeness, we provide statements of (33) and Corollary 2 with exact constants in Proposition 9. We compare below Corollary 2 to Mou et al. [26, Theorem 3]. It has the same leading term, but makes a covariance matrix appear which is different from ours. Namely, the leading term in Mou et al. [26, Theorem 3] is the asymptotic covariance matrix associated with the $\{\theta_k\}_{k\in\mathbb{N}}$ when considered as a Markov chain: $\Sigma_\varepsilon^{(\alpha)} = n^{-1}\lim_{n\to+\infty}\mathbb{E}[\sum_{i=1}^n(\theta_i-\theta^\star)(\theta_i-\theta^\star)^\top]$. Durmus et al. [12, Proposition 6] shows that $\|\Sigma_\varepsilon^{(\alpha)}-\Sigma_\varepsilon\| = \mathcal{O}(\alpha)$ with $\alpha\to 0$. This explains why, compared to Mou et al. [26, Theorem 3], Corollary 2 has an additional term of order $\alpha^{1/2}$. It is worth noting that the conclusions of Mou et al. [26, Theorem 3] regarding the choice of the optimal step size and the resulting high probability bounds differ slightly from ours, since in their optimization the dependence on the step size α in the covariance $\mathrm{Tr}(\Sigma_\varepsilon^{(\alpha)})$ is omitted. Corollary 2 accounts for this additional factor in the optimization, leading to an optimal choice for α of order $n^{-1/2}$ and a residual term in $n^{-1/4}$, while the optimal choice of α derived by the authors according to Mou et al. [26, Theorem 3] is of order $n^{-1/3}$, leading to a residual term in $n^{-1/3}$. Moreover, Corollary 2 improves the scaling of the residual term with respect to $\log(1/\delta)$, and shows exponential forgetting of the initial condition in contrast to Mou et al. [26, Theorem 3]. Finally, an inspection of the proof of Mou et al. [26, Theorem 3] shows that it relies heavily on results from Joulin and Ollivier [22]. However, the application of these results requires very strong log-Sobolev conditions on the noise distribution $(\varepsilon(Z_n))_{n\in\mathbb{N}}$, e.g., Gaussian distribution, and these results do not apply to the general framework considered here.

4. Finite-time Moment and High-probability Bounds in the Markovian Noise Setting We now consider the Markov case. Let $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ be a Polish space endowed with its Borel σ-field denoted by $\mathcal{Z}$ and let $(\mathsf{Z}^{\mathbb{N}},\mathcal{Z}^{\otimes\mathbb{N}})$ be the corresponding canonical space. Consider a Markov kernel Q on $\mathsf{Z}\times\mathcal{Z}$ and denote by $\mathbb{P}_\xi$ and $\mathbb{E}_\xi$ the corresponding probability distribution and expectation with initial distribution ξ. Without loss of generality, assume that $(Z_k)_{k\in\mathbb{N}}$ is the associated canonical process. By construction, for any $\mathsf{A}\in\mathcal{Z}$, $\mathbb{P}_\xi(Z_k\in\mathsf{A}\,|\,Z_{k-1}) = \mathrm{Q}(Z_{k-1},\mathsf{A})$, $\mathbb{P}_\xi$-a.s. In the case $\xi=\delta_z$, $z\in\mathsf{Z}$, $\mathbb{P}_\xi$ and $\mathbb{E}_\xi$ are denoted by $\mathbb{P}_z$ and $\mathbb{E}_z$. Consider the following assumption.

UGE 1. The Markov kernel Q is uniformly geometrically ergodic, that is, there exists $t_{\mathrm{mix}}\in\mathbb{N}^*$ such that for all $k\in\mathbb{N}^*$,

$$\Delta(\mathrm{Q}^k) = \sup_{z,z'\in\mathsf{Z}}(1/2)\|\mathrm{Q}^k(z,\cdot)-\mathrm{Q}^k(z',\cdot)\|_{\mathsf{TV}} \le (1/4)^{\lfloor k/t_{\mathrm{mix}}\rfloor}\,. \tag{34}$$

Here, t_{mix} is the mixing time of Q. With (34) it is easy to see that

$$\textstyle\sum_{k=0}^{\infty}\Delta(\mathrm{Q}^k) = \sum_{\ell=0}^{t_{\mathrm{mix}}-1}\sum_{r=0}^{\infty}\Delta(\mathrm{Q}^{\ell+rt_{\mathrm{mix}}}) \le (4/3)t_{\mathrm{mix}}\,. \tag{35}$$

UGE 1 implies that Q has a unique invariant distribution which is denoted by π. **UGE** 1 is equivalent to the condition that Q satisfies a uniform minorization condition (see Douc et al. [11, Theorem 18.2.5], i.e., there exist a probability measure ν, such that, for all $z\in\mathsf{Z}$, $\mathsf{A}\in\mathcal{Z}$, $\mathrm{Q}^{t_{\mathrm{mix}}}(z,\mathsf{A})\ge(3/4)\nu(\mathsf{A})$.

Under A 1, we define the quantity

$$\alpha_\infty^{(\mathrm{M})} = \Big[\alpha_\infty\wedge\kappa_Q^{-1/2}\,\mathrm{C}_{\mathbf{A}}^{-1}\wedge a/(6\mathrm{e}\kappa_Q\,\mathrm{C}_{\mathbf{A}})\Big]\times\lceil 8\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}/a\rceil^{-1}\,, \tag{36}$$

where α_∞, a,κ_Q are defined in (6) and (16), respectively. For any $q\ge 2$, we introduce

$$\mathrm{C}_{\mathbf{\Gamma}} = 4(\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}+a/6)^2\times\lceil 8\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}/a\rceil\,.$$

Now we use $\alpha_\infty^{(\mathrm{M})}$ and $\mathrm{C}_{\mathbf{\Gamma}}$ to define, for $q\ge 2$,

$$\alpha_{q,\infty}^{(\mathrm{M})} = \alpha_\infty^{(\mathrm{M})}\wedge\mathrm{c}_{\mathbf{A}}^{(\mathrm{M})}/q\,,\quad \mathrm{c}_{\mathbf{A}}^{(\mathrm{M})} = a/\{12\,\mathrm{C}_{\mathbf{\Gamma}}\}\,. \tag{37}$$

We will see that $\alpha_{q,\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}$ is a natural counterpart of the stability threshold $\alpha_{q,\infty}$ from (17). Now we aim to extend the exponential stability result for product of random matrices (cf. Theorem 1) to **UGE** 1 scenario, provided that $\alpha \in (0; \alpha_{q,\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$.

Theorem 3. *Assume A 1 and **UGE** 1. Then, for any $p, q \in \mathbb{N}$, $2 \le p \le q$, $\alpha \in (0, \alpha_{\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$, $n \in \mathbb{N}$, and any probability distribution ξ on $(\mathsf{Z}, \mathcal{Z})$, it holds*

$$\mathbb{E}_{\xi}^{1/p}\left[\|\Gamma_{1:n}^{(\alpha)}\|^p\right] \le \sqrt{\kappa_Q} \mathrm{e}^2 d^{1/q} \exp\{-n(a\alpha/6 + (q-1)\,\mathrm{C}_{\mathbf{\Gamma}}\,\alpha^2)\}\,,$$

where $\alpha_{\infty}^{(\mathrm{M})}$ is defined in (36).

Proof. Our proof is based on a simplification of the arguments in Durmus et al. [13] together with a new result about the product of general dependent random matrices based on Huang et al. [18]. Denote by $h \in \mathbb{N}$ a block length, the value of which is determined later. Define the sequence $j_0 = 0$, $j_{\ell+1} = \min(j_\ell + h, n)$. By construction $j_{\ell+1} - j_\ell \le h$. Let $N = \lceil n/h \rceil$. Now we introduce the decomposition

$$\Gamma_{1:n}^{(\alpha)} = \prod_{\ell=1}^{N} \mathbf{Y}_\ell\,, \quad \text{where} \quad \mathbf{Y}_\ell = \prod_{i=j_{\ell-1}}^{j_\ell} (\mathrm{I} - \alpha \mathbf{A}(Z_i))\,, \quad \ell \in \{1, \dots, N\}\,.$$

Using a crude bound $\|\mathbf{Y}_N\| \le (1 + \alpha\,\mathrm{C}_{\mathbf{A}})^h$, we get

$$\mathbb{E}_{\xi}^{1/p}[\|\Gamma_{1:n}^{(\alpha)}\|^p] \le (1 + \alpha\,\mathrm{C}_{\mathbf{A}})^h \mathbb{E}_{\xi}^{1/p}[\|\textstyle\prod_{\ell=1}^{N-1} \mathbf{Y}_\ell\|^p]\,.$$

Now we aim to bound $\mathbb{E}_{\xi}^{1/p}[\|\prod_{\ell=1}^{N-1} \mathbf{Y}_\ell\|^p]$ with the technique introduced in Proposition 13. To do so, we define, for $\ell \in \{1, \dots, N-1\}$, the filtration $\mathcal{H}_\ell = \sigma(Z_k : k \le j_\ell)$ and establish almost sure bounds on $\|\mathbb{E}_{\xi}^{\mathcal{H}_{\ell-1}}[\mathbf{Y}_\ell]\|_Q$ and $\|\mathbf{Y}_\ell - \mathbb{E}_{\xi}^{\mathcal{H}_{\ell-1}}[\mathbf{Y}_\ell]\|_Q$ for $\ell \in \{1, \dots, N-1\}$. More precisely, by the Markov property, it is sufficient to show that there exist $\mathtt{m} \in (0, 1]$ and $\sigma > 0$ such that for any probabilities ξ, ξ' on $(\mathsf{Z}, \mathcal{Z})$,

$$\|\mathbb{E}_{\xi'}[\mathbf{Y}_1]\|_Q^2 \le 1 - \mathtt{m} \text{ and } \|\mathbf{Y}_1 - \mathbb{E}_{\xi'}[\mathbf{Y}_1]\|_Q \le \sigma\,, \qquad \mathbb{P}_\xi\text{-a.s.}\,. \tag{38}$$

Such bounds require the blocking procedure, since (38) not necessarily holds with $h = 1$. Set

$$h = \lceil 8\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}/a \rceil t_{\mathrm{mix}}\,.$$

Applying Lemma 1 and Lemma 2, we show that (38) hold with $\mathtt{m} = a\alpha h/6$ and $\sigma = \mathrm{C}_\sigma\,\alpha h$, with $\mathrm{C}_\sigma = 2(\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}} + a/6)$. Then, applying Proposition 13,

$$\begin{aligned}
\mathbb{E}_{\xi}^{1/p}\left[\|\Gamma_{1:n}^{(\alpha)}\|^p\right] &\le \mathbb{E}_{\xi}^{1/q}\left[\|\Gamma_{1:n}^{(\alpha)}\|^q\right] \le \sqrt{\kappa_Q} d^{1/q} \mathrm{e}^{\alpha\,\mathrm{C}_{\mathbf{A}}\,h} \textstyle\prod_{\ell=1}^{N-1} (1 - a\alpha h/6 + (q-1)\,\mathrm{C}_\sigma^2\,\alpha^2 h^2) \\
&\le \sqrt{\kappa_Q} d^{1/q} \mathrm{e}^{\alpha\,\mathrm{C}_{\mathbf{A}}\,h} \mathrm{e}^{-a\alpha h(N-1)/6 + (q-1)\alpha^2\,\mathrm{C}_\sigma^2\,h^2(N-1)} \\
&\le \sqrt{\kappa_Q} d^{1/q} \mathrm{e}^{\alpha h(\mathrm{C}_{\mathbf{A}} + a/6)} \mathrm{e}^{-a\alpha n/6 + (q-1)\alpha^2 n\,\mathrm{C}_\sigma^2\,h} \\
&\le \sqrt{\kappa_Q} \mathrm{e}^2 d^{1/q} \mathrm{e}^{-a\alpha n/6 + (q-1)\,\mathrm{C}_{\mathbf{\Gamma}}\,\alpha^2 n}\,.
\end{aligned}$$

Here we used that by definition of h and since $\alpha \in (0, \alpha_{\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$, $\alpha h\,\mathrm{C}_{\mathbf{A}} \le 1$, and $\alpha h a/6 \le 1$ by (67).

Corollary 3. *Assume A 1 and **UGE** 1. For any $2 \le p \le q$, $\alpha \in (0, \alpha_{q,\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$ with $\alpha_{q,\infty}^{(\mathrm{M})}$ defined in (37), probability distribution ξ on $(\mathsf{Z}, \mathcal{Z})$, and $n \in \mathbb{N}$,*

$$\mathbb{E}_{\xi}^{1/p}\left[\|\Gamma_{1:n}^{(\alpha)}\|^p\right] \le \sqrt{\kappa_Q} \mathrm{e}^2 d^{1/q} \mathrm{e}^{-a\alpha n/12}\,.$$

Similarly to i.i.d. setting, decomposition (11) applied with $L=0$ is enough to bound the p-th moment of LSA error $\|\theta_n-\theta^\star\|$. We begin with bounding $\mathbb{E}_\xi^{1/p}\big[\|J_n^{(0)}\|^p\big]$.

PROPOSITION 6. *Assume A 1, A 2, and **UGE** 1. Then, for any $\alpha\in(0,\alpha_\infty]$, $p\geq 2$, probability ξ on $(\mathsf{Z},\mathcal{Z})$, and $n\in\mathbb{N}$, it holds*

$$\mathbb{E}_\xi^{1/p}\big[\|J_n^{(0)}\|^p\big]\leq \mathsf{D}_1^{(\mathrm{M})}\sqrt{\alpha a p t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}}\,, \tag{39}$$

where

$$\mathsf{D}_1^{(\mathrm{M})}=2^{7/2}\kappa_Q^{1/2}\,\mathrm{C}_\varepsilon\,a^{-1}\{\mathrm{e}^{-1/4}+\sqrt{2\pi\mathrm{e}}\,\mathrm{C}_{\mathbf{A}}\,a^{-1}\}\,. \tag{40}$$

Proof. By definition (19), $J_n^{(0)}$ is a linear statistics of the Markov chain $(Z_k)_{k\in\mathbb{N}}$. Thus the desired result follows from a Mac-Diarmid type inequality under **UGE** 1 (see Paulin [30, Corollary 2.10]). Detailed argument is provided in Section D.2. Note that the bound (39) is equivalent to the one established in Proposition 2 up to an additional $\sqrt{t_{\mathrm{mix}}}$ factor.

PROPOSITION 7. *Assume A 1, A 2, and **UGE** 1. Let $2\leq p\leq q/2$ and $\alpha_{q,\infty}^{(\mathrm{M})}$ be defined in (37). Then, for any $\alpha\in(0,\alpha_{q,\infty}^{(\mathrm{M})}t_{\mathrm{mix}}^{-1}]$, $\theta_0\in\mathbb{R}^d$, probability ξ on $(\mathsf{Z},\mathcal{Z})$, and $n\in\mathbb{N}$, it holds*

$$\mathbb{E}_\xi^{1/p}\left[\|\theta_n-\theta^\star\|^p\right]\leq\sqrt{\kappa_Q}\mathrm{e}^2d^{1/q}\mathrm{e}^{-\alpha a n/12}\|\theta_0-\theta^\star\|+\mathsf{D}_2^{(\mathrm{M})}d^{1/q}\sqrt{\alpha a p t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}}\,,$$

where $\mathsf{D}_2^{(\mathrm{M})}=\mathsf{D}_1^{(\mathrm{M})}(1+24\sqrt{2}\mathrm{e}^2\sqrt{\kappa_Q}\,\mathrm{C}_{\mathbf{A}}\,a^{-1})$ and $\mathsf{D}_1^{(\mathrm{M})}$ is defined in (40).

Proof. Proceeding as in (21), we get

$$\mathbb{E}_\xi^{1/p}\left[\|\theta_n-\theta^\star\|^p\right]\leq\mathbb{E}_\xi^{1/p}\left[\|\Gamma_{1:n}^{(\alpha)}(\theta_0-\theta^\star)\|^p\right]+\mathbb{E}_\xi^{1/p}\left[\|J_n^{(0)}\|^p\right]+\mathbb{E}_\xi^{1/p}\left[\|H_n^{(0)}\|^p\right]\,.$$

The first two terms are bounded using Corollary 3 and Proposition 6, respectively. Regarding the last one, the recurrence (8), $H_n^{(0)}=-\alpha\sum_{j=1}^n\Gamma_{j+1:n}^{(\alpha)}\tilde{\mathbf{A}}(Z_j)J_{j-1}^{(0)}$, and Minkowski's inequality yields

$$\mathbb{E}_\xi^{1/p}\left[\|H_n^{(0)}\|^p\right]\leq\alpha\sum_{j=1}^n\left\{\mathbb{E}_\xi\left[\|\Gamma_{j+1:n}^{(\alpha)}\|^{2p}\right]\right\}^{1/2p}\left\{\mathbb{E}_\xi\left[\|\tilde{\mathbf{A}}(Z_j)J_{k-1}^{(0)}\|^{2p}\right]\right\}^{1/2p}\,.$$

Using Proposition 6 and $\mathrm{e}^{-x}\leq 1-x/2$, valid for $x\in[0,1]$, we get

$$\mathbb{E}_\xi^{1/p}\left[\|H_n^{(0)}\|^p\right]\leq\alpha d^{1/q}\mathrm{e}^2\sqrt{\kappa_Q}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{D}_1^{(\mathrm{M})}\sqrt{2\alpha a p t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}}\sum_{j=1}^n(1-a\alpha/24)^n\,.$$

This completes the proof.

Proposition 7 improves Mou et al. [27, Proposition 1]. First, we obtain a better scaling with respect to p for the fluctuation term. Indeed, Mou et al. [27, Proposition 1] implies that this term scales with $p^{3/2}$, while we obtain $p^{1/2}$. Moreover, the constraints on the step size α are relaxed. Proposition 7 holds for $\alpha_{p,\infty}^{(\mathrm{M})}\approx 1/[p(1+\log(d))]$, while Mou et al. [27, Proposition 1] requires that $\alpha\lesssim 1/[p^3d]$. We state below a counterpart of Proposition 5:

PROPOSITION 8. *Assume A 1, A 2, and **UGE** 1. Then, for any $2\leq p\leq q/2$, $\alpha\in(0,\alpha_{q,\infty}^{(\mathrm{M})}t_{\mathrm{mix}}^{-1}]$, probability ξ on $(\mathsf{Z},\mathcal{Z})$, and $n\in\mathbb{N}$, it holds*

$$\mathbb{E}_\xi^{1/p}[\|H_n^{(0)}\|^p]\leq d^{1/q}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}(\alpha a t_{\mathrm{mix}})\left\{\mathsf{D}_3^{(\mathrm{M})}\sqrt{\log(1/\alpha a)}p^2+\mathsf{D}_4^{(\mathrm{M})}(\alpha a t_{\mathrm{mix}})^{1/2}p^{1/2}\right\}\,,$$

where $\mathsf{D}_3^{(\mathrm{M})}$ and $\mathsf{D}_4^{(\mathrm{M})}$ are given in (75).

Proof. Expanding the recurrence (10) yields $J_n^{(1)}=\alpha^2\sum_{\ell=1}^{n-1}S_{\ell+1:n}\varepsilon(Z_\ell)$, where

$$S_{\ell+1:n}=\sum_{k=\ell+1}^n(\mathrm{I}-\alpha\bar{\mathbf{A}})^{n-k}\tilde{\mathbf{A}}(Z_k)(\mathrm{I}-\alpha\bar{\mathbf{A}})^{k-1-\ell}\,. \tag{41}$$

Unlike the i.i.d.-noise scenario, $J_n^{(1)}$ is no longer a martingale, so we cannot directly apply Rosenthal-type inequalities to upper bound $\mathbb{E}_\xi^{1/p}[\|J_n^{(1)}\|^p]$. Instead, we rely on Berbee's lemma (Rio [34, Lemma 5.1]). For a detailed argument, see Section D.3. We emphasise that the constants $\mathsf{D}_3^{(\mathrm{M})}$ and $\mathsf{D}_4^{(\mathrm{M})}$ do not depend on α or t_{mix}.

With the estimates above, we are ready to state and prove the Markov counterpart of Theorem 2. Under A 2 and **UGE** 1, we define the asymptotic covariance matrix

$$\Sigma_{\varepsilon}^{(\mathrm{M})} = \mathbb{E}_{\pi}[\varepsilon(Z_0)\varepsilon(Z_0)^{\top}] + 2\sum_{\ell=0}^{\infty}\mathbb{E}_{\pi}[\varepsilon(Z_0)\varepsilon(Z_\ell)^{\top}]\,. \tag{42}$$

We also introduce the fluctuation and transient components of the LSA-PR error

$$\begin{aligned} R^{(\mathrm{fl})}_{n,p,\alpha,t_{\mathrm{mix}}} &= \frac{4\,\mathsf{D}_2^{(\mathrm{M})}\mathrm{e}^{1/p}\sqrt{apt_{\mathrm{mix}}}}{\sqrt{\alpha n}} + \frac{2^{1/2}\,\mathrm{D}_{\mathrm{Ros},1}\,\mathrm{C}_{\varepsilon}\,pt_{\mathrm{mix}}^{3/5}}{n^{1/10}} + \\ &+ 4\mathrm{e}^{1/p}\big(\mathsf{D}_3^{(\mathrm{M})}\alpha a t_{\mathrm{mix}}\sqrt{\log(1/\alpha a)}p^2 + \mathsf{D}_4^{(\mathrm{M})}(\alpha a t_{\mathrm{mix}})^{3/2}p^{1/2}\big)(\alpha^{-1}n^{-1/2} + n^{1/2}\,\mathrm{C}_{\mathbf{A}})\,; \\ R^{(\mathrm{tr})}_{n,p,\alpha,t_{\mathrm{mix}}} &= \mathrm{e}^{2+1/p}\kappa_Q^{1/2}\left(\frac{2}{\alpha n^{1/2}} + 2^{-1/2}n^{1/2}\,\mathrm{C}_{\mathbf{A}}\right)\,, \end{aligned} \tag{43}$$

Here $\mathrm{D}_{\mathrm{Ros},1}$ is given in (88) and is a universal constant. The quantities above corresponds to the fluctuation and transient terms in (29), respectively.

Theorem 4. *Assume A 1, A 2, and* ***UGE*** *1. Then, for any $p \geq 2$, step size $\alpha \in (0, \alpha_{\infty}^{(\mathrm{M})}t_{\mathrm{mix}}^{-1} \wedge \mathrm{c}_{\mathbf{A}}^{(\mathrm{M})}\,t_{\mathrm{mix}}^{-1}/\{p(1+\log d)\})$, $n \geq 4$, $\theta_0 \in \mathbb{R}^d$, and probability ξ on $(\mathsf{Z}, \mathcal{Z})$, it holds*

$$\begin{aligned} (n/2)^{1/2}\mathbb{E}_{\xi}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n} - \theta^{\star}\right)\|^p\right] &\leq \mathrm{C}_{\mathsf{Rm},1}\{\mathrm{Tr}\,\Sigma_{\varepsilon}^{(\mathrm{M})}\}^{1/2}p^{1/2} + \{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}R^{(\mathrm{fl})}_{n,p,\alpha,t_{\mathrm{mix}}} \\ &+ (8/3)\,\mathrm{C}_{\varepsilon}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}n^{1/5}t_{\mathrm{mix}}^{2/5}\exp\left\{-\frac{n^{1/5}\ln 2}{(2t_{\mathrm{mix}})^{1/5}}\right\} + R^{(\mathrm{tr})}_{n,p,\alpha,t_{\mathrm{mix}}}\|\theta_0 - \theta^{\star}\|\exp\left\{-\frac{\alpha a n}{24}\right\}\,, \end{aligned}$$

where $\mathrm{C}_{\mathsf{Rm},1}$ is defined in Section A.

Proof. Let $p \geq 2$ and $q \geq p$ be a number to be fixed later. In addition assume that $\alpha \in (0, \alpha_{q,\infty}t_{\mathrm{mix}}^{-1}]$, and set $n_0 = \lceil n/2 \rceil$. Below we use shorthand notations $\tilde{\mathbf{A}}_t, \mathbf{A}_t, \varepsilon_t$ for $\tilde{\mathbf{A}}(Z_t), \mathbf{A}(Z_t)$, and $\varepsilon(Z_t)$, respectively. Proceeding as in (30) and (31), we decompose the p-th moment of LSA-PR error as

$$\begin{aligned} &(n-n_0)\mathbb{E}_{\xi}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{n_0,n} - \theta^{\star}\right)\|^p\right] \leq \mathbb{E}_{\xi}^{1/p}\big[\|\textstyle\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\big] + T_1^{(\mathrm{M})} + T_2^{(\mathrm{M})} + T_3^{(\mathrm{M})} \\ &T_1^{(\mathrm{M})} = \alpha^{-1}\mathbb{E}_{\xi}^{1/p}[\|\theta_{n_0} - \theta_n\|^p]\,, \qquad T_2^{(\mathrm{M})} = \mathbb{E}_{\xi}^{1/p}\big[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}\tilde{\theta}_t^{(\mathrm{tr})}\|^p\big]\,, \\ &T_3^{(\mathrm{M})} = \mathbb{E}_{\xi}^{1/p}\big[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\|^p\big] + \mathbb{E}_{\xi}^{1/p}\big[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}H_t^{(0)}\|^p\big]\,. \end{aligned} \tag{44}$$

Now we bound each term in the decomposition (44). We begin with the first term. Applying Corollary 5, we get

$$\begin{aligned} \mathbb{E}_{\xi}^{1/p}\big[\|\sum_{t=n_0}^{n-1}\varepsilon_{t+1}\|^p\big] &\leq \mathrm{C}_{\mathsf{Rm},1}\,p^{1/2}(n-n_0)^{1/2}\{\mathrm{Tr}\,\Sigma_{\varepsilon}^{(\mathrm{M})}\}^{1/2} \\ &+ \mathrm{D}_{\mathrm{Ros},1}\,\mathrm{C}_{\varepsilon}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}p(n-n_0)^{2/5}t_{\mathrm{mix}}^{3/5} + (8/3)\,\mathrm{C}_{\varepsilon}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}(n-n_0)^{3/5}t_{\mathrm{mix}}^{2/5}2^{-\{(n-n_0)/t_{\mathrm{mix}}\}^{1/5}}\,. \end{aligned}$$

Applying Proposition 7 and Minkowski's inequality, we get

$$T_1^{(\mathrm{M})} \leq 2\alpha^{-1}\sqrt{\kappa_Q}\mathrm{e}^2 d^{1/q}\mathrm{e}^{-\alpha a n_0/12}\|\theta_0 - \theta^{\star}\| + 2\mathsf{D}_2^{(\mathrm{M})}d^{1/q}\alpha^{-1/2}\sqrt{apt_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}}\,.$$

Applying Corollary 3, Minkowski's inequality, and using A 1, (4), we get

$$T_2^{(\mathrm{M})} \leq (n-n_0)\sqrt{\kappa_Q}\mathrm{e}^2 d^{1/q}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{e}^{-a\alpha n_0/12}\|\theta_0 - \theta^{\star}\|\,.$$

It remains to proceed with $T_3^{(\mathrm{M})}$. Using the representation (8),

$$\textstyle\sum_{t=n_0}^{n-1}H_{t+1}^{(0)} = \sum_{t=n_0}^{n-1}\{\mathrm{I} - \alpha\mathbf{A}(Z_{t+1})\}H_t^{(0)} - \alpha\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\,,$$

which yields

$$\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)} = \alpha^{-1}(H_{n_0}^{(0)} - H_n^{(0)}) - \sum_{t=n_0}^{n-1}\mathbf{A}(Z_{t+1})H_t^{(0)}\,.$$

Applying again Minkowski's inequality, we get

$$\mathbb{E}_\xi^{1/p}\big[\|\textstyle\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)}\|^p\big] \leq \{2\alpha^{-1}+(n-n_0)\,\mathrm{C}_{\mathbf{A}}\}\sup_{t\in\mathbb{N}^*}\mathbb{E}_\xi^{1/p}\big[\|H_t^{(0)}\|^p\big]\,.$$

Now it remains to combine the bounds above in (44), use Proposition 8, and set $q=p(1+\log d)$, $n_0=\lceil n/2\rceil$.

Remark 1. It is important to highlight that the bound we establish in the proof of Theorem 4 for $T_3^{(\mathrm{M})}$ cannot be improved with further expansion of $H_t^{(0)}$. Contrary to the i.i.d. scenario, the term $\sum_{t=n_0}^{n-1}\tilde{\mathbf{A}}_{t+1}J_t^{(0)}$ is no longer a martingale, moreover, in general $\mathbb{E}_\xi[\tilde{\mathbf{A}}_{t+1}J_t^{(0)}]\neq 0$. As shown in Lemma 6,

$$\|\mathbb{E}_\xi[\tilde{\mathbf{A}}_{t+1}J_t^{(0)}]\| \leq \mathsf{D}_4^{(\mathrm{M})}\alpha a t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,.$$

This bound is of the same order w.r.t. step size α (up to the factor $\sqrt{\log 1/\alpha a}$), as the bound for $H_t^{(0)}$ obtained in Proposition 8. Therefore, it is not possible to obtain better bounds by further expanding the $H_t^{(0)}$ term.

Proceeding as in (33), we refine the bound of Theorem 4 under the special choice of the step size α. The fluctuation error term (43) suggests that α should scale with n as $n^{-2/3}$ and therefore we set

$$\alpha^{(\mathrm{M})}(n,d,p,t_{\mathrm{mix}}) = \Big(\alpha_\infty^{(\mathrm{M})}\wedge \mathrm{c}_{\mathbf{A}}^{(\mathrm{M})}/\{1+\log d\}\Big)\,(pn^{2/3}t_{\mathrm{mix}}^{1/3})^{-1}\,. \tag{45}$$

With the step size above, Theorem 4 implies, for $n\geq t_{\mathrm{mix}}$, the following p-th moment bound:

$$\begin{aligned}\mathbb{E}_\xi^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar\theta_{n_0,n}-\theta^\star\right)\|^p\right] &\lesssim \frac{\{\operatorname{Tr}\Sigma_\varepsilon^{(\mathrm{M})}\}^{1/2}p^{1/2}}{n^{1/2}} + \{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\left(\frac{t_{\mathrm{mix}}^{3/5}p}{n^{3/5}}+\frac{t_{\mathrm{mix}}p^2}{n}\right) \\ &+p(t_{\mathrm{mix}}/n)^{2/5}\exp\left\{-\frac{n^{1/5}\ln 2}{(2t_{\mathrm{mix}})^{1/5}}\right\}+pn^{1/2}\|\theta_0-\theta^\star\|\exp\left\{-\frac{(\alpha_\infty^{(\mathrm{M})}\wedge \mathrm{c}_{\mathbf{A}}^{(\mathrm{M})})n^{1/3}}{24pt_{\mathrm{mix}}^{1/3}(1+\log d)}\right\}\,,\end{aligned} \tag{46}$$

where $\lesssim$ stands for inequality up to a constant, depending on κ_Q, a, $\mathrm{C}_{\mathbf{A}}$, C_ε, and polylogarithmic factors in d and n. Similarly to Corollary 2, the bound above can be reformulated as a high-probability bound using the Markov inequality.

Corollary 4. *Assume A 1, A 2, **UGE** 1, and set $\delta\in(0,1)$. Then, for any $\theta_0\in\mathbb{R}^d$, sample size $n\in\mathbb{N}^*$, $n\geq 4\vee t_{\mathrm{mix}}$, step size $\alpha=\alpha^{(\mathrm{M})}(n,d,\log(3\mathrm{e}/\delta),t_{\mathrm{mix}})$ defined in (45), it holds with probability at least $1-\delta$, that*

$$(n/2)^{1/2}\|\bar{\mathbf{A}}\left(\bar\theta_{\lceil n/2\rceil,n}-\theta^\star\right)\| \leq 3\mathrm{e}\,\mathrm{C}_{\mathsf{Rm},1}\sqrt{\{\operatorname{Tr}\Sigma_\varepsilon^{(\mathrm{M})}\}\log(3\mathrm{e}/\delta)}+c_1^{(\mathrm{M})}R^{(\mathrm{HP})}(n,\theta_0,\delta,t_{\mathrm{mix}})\,,$$

where

$$\begin{aligned}R^{(\mathrm{HP})}(n,\theta_0,\delta,t_{\mathrm{mix}}) &= \{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\log(3\mathrm{e}/\delta)\{n^{-1/10}t_{\mathrm{mix}}^{3/5}+n^{-1/2}t_{\mathrm{mix}}\log(3\mathrm{e}/\delta)\} \\ &+(n^{1/6}t_{\mathrm{mix}}^{1/3}\log(3\mathrm{e}/\delta)+n^{1/2})\|\theta_0-\theta^\star\|\exp\left\{-\frac{(\alpha_\infty^{(\mathrm{M})}\wedge \mathrm{c}_{\mathbf{A}}^{(\mathrm{M})})n^{1/3}}{24t_{\mathrm{mix}}^{1/3}(1+\log d)\log(3\mathrm{e}/\delta)}\right\} \\ &+n^{1/10}t_{\mathrm{mix}}^{2/5}\log(3\mathrm{e}/\delta)\exp\left\{-\frac{n^{1/5}\ln 2}{(2t_{\mathrm{mix}})^{1/5}}\right\}\,,\end{aligned}$$

and $c_1^{(\mathrm{M})}$ depends only on κ_Q, a, $\mathrm{C}_{\mathbf{A}}$, C_ε, and polylogarithmic factors in d in n.

Theorem 4 generalizes and improves the results of Mou et al. [27, Theorem 1]. First, Mou et al. [27, Theorem 1] only consider the mean square error, while in Theorem 4 we derive bounds for arbitrary p-th moments of the LSA-PR error. These bounds are further used to derive high probability bounds in Corollary 4. Second, the refined bound (46) for $p=2$ yields the same leading term of order $\{\operatorname{Tr}\Sigma_\varepsilon^{(\mathrm{M})}\}^{1/2}n^{-1/2}$ and improves the dependence of the residual term on the dimension. Namely, for comparison with Mou et al. [27, Theorem 1], assume that $\operatorname{Tr}\Sigma_\varepsilon \approx d$. This leads to a residual term with dependence of order $d^{1/2}$ in Theorem 4 instead of $d^{4/3}$ in Mou et al. [27, Theorem 1]. Moreover, the step size α in (46) scales with d as $(1+\log d)^{-1}$, unlike $d^{-1/3}$ in Mou et al. [27, Theorem 1].

Appendix A: Notations and Constants Denote $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$ and $\mathbb{N}_- = \mathbb{Z}\setminus\mathbb{N}^*$. Let $d \in \mathbb{N}^*$ and Q be a symmetric positive definite $d\times d$ matrix. For $x \in \mathbb{R}^d$, we denote $\|x\|_Q = \{x^\top Q x\}^{1/2}$. For brevity, we set $\|x\| = \|x\|_{\mathrm{I}_d}$. We denote $\|A\|_Q = \max_{\|x\|_Q=1}\|Ax\|_Q$, and the subscriptless norm $\|A\| = \|A\|_{\mathrm{I}}$ is the standard spectral norm. For a function $g: \mathsf{Z}\to\mathbb{R}^d$, we denote $\|g\|_\infty := \sup_{z\in\mathsf{Z}}\|g(z)\|$.

We denote $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \,|\, \|x\| = 1\}$. Let $A_1,\ldots,A_N$ be d-dimensional matrices. We denote $\prod_{\ell=i}^{j} A_\ell = A_j \ldots A_i$ if $i \le j$ and by convention $\prod_{\ell=i}^{j} A_\ell = \mathrm{I}$ if $i > j$. We say that a centered random variable (r.v.) X is subgaussian with variance proxy factor σ^2 and denote $X \in \mathrm{SG}(\sigma^2)$ if for all $\lambda \in \mathbb{R}$, $\log\mathbb{E}[\mathrm{e}^{\lambda X}] \le \lambda^2\sigma^2/2$.

The readers can refer to the following table on the variables and constants that are used across the paper for references.

Variable	Description	Reference
Q	Solution of Lyapunov equation for $\bar{\mathbf{A}}$	Proposition 1
κ_Q	$\lambda_{\mathsf{min}}^{-1}(Q)\lambda_{\mathsf{max}}(Q)$	Proposition 1
a	Real part of minimum eigenvalue of $\bar{\mathbf{A}}$	Proposition 1
$\Gamma_{m:n}^{(\alpha)}$	Product of random matrices with step size α	(3)
$\varepsilon(Z_n)$	Noise in LSA procedure	(2)
$\tilde{\theta}_n^{(\mathsf{tr})}, \tilde{\theta}_n^{(\mathsf{fl})}$	Transient and flutuation terms of LSA error	(4)
$\alpha_{p,\infty}$ (resp. $\alpha_{p,\infty}^{(M)}$)	Stability threshold for $\Gamma_{m:n}^{(\alpha)}$ to have bounded p-th moment under IND 1 (resp. **UGE** 1)	(17)
$J_n^{(0)}$	Dominant term in $\tilde{\theta}_n^{(\mathsf{fl})}$	(8)
$H_n^{(0)}$	Residual term $\tilde{\theta}_n^{(\mathsf{fl})} - J_n^{(0)}$	(8)
$J_n^{(\ell)}, H_n^{(\ell)}$, $\ell \ge 1$	Stochastic terms from expansion of $H_n^{(0)}$	(9)-(10)
Σ_ε	Noise covariance $\mathbb{E}[\varepsilon_1\varepsilon_1^\top]$	A 2
$\Sigma_\varepsilon^{(\mathrm{M})}$	Asymptotic covariance matrix under Markovian noise	(42)
$\mathrm{C}_{\mathsf{Rm},1} = 60\mathrm{e}$	Constant in martingale Rosenthal's inequality	[31, Theorem 4.1]
$\mathrm{C}_{\mathsf{Rm},2} = 60$	Constant in martingale Rosenthal's inequality	[31, Theorem 4.1]
$\mathrm{D}_{\mathrm{Ros},1}$	Constant in Rosenthal's inequallity under **UGE** 1	(88)

Appendix B: Independent case bounds In the lemmas below we use shorthand notation $\tilde{\mathbf{A}}_n, \mathbf{A}_n, \varepsilon_n$ for $\tilde{\mathbf{A}}(Z_n), \mathbf{A}(Z_n)$, and $\varepsilon(Z_n)$, respectively, where $\varepsilon(z): \mathbb{Z}\to\mathbb{R}^d$ is defined in (2).

B.1. Proof of Proposition 2 With decomposition (8), we expand $J_n^{(0)}$ as

$$J_n^{(0)} = \alpha\sum_{j=1}^n \left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-j}\varepsilon_j =: \alpha\sum_{j=1}^n \eta_{n,j}\,, \text{ where } \eta_{n,j} = \left(\mathrm{I}-\alpha\bar{\mathbf{A}}\right)^{n-j}\varepsilon_j\,.$$

Proposition 1 implies that $\|\left(\mathrm{I}-\alpha\bar{\mathbf{A}}\right)^{n-j}\| \le \kappa_Q^{1/2}(1-\alpha a)^{(n-j)/2}$. Hence, using the Hoeffding-type bound of Lemma 11 and assumption A 2, we get for any $t \ge 0$ that

$$\mathbb{P}(\|J_n^{(0)}\| \ge t) \le 2\exp\{-t^2/(2\sigma_{\alpha,n}^2)\}\,,$$

where, with Σ_ε defined in (15),

$$\sigma_{\alpha,n}^2 = \alpha^2\kappa_Q\,\mathrm{C}_\varepsilon^2\operatorname{Tr}(\Sigma_\varepsilon)\sum_{j=1}^n(1-\alpha a)^{n-j} \le \alpha\kappa_Q\operatorname{Tr}(\Sigma_\varepsilon)\,\mathrm{C}_\varepsilon^2/a\,.$$

Combining this result with the moment bound of Lemma 10 yields (18).

B.2. Proof of Proposition 5

B.2.1. Moment bounds for $J_n^{(1)}$ We begin with the bound (27). Expanding the recurrence (10) with $\ell = 1$ and using that $J_{k-1}^{(0)} = -\alpha \sum_{i=1}^{k-1} (\mathrm{I} - \alpha \bar{\mathbf{A}})^{k-i-1} \varepsilon_i$ yields

$$J_n^{(1)} = \alpha^2 \sum_{i=1}^{n-1} S_{i+1:n}^{(1)} \varepsilon_i, \quad \text{where} \quad S_{i+1:n}^{(1)} = \sum_{k=i+1}^{n} (\mathrm{I} - \alpha \bar{\mathbf{A}})^{n-k} \tilde{\mathbf{A}}_k (\mathrm{I} - \alpha \bar{\mathbf{A}})^{k-1-i} . \tag{47}$$

Recall for $k \in \mathbb{N}$, $\mathcal{F}_k = \sigma(Z_s : 1 \leq s \leq k)$, $\mathcal{F}_0 = \{\emptyset, \mathsf{Z}\}$. It is easy to check that the sequence $\{S_{i+1:n}^{(1)} \varepsilon_i\}_{i=1}^{n-1}$ is a martingale-difference with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{N}}$: $\mathbb{E}[S_{i+1:n}^{(1)} \varepsilon_i | \mathcal{F}_{i-1}] = 0$. Applying the Burkholder inequality Osekowski [29, Theorem 8.6] and the Minkowski inequality, we get

$$\begin{aligned} \mathbb{E}[\|J_n^{(1)}\|^p] &\leq p^p \alpha^{2p} \mathbb{E}[(\textstyle\sum_{i=1}^{n-1} \|S_{i+1:n}^{(1)} \varepsilon_i\|^2)^{p/2}] \\ &\leq p^p \alpha^{2p} (\textstyle\sum_{i=1}^{n-1} \mathbb{E}^{2/p}[\|S_{i+1:n}^{(1)} \varepsilon_i\|^p])^{p/2} . \end{aligned} \tag{48}$$

Let us denote $v_i = \varepsilon_i / \|\varepsilon_i\|$. Then, using IND 1, we get

$$\mathbb{E}[\|S_{i+1:n}^{(1)} \varepsilon_i\|^p] = \mathbb{E}\Big[\|\varepsilon_i\|^p \mathbb{E}^{\mathcal{F}_i}\Big[\|S_{i+1:n}^{(1)} v_i\|^p\Big]\Big] \leq \mathbb{E}[\|\varepsilon_i\|^p] \sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}[\|S_{i+1:n}^{(1)} u\|^p] .$$

A 1 and Proposition 1 imply that $\|(\mathrm{I} - \alpha \bar{\mathbf{A}})^{n-k} \tilde{\mathbf{A}}_k (\mathrm{I} - \alpha \bar{\mathbf{A}})^{k-1-i}\| \leq \kappa_Q \,\mathrm{C}_{\mathbf{A}} (1 - \alpha a)^{(n-i-1)/2}$. Hence, applying Lemma 11, we get for any $t \geq 0$ and $u \in \mathbb{S}^{d-1}$ that

$$\mathbb{P}\big(\|S_{i+1:n}^{(1)} u\| \geq t\big) \leq 2 \exp\left\{ -\frac{t^2}{2\kappa_Q^2 \,\mathrm{C}_{\mathbf{A}}^2 (n-i)(1-\alpha a)^{n-i-1}} \right\} .$$

Applying Lemma 10, we get for any $u \in \mathbb{S}^{d-1}$

$$\mathbb{E}^{2/p}[\|S_{i+1:n}^{(1)} u\|^p] \leq 2p \,\mathrm{C}_{\mathbf{A}}^2 \,\kappa_Q^2 (n-i)(1-\alpha a)^{n-i-1} . \tag{49}$$

Combining (48), (49), and A 2, we get

$$\begin{aligned} \mathbb{E}^{1/p}[\|J_n^{(1)}\|^p] &\leq 2\{\mathrm{Tr}\, \Sigma_\varepsilon\}^{1/2} p^{3/2} \alpha^2 \,\mathrm{C}_{\mathbf{A}} \,\kappa_Q \,\mathrm{C}_\varepsilon (\textstyle\sum_{i=1}^{n-1} (n-i)(1-\alpha a)^{n-i-1})^{1/2} \\ &\leq \mathsf{D}_3 \alpha a p^{3/2} \{\mathrm{Tr}\, \Sigma_\varepsilon\}^{1/2} , \end{aligned} \tag{50}$$

where D_3 is defined in (27). In the above we have used that $\sum_{k=1}^{\infty} k \rho^{k-1} = (1-\rho)^{-2}$ for $\rho \in [0,1)$ together with $\alpha a \leq 1/2$.

B.2.2. Moment bounds for $H_n^{(1)}$ The decomposition (10) implies that

$$H_n^{(1)} = -\alpha \textstyle\sum_{\ell=1}^{n} \Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell J_{\ell-1}^{(1)} .$$

Hence, using Minkowski's inequality together with IND 1,

$$\mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] \leq \alpha \textstyle\sum_{\ell=1}^{n} \mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)} \tilde{\mathbf{A}}_\ell\|^p] \mathbb{E}^{1/p}[\|J_{\ell-1}^{(1)}\|^p] .$$

Applying Theorem 1 and (50), we get using the definition (28) of D_4

$$\begin{aligned} \mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] &\leq \kappa_Q^{1/2} \,\mathrm{C}_{\mathbf{A}} \,\mathsf{D}_3 \alpha^2 a d^{1/q} \{\mathrm{Tr}\, \Sigma_\varepsilon\}^{1/2} p^{3/2} \sum_{\ell=1}^{n} (1 - \alpha a/4)^n \\ &\leq \mathsf{D}_4 d^{1/q} \{\mathrm{Tr}\, \Sigma_\varepsilon\}^{1/2} \alpha a p^{3/2} . \end{aligned}$$

B.3. Version of Corollary 2 with exact constants

Proposition 9. *Under the assumptions of Theorem 2 with the step size $\alpha = \alpha(n,d,p)$ specified in (32), it holds that*

$$
\begin{aligned}
(n/2)^{1/2}\mathbb{E}^{1/p}\left[\|\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n}-\theta^\star\right)\|^p\right] &\leq \mathrm{C}_{\mathsf{Rm},1}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}p^{1/2} \\
&+ \mathrm{e}^{1/p}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\left(\frac{c_3(1+\log d)^{1/2}p}{n^{1/4}}+\frac{c_4 p^{3/2}}{n^{1/2}}\right) \\
&+ \mathrm{e}^{1/p}c_5(1+\log d)p\|\theta_0-\theta^\star\|\exp\left\{-\frac{(\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}})\sqrt{n}}{8p(1+\log d)}\right\}\,,
\end{aligned}
\tag{51}
$$

where c_3, c_4 and c_5 are given by

$$
\begin{aligned}
&c_3 = \frac{4\mathsf{D}_2}{(\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}})^{1/2}} + \sqrt{2}\,\mathrm{C}_{\mathsf{Rm},2}\,\mathrm{C}_\varepsilon + (\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}})^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{D}_1\,,\; c_5 = \kappa_Q^{1/2}\left(\frac{2\sqrt{2}}{\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}}}+\mathrm{C}_{\mathbf{A}}\right)\,, \\
&c_4 = \mathrm{C}_{\mathbf{A}}(\mathsf{D}_3+\mathsf{D}_4)a(\alpha_\infty\wedge \mathrm{c}_{\mathbf{A}})\,.
\end{aligned}
$$

Moreover, Corollary 2 holds with

$$
c_2 = 3\mathrm{e}\left((c_3+c_4)(1+\log d)^{1/2}\vee c_5(1+\log d)\right)\,. \tag{52}
$$

Proof. The bound (51) automatically follows from Theorem 2 after substituting the step size $\alpha(n,d,p)$. Now Corollary 2 with c_2 defined in (52) follows from the Markov inequality applied with $p = \log(3\mathrm{e}/\delta) > 2$.

Appendix C: Independent case bounds under subgaussian noise assumption Assumption A 2 can be relaxed to a subgaussian-type conditions on the noise variable $\varepsilon(Z)$. Consider the following assumption:

A 3. For any $u \in \mathbb{S}^{d-1}$, and $\lambda \in \mathbb{R}$, $\log\{\mathbb{E}[\exp(\lambda u^\top \varepsilon(Z))]\} \leq \lambda^2\sigma_\varepsilon^2/2$, where Z is a random variable with distribution π.

Note that A 2 implies A 3, and A 3 can be written more concisely as $u^\top\varepsilon(Z) \in \mathrm{SG}(\sigma_\varepsilon^2)$ for any $u \in \mathbb{S}^{d-1}$. For instance, this condition holds when $\varepsilon(Z_{t+1})$ is an outer product of sub-Gaussian random variables in the canonical coordinates; see Mou et al. [27, Assumption 2][1]. Note that, for any $u \in \mathbb{S}^{d-1}$, and $t \geq 0$,

$$
\mathbb{P}(|u^\top\varepsilon(Z)| \geq t) \leq 2\exp(-t^2/(2\sigma_\varepsilon^2))\,. \tag{53}
$$

Below we state the counterpart of Proposition 2 and Proposition 3.

Proposition 10. *Assume A 1, IND 1 and A 3. Then, for any $\alpha \in (0,\alpha_\infty]$, $p \geq 2$, $u \in \mathbb{S}^d$ and $n \in \mathbb{N}$,*

$$
\mathbb{E}^{1/p}\left[\left|u^\top J_n^{(0)}\right|^p\right] \leq \mathsf{D}_1\sqrt{\alpha a p\sigma_\varepsilon^2}\,, \tag{54}
$$

where D_1 is given in (18). Moreover, for any $p,q \in \mathbb{N}$, $2 \leq p \leq q$, $\alpha \in (0,\alpha_{q,\infty}]$, $n \in \mathbb{N}$, $u \in \mathbb{S}^d$ and $\theta_0 \in \mathbb{R}^d$,

$$
\mathbb{E}^{1/p}\left[|u^\top(\theta_n-\theta^\star)|^p\right] \leq d^{1/q}\kappa_Q^{1/2}\left(1-\alpha a/4\right)^n\|\theta_0-\theta^\star\| + \mathsf{D}d^{1/q}\sqrt{\alpha a p\sigma_\varepsilon^2}\,, \tag{55}
$$

where the constant D is given by

$$
\mathsf{D} = (2\kappa_Q)^{1/2}a^{-1}(1+4\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,a^{-1})
$$

[1] The condition can be further relaxed to cover heavier-tail setting in which $\varepsilon(Z_{t+1})$ has only a finite number of moments or is sub-exponential (instead of sub-gaussian).

Proof. We first show the bound (54). Expanding (8), we get for any $u \in \mathbb{S}^{d-1}$, that

$$u^\top J_n^{(0)} = \alpha \sum_{j=1}^n \eta_{n,j} \;, \text{ where } \eta_{n,j} = u^\top \left(\mathrm{I} - \alpha \bar{\mathbf{A}}\right)^{n-j} \varepsilon_j \;. \tag{56}$$

Note that $\{\eta_{n,j}\}_{j=1}^n$ are sub-Gaussian random variables. With Proposition 1, for any $\lambda \in \mathbb{R}$,

$$\log \mathbb{E}[\exp\{\lambda \eta_{n,j}\}] \le (1/2)\lambda^2 \|u^\top \left(\mathrm{I} - \alpha \bar{\mathbf{A}}\right)^{n-j} \|^2 \sigma_\varepsilon^2 \le (1/2)\lambda^2 \kappa_Q (1-\alpha a)^{n-j} \sigma_\varepsilon^2 \;.$$

Hence, $\eta_{n,j} \in \mathrm{SG}(\sigma_{n,j}^2)$, where $\sigma_{n,j}^2 = \kappa_Q(1-\alpha a)^{n-j}\sigma_\varepsilon^2$. IND 1 and (56) imply that $u^\top J_n^{(0)}$ is also sub-Gaussian random variable, that is,

$$u^\top J_n^{(0)} \in \mathrm{SG}(\sigma_{\alpha,n}^2) \;, \quad \sigma_{\alpha,n}^2 = \alpha^2 \sum_{j=1}^n \sigma_{n,j}^2 \le a^{-1} \kappa_Q \sigma_\varepsilon^2 \alpha \;.$$

Using (53) and applying Lemma 10, we obtain for $p \ge 2$ that

$$\mathbb{E}^{1/p}\left[\left|u^\top J_n^{(0)}\right|^p\right] \le \mathsf{D}_1 \sigma_\varepsilon \sqrt{\alpha a p} \,, \text{ where } \mathsf{D}_1 = 2\kappa_Q^{1/2} a^{-1} \;.$$

Now the proof of the bound (55) follows the same line as the proof of Proposition 3 and is omitted.

PROPOSITION 11. *Assume A 1, IND 1 and A 3. Then, for any $n \in \mathbb{N}$, $\alpha \in (0, \alpha_\infty \wedge [\mathrm{c}_{\mathbf{A}} / \{2 + 2\log d\}])$, $\theta_0 \in \mathbb{R}^d$, $u \in \mathbb{S}^{d-1}$, it holds*

$$(n/2)\mathbb{E}\left[|u^\top \bar{\mathbf{A}} \left(\bar{\theta}_{\lceil n/2 \rceil, n} - \theta^\star\right)|^2\right] \le 4 u^\top \Sigma_\varepsilon u + \Delta_{n,\alpha}^{(\mathrm{fl})} + \Delta_{n,\alpha}^{(\mathrm{tr})} \|\theta_0 - \theta^\star\|^2 (1-\alpha a/4)^n / n \,,$$

where $\Delta_{n,\alpha}^{(\mathrm{fl})}, \Delta_{n,\alpha}^{(\mathrm{tr})}$ are given in (24).

Proof. The proof follows the same line as Proposition 4 using Proposition 10 instead of Proposition 2 and Proposition 3.

PROPOSITION 12. *Assume A 1, A 3, and IND 1. Then, for any $\alpha \in (0, \alpha_\infty]$, $p \ge 2$, $u \in \mathbb{S}^{d-1}$ and $n \in \mathbb{N}$, it holds*

$$\mathbb{E}^{1/p}\left[|u^\top J_n^{(1)}|^p\right] \le \mathsf{D}_3 \alpha a p^2 \sigma_\varepsilon \;, \text{ where } \mathsf{D}_3 = 4\kappa_Q \,\mathrm{C}_{\mathbf{A}} / a^2 \;. \tag{57}$$

Moreover, for any $2 \le p \le q$ and $\alpha \in (0, \alpha_{q,\infty}]$, $n \in \mathbb{N}$,

$$\mathbb{E}^{1/p}\left[|u^\top H_n^{(1)}|^p\right] \le \mathsf{D}_4 \alpha a p^2 d^{1/q} \sigma_\varepsilon \;, \text{ where } \mathsf{D}_4 = 4\kappa_Q^{1/2} \,\mathrm{C}_{\mathbf{A}} \,\mathsf{D}_3 / a^2 \;. \tag{58}$$

Proof. We begin with the bound (57). We use (47). The sequence $\{S_{i+1:n}^{(1)} \varepsilon_i\}_{i=1}^{n-1}$ is a martingale-difference with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{N}}$: Applying Burkholder's inequality Osekowski [29, Theorem 8.6] and Minkowski's inequality, we get

$$\begin{aligned} \mathbb{E}[|u^\top J_n^{(1)}|^p] &\le p^p \alpha^{2p} \mathbb{E}[(\textstyle\sum_{i=1}^{n-1} (u^\top S_{i+1:n}^{(1)} \varepsilon_i)^2)^{p/2}] \\ &\le p^p \alpha^{2p} (\textstyle\sum_{i=1}^{n-1} \mathbb{E}^{2/p}[|u^\top S_{i+1:n}^{(1)} \varepsilon_i|^p])^{p/2} \;. \end{aligned} \tag{59}$$

Set $v_{i+1:n} = [S_{i+1:n}^{(1)}]^T u$. Then, using IND 1, we get

$$\mathbb{E}[|v_{i+1:n}^\top \varepsilon_i|^p] \le \mathbb{E}\left[\|v_{i+1:n}\|^p\right] \sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}[|v^\top \varepsilon_i|^p] \,.$$

Using the same arguments as in Proposition 5, we get

$$\mathbb{P}\big(\|v_{i+1:n}\| \ge t\big) \le 2\exp\left\{-\frac{t^2}{2\kappa_Q^2 \,\mathrm{C}_{\mathbf{A}}^2 (n-i)(1-\alpha a)^{n-i-1}}\right\} \,.$$

Hence, applying Lemma 10, we get for any $u \in \mathbb{S}^{d-1}$

$$\mathbb{E}^{2/p}[\|v_{i+1:n}\|^p] \leq 4p\,\mathrm{C}_{\mathbf{A}}^2\,\kappa_Q^2(n-i)(1-\alpha a)^{n-i-1}\,. \tag{60}$$

Combining (59), (60), and A 2, we get

$$\begin{aligned}\mathbb{E}^{1/p}[|u^\top J_n^{(1)}|^p] &\leq 4\sigma_\varepsilon p^2\alpha^2\,\mathrm{C}_{\mathbf{A}}\,\kappa_Q(\textstyle\sum_{i=1}^{n-1}(n-i)(1-\alpha a)^{n-i-1})^{1/2} \\ &\leq \mathsf{D}_3\alpha a p^2\sigma_\varepsilon\,.\end{aligned} \tag{61}$$

We now consider (58). Recall that $H_n^{(1)} = -\alpha\sum_{\ell=1}^n \Gamma_{\ell+1:n}^{(\alpha)}\tilde{\mathbf{A}}_\ell J_{\ell-1}^{(1)}$. Hence, using Minkowski's inequality together with IND 1,

$$\mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] \leq \alpha\textstyle\sum_{\ell=1}^n \mathbb{E}^{1/p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\tilde{\mathbf{A}}_\ell\|^p]\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{1/p}[|u^\top J_{\ell-1}^{(1)}|^p]\,.$$

Applying Theorem 1 and (61), we get using the definition (28) of D_4

$$\mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] \leq \kappa_Q^{1/2}d^{1/q}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{D}_3\alpha^2\sigma_\varepsilon p^2\textstyle\sum_{\ell=1}^n(1-\alpha a/4)^n \leq \mathsf{D}_4 d^{1/q}\sigma_\varepsilon\alpha a p^2\,.$$

Using the bounds of Proposition 11, we obtain the p-th moment error bound for LSA-PR procedure similarly to Theorem 2. Proceeding as in (29), we introduce the fluctuation and transient components of the LSA-PR error

$$\begin{aligned}\Delta_{n,p,\alpha}^{(\mathrm{fl})} &= \frac{4\mathrm{e}^{1/p}\mathsf{D}_2 p^{1/2}}{(\alpha n)^{1/2}} + \mathrm{e}^{1/p}\,\mathrm{C}_{\mathbf{A}}(\mathsf{D}_3+\mathsf{D}_4)\alpha a p^3 + \frac{3\sqrt{2}\,\mathrm{C}_{\mathsf{Rm},2}\sqrt{\log\{\mathrm{e}n\}}p^{3/2}}{n^{1/2}} \\ &\quad + \mathrm{C}_{\mathbf{A}}\,\mathsf{D}_1\alpha^{1/2}p^{3/2}\,, \\ \Delta_{n,p,\alpha}^{(\mathrm{tr})} &= \mathrm{e}^{1/p}\kappa_Q^{1/2}(2\sqrt{2}/(\alpha n^{1/2}) + 2^{-1/2}n^{1/2}\,\mathrm{C}_{\mathbf{A}})\,.\end{aligned} \tag{62}$$

THEOREM 5. *Assume Assume A 1, IND 1, and A 3. Then, for any $n \in \mathbb{N}$, $p \geq 2$, $\alpha \in (0, \alpha_\infty \wedge \mathrm{c}_{\mathbf{A}}/\{p(1+\log d)\})$, $\theta_0 \in \mathbb{R}^d$, $u \in \mathbb{S}_{d-1}$, it holds*

$$\begin{aligned}(n/2)^{1/2}\mathbb{E}^{1/p}\left[|u^\top\bar{\mathbf{A}}\left(\bar{\theta}_{\lceil n/2\rceil,n}-\theta^\star\right)|^p\right] &\leq \mathrm{C}_{\mathsf{Rm},1}\{u^\top\Sigma_\varepsilon u\}^{1/2}p^{1/2} + \sigma_\varepsilon\Delta_{n,p,\alpha}^{(\mathrm{fl})} \\ &\quad + \Delta_{n,p,\alpha}^{(\mathrm{tr})}(1-\alpha a/4)^{n/2}\|\theta_0-\theta^\star\|\,,\end{aligned}$$

where $\mathrm{C}_{\mathsf{Rm,i}}$, $i = 1,2$ are defined in Section A.

Proof. The proof follows the lines of Theorem 2 and is omitted. The only difference with the mentioned proof is related with the term $\mathbb{E}^{1/p}\left[|\sum_{t=n_0}^{n-1}u^\top\varepsilon_{t+1}|^p\right]$. Application of Rosenthal's inequality yields

$$\begin{aligned}\mathbb{E}^{1/p}\left[|\textstyle\sum_{t=n_0}^{n-1}u^\top\varepsilon_{t+1}|^p\right] &\leq \mathrm{C}_{\mathsf{Rm},1}\,p^{1/2}(n-n_0)^{1/2}\{u^\top\Sigma_\varepsilon u\}^{1/2} \\ &\quad + \mathrm{C}_{\mathsf{Rm},2}\,p\,\mathbb{E}^{1/p}[\max_{t\in\{n_0,\ldots,n-1\}}|u^\top\varepsilon_{t+1}|^p]\,.\end{aligned}$$

Since $u^\top\varepsilon_{t+1} \in \mathrm{SG}(\sigma_\varepsilon^2)$ for any $t \in \mathbb{N}^*$, we obtain using Durmus et al. [12, Lemma 4], that

$$\mathbb{E}^{1/p}[\max_{t\in\{n_0,\ldots,n-1\}}|u^\top\varepsilon_{t+1}|^p] \leq 3\sigma_\varepsilon p^{1/2}\sqrt{1+\log\{n-n_0\}}\,.$$

This modification affects the fluctuation term $\Delta_{n,p,\alpha}^{(\mathrm{fl})}$ in (62).

Appendix D: Markov case bounds

D.1. Proof of Theorem 3 We first provide a result on the product of dependent random matrices. The proof is based on Huang et al. [18]. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\ell\}_{\ell\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space. For the matrix $B \in \mathbb{R}^{d\times d}$ we denote by $(\sigma_\ell(B))_{\ell=1}^d$ its singular values. For $q \geq 1$, the Shatten q-norm is denoted by $\|B\|_q = \{\sum_{\ell=1}^d \sigma_\ell^q(B)\}^{1/q}$. For $q,p \geq 1$ and a random matrix $\mathbf{X}$ we write $\|\mathbf{X}\|_{q,p} = \{\mathbb{E}[\|\mathbf{X}\|_q^p]\}^{1/p}$.

Proposition 13. *Let $\{\mathbf{Y}_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of random matrices adapted to the filtration $\{\mathfrak{F}_\ell\}_{\ell\in\mathbb{N}}$ and P be a positive definite matrix. Assume that for each $\ell \in \mathbb{N}^*$ there exist $\mathtt{m}_\ell \in (0,1]$ and $\sigma_\ell > 0$ such that*

$$\|\mathbb{E}^{\mathfrak{F}_{\ell-1}}[\mathbf{Y}_\ell]\|_P^2 \leq 1 - \mathtt{m}_\ell \text{ and } \|\mathbf{Y}_\ell - \mathbb{E}^{\mathfrak{F}_{\ell-1}}[\mathbf{Y}_\ell]\|_P \leq \sigma_\ell \quad \mathbb{P}\text{-a.s.}.$$

Define $\mathbf{Z}_n = \prod_{\ell=0}^n \mathbf{Y}_\ell = \mathbf{Y}_n\mathbf{Z}_{n-1}$, for $n \geq 1$. Then, for any $2 \leq p \leq q$ and $n \geq 1$,

$$\|\mathbf{Z}_n\|_{q,p}^2 \leq \kappa_P \prod_{\ell=1}^n (1 - \mathtt{m}_\ell + (q-1)\sigma_\ell^2)\|P^{1/2}\mathbf{Z}_0 P^{-1/2}\|_{q,p}^2 ,$$

where $\kappa_P = \lambda_{\max}(P)/\lambda_{\min}(P)$ and $\lambda_{\max}(P), \lambda_{\min}(P)$ correspond to the largest and smallest eigenvalues of P.

Proof. Let $n \in \mathbb{N}^*$ and $2 \leq p \leq q$. We begin with the decomposition

$$\mathbf{Z}_n = \mathbf{Y}_n\mathbf{Z}_{n-1} = (\mathbf{Y}_n - \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n])\mathbf{Z}_{n-1} + \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n]\mathbf{Z}_{n-1} .$$

Let us define $f_P : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ as $f_P(B) = P^{1/2}BP^{-1/2}$. Therefore, for any $n \in \mathbb{N}$, it holds $f_P(\mathbf{Z}_n) = \mathbf{A}_n + \mathbf{B}_n$, where

$$\mathbf{A}_n = f_P((\mathbf{Y}_n - \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n])\mathbf{Z}_{n-1}) , \quad \mathbf{B}_n = f_P(\mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n])f_P(\mathbf{Z}_{n-1}) .$$

Since $\mathbb{E}^{\mathbf{B}_n}[\mathbf{A}_n] = \mathbb{E}^{\mathbf{B}_n}[\mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{A}_n]] = 0$, Huang et al. [18, Proposition 4.3] implies that

$$\|f_P(\mathbf{Z}_n)\|_{q,p}^2 \leq \|\mathbf{B}_n\|_{q,p}^2 + (q-1)\|\mathbf{A}_n\|_{q,p}^2 . \tag{63}$$

It remains to bound the two terms on the right-hand side. To this end, we use Hiai and Petz [17, Theorem 6.20] which implies that for any $B_1, B_2 \in \mathbb{R}^{d\times d}$,

$$\|B_1B_2\|_{q,p} \leq \|B_1\|\|B_2\|_{q,p} . \tag{64}$$

Combining (64) with $\|B\|_P = \|f_P(B)\|$, and $\|\mathbf{Y}_n - \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n]\|_P \leq \sigma_n$, we get

$$\begin{aligned}\|\mathbf{A}_n\|_{q,p} &= \left(\mathbb{E}\left[\|f_P(\mathbf{Y}_n - \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n])f_P(\mathbf{Z}_{n-1})\|_q^p\right]\right)^{1/p} \\ &\leq \left(\mathbb{E}\left[\|\mathbf{Y}_n - \mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n]\|_P^p\|f_P(\mathbf{Z}_{n-1})\|_q^p\right]\right)^{1/p} \leq \sigma_n\|f_P(\mathbf{Z}_{n-1})\|_{q,p} .\end{aligned} \tag{65}$$

Similarly, applying $\|\mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n]\|_P^2 \leq 1 - \mathtt{m}_n$

$$\begin{aligned}\|\mathbf{B}_n\|_{q,p}^2 &= \left(\mathbb{E}\left[\|f_P(\mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n])f_P(\mathbf{Z}_{n-1})\|_q^p\right]\right)^{2/p} \\ &\leq \left(\mathbb{E}\left[\|\mathbb{E}^{\mathfrak{F}_{n-1}}[\mathbf{Y}_n]\|_P^p\|f_P(\mathbf{Z}_{n-1})\|_q^p\right]\right)^{2/p} \leq (1 - \mathtt{m}_n)\|f_P(\mathbf{Z}_{n-1})\|_{q,p}^2 .\end{aligned} \tag{66}$$

Combining (65) and (66) in (63) yields

$$\|f_P(\mathbf{Z}_n)\|_{q,p}^2 \leq (1 - \mathtt{m}_n + (q-1)\sigma_n^2)\|f_P(\mathbf{Z}_{n-1})\|_{q,p}^2 \leq \prod_{i=1}^n (1 - \mathtt{m}_i + (q-1)\sigma_i^2)\|f_P(\mathbf{Z}_0)\|_{q,p}^2 .$$

The proof is completed using (64) which implies that

$$\|\mathbf{Z}_n\|_{q,p} = \|P^{-1/2}f_P(\mathbf{Z}_n)P^{1/2}\|_{q,p} \leq \sqrt{\kappa_P}\|f_P(\mathbf{Z}_n)\|_{q,p} .$$

In the lemmas below we aim to prove the bound (38). Recall that $\mathbf{Y}_1 = \prod_{i=1}^{h}(\mathrm{I} - \alpha \mathbf{A}(Z_i))$.

LEMMA 1. *Assume A 1 and **UGE** 1. Then for any $\alpha \in (0, \alpha_\infty^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$ with $\alpha_\infty^{(\mathrm{M})}$ defined in (36), and any probability ξ on $(\mathsf{Z}, \mathcal{Z})$,*

$$\|\mathbb{E}_\xi[\mathbf{Y}_1]\|_Q^2 \le 1 - a\alpha h/6\,, \quad \textit{where } h = 1 \vee \lceil 8\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\, t_{\mathrm{mix}}/a \rceil\,. \tag{67}$$

Proof. We decompose the matrix product $\mathbf{Y}_1$ as follows:

$$\mathbf{Y}_1 = \mathrm{I} - \alpha h \bar{\mathbf{A}} - \mathbf{S}_1 + \mathbf{R}_1\,, \tag{68}$$

where $\mathbf{S}_1 = \alpha \sum_{k=1}^{h}\{\mathbf{A}(Z_k) - \bar{\mathbf{A}}\}$ is linear statistics in $\{\mathbf{A}(Z_k)\}_{k=1}^{h}$, and the remainder $\mathbf{R}_1$ collects the higher-order terms in the products

$$\mathbf{R}_1 = \sum_{r=2}^{h}(-1)^r \alpha^r \sum_{(i_1,\ldots,i_r)\in \mathsf{I}_r^\ell} \prod_{u=1}^{r} \mathbf{A}(Z_{i_u})\,.$$

with $\mathsf{I}_r^\ell = \{(i_1,\ldots,i_r) \in \{1,\ldots,h\}^r : i_1 < \cdots < i_r\}$. Using $\|M\|_Q = \|Q^{1/2} M Q^{-1/2}\|$, it is straightforward to check that $\mathbb{P}$-a.s. it holds

$$\|\mathbf{R}_1\|_Q \le \sum_{r=2}^{h}(\alpha\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}})^r \binom{h}{r} \le (\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\alpha h)^2 (1 + \kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\alpha)^h = T_2\,. \tag{69}$$

On the other hand, using **UGE** 1, we have for any $k \in \mathbb{N}^*$, that

$$\|\mathbb{E}_\xi[\mathbf{A}(Z_k) - \bar{\mathbf{A}}]\| = \sup_{u,v\in\mathbb{S}^{d-1}} [\mathbb{E}_\xi[u^\top \mathbf{A}(Z_k) v] - u^\top \bar{\mathbf{A}} v] \le \mathrm{C}_{\mathbf{A}}\,\Delta(\mathrm{Q}^k)\,.$$

Hence, with the triangle inequality and (35),

$$\begin{aligned}\|\mathbb{E}_\xi[\mathbf{S}_1]\|_Q &\le \alpha\kappa_Q^{1/2} \textstyle\sum_{k=1}^{h} \|\mathbb{E}_\xi[\mathbf{A}(Z_k) - \bar{\mathbf{A}}]\| \le \alpha\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}} \sum_{k=1}^{h} \Delta(\mathrm{Q}^k) \\ &\le (4/3)\alpha t_{\mathrm{mix}} \kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}} = T_1\,.\end{aligned}$$

This result combined with (69) in (68) implies that

$$\|\mathbb{E}_\xi[\mathbf{Y}_1]\|_Q \le \|\mathrm{I} - \alpha h \bar{\mathbf{A}}\|_Q + T_1 + T_2\,.$$

First, by definition (67) of h , we have

$$T_1 \le \alpha a h/6\,. \tag{70}$$

With the definition of $\alpha_\infty^{(\mathrm{M})}$ in (36), $\alpha \le \alpha_\infty^{(\mathrm{M})} \le (\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\, h)^{-1} \wedge [a/(6\mathrm{e}\kappa_Q\,\mathrm{C}_{\mathbf{A}}^2\, h)]$, and

$$T_2 \le (\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\alpha h)^2 \mathrm{e} \le \alpha a h/6\,. \tag{71}$$

Finally, Proposition 1 implies that, for $\alpha h \le \alpha_\infty$,

$$\|\mathrm{I} - \alpha h \bar{\mathbf{A}}\|_Q \le 1 - \alpha a h/2\,. \tag{72}$$

Combining (70), (71), and (72) yield $\|\mathbb{E}_\xi[\mathbf{Y}_1]\|_Q \le 1 - \alpha a h/6$, and the statement follows.

LEMMA 2. *Assume A 1 and **UGE** 1, and let $\alpha \in (0, \alpha_\infty^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$. Then, for any probability ξ on $(\mathsf{Z}, \mathcal{Z})$, we have*

$$\|\mathbf{Y}_1 - \mathbb{E}_\xi[\mathbf{Y}_1]\|_Q \le \mathrm{C}_\sigma\, \alpha h\,, \ \textit{where } \mathrm{C}_\sigma = 2(\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}} + a/6)\,,$$

and h is given in (67).

Proof. Using (68), we obtain

$$\|\mathbf{Y}_1 - \mathbb{E}_\xi[\mathbf{Y}_1]\|_Q \le \alpha \sum_{k=1}^h \|\mathbf{A}(Z_k) - \mathbb{E}_\xi[\mathbf{A}(Z_k)]\|_Q + \|\mathbf{R}_1 - \mathbb{E}_\xi[\mathbf{R}_1]\|_Q \,.$$

Applying the definition of $\mathbf{R}_1$ in (69), the definition of h,$\alpha_\infty^{(\mathrm{M})}$, and T_2 in (71), we get from the above inequalities

$$\|\mathbf{Y}_1 - \mathbb{E}_\xi[\mathbf{Y}_1]\|_Q \le 2\alpha\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,h + \alpha a h/3\,,$$

and the statement follows.

D.2. Proof of Proposition 6 We first apply the Abel transform to $J_n^{(0)}$. Using the representation (8), we obtain that

$$\begin{aligned} J_n^{(0)} &= \alpha \sum_{j=1}^n \left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-j} \varepsilon(Z_j) \\ &= \alpha \left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-1} \sum_{k=1}^n \varepsilon(Z_k) - \alpha^2 \sum_{j=1}^{n-1} \left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-j-1} \bar{\mathbf{A}} \sum_{k=j+1}^n \varepsilon(Z_k)\,. \end{aligned} \tag{73}$$

Note that $\pi(\varepsilon) = 0$, and for any $z, z' \in \mathsf{Z}$, A 2 implies $\|\varepsilon(z) - \varepsilon(z')\| \le 2\,\mathrm{C}_\varepsilon \sqrt{\mathrm{Tr}\,\Sigma_\varepsilon}$. Hence, applying Lemma 12, we get for any $j \in \mathbb{N}$ and $t > 0$, that

$$\mathbb{P}_\xi\left(\|\sum_{k=j+1}^n \varepsilon(Z_k)\| \ge t\right) \le 2\exp\left\{-t^2/(2\beta_{n-j}^2)\right\}\,, \tag{74}$$

where for $\ell \in \mathbb{N}^*$,

$$\beta_\ell = 8\,\mathrm{C}_\varepsilon \sqrt{\ell t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_\varepsilon\}}\,.$$

Lemma 10 and (74) imply that, for any $p \ge 2$,

$$\mathbb{E}_\xi^{1/p}\left[\|\sum_{k=j+1}^n \varepsilon(Z_k)\|^p\right] \le 2^{7/2}\,\mathrm{C}_\varepsilon \sqrt{(n-j)p t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_\varepsilon\}}\,.$$

Then, applying Minkowski's inequality to (73), we get

$$\begin{aligned} \mathbb{E}_\xi^{1/p}\left[\|J_n^{(0)}\|^p\right] &\le 2^{7/2}\,\mathrm{C}_\varepsilon\,\alpha \|\left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-1}\| \sqrt{np t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_\varepsilon\}} \\ &\quad + 2^{7/2}\,\mathrm{C}_\varepsilon\,\alpha^2 \sum_{j=1}^{n-1} \|\left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-j-1} \bar{\mathbf{A}}\| \sqrt{(n-j)p t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_\varepsilon\}}\,. \end{aligned}$$

Using A 1 and Proposition 1, for $j \in \{1, \dots, n\}$, $\|\left(\mathrm{I} - \alpha\bar{\mathbf{A}}\right)^{n-j}\| \le \sqrt{\kappa_Q}(1 - \alpha a)^{(n-j)/2}$. Note also that, since $a\alpha \le 1/2$,

$$\begin{aligned} \sum_{j=1}^{n-1}(1-\alpha a)^{(n-j-1)/2}\sqrt{n-j} &\le \mathrm{e}^{\alpha a} \sum_{k=1}^{n-1} \exp\{-\alpha a(k+1)/2\}\sqrt{k} \\ &\le \frac{2^{3/2}\mathrm{e}^{\alpha a}}{(\alpha a)^{3/2}} \int_0^{+\infty} \exp\{-y\}\sqrt{y}\,\mathrm{d}y \le \frac{2^{1/2}\pi^{1/2}\mathrm{e}^{1/2}}{(\alpha a)^{3/2}}\,. \end{aligned}$$

It remains to combine the previous bounds with an elementary inequality, using $\alpha a \le 1/2$, for any $x > 0$,

$$(1-\alpha a)^{(x-1)/2}\sqrt{x} \le \mathrm{e}^{\alpha a/2}\exp\{-\alpha a x/2\}\sqrt{x} \le \frac{\mathrm{e}^{1/4}}{(\alpha a)^{1/2}} \sup_{u \ge 0}\{u\mathrm{e}^{-u}\}^{1/2} \le \frac{1}{(\alpha a)^{1/2}\mathrm{e}^{1/4}}\,.$$

Combining the bounds above yield (39) with the constant $\mathsf{D}_1^{(\mathrm{M})}$ defined in (40).

D.3. Proof of Proposition 8 Using (10), we decompose $H_n^{(0)} = J_n^{(1)} + H_n^{(1)}$. Now the statement follows from Minkowski's inequality, Lemma 3 and Lemma 5. The constants $\mathsf{D}_3^{(\mathrm{M})}, \mathsf{D}_4^{(\mathrm{M})}$ are given by

$$\begin{aligned} \mathsf{D}_3^{(\mathrm{M})} &= \mathsf{D}_{J,1}^{(\mathrm{M})} \left(1 + 96 a^{-1} \mathrm{C}_{\mathbf{A}} \, \mathrm{e}^2 \kappa_Q^{1/2}\right) , \\ \mathsf{D}_4^{(\mathrm{M})} &= \mathsf{D}_{J,2}^{(\mathrm{M})} \left(1 + 48 a^{-1} \mathrm{C}_{\mathbf{A}} \, \mathrm{e}^2 \kappa_Q^{1/2}\right) . \end{aligned} \tag{75}$$

with $\mathsf{D}_{J,1}^{(\mathrm{M})}$ and $\mathsf{D}_{J,2}^{(\mathrm{M})}$ defined in (76).

LEMMA 3. *Assume A1, A2, and* ***UGE*** *1. Then, for any $\alpha \in (0, \alpha_\infty]$, $p \geq 2$ any initial probability ξ on $(\mathsf{Z}, \mathcal{Z})$, it holds that*

$$\mathbb{E}_\xi^{1/p}[\|J_n^{(1)}\|^p] \leq \{\operatorname{Tr} \Sigma_\varepsilon\}^{1/2} (\alpha a t_{\mathrm{mix}}) \{\mathsf{D}_{J,1}^{(\mathrm{M})} \sqrt{\log(1/\alpha a)} p^2 + \mathsf{D}_{J,2}^{(\mathrm{M})} (\alpha a t_{\mathrm{mix}})^{1/2} p^{1/2}\} ,$$

where

$$\begin{aligned} \mathsf{D}_{J,1}^{(\mathrm{M})} &= 64 \kappa_Q \, \mathrm{C}_{\mathbf{A}} \, \mathrm{C}_\varepsilon \, a^{-2} \left((\sqrt{2} + \kappa_Q^{1/2})/\sqrt{2 \log 2} + 2 \pi^{1/2} \kappa_Q^{1/2} + \kappa_Q^{1/2}/\sqrt{\log 2}\right) \\ \mathsf{D}_{J,2}^{(\mathrm{M})} &= \frac{128}{3} \kappa_Q^{3/2} \, \mathrm{C}_{\mathbf{A}} \, \mathrm{C}_\varepsilon \, a^{-2} . \end{aligned} \tag{76}$$

To proceed with Berbee's lemma, we consider the extended measurable space $\tilde{\mathsf{Z}}_{\mathbb{N}} = \mathsf{Z}^{\mathbb{N}} \times [0,1]$ endowed with the σ-field $\tilde{\mathcal{Z}}_{\mathbb{N}} = \mathcal{Z}^{\otimes \mathbb{N}} \otimes \mathcal{B}([0,1])$. For any probability measure ξ on $(\mathsf{Z}, \mathcal{Z})$, we consider the probability measure $\tilde{\mathbb{P}}_\xi = \mathbb{P}_\xi \otimes \mathbf{Unif}([0,1])$ and denote by $\tilde{\mathbb{E}}_\xi$ the corresponding expectation. Finally, we denote by $(\tilde{Z}_k)_{k \in \mathbb{N}}$ the canonical process $\tilde{Z}_k : ((z_i)_{i \in \mathbb{N}}, u) \in \tilde{\mathsf{Z}}_{\mathbb{N}} \mapsto z_k$ and $U : ((z_i)_{i \in \mathbb{N}}, u) \in \tilde{\mathsf{Z}}_{\mathbb{N}} \mapsto u$. Under $\tilde{\mathbb{P}}_\xi$, by construction $\{\tilde{Z}_k\}_{k \in \mathbb{N}}$ is a Markov chain with initial distribution ξ and Markov kernel Q independent of U. The distribution of U under $\tilde{\mathbb{P}}_\xi$ is uniform over $[0,1]$.

LEMMA 4. *Assume* ***UGE*** *1, let $m \in \mathbb{N}^*$ and ξ be a probability measure on $(\mathsf{Z}, \mathcal{Z})$. Then, there exists a random process $(\tilde{Z}_k^\star)_{k \in \mathbb{N}}$ defined on $(\tilde{\mathsf{Z}}_{\mathbb{N}}, \tilde{\mathcal{Z}}_{\mathbb{N}}, \tilde{\mathbb{P}}_\xi)$ such that for any $k \in \mathbb{N}$,*
(a) $\tilde{Z}_k^\star$ is independent of $\tilde{\mathcal{F}}_{k+m} = \sigma\{\tilde{Z}_\ell : \ell \geq k+m\}$;
(b) $\tilde{\mathbb{P}}_\xi(\tilde{Z}_k^\star \neq \tilde{Z}_k) \leq \Delta(\mathrm{Q}^m)$;
(c) the random variables $\tilde{Z}_k^\star$ and $\tilde{Z}_k$ have the same distribution under $\tilde{\mathbb{P}}_\xi$.

Proof. Berbee's lemma Rio [34, Lemma 5.1] ensures that for any k, there exists $\tilde{Z}_k^\star$ satisfying (a), (c) and $\tilde{\mathbb{P}}_\xi(\tilde{Z}_k^\star \neq \tilde{Z}_k) = \beta_\xi(\sigma(\tilde{Z}_k), \tilde{\mathcal{F}}_{k+m})$. In this formulae for two σ-fields $\mathfrak{F}$, $\mathfrak{G}$ on $\tilde{\mathcal{Z}}_{\mathbb{N}}$,

$$\beta_\xi(\mathfrak{F}, \mathfrak{G}) = \frac{1}{2} \sup \sum_{i \in \mathsf{I}} \sum_{j \in \mathsf{J}} |\tilde{\mathbb{P}}_\xi(\mathsf{A}_i \cap \mathsf{B}_j) - \tilde{\mathbb{P}}_\xi(\mathsf{A}_i) \tilde{\mathbb{P}}_\xi(\mathsf{B}_j)| ,$$

and the supremum is taken over all pairs of partitions $\{\mathsf{A}_i\}_{i \in \mathsf{I}} \in \mathfrak{F}^{\mathsf{I}}$ and $\{\mathsf{B}_j\}_{j \in \mathsf{J}} \in \mathfrak{G}^{\mathsf{J}}$ of $\tilde{\mathsf{Z}}_{\mathbb{N}}$ with I and J finite. Applying Douc et al. [11, Theorem 3.3] with **UGE** 1 completes the proof.

Proof. Recall that $J_n^{(1)} = \alpha^2 \sum_{\ell=1}^{n-1} S_{\ell+1:n} \varepsilon(Z_\ell)$, where $S_{\ell+1:n}$ is given in (41). We first set a constant block size $m \in \mathbb{N}^\star, m \geq t_{\mathrm{mix}}$ (to be determined later). In order to proceed with $S_{\ell+1:n} \varepsilon(Z_\ell)$, we split $S_{\ell+1:n}$ into a part measurable with respect to $\mathcal{F}_{\ell+m}^n = \sigma(Z_k : k \geq m + \ell)$ and a remainder term. Indeed, using its definition (41),

$$S_{\ell+1:n} = (\mathrm{I} - \alpha \bar{\mathbf{A}})^{n-m-\ell} S_{\ell+1:\ell+m} + S_{\ell+m+1:n} (\mathrm{I} - \alpha \bar{\mathbf{A}})^m .$$

Let $N = \lfloor (n-1)/m \rfloor$. With these notations, we can decompose $J_n^{(1)}$ as a sum of three terms: $J_n^{(1)} = T_1 + T_2 + T_3$, with

$$\begin{aligned} T_1 &= \alpha^2 \sum_{\ell=1}^{m(N-1)} (\mathrm{I} - \alpha \bar{\mathbf{A}})^{n-m-\ell} S_{\ell+1:\ell+m} \varepsilon(Z_\ell) \\ T_2 &= \alpha^2 \sum_{\ell=1}^{m(N-1)} S_{\ell+m+1:n} (\mathrm{I} - \alpha \bar{\mathbf{A}})^m \varepsilon(Z_\ell) , \quad T_3 = \alpha^2 \sum_{\ell=m(N-1)+1}^{n-1} S_{\ell+1:n} \varepsilon(Z_\ell) . \end{aligned}$$

We bound the terms T_1, T_2 and T_3 separately. Using Minkowski's inequality together with Proposition 1, Lemma 7, and the definition (41), we get

$$
\begin{aligned}
\mathbb{E}_\xi^{1/p}\left[\|T_1\|^p\right] &\le \alpha^2 \sum_{\ell=1}^{m(N-1)} \kappa_Q{}^{1/2}(1-\alpha a)^{(n-m-\ell)/2}\mathbb{E}_\xi^{1/p}\left[\|S_{\ell+1:\ell+m}\varepsilon(Z_\ell)\|^p\right] \\
&\le 16\alpha^2\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\,\mathsf{D}_S^{(\mathrm{M})}\sqrt{m t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}p}\sum_{\ell=1}^{m(N-1)}(1-\alpha a)^{(n-\ell-1)/2} \\
&\le 32\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\,a^{-1}\alpha\sqrt{m t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}p}\,,
\end{aligned}
$$

where for the last inequality, we additionally used that $\sqrt{1-x} \le 1-x/2$ for $x \in [0,1]$. Similarly, with Minkowski's inequality and Lemma 7, we bound T_3:

$$
\begin{aligned}
\mathbb{E}_\xi^{1/p}\left[\|T_3\|^p\right] &\le \alpha^2 \sum_{\ell=m(N-1)+1}^{n-1} \mathbb{E}_\xi^{1/p}\left[\|S_{\ell+1:n}\varepsilon(Z_\ell)\|^p\right] \\
&\le 16\sqrt{2}\alpha^2\kappa_Q\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\,\sqrt{m t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}p}\sum_{\ell=m(N-1)+1}^{n-1}(1-\alpha a)^{(n-\ell-1)/2} \\
&\le 32\sqrt{2}\kappa_Q\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\,a^{-1}\alpha\sqrt{m t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}p}\,.
\end{aligned}
$$

In the bound above we used that $n-1-m(N-1) \le 2m$. Combining the above,

$$
\mathbb{E}_\xi^{1/p}\left[\|T_1\|^p\right] + \mathbb{E}_\xi^{1/p}\left[\|T_3\|^p\right] \le c_1^{(\mathrm{M})}\alpha a\sqrt{m t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}p}\,, \tag{77}
$$

where $c_1^{(\mathrm{M})} = 32\kappa_Q\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\,a^{-2}(\sqrt{2}+\kappa_Q^{1/2})$. It remains to bound $\mathbb{E}_\xi^{1/p}[\|T_2\|^p]$. We switch to the extended space $(\tilde{\mathsf{Z}}_{\mathbb{N}}, \tilde{\mathcal{Z}}_{\mathbb{N}}, \tilde{\mathbb{P}}_\xi)$,and, using Lemma 4, we get that $\mathbb{E}_\xi^{1/p}[\|T_2\|^p] = \tilde{\mathbb{E}}_\xi^{1/p}\|\tilde{T}_2\|^p]$ with $\tilde{T}_2 = \alpha^2\sum_{\ell=1}^{m(N-1)}\tilde{S}_{\ell+m+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m\varepsilon(\tilde{Z}_\ell)$. Here $\tilde{S}_{\ell+m+1:n}$ is a counterpart of $S_{\ell+m+1:n}$ defined on the extended space, that is,

$$
\tilde{S}_{\ell+m+1:n} = \textstyle\sum_{k=\ell+m+1}^{n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^{n-k}\tilde{\mathbf{A}}(\tilde{Z}_k)(\mathrm{I}-\alpha\bar{\mathbf{A}})^{k-1-\ell}\,.
$$

We further decompose $\tilde{T}_2 = \tilde{T}_{2,1} + \tilde{T}_{2,2}$, where

$$
\begin{aligned}
\tilde{T}_{2,1} &= \alpha^2\sum_{k=0}^{N-2}\sum_{i=1}^{m}\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m\varepsilon(\tilde{Z}^\star_{km+i})\,, \\
\tilde{T}_{2,2} &= \alpha^2\sum_{k=0}^{N-2}\sum_{i=1}^{m}\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m\{\varepsilon(\tilde{Z}_{km+i})-\varepsilon(\tilde{Z}^\star_{km+i})\}\,.
\end{aligned} \tag{78}
$$

We begin with bounding $\tilde{T}_{2,2}$. Set $V_\ell = \varepsilon(\tilde{Z}_\ell) - \varepsilon(\tilde{Z}^\star_\ell)$ and $\tilde{\mathcal{F}}^\star_\ell = \sigma(\tilde{Z}_k, \tilde{Z}^\star_k : k \le \ell)$. Using Lemma 4 we get with the convention $0/0 = 0$,

$$
\begin{aligned}
&\tilde{\mathbb{E}}_\xi^{1/p}[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m V_{km+i}\|^p] \\
&\quad= \tilde{\mathbb{E}}_\xi^{1/p}[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m V_{km+i}\mathbf{1}_{\{\tilde{Z}_{km+i}\neq\tilde{Z}^\star_{km+i}\}}\|^p] \\
&\quad\le \tilde{\mathbb{E}}_\xi^{1/p}\left[\|V_{km+i}\|^p\tilde{\mathbb{E}}^{\tilde{\mathcal{F}}^\star_{km+i}}\left[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m V_{km+i}/\|V_{km+i}\|\|^p\right]\right] \\
&\quad\le \tilde{\mathbb{E}}_\xi^{1/p}\left[\|V_{km+i}\|^p\sup_{u\in\mathbb{S}^{d-1},\,\xi'\in\mathcal{P}(\mathsf{Z})}\tilde{\mathbb{E}}_{\xi'}[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m u\|^p]\right]\,,
\end{aligned}
$$

where $\mathcal{P}(\mathsf{Z})$ is the set of probability measure on $(\mathsf{Z},\mathcal{Z})$. Applying Lemma 8 and Proposition 1, for any $u\in\mathbb{S}^{d-1}$ and probability measure ξ',

$$\begin{aligned}\tilde{\mathbb{E}}^{1/p}_{\xi'}[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m u\|^p] &= \mathbb{E}^{1/p}_{\xi'}[\|S_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m u\|^p]\\ &\le 16\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}[(n-(k+1)m-i)t_{\mathrm{mix}}(1-\alpha a)^{n-km-i-1}p]^{1/2}\,.\end{aligned}$$

Moreover, under A 2 and **UGE** 1, $\|V_{km+i}\|\le 2\,\mathrm{C}_{\varepsilon}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}\mathbf{1}\{\tilde{Z}_{km+i}\neq\tilde{Z}^{\star}_{km+i}\}$, and $\tilde{\mathbb{P}}_{\xi}(\tilde{Z}^{\star}_{km+i}\neq\tilde{Z}_{km+i})\le\Delta(\mathrm{Q}^m)\le(1/4)^{\lfloor m/t_{\mathrm{mix}}\rfloor}$ by Lemma 4 and **UGE** 1. Combining the bounds above,

$$\begin{aligned}\tilde{\mathbb{E}}^{1/p}_{\xi}[\|\tilde{S}_{(k+1)m+i+1:n}(\mathrm{I}-\alpha\bar{\mathbf{A}})^m V_{km+i}\|^p] &\le 32\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_{\varepsilon}(1/4)^{(1/p)\lfloor m/t_{\mathrm{mix}}\rfloor}\times\\ &\big[(n-(k+1)m-i)(1-\alpha a)^{(n-km-i-1)}t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}p\big]^{1/2}\,.\end{aligned}\tag{79}$$

Substituting (79) into the definition (78) of $\tilde{T}_{2,2}$, and using

$$\textstyle\sum_{\ell=1}^{m(N-1)}\sqrt{n-\ell}(1-\alpha a)^{(n-\ell+1)/2}\le\int_0^{+\infty}t^{1/2}\mathrm{e}^{-\alpha a t/2}\mathrm{d}t=2^{3/2}(a\alpha)^{-3/2}\Gamma(3/2)\,,$$

we get

$$\tilde{\mathbb{E}}^{1/p}_{\xi}\big[\|\tilde{T}_{2,2}\|^p\big]\le c_2^{(\mathrm{M})}(\alpha a)^{1/2}(1/4)^{(1/p)\lfloor m/t_{\mathrm{mix}}\rfloor}\sqrt{t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}p}\,,\tag{80}$$

where $c_2^{(\mathrm{M})}=64\pi^{1/2}\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_{\varepsilon}\,a^{-2}$. To obtain (80) we have additionally used that $m\ge1$ and $\alpha a\le 1/2$.

Now we bound $\tilde{T}_{2,1}$. Define the function $g(z):\mathsf{Z}\mapsto\mathbb{R}^d$, $g(z)=(\mathrm{I}-\alpha\bar{\mathbf{A}})^m\varepsilon(z)$. A2 and Proposition 1 imply $\|g\|_{\infty}\le\kappa_Q^{1/2}(1-\alpha a)^{m/2}\,\mathrm{C}_{\varepsilon}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}^{1/2}$ and $\pi(g)=0$. Then we apply Lemma 4 and Lemma 9, and obtain

$$\begin{aligned}\tilde{\mathbb{E}}^{1/p}_{\xi}\big[\|\tilde{T}_{2,1}\|^p\big]\le\alpha^2\sum_{i=1}^{m}\Bigg[&2p\|g\|_{\infty}\big\{\textstyle\sum_{k=0}^{N-2}\sup_{u\in\mathbb{S}^{d-1}}\tilde{\mathbb{E}}^{2/p}_{\xi}[\|\tilde{S}_{(k+1)m+i+1:n}u\|^p]\big\}^{1/2}\\ &+\sum_{k=0}^{N-2}\|\xi\mathrm{Q}^{km+i}g\|\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{1/p}_{\xi}\big[\|S_{(k+1)m+i+1:n}u\|^p\big]\Bigg]\,.\end{aligned}$$

Assumption **UGE** 1 with $\pi(g)=0$ implies $\|\xi\mathrm{Q}^{km+i}g\|\le\Delta(\mathrm{Q}^{km+i})\|g\|_{\infty}$. Combining it with Lemma 8,

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{k=0}^{N-2}\|\xi\mathrm{Q}^{km+i}g\|\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{1/p}_{\xi}\big[\|S_{(k+1)m+i+1:n}u\|^p\big]\\ &\quad\le 16\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_{\varepsilon}(1-\alpha a)^{(m-1)/2}\sup_{x\ge1}\{x(1-\alpha a)^x\}^{1/2}\sqrt{t_{\mathrm{mix}}\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}p}\sum_{\ell=0}^{+\infty}\Delta(\mathrm{Q}^{\ell})\\ &\quad\le\frac{64}{3\mathrm{e}^{1/2}}(a\alpha)^{-1/2}\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_{\varepsilon}(1-\alpha a)^{(m-1)/2}t_{\mathrm{mix}}^{3/2}\sqrt{\{\mathrm{Tr}\,\Sigma_{\varepsilon}\}p}\,,\end{aligned}$$

where we have used for the last inequality (35), $a\alpha\le1/2$, and $\sup_{x\ge1}\{x(1-\alpha a)^x\}^{1/2}\le\mathrm{e}^{-1/2}(a\alpha)^{-1/2}$. Jensen's inequality together with Lemma 8 yields

$$\begin{aligned}&\sum_{i=1}^{m}\big\{\textstyle\sum_{k=0}^{N-2}\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{2/p}_{\xi}[\|S_{(k+1)m+i+1:n}u\|^p]\big\}^{1/2}\\ &\quad\le\sqrt{m}\big\{\textstyle\sum_{\ell=1}^{m(N-1)}\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{2/p}_{\xi}[\|S_{\ell+m+1:n}u\|^p]\big\}^{1/2}\\ &\quad\le16\kappa_Q\,\mathrm{C}_{\mathbf{A}}(mt_{\mathrm{mix}}p)^{1/2}\big\{\textstyle\sum_{\ell=1}^{m(N-1)}(n-\ell-m)(1-\alpha a)^{n-\ell-m-1}\big\}^{1/2}\\ &\quad\le16\sqrt{2}\kappa_Q\,\mathrm{C}_{\mathbf{A}}(mt_{\mathrm{mix}}p)^{1/2}(\alpha a)^{-1}\,.\end{aligned}$$

Combining the bounds above with $\|g\|_\infty \le \kappa_Q^{1/2}(1-\alpha a)^{m/2} \mathrm{C}_\varepsilon \{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}$, we get

$$\begin{aligned}\tilde{\mathbb{E}}_\xi^{1/p}\big[\|\tilde{T}_{2,1}\|^p\big] \le [32\sqrt{2}\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\, a^{-2}]\sqrt{mt_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}}\alpha a p^{3/2} \\ + [(64/3\mathrm{e}^{-1/2})\kappa_Q^{3/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\, a^{-2}]\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}(\alpha a t_{\mathrm{mix}})^{3/2}p^{1/2}\,.\end{aligned} \tag{81}$$

Now the proof is completed combining (77), (80), and (81), setting

$$m = t_{\mathrm{mix}}\left\lceil \frac{p\log(1/\alpha a)}{2\log(2)}\right\rceil ,$$

and using $p^{1/2} \le p$ and $t_{\mathrm{mix}}^{1/2} \le t_{\mathrm{mix}}$. Indeed, with this choice of m, $(1/4)^{(1/p)\lfloor m/t_{\mathrm{mix}}\rfloor} \le \sqrt{\alpha a}$, $m \ge t_{\mathrm{mix}}$. In addition, note that $m \le 2t_{\mathrm{mix}} p\log(1/\alpha a)/(2\log(2))$ using $\alpha a \le 1/2$ and $p \ge 2$.

Lemma 5. *Assume A 1, A 2, and* ***UGE*** *1. Let $2 \le p \le q/2$. Then, for any $\alpha \in (0, \alpha_{q,\infty}^{(\mathrm{M})} t_{\mathrm{mix}}^{-1}]$ with $\alpha_{q,\infty}^{(\mathrm{M})}$ defined in (37), and any initial probability ξ on $(\mathsf{Z},\mathcal{Z})$, it holds that*

$$\mathbb{E}_\xi^{1/p}[\|H_n^{(1)}\|^p] \le d^{1/q}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}(\alpha a t_{\mathrm{mix}})\big[\mathsf{D}_{H,1}^{(\mathrm{M})}\sqrt{\log(1/\alpha a)}p^2 + \mathsf{D}_{H,2}^{(\mathrm{M})}(\alpha a t_{\mathrm{mix}})^{1/2}p^{1/2}\big]\,,$$

where

$$\mathsf{D}_{H,1}^{(\mathrm{M})} = 96a^{-1}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{e}^2\kappa_Q^{1/2}\mathsf{D}_{J,1}^{(\mathrm{M})}\,, \qquad \mathsf{D}_{H,2}^{(\mathrm{M})} = 48a^{-1}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{e}^2\kappa_Q^{1/2}\mathsf{D}_{J,2}^{(\mathrm{M})}\,.$$

Proof. The decomposition (10) implies

$$H_n^{(1)} = -\alpha\textstyle\sum_{\ell=1}^n \Gamma_{\ell+1:n}^{(\alpha)}\tilde{\mathbf{A}}(Z_\ell)J_{\ell-1}^{(1)}\,.$$

Hence, with Minkowski's and Holder's inequalities,

$$\mathbb{E}_\xi^{1/p}[\|H_n^{(1)}\|^p] \le \alpha\textstyle\sum_{\ell=1}^n \mathbb{E}_\xi^{1/2p}[\|\Gamma_{\ell+1:n}^{(\alpha)}\tilde{\mathbf{A}}(Z_\ell)\|^{2p}]\mathbb{E}_\xi^{1/2p}[\|J_{\ell-1}^{(1)}\|^{2p}]\,.$$

Applying Corollary 3 and Lemma 3,

$$\begin{aligned}\mathbb{E}_\xi^{1/p}[\|H_n^{(1)}\|^p] \le 4\mathrm{e}^2\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathsf{D}_{J,1}^{(\mathrm{M})}t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}d^{1/q}p^2\alpha^2 a\sqrt{\log(1/\alpha a)}\sum_{\ell=1}^n \mathrm{e}^{-a\alpha n/12} \\ + 2\mathrm{e}^2\kappa_Q^{1/2}d^{1/q}\mathsf{D}_{J,2}^{(\mathrm{M})}\{\operatorname{Tr}\Sigma_\varepsilon p\}^{1/2}(\alpha a t_{\mathrm{mix}})^{3/2}\sum_{\ell=1}^n \mathrm{e}^{-a\alpha n/12}\,.\end{aligned}$$

Now the proof follows from elementary bound $\mathrm{e}^{-x} \le 1 - x/2$, $x \in [0,1]$.

D.4. Auxiliary lemma for Theorem 4

Lemma 6. *Assume A 1, A 2, and* ***UGE*** *1. Then, for any $\alpha \in (0, \alpha_\infty]$, $t \in \mathbb{N}^\star$ and initial probability ξ on $(\mathsf{Z},\mathcal{Z})$, it holds that*

$$\|\mathbb{E}_\xi[\tilde{\mathbf{A}}(Z_{t+1})J_t^{(0)}]\| \le \mathsf{D}_4^{(\mathrm{M})}\alpha a t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,, \tag{82}$$

where $\mathsf{D}_4^{(\mathrm{M})} = (4/3)\kappa_Q^{1/2}\,\mathrm{C}_\varepsilon\,\mathrm{C}_{\mathbf{A}}\,a^{-1}$.

Proof. Using (8), we get

$$\|\mathbb{E}_\xi[\tilde{\mathbf{A}}(Z_{t+1})J_t^{(0)}]\| = \sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}_\xi[\alpha u^\top\tilde{\mathbf{A}}(Z_{t+1})\textstyle\sum_{j=1}^t(\mathrm{I}-\alpha\bar{\mathbf{A}})^{t-j}\varepsilon(Z_j)]\,.$$

Define for $z \in \mathsf{Z}$ and $j \in \{1,\dots,t\}$, the function $g_{j,t}(z): \mathsf{Z} \mapsto \mathbb{R}^d$ as

$$g_{j,t}(z) = \textstyle\int_{\mathsf{Z}}\tilde{\mathbf{A}}(z')(\mathrm{I}-\alpha\bar{\mathbf{A}})^{t-j}\varepsilon(z)\mathrm{Q}^{t-j+1}(z,\mathrm{d}z')$$

Using that $\pi(\tilde{\mathbf{A}}) = 0$ together with Proposition 1 and **UGE 1**, for any $u \in \mathbb{S}^{d-1}$,

$$\left|u^\top g_{j,t}(z)\right| \le \kappa_Q^{1/2}(1-\alpha a)^{(t-j)/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\Delta(\mathrm{Q}^{t-j+1})\,.$$

Using the Markov property of $(Z_k)_{k\in\mathbb{N}}$ and the definition of t_{mix} (see **UGE 1**), we get from the previous bound that

$$\begin{aligned}\left|\mathbb{E}_\xi[\alpha u^\top \tilde{\mathbf{A}}(Z_{t+1})\textstyle\sum_{j=1}^{t}(\mathrm{I}-\alpha\bar{\mathbf{A}})^{t-j}\varepsilon(Z_j)]\right| &\le \alpha\kappa_Q^{1/2}\,\mathrm{C}_{\mathbf{A}}\,\mathrm{C}_\varepsilon\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\textstyle\sum_{\ell=0}^{\infty}\Delta(\mathrm{Q}^\ell)\\ &\le \mathsf{D}_4^{(\mathrm{M})}\alpha a t_{\mathrm{mix}}\{\operatorname{Tr}\Sigma_\varepsilon\}^{1/2}\,,\end{aligned}$$

and (82) follows.

Appendix E: Technical bounds: Markov case Recall that $S_{\ell+1:\ell+m}$ is defined, for $\ell, m \in \mathbb{N}^*$, as

$$S_{\ell+1:\ell+m} = \textstyle\sum_{k=\ell+1}^{\ell+m}\mathbf{B}_k(Z_k)\,,\ \text{with } \mathbf{B}_k(z) = (\mathrm{I}-\alpha\bar{\mathbf{A}})^{\ell+m-k}\tilde{\mathbf{A}}(z)(\mathrm{I}-\alpha\bar{\mathbf{A}})^{k-1-\ell}\,. \tag{83}$$

Lemma 7. *Assume A 1, A 2, and **UGE 1**. Then, for any $p \ge 2$, any initial probability ξ on $(\mathsf{Z},\mathcal{Z})$, $\ell, m \in \mathbb{N}^*$, it holds that*

$$\mathbb{E}_\xi^{1/p}[\|S_{\ell+1:\ell+m}\varepsilon(Z_\ell)\|^p] \le \mathsf{D}_S^{(\mathrm{M})} m^{1/2}(1-\alpha a)^{(m-1)/2}\sqrt{t_{\mathrm{mix}} p \operatorname{Tr}\Sigma_\varepsilon}\,,$$

where $\mathsf{D}_S^{(\mathrm{M})} = 16\kappa_Q\,\mathrm{C}_\varepsilon\,\mathrm{C}_{\mathbf{A}}$.

Proof. Now, with $\mathcal{F}_\ell = \sigma\{Z_j, j \le \ell\}$, it holds that

$$\begin{aligned}\mathbb{E}_\xi^{1/p}[\|S_{\ell+1:\ell+m}\varepsilon(Z_\ell)\|^p] &= \mathbb{E}_\xi^{1/p}\left[\|\varepsilon(Z_\ell)\|^p\mathbb{E}^{\mathcal{F}_\ell}\left[S_{\ell+1:\ell+m}\varepsilon(Z_\ell)/\|\varepsilon(Z_\ell)\|\right]\right]\\ &\le \mathbb{E}_\xi^{1/p}\left[\|\varepsilon(Z_\ell)\|^p \sup_{u\in\mathbb{S}^{d-1},\,\xi'\in\mathcal{P}(\mathsf{Z})}\mathbb{E}_{\xi'}[\|S_{\ell+1:\ell+m}u\|^p]\right]\,,\end{aligned}$$

where $\mathcal{P}(\mathsf{Z})$ denotes the set of probability measure on $(\mathsf{Z},\mathcal{Z})$. Combining the above bounds with Lemma 8 and A 2 yields the statement.

Lemma 8. *Assume A 1, A 2, and **UGE 1**. For any $\ell, m \in \mathbb{N}^*$, $t \ge 0$, $u \in \mathbb{S}^{d-1}$, and initial probability ξ on $(\mathsf{Z},\mathcal{Z})$, it holds that*

$$\mathbb{P}_\xi\left(\|S_{\ell+1:\ell+m}u\| \ge t\right) \le 2\exp\left\{-\frac{t^2}{2\gamma_m^2}\right\}\,,\ \text{where } \gamma_m = 8\kappa_Q\,\mathrm{C}_{\mathbf{A}}[m t_{\mathrm{mix}}(1-\alpha a)^{m-1}]^{1/2}\,.$$

Moreover,

$$\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}_\xi^{1/p}[\|S_{\ell+1:\ell+m}u\|^p] \le 16\kappa_Q\,\mathrm{C}_{\mathbf{A}}[m t_{\mathrm{mix}}(1-\alpha a)^{m-1}p]^{1/2}\,.$$

Proof. Define $g_k(z) : \mathbb{Z} \mapsto \mathbb{R}^d$ as $g_k(z) = \mathbf{B}_k(z)u$ where $\mathbf{B}_k$ is given in (83). Note that under A 1 and applying Proposition 1, $\pi(g_k) = 0$ and $\sup_{z\in\mathsf{Z}}\|g_k(z)\| \le \kappa_Q\,\mathrm{C}_{\mathbf{A}}(1-\alpha a)^{(m-1)/2}$ for any $k \in \{\ell+1,\ldots,\ell+m\}$. The proof then follows from Lemma 12 and Lemma 10.

Lemma 9. *Let $(\Omega,\mathfrak{G},\mathbb{P})$ be a probability space, $\{W_k, W_k^\star\}_{k\in\mathbb{N}}$ be a sequence of Z^2-valued random variables, and $\{\check{\mathbf{A}}_k\}_{k\in\{2,\ldots,N+1\}}$ be a sequence of $d\times d$ random matrices. Denote $\mathfrak{G}_k = \sigma(W_\ell, \ell \ge k)$ for $k \in \mathbb{N}^*$. Assume that for $k \in \mathbb{N}^*$, that $\check{\mathbf{A}}_k$ is $\mathfrak{G}_k$-measurable and $\sigma(W_k^\star)$ and $\mathfrak{G}_{k+1}$ are independent. Then, for any family of measurable functions $\{g_k\}_{k=1}^N$ from Z to $\mathbb{R}^d$, with $\max_{k\in\{1,\ldots,N\}}\|g_k\|_\infty \le 1$, and $p \ge 2$,*

$$\begin{aligned}&\mathbb{E}^{1/p}\left[\|\textstyle\sum_{k=1}^{N}\check{\mathbf{A}}_{k+1}g_k(W_k^\star)\|^p\right]\\ &\quad\le 2p\left\{\textstyle\sum_{k=1}^{N}\sup_{u\in\mathbb{S}^{d-1}}\mathbb{E}^{2/p}\left[\|\check{\mathbf{A}}_{k+1}u\|^p\right]\right\}^{1/2} + \mathbb{E}^{1/p}\left[\|\textstyle\sum_{k=1}^{N}\check{\mathbf{A}}_{k+1}\,\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\|^p\right]\,.\end{aligned}$$

Proof. Applying Minkowski's inequality,

$$\mathbb{E}^{1/p}\big[\|\sum_{k=1}^{N}\check{\mathbf{A}}_{k+1}g_k(W_k^\star)\|^p\big] \le \mathbb{E}^{1/p}\big[\|\sum_{k=1}^{N}\check{\mathbf{A}}_{k+1}\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\|^p\big] + \mathbb{E}^{1/p}\big[\|\sum_{k=1}^{N}\check{\mathbf{A}}_{k+1}\{g_k(W_k^\star)-\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\}\|^p\big]\,.$$

The sequence $\{\check{\mathbf{A}}_k\big(g_k(W_k^\star)-\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\big)\}_{k=1}^N$ is a reversed martingale difference sequence with respect to $\{\mathfrak{G}_k\}_{k\ge1}$. Hence, applying the Burkholder inequality (see Osekowski [29, Theorem 8.6]), we obtain

$$\mathbb{E}^{1/p}\big[\|\sum_{k=1}^{N}\check{\mathbf{A}}_k\{g_k(W_k^\star)-\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\}\|^p\big] \le p\big(\sum_{k=1}^{N}\mathbb{E}^{2/p}[\|\check{\mathbf{A}}_{k+1}\{g_k(W_k^\star)-\mathbb{E}^{\mathfrak{G}_{k+1}}[g_k(W_k^\star)]\}\|^p]\big)^{1/2}\,.$$

E.1. Rosenthal inequality under UGE 1 In this section we derive a sharp Rosenthal inequality for the Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ under **UGE** 1. We preface the proof by some definitions and properties of coupling. Let $(\mathsf{X},\mathcal{X})$ be a measurable space. In all this section, $\mathbb{Q}$ and $\mathbb{Q}'$ denote two probability measures on the canonical space $(\mathsf{X}^{\mathbb{N}},\mathcal{X}^{\otimes\mathbb{N}})$. Fix $x^*\in\mathsf{X}$. For any X-valued stochastic process $X=\{X_n\}_{n\in\mathbb{N}}$ and any $\bar{\mathbb{N}}$-valued random variable T, define the X-valued stochastic process $\mathrm{S}_T X$ by $\mathrm{S}_T X=\{X_{T+k},\ k\in\mathbb{N}\}$ on $\{T<\infty\}$ and $\mathrm{S}_T X=(x^*,x^*,x^*,\ldots)$ on $\{T=\infty\}$. For any measure $\mathbb{Q}$ on $(\mathsf{X}^{\mathbb{N}},\mathcal{X}^{\otimes\mathbb{N}})$ and any σ-field $\mathcal{G}\subset\mathcal{X}^{\otimes\mathbb{N}}$, we denote by $(\mu)_{\mathcal{G}}$ the restriction of the measure μ to $\mathcal{G}$. Moreover, for all $n\in\mathbb{N}$, define the σ-field $\mathcal{G}_n=\{\mathrm{S}_n^{-1}(A):A\in\mathcal{X}^{\otimes\mathbb{N}}\}$. We say that $(\Omega,\mathcal{F},\mathbb{P},X,X',T)$ is an *exact coupling* of $(\mathbb{Q},\mathbb{Q}')$ (see Douc et al. [11, Definition 19.3.3]), if

- for all $A\in\mathcal{X}^{\otimes\mathbb{N}}$, $\mathbb{P}(X\in A)=\mathbb{Q}(A)$ and $\mathbb{P}(X'\in A)=\mathbb{Q}'(A)$,
- $\mathrm{S}_T X=\mathrm{S}_T X'$, $\mathbb{P}-$a.s.

The integer-valued random variable T is a coupling time. An exact coupling $(\Omega,\mathcal{F},\mathbb{P},X,X',T)$ of $(\mathbb{Q},\mathbb{Q}')$ is *maximal* (see Douc et al. [11, Definition 19.3.5]) if for all $n\in\mathbb{N}$,

$$\big\|(\mathbb{Q})_{\mathcal{G}_n}-(\mathbb{Q}')_{\mathcal{G}_n}\big\|_{\mathrm{TV}}=2\mathbb{P}(T>n)\,.$$

Assume that $(\mathsf{X},\mathcal{X})$ is a complete separable metric space and let $\mathbb{Q}$ and $\mathbb{Q}'$ denote two probability measures on $(\mathsf{X}^{\mathbb{N}},\mathcal{X}^{\otimes\mathbb{N}})$. Then, there exists a maximal exact coupling of $(\mathbb{Q},\mathbb{Q}')$.

We now turn to the special case of Markov chains. Let P be a Markov kernel on $(\mathsf{X},\mathcal{X})$. Denote by $\{X_n\}_{n\in\mathbb{N}}$ the coordinate process and define as before $\mathcal{G}_n=\{\mathrm{S}_n^{-1}(A):A\in\mathcal{X}^{\otimes\mathbb{N}}\}$. By Douc et al. [11, Lemma 19.3.6], for any probabilities μ,ν on $(\mathsf{X},\mathcal{X})$, we have

$$\big\|(\mathbb{P}_\mu)_{\mathcal{G}_n}-(\mathbb{P}_\nu)_{\mathcal{G}_n}\big\|_{\mathrm{TV}}=\|\mu\mathrm{P}^n-\nu\mathrm{P}^n\|_{\mathrm{TV}}\,.$$

Moreover, if $(\mathsf{X},\mathcal{X})$ is Polish, then, there exists a maximal and exact coupling of $(\mathbb{P}_\mu,\mathbb{P}_\nu)$; see Douc et al. [11, Theorem 19.3.9].

We apply this construction for the Markov kernel Q defined on the complete separable metric space $(\mathsf{Z},\mathsf{d}_{\mathsf{Z}})$. For any two probabilities ξ,ξ' on $(\mathsf{Z},\mathcal{Z})$, there exists a maximal exact coupling $(\Omega,\mathcal{F},\tilde{\mathbb{P}}_{\xi,\xi'},Z,Z',T)$ of $\mathbb{P}_\xi^{\mathrm{Q}}$ and $\mathbb{P}_{\xi'}^{\mathrm{Q}}$, that is,

$$\|\xi\mathrm{Q}^n-\xi'\mathrm{Q}^n\|_{\mathrm{TV}}=2\mathbb{P}(T>n)\,. \tag{84}$$

We write $\tilde{\mathbb{E}}_{\xi,\xi'}$ for the expectation with respect to $\tilde{\mathbb{P}}_{\xi,\xi'}$.

Under **UGE** 1, it is known that $n^{-1/2}\sum_{i=0}^{n-1}\{f(Z_i)-\pi(f)\}$ converges in distribution to the zero-mean Gaussian law with variance

$$\sigma_\pi^2(f)=\lim_{n\to\infty}n^{-1}\mathbb{E}_\pi\Big[\Big\|\sum_{i=0}^{n-1}\{f(Z_i)-\pi(f)\}\Big\|^2\Big]\,. \tag{85}$$

We first start this section with a preliminary result.

THEOREM 6. *Assume **UGE** 1. Then, for any measurable function $f : \mathsf{Z} \to \mathbb{R}^d$, $\|f\|_\infty \leq 1$, $\kappa \in \mathbb{N}^*$, $p \geq 2$, $q \in \mathbb{N}^*$, it holds*

$$\mathbb{E}_\pi^{2/p}[\| \sum_{i=0}^{q-1} f(Z_{i\kappa t_{\mathrm{mix}}}) - \pi(f)\|^{p/2}] \leq (16/3)\{1 + pq^{1/2}\} .$$

Proof. Without loss of generality, we assume that $\pi(f) = 0$ and for notational conciseness we set $t_m = t_{\mathrm{mix}}$. Note that the function $g_\kappa = \sum_{i=0}^{+\infty} \mathrm{Q}^{i\kappa t_m} f$ is well-defined under **UGE** 1. Moreover, (34) implies that $\|g_\kappa\|_\infty \leq 2/(1-2^{-2\kappa}) \leq 8/3$ by definition of t_m. The function g_κ is a solution to the Poisson equation associated with the κt_m-th iterates of Q: $g_\kappa - \mathrm{Q}^{\kappa t_m} g_\kappa = f$. Therefore, we write

$$\sum_{i=0}^{q-1} f(Z_{i\kappa t_m}) = \{g_\kappa(Z_0) - g_\kappa(Z_{qt_m})\} + \sum_{i=1}^{q} \Delta M_i^{g_\kappa} ,$$

where we have set, for $i \in \mathbb{N}^*$, $\Delta M_i^{g_\kappa} = g_\kappa(Z_{i\kappa t_m}) - \mathrm{Q}^{\kappa t_m} g_\kappa(Z_{(i-1)t_m})$. By construction, $\{\Delta M_i^{g_\kappa}\}_{i\in\mathbb{N}^*}$ is a sequence of $\{\mathcal{F}_{i\kappa t_m}\}_{i\in\mathbb{N}^*}$-martingale increments, and $\|\Delta M_i^{g_\kappa}\|_\infty \leq 2\|g_\kappa\|_\infty$. Minkowski's inequality implies

$$\mathbb{E}_\pi^{2/p}[\| \sum_{i=0}^{q-1} f(Z_{it_m})\|^{p/2}] \leq \mathbb{E}_\pi^{2/p}[\| \sum_{i=1}^{q} \Delta M_i^{g_\kappa}\|^{p/2}] + 2\|g_\kappa\|_\infty .$$

Using Osekowski [29, Theorem 8.6], Lyapunov's inequality, we get that

$$\begin{aligned}\mathbb{E}_\pi^{2/p}\big[\| \textstyle\sum_{i=1}^{q} \Delta M_i^{f}\|^{p/2}\big] &\leq \mathbb{E}_\pi^{1/p}\big[\| \textstyle\sum_{i=1}^{q} \Delta M_i^{g_\kappa}\|^{p}\big] \\ &\leq p\mathbb{E}_\pi^{1/p}\Big[\big(\textstyle\sum_{i=1}^{q} \|\Delta M_i^{g_\kappa}\|^2\big)^{p/2}\Big] \leq 2\|g_\kappa\|_\infty pq^{1/2} .\end{aligned}$$

Combining the previous inequalities completes the proof.

Now we prove a version of Rosenthal inequality under the stationary distribution π.

THEOREM 7. *Assume **UGE** 1. Then, for any measurable function $f : \mathsf{Z} \to \mathbb{R}^d$, $\|f\|_\infty \leq 1$, $p \geq 2$, and $n \geq t_{\mathrm{mix}}$, it holds*

$$\mathbb{E}_\pi^{1/p}[\| \sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] \leq \mathrm{C}_{\mathsf{Rm},1}\, p^{1/2} n^{1/2} \sigma_\pi(f) + \mathrm{C}_{\mathsf{Ros},1}^{(\mathrm{M})}\, pn^{2/5} t_{\mathrm{mix}}^{3/5} + (8/3) n^{3/5} t_{\mathrm{mix}}^{2/5} 2^{-(n/t_{\mathrm{mix}})^{1/5}} ,$$

where

$$\mathrm{C}_{\mathsf{Ros},1}^{(\mathrm{M})} = \mathrm{C}_{\mathsf{Rm},1}\{(16 \ln 4/\sqrt{3}) + 2(16/3)^{3/2}\, \mathrm{C}_{\mathsf{Rm},1}\} + (16/3)\, \mathrm{C}_{\mathsf{Rm},2} + 19/3 ,$$

$\mathrm{C}_{\mathsf{Rm},1}, \mathrm{C}_{\mathsf{Rm},2}$ *are given in Section A and $\sigma_\pi^2(f)$ is defined in (85).*

Proof. Without loss of generality, we assume that $\pi(f) = 0$ and for notational conciseness we set $t_m = t_{\mathrm{mix}}$. We also introduce an additional integer κ to be fixed later. First, we note that, using Minkowski's inequality and $\|f\|_\infty \leq 1$, we obtain that

$$\mathbb{E}_\pi^{1/p}[\| \sum_{i=0}^{n-1} f(Z_i)\|^p] \leq \mathbb{E}_\pi^{1/p}[\| \sum_{i=0}^{\lfloor n/\kappa t_m \rfloor \kappa t_m - 1} f(Z_i)\|^p] + \kappa t_m .$$

Define $q_\kappa = \lfloor n/\kappa t_m \rfloor$ and consider now $n_\kappa = \kappa t_m q_\kappa$. Proceeding as in Theorem 6, we define

$$g_\kappa = \sum_{\ell=0}^{\infty} \mathrm{Q}^{\ell\kappa t_m} f , \tag{86}$$

which satisfies $\|g_\kappa\|_\infty \leq 2/(1-2^{-2\kappa}) \leq 8/3$ since $\|f\|_\infty \leq 1$. The function g_κ is a solution to the Poisson equation associated with the κt_m-th iterate $\mathrm{Q}^{\kappa t_m}$, i.e., $g_\kappa - \mathrm{Q}^{\kappa t_m} g_\kappa = f$. We consider the decomposition

$$\sum_{i=0}^{n_\kappa - 1} f(Z_i) = \sum_{r=0}^{\kappa t_m - 1} \sum_{q=1}^{q_\kappa} \{g_\kappa(Z_{q\kappa t_m + r}) - Q^{\kappa t_m} g_\kappa(Z_{(q-1)\kappa t_m + r})\} + \sum_{r=0}^{\kappa t_m - 1} \{g_\kappa(Z_r) - g_\kappa(Z_{q_\kappa \kappa t_m + r})\} .$$

Set, for $q \in \mathbb{N}^*$, $\Delta M_q^{g_\kappa} = \sum_{r=0}^{t_m-1} \{g_\kappa(Z_{qt_m+r}) - \mathrm{Q}^{t_m} g_\kappa(Z_{(q-1)t_m+r})\}$. By construction $\{\Delta M_q^{g_\kappa}\}_{q\in\mathbb{N}}$ is a martingale increment sequence with respect to the filtration $\{\mathcal{F}_{q\kappa t_m}\}_{q\in\mathbb{N}}$. Note that $\|\Delta M_q^{g_\kappa}\| \leq 2\|g_\kappa\|_\infty \kappa t_m \leq (16/3)\kappa t_m$. We get using Minkowski's inequality that

$$\mathbb{E}_\pi^{1/p}\big[\| \textstyle\sum_{i=0}^{n_\kappa-1} f(Z_i)\|^p\big] \leq A_1 + (16/3)\kappa t_m\,, \text{ with } A_1 = \mathbb{E}_\pi^{1/p}\big[\| \textstyle\sum_{q=1}^{q_\kappa} \Delta M_q^g \|^p\big]\,.$$

Applying the Pinelis [31, Theorem 4.1] version of the Rosenthal inequality yields

$$A_1 \leq \mathrm{C}_{\mathsf{Rm},1}\, p^{1/2} A_2 + (16/3)\,\mathrm{C}_{\mathsf{Rm},2}\, \kappa t_m p\,, \quad A_2 = \mathbb{E}_\pi^{1/p}\big[\| \textstyle\sum_{q=0}^{q_\kappa-1} \bar{g}_\kappa(Z_{q\kappa t_m})\|^{p/2}\big]$$

where we have set $\bar{g}_\kappa(z) = \mathbb{E}_z[\|\Delta M_q^{g_\kappa}\|^2]$. Applying Minkowski's inequality again, we obtain

$$A_2^2 \leq q_\kappa \pi(\bar{g}_\kappa) + \mathbb{E}_\pi^{2/p}\big[\| \textstyle\sum_{q=0}^{q_\kappa-1} \{\bar{g}_\kappa(Z_{qt_m}) - \pi(\bar{g}_\kappa)\}\|^{p/2}\big]\,.$$

Then, since $\|\bar{g}_\kappa\|_\infty \leq (16/3)^2\kappa^2 t_m^2$, Theorem 6 implies that

$$\mathbb{E}_\pi^{1/p}\big[\| \textstyle\sum_{q=0}^{q_\kappa-1} \{\bar{g}_\kappa(Z_{qt_m}) - \pi(\bar{g}_\kappa)\}\|^{p/2}\big] \leq (16/3)^{3/2} \kappa t_m \{1 + p^{1/2} q_\kappa^{1/4}\}\,.$$

Using that $n_\kappa = q_\kappa \kappa t_m$, we finally get

$$A_2 \leq \{q_\kappa \pi(\bar{g}_\kappa)\}^{1/2} + (16/3)^{3/2} p^{1/2} n^{1/4} (\kappa t_m)^{3/4} + (16/3)^{3/2}\kappa t_m\,. \tag{87}$$

Finally, we establish the identity $q_\kappa \pi(\bar{g}_\kappa) \sim n\sigma_\pi^2(f)$. To this end, we first rewrite $q_\kappa\pi(\bar{g}_\kappa)$ into a more explicit form. Note first that

$$\pi(\bar{g}_\kappa) = A_3 + 2A_4$$

where, using $\pi\mathrm{Q} = \pi$,

$$\begin{aligned}
A_3 &= \sum_{r=0}^{\kappa t_m - 1} \mathbb{E}_\pi[\|g_\kappa(Z_{\kappa t_m + r}) - \mathrm{Q}^{\kappa t_m} g_\kappa(Z_r)\|^2] = \kappa t_m \mathbb{E}_\pi[\|g_\kappa(Z_{\kappa t_m}) - \mathrm{Q}^{\kappa t_m} g_\kappa(Z_0)\|^2]\,,\\
A_4 &= \sum\nolimits_{0\leq r_1 < r_2 \leq \kappa t_m - 1} \mathbb{E}_\pi[\{g_\kappa(Z_{\kappa t_m + r_1}) - \mathrm{Q}^{\kappa t_m} g_\kappa(Z_{r_1})\}^\top \{g_\kappa(Z_{\kappa t_m + r_2}) - \mathrm{Q}^{\kappa t_m} g_\kappa(Z_{r_2})\}]\,.
\end{aligned}$$

Consider first A_3. Using that $\mathrm{Q}^{\kappa t_m} g_\kappa = g_\kappa - f$, we get

$$A_3 = \kappa t_m\{\pi(\|g_\kappa\|^2) - \pi(\|\mathrm{Q}^{\kappa t_m} g_\kappa\|^2)\} = \kappa t_m\{2\pi(f^\top g_\kappa) - \pi(\|f\|^2)\}\,.$$

Consider now A_4. Using the Markov property, $\pi\mathrm{Q} = \pi$, we obtain

$$\begin{aligned}
A_4 &= \textstyle\sum_{r=1}^{\kappa t_m - 1} (\kappa t_m - r)\big\{\pi(g_\kappa^\top \mathrm{Q}^r g_\kappa) - \pi(\{\mathrm{Q}^{\kappa t_m} g_\kappa\}^\top \mathrm{Q}^{\kappa t_m - r} g_\kappa)\big\}\\
&= \textstyle\sum_{r=1}^{\kappa t_m - 1} (\kappa t_m - r)\pi(f^\top \mathrm{Q}^r g_\kappa) + \sum_{r=1}^{\kappa t_m - 1}(\kappa t_m - r)\pi(\{\mathrm{Q}^{\kappa t_m} g_\kappa\}^\top Q^r g_\kappa)\\
&\qquad - \textstyle\sum_{r=1}^{\kappa t_m - 1}(\kappa t_m - r)\pi(\{\mathrm{Q}^{\kappa t_m} g_\kappa\}^\top \mathrm{Q}^{\kappa t_m - r} g_\kappa)\\
&= \textstyle\sum_{r=1}^{\kappa t_m - 1} (\kappa t_m - r)\pi(f^\top \mathrm{Q}^r g_\kappa) + \sum_{r=1}^{\kappa t_m - 1}(\kappa t_m - 2r)\pi(\{\mathrm{Q}^{\kappa t_m} g_\kappa\}^\top Q^r g_\kappa)\,.
\end{aligned}$$

Therefore, using that $\sigma_\pi^2(f) = 2\sum_{r=0}^{\kappa t_m - 1} \pi(f^\top Q^r g_\kappa) - \pi(\|f\|^2)$, we obtain that

$$\pi(\bar{g}_\kappa) = \kappa t_m \sigma_\pi^2(f) - 2\textstyle\sum_{r=0}^{\kappa t_m - 1} r\pi(f^\top \mathrm{Q}^r g_\kappa) + \kappa t_m \sum_{r=1}^{\kappa t_m - 1}(1 - \frac{2r}{\kappa t_m})\pi(\mathrm{Q}^{\kappa t_m} g_\kappa^\top Q^r g_\kappa)\,.$$

Note that (86) implies that $\pi(g_\kappa) = 0$, hence, $\|Q^r g_\kappa\|_\infty \leq \|g_\kappa\|_\infty (1/4)^{\lfloor r/t_m \rfloor}$, and

$$\Big|\kappa t_m \textstyle\sum_{r=1}^{\kappa t_m - 1}(1 - \frac{2r}{\kappa t_m})\pi(\mathrm{Q}^{\kappa t_m} g_\kappa^\top Q^r g_\kappa)\Big| \leq (\kappa t_m)^2 (1/4)^\kappa \|g_\kappa\|_\infty^2 \leq (8/3\kappa t_m)^2 (1/4)^\kappa\,.$$

Similarly, using $\|f\|_\infty \leq 1$,

$$\Big|\sum_{r=0}^{\kappa t_m - 1} r\pi(f^\top \mathrm{Q}^r g_\kappa)\Big| \leq \|g_\kappa\|_\infty \sum_{r=0}^{\infty} r(1/4)^{\lfloor r/t_m \rfloor} \leq 4\|g_\kappa\|_\infty (1-(1/4)^{1/t_m})^{-2} \leq c_1 t_m^2 \,,$$

where $c_1 = 16\cdot(8/3)\cdot(\ln 4)^2$. In the last inequality we used that $1-\mathrm{e}^{-x} \geq 1 - x/2$ for $x \in [0;\log 4]$. Combining the above inequalities, we obtain that

$$\pi(\bar{g}_\kappa) \leq \kappa t_m \sigma_\pi^2(f) + 2c_1 t_m^2 + (8/3\kappa t_m)^2 (1/4)^\kappa \,.$$

Substituting into (87) together with $n_\kappa = \kappa t_m q_\kappa \leq n$ yields

$$A_2 \leq n^{1/2}\sigma_\pi(f) + (16\ln 4/\sqrt{3}) n^{1/2}(t_m/\kappa)^{1/2} + (8/3)n^{1/2}(\kappa t_m)^{1/2}(1/4)^{\kappa/2} + (16/3)^{3/2} p^{1/2} n^{1/4} (\kappa t_m)^{3/4} + (16/3)^{3/2}\kappa t_m \,.$$

Optimizing the above expression with respect to κ yields $\kappa = (n/t_m)^{1/5}$. Then the previous bound implies

$$A_2 \leq n^{1/2}\sigma_\pi(f) + (16\ln 4/\sqrt{3}) n^{2/5} t_m^{3/5} + (8/3) n^{3/5} t_m^{2/5} (1/4)^{(n/t_m)^{1/5}/2} + (16/3)^{3/2} p^{1/2} n^{2/5} t_m^{3/5} + (16/3)^{3/2} n^{1/5} t_m^{4/5} \,.$$

Corollary 5. *Assume* ***UGE*** *1. Then, for any measurable function $f : \mathsf{Z} \to \mathbb{R}^d$, $\|f\|_\infty \leq 1$, $p \geq 2$, $n \geq t_{\mathrm{mix}}$, and initial probability ξ on $(\mathsf{Z}, \mathcal{Z})$, it holds*

$$\mathbb{E}_\xi^{1/p}[\|\sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] \leq \mathrm{C}_{\mathsf{Rm},1}\, p^{1/2} n^{1/2} \sigma_\pi(f) + \mathrm{D}_{\mathrm{Ros},1}\, p n^{2/5} t_{\mathrm{mix}}^{3/5} + (8/3) n^{3/5} t_{\mathrm{mix}}^{2/5} 2^{-(n/t_{\mathrm{mix}})^{1/5}} \,,$$

where

$$\mathrm{D}_{\mathrm{Ros},1} = \mathrm{C}_{\mathsf{Ros},1}^{(\mathrm{M})} + 12\sqrt{2}/\ln 4 \,. \tag{88}$$

Proof. With the triangle inequality and maximal exact coupling construction (84), we obtain

$$\mathbb{E}_\xi^{1/p}[\|\sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] \leq \mathbb{E}_\pi^{1/p}[\|\sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] + \{\tilde{\mathbb{E}}_{\xi,\pi}[\|\sum_{i=0}^{n-1}\big(f(Z_i) - f(Z_i')\big)\|^p]\}^{1/p} \,.$$

The first term is bounded with Theorem 7. Moreover, with (84) and $\|f\|_\infty \leq 1$, we get

$$\|\sum_{i=0}^{n-1}\big(f(Z_i) - f(Z_i')\big)\|^p \leq 2^p \Big(\sum_{i=0}^{n-1} \mathbf{1}_{\{Z_i \neq Z_i'\}}\Big)^p = 2^p \Big(\sum_{i=0}^{n-1} \mathbf{1}_{\{T > i\}}\Big)^p \leq 2^p \Big(\sum_{k=1}^{n-1} k \mathbf{1}_{\{T=k\}} + n\mathbf{1}_{\{T>n\}}\Big)^p \leq 2^p T^p \,.$$

We obtain combining the previous bounds that

$$\mathbb{E}_\xi^{1/p}[\|\sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] \leq \mathbb{E}_\pi^{1/p}[\|\sum_{i=0}^{n-1} f(Z_i) - \pi(f)\|^p] + 2\{\tilde{\mathbb{E}}_{\xi,\pi}\big[T^p\big]\}^{1/p} \,.$$

Assumption **UGE** 1 implies that $\Delta(\mathrm{Q}^k) \leq 4(1/4)^{k/t_{\mathrm{mix}}}$ for any $k \in \mathbb{N}$. Hence, setting $\rho = (1/4)^{1/t_{\mathrm{mix}}}$, we get

$$\tilde{\mathbb{E}}_{\xi,\pi}\big[T^p\big] = 1 + \sum_{k=2}^{\infty}\{k^p - (k-1)^p\}\tilde{\mathbb{P}}_{\xi,\pi}\big(T > k-1\big) \leq 1 + \sum_{k=2}^{\infty}\{k^p - (k-1)^p\}\Delta(\mathrm{Q}^{k-1}) \leq 1 + 4\rho^{-1}\big(1-\rho\big)\sum_{k=1}^{\infty} k^p \rho^k \,.$$

Now we use the upper bound, for $\rho \in (0,1)$,

$$\sum_{k=1}^{\infty} k^p \rho^k \leq \rho^{-1}\int_0^{+\infty} x^p \rho^x \,\mathrm{d}x \leq \rho^{-1}(\ln \rho^{-1})^{-p-1}\,\Gamma(p+1) \,.$$

Combining the bounds above and the elementary inequality $1 - \rho \leq \ln \rho^{-1}$, we obtain

$$\tilde{\mathbb{E}}_{\xi,\pi}\big[T^p\big] \leq 1 + 4\rho^{-2}\left(\ln \rho^{-1}\right)^{-p}\Gamma(p+1) \leq 1 + 64(t_{\mathrm{mix}}/\ln 4)^p \Gamma(p+1) \,.$$

To complete the proof, we use an upper bound $\Gamma(p+1) \leq (p+1)^{p+1/2}\mathrm{e}^{-p}$ due to Guo et al. [16, Theorem 2] and apply an elementary inequality $(p+1)^{1/2} \leq 2^{p/2}$.

Appendix F: Technical lemmas We start by a standard moment bounds for sub-Gaussian random variable, which is proven for completeness.

LEMMA 10. *Let X be an $\mathbb{R}^d$-valued random variable satisfying $\mathbb{P}(\|X\| \geq t) \leq 2\exp(-t^2/(2\sigma^2))$ for any $t \geq 0$ and some $\sigma^2 > 0$. Then, for any $p \geq 2$, it holds that $\mathbb{E}[\|X\|^p] \leq 2p^{p/2}\sigma^p$.*

Proof. Using Fubini's theorem and the change of variable formula,

$$\mathbb{E}[\|X\|^p] = \int_0^\infty pt^{p-1}\mathbb{P}(\|X\| \geq t)\,\mathrm{d}t = p2^{p/2}\sigma^p\mathbf{\Gamma}(p/2)\,,$$

where $\mathbf{\Gamma}$ is the Gamma function. It remains to apply the bound $\mathbf{\Gamma}(p/2) \leq (p/2)^{p/2-1}$, which holds for $p \geq 2$ due to Anderson and Qiu [2, Theorem 1.5].

Now we present the general version of Hoeffding inequality for martingale-difference sequences, taking values in Banach spaces. This result is due to Pinelis [31, Theorem 3.5]. Below we specify this inequality to the special case of sum of zero-mean independent random vectors.

LEMMA 11. *Let $X_1, \ldots, X_n \in \mathbb{R}^d$ be independent random vectors satisfying $\|X_i\| \leq \beta_i$ $\mathbb{P}$-a.s. and $\mathbb{E}[X_i] = 0$, $i \in \{1, \ldots, n\}$. Then, for any $t \geq 0$, it holds*

$$\mathbb{P}\left(\|\sum_{i=1}^n X_i\| \geq t\right) \leq 2\exp\left\{-\frac{t^2}{2\sum_{j=1}^n \beta_j^2}\right\}\,.$$

The result above can be generalized for bounded $\mathbb{R}^d$-valued functions of the Markov chains with kernel satisfying **UGE** 1.

LEMMA 12. *Assume **UGE** 1. Let $\{g_i\}_{i=1}^n$ be a family of measurable functions from Z to $\mathbb{R}^d$ such that $\|g\|_\infty = \max_{i\in\{1,\ldots,n\}} \|g_i\|_\infty < \infty$ and $\pi(g_i) = 0$ for any $i \in \{1, \ldots, n\}$. Then, for any initial probability ξ on $(\mathsf{Z}, \mathcal{Z})$, $n \in \mathbb{N}$, $t \geq 0$, it holds*

$$\mathbb{P}_\xi\left(\|\sum_{i=1}^n g_i(Z_i)\| \geq t\right) \leq 2\exp\left\{-\frac{t^2}{2u_n^2}\right\}, \text{ where } u_n = 8\|g\|_\infty\sqrt{n}\sqrt{t_{\mathrm{mix}}}\,. \tag{89}$$

Proof. The function $\varphi(z_1, \ldots, z_n) := \|\sum_{i=1}^n g_i(z_i)\|$ on Z^n satisfies the bounded differences property. Hence, since $(1/2)\sup_{z,z'\in\mathsf{Z}} \|\mathrm{Q}^{t_{\mathrm{mix}}}(z,\cdot) - \mathrm{Q}^{t_{\mathrm{mix}}}(z',\cdot)\|_{\mathsf{TV}} \leq 1/4$ by definition of t_{mix} under **UGE** 1, applying Paulin [30, Corollary 2.10], we get for $t \geq \mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$,

$$\mathbb{P}_\xi\left(\|\sum_{i=1}^n g_i(Z_i)\| \geq t\right) \leq \exp\left\{-\frac{2(t - \mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|])^2}{9n\|g\|_\infty^2 t_{\mathrm{mix}}}\right\}\,.$$

It remains to upper bound $\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$. Note that

$$\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|^2] = \sum_{i=1}^n \mathbb{E}_\xi[\|g_i(Z_i)\|^2] + 2\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-k} \mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]\,.$$

and, using **UGE** 1 and $\pi(g_{k+\ell}) = 0$, we obtain

$$\left|\mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]\right| = \left|\int_{\mathsf{Z}} g_k(z)^\top\left(\mathrm{Q}^\ell g_{k+\ell}(z) - \pi(g_{k+\ell})\right)\xi\mathrm{Q}^k(\mathrm{d}z)\right| \leq \|g\|_\infty^2\Delta(\mathrm{Q}^\ell)\,.$$

Together with (35), this implies

$$\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-k} |\mathbb{E}_\xi[g_k(Z_k)^\top g_{k+\ell}(Z_{k+\ell})]| \leq \sum_{k=1}^{n-1} \|g\|_\infty^2\Delta(\mathrm{Q}^\ell) \leq (4/3)|g|_\infty^2 t_{\mathrm{mix}} n\,.$$

Combining the bounds above, we upper bound $\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|]$ as

$$\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|] \leq \left\{\mathbb{E}_\xi[\|\sum_{i=1}^n g_i(Z_i)\|^2]\right\}^{1/2} \leq 2\sqrt{n}\|g\|_\infty\sqrt{t_{\mathrm{mix}}} =: v_n\,.$$

Plugging this result in (89), we obtain that

$$\mathbb{P}_{\xi}\left(\|\sum_{i=1}^{n} g_i(Z_i)\| \geq t\right) \leq \begin{cases} 1, & t < v_n, \\ \exp\left\{-\frac{2(t-v_n)^2}{3v_n^2}\right\}, & t \geq v_n\,. \end{cases} \tag{90}$$

Now it is easy to see that right-hand side of (90) is upper bounded by $2\exp\{-t^2/(8v_n^2)\}$ for any $t \geq 0$, and the statement follows.

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