GLOBAL CONVERGENCE OF FOUR-LAYER MATRIX FACTORIZATION UNDER RANDOM INITIALIZATION

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ABSTRACT

Gradient descent dynamics on the deep matrix factorization problem is extensively studied as a simplified theoretical model for deep neural networks. Although the convergence theory for two-layer matrix factorization is well-established, no global convergence guarantee for general deep matrix factorization under random initialization has been established to date. To bridge this gap, we provide a polynomial-time global convergence guarantee for randomly initialized gradient descent on four-layer matrix factorization, given certain conditions on the target matrix and a standard balanced regularization term. Our analysis employs new techniques to show saddle-avoidance properties of gradient decent dynamics, and extends previous theories to characterize the eigenvalue change of layer weights.

1 Introduction

This paper investigates matrix factorization, a fundamental problem in non-convex optimization. This topic, in its canonical form, seeks to optimize the following objective:

$$\mathcal{L}(W_1, \dots, W_N) := \frac{1}{2} \|W_N \cdots W_1 - \Sigma\|_F^2 + \mathcal{L}_{reg}(W_1, \dots, W_N)$$
 (1)

where $W_j \in \mathbb{F}^{d \times d}$ denotes the j^{th} layer weight matrix, $\Sigma \in \mathbb{F}^{d \times d}$ denotes the target matrix and \mathcal{L}_{reg} is a (optional) regularizer. Here $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ as we consider both real and complex matrices in this paper. Following a long line of works (Arora et al., 2019a; Jiang et al., 2023; Ye & Du, 2021; Chou et al., 2024), we aim to understand the dynamics of gradient descent (GD) on this problem:

$$j = 1, ..., N : W_j(t+1) = W_j(t) - \eta \nabla_{W_j} \mathcal{L}(W_1(t), ..., W_N(t)),$$
 (2)

where $\eta \in \mathbb{R}^+$ is the learning rate.

While global convergence guarantee for the case of two-layer matrix factorization (N=2) is well studied (Du et al., 2018; Ye & Du, 2021; Jiang et al., 2023), the deep matrix factorization problem, i.e., the N>2 case is less explored. While the model representation power is independent of depth N, the deep matrix factorization problem is naturally motivated by the goal of understanding benefits of depth in deep learning (see, e.g., Arora et al. (2019b)). A long line of previous works (Hardt & Ma, 2016; Arora et al., 2019b;a; Wang & Jacot, 2023) studies this regime as it directly captures Deep Linear Networks (DLN), the simplest type of deep neural networks. However, a general global convergence guarantee is still missing. Therefore, the following open research question can be naturally asked:

Can we prove global convergence of GD for matrix factorization problem (1) with N > 2 layers?

In this paper, we provide a positive answer to the question above. Specifically, we consider 4-layer matrix factorization (N=4) with the standard balancing regularization term (see Park et al. (2017); Ge et al. (2017); Zheng & Lafferty (2016)) as

$$\mathcal{L}(W_1, W_2, W_3, W_4) := \frac{1}{2} \|W_4 W_3 W_2 W_1 - \Sigma\|_F^2 + \frac{1}{4} a \left(\sum_{j=1}^3 \|W_j W_j^H - W_{j+1}^H W_{j+1}\|_F^2 \right),$$

where W_j^H denotes the Hermitian transpose of W_j and $a \in \mathbb{R}^+$ is a hyperparameter. We consider both real $(\mathbb{F} = \mathbb{R})$ and complex $(\mathbb{F} = \mathbb{C})$ setting with random Gaussian initialization and prove global convergence of gradient descent. Our main result can be summarized as follows:

Theorem 1 (Main theorem, informal). For four-layer matrix factorization under gradient descent, random Gaussian initialization with scaling factor $\epsilon \leq \sigma_1^{1/4}(\Sigma)/poly(c_1,c_2,d)$, regularization factor $a \geq \sigma_1(\Sigma) \cdot poly\left(c_1,c_2,d,\ln\left(\sigma_1^{1/4}(\Sigma)/\epsilon\right)\right)$, then for target matrix Σ with identical singular values, there exists learning rate $\eta = O\left(1/\left[\sigma_1^{3/2}(\Sigma) \cdot poly\left(a/\sigma_1(\Sigma),c_1,c_2,d,\sigma_1^{1/4}(\Sigma)/\epsilon\right)\right]\right)$ and convergence time $T(\delta,\eta) = \eta^{-1}\sigma_1^{-3/2}(\Sigma) \cdot poly\left(c_1,c_2,d,\sigma_1^{1/4}(\Sigma)/\epsilon,\ln\left(\frac{d\sigma_1^2(\Sigma)}{\delta}\right)\right)$, such that for any $\delta > 0$, (1) with high probability over the complex initialization (2) with probability close to $\frac{1}{2}$ over the real initialization, when $t > T(\delta)$, $\mathcal{L}(t) < \delta$.

The formal version of Theorem 1 is stated in Theorem 48 in Appendix. **Remark 1.** A natural question is why the convergence guarantee in the real case holds only with probability close to $\frac{1}{2}$, but not 1. For the other $\frac{1}{2}$ probability, Theorem 2 presents a special case considering gradient flow under the strict balance condition (which can be viewed as the limit as $a \to +\infty$), showing that the optimization process does not converge to a global minimum in finite time (and hence converges to a saddle point).

Main contributions. Our major contributions can summarized as follows:

- We prove global convergence of GD for 4-layer matrix factorization under random Gaussian initialization. To the best of our knowledge, this is the first global convergence result for general deep linear networks under random initialization beyond the NTK regime in Du & Hu (2019). This result might provide new insights towards understanding the training dynamics of general deep neural networks.
- We construct a novel three-stage convergence analysis of gradient descent dynamics, consisting of an alignment stage, a saddle-avoidance stage, and a local convergence stage. We also develop new techniques to show GD dynamics avoids saddle points and to characterize layer matrix eigenvalue changes, which we believe are of independent interest for deep linear networks analysis.

Challenges and techniques. Our analysis employs the following key techniques:

- Initialization analysis. To guarantee that gradient descent makes progress, it is necessary to establish a monotonically increasing lower bound for the singular values of the weight matrices. This, in turn, requires analyzing the smallest singular value of a newly introduced term (namely $W+WW^H$, where $W=W_4W_3W_2W_1$), at initialization. This analysis utilizes tools from random matrix theory, particularly the concept of the Circular Ensembles. The detailed proof is given in Appendix B.
- Regularity condition of each layer. To bridge the initialization with the subsequent training
 dynamics, we need to ensure that key matrix properties evolve in a controlled manner even
 during the rapid changes in the alignment stage. We prove that despite significant updates,
 the weight matrices retain certain spectral properties from their initial state. A delicate
 analysis of the smooth evolution of the extreme singular values and the limiting behavior
 of the Hermitian term after the regularization term converges is provided in Section 5.2.1
 and 5.2.2.
- Saddle avoidance. To avoid convergence to a saddle point, it is essential to prevent the smallest singular values of the weight matrices from decaying to zero, as such decay would cause the gradient norm to vanish. To this end, we construct a hermitian term providing lower-bounds for these singular values, along with a skew-hermitian error. During the optimization, the skew-hermitian error is approximately non-increasing, which in turn ensures that the minimum singular value of the hermitian term is non-decreasing. This mechanism provides a persistent lower bound, thereby effectively avoiding saddle points.
- Bound of eigenvalue change. Finally, to translate the continuous-time intuition into rigorous guarantees for the discrete gradient descent algorithm, we develop new perturbation bounds for eigenvalues. In continuous time, the time derivatives of eigenvalues are directly characterized by the derivatives of the matrix. In discrete time, however, eigenvalue changes depend on the spectral gap in general, requiring a fine-grained, problem-specific

analysis. Similar challenge are noted in Lemma 3.2 of Ye & Du (2021). We address this issue in Lemma 28 and 29 in Appendix C.4.

These techniques form a cohesive proof strategy: the initialization analysis provides a favorable starting point; the regularity analysis ensures controlled dynamics throughout training; the saddle avoidance mechanism guarantees persistent progress; and the discrete-time perturbation bounds rigorously translate these insights into a full global convergence proof.

2 Related works

For two-layer matrix factorization, the global convergence of symmetric case has been established under various settings (Jain et al., 2017; Li et al., 2019; Chen et al., 2019). For asymmetric matrix factorization case with objective $\mathcal{L} = \frac{1}{2} \|UV^{\top} - \Sigma\|_F^2$, the following homogeneity issue occurs: the prediction result remains the same if one layer is multiplied by a positive constant while the other is divided by the same, introducing significant challenges in convergence analyzing (Lee et al. (2016), Proposition 4.11). Tu et al. (2016) and Ge et al. (2017) tackles this problem by manually adding a regularization term on the objective function. Du et al. (2018) discovers that gradient descent automatically balances the magnitudes of layers under small initialization, providing analysis of global convergence with polynomial time under decayed learning rate, while removing the regularization term. Ye & Du (2021) extends the convergence analysis to constant learning rate.

Kawaguchi (2016) analyzes landscape for general DLN, showing there exists saddle points with no negative eigenvalues of Hessian for depth over three. Bartlett et al. (2018) analyzes the dynamic under identity initialization, proving polynomial convergence with target matrix near initialization or symmetric positive definite, but such initialization fails to converge when target matrix is symmetric and has a negative eigenvalue. Arora et al. (2019a) provides global convergence proof under specific deep linear neural network structures and initialization scheme, requiring the initial loss to be smaller than the loss of any rank-deficient solution. Ji & Telgarsky (2019) conducted the proof of convergence on general deep neural networks with similar requirements on the initial loss. Arora et al. (2019b) simplifies the training dynamics of deep linear neural network into the dynamic of singular values and singular vectors of product matrix under balanced initialization, providing theoretical illustration of local convergence when singular vectors are stationary. Du & Hu (2019) proves global convergence for wide linear networks under the neural tangent kernel (NTK) regime. More recent works focus on GD dynamics under (approximately) balanced initialization schemes (Min et al., 2023) or the 2-layer case (Min et al., 2021; Xiong et al., 2023; Tarmoun et al., 2021). Chizat et al. (2024) studies the infinite-width limit of DLN in the mean field regime. However, none of these results imply a global convergence guarantee for general DLN with N>2 under random initialization.

3 PRELIMINARIES

Notations. Denote the complex conjugate of M as \bar{M} and adjoint of M as M^H , \mathbb{N} as the set of non-negative integers, and \mathbb{N}^* as the set of positive integers. For $k_1 < k_2 \in \mathbb{N}$, $\prod_{j=k_2}^{k_1} M_j = M_{k_2} M_{k_2-1} \cdots M_{k_1}$. $x \sim \mathcal{N}(0,1)_{\mathbb{C}}$ means that the real and imaginary parts are independently sampled from Gaussian distribution with variance $\frac{1}{2}$: $\Re x$, $\Im x \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0,1/2)$. $Q \sim U(d,\mathbb{C})$ or $O(d,\mathbb{R})$ means Q is drawn from the unique uniform distribution (Haar measure) on the unitary or orthogonal group, implying its distribution is unitarily/orthogonally invariant. Consider general N-layer matrix factorization, for simplicity we define the following notations:

$$W_{\prod_{L},j} := \prod_{k=N}^{j} W_{k}, W_{\prod_{R},j} := \prod_{k=j}^{1} W_{k}, W := \prod_{k=N}^{1} W_{k} = W_{\prod_{L},1} = W_{\prod_{R},N}, \tag{3}$$

W is referred to as *product matrix*. The loss is written by $\mathcal{L}(W_1, \cdots, W_N) = \mathcal{L}_{ori} + \mathcal{L}_{reg}$, where $\mathcal{L}_{ori} = \frac{1}{2} \|\Sigma - W\|_F^2$, $\mathcal{L}_{reg} = \frac{1}{4} a \left(\sum_{j=1}^{N-1} \|\Delta_{j,j+1}\|_F^2 \right)$.

Algorithmic setup. For the real case $(W_j \in \mathbb{R}^{d \times d})$, GD dynamics is canonical and described by equation 2. Under complex field $(W_j \in \mathbb{C}^{d \times d})$, for simplicity and coherence we define $\nabla_M = \frac{\partial}{\partial \Re M} + i \frac{\partial}{\partial \Im M}$, which is two times of Wirtinger derivative with \overline{M} : $\frac{\partial}{\partial \overline{M}} = \frac{1}{2} \left(\frac{\partial}{\partial \Re M} + i \frac{\partial}{\partial \Im M} \right)$. By following the updating rule of complex neural networks (see Guberman (2016)), the gradient can be uniformly represented by

$$\nabla_{W_{j}} \mathcal{L} = \nabla_{W_{j}} \mathcal{L}_{ori} + \nabla_{W_{j}} \mathcal{L}_{reg}$$

$$\nabla_{W_{j}} \mathcal{L}_{ori} = -W_{\prod_{L}, j+1}^{H} (\Sigma - W) W_{\prod_{R}, j-1}^{H}, \nabla_{W_{j}} \mathcal{L}_{reg} = -aW_{j} \Delta_{j-1, j} + a\Delta_{j, j+1} W_{j},$$
(4)

Under gradient flow, $\frac{\mathrm{d}W_j}{\mathrm{d}t} = -\nabla_{W_j}\mathcal{L}$; under gradient descent, $W_j(t+1) = W_j(t) - \eta\nabla_{W_j}\mathcal{L}(t)$.

Reduction to diagonal target. Following the simplification process of Section 2.1 in Ye & Du (2021), suppose the singular value decomposition of Σ is $\Sigma = U_\Sigma \Sigma' V_\Sigma^H$, by applying the following transformation $W_1 \leftarrow W_1 V_\Sigma$ and $W_N \leftarrow U_\Sigma^H W_N$, the dynamics remain the same form, while the distributions of W_j under our initialization schemes remain the same. Hence without loss of generality, we assume the target matrix is diagonal with real and non-negative entries throughout our analysis. Detailed analysis is presented in Appendix A.

For some of the results, we further require target matrix to be an identity matrix scaled by a positive constant $\Sigma = \sigma_1(\Sigma)I$, which is equivalent to requiring the singular values of target matrix are identical.

Balancedness. Following a long line of works (Arora et al., 2019a;b; Du et al., 2018), we define the balance error between layer j and j + 1 as

$$\Delta_{j,j+1} := \begin{cases} W_j W_j^H - W_{j+1}^H W_{j+1} &, j \in [1, N-1] \cap \mathbb{N}^* \\ O^{d \times d} &, j \in \{0, N\} \end{cases}$$
 (5)

As discussed in Definition 1 of Arora et al. (2019a), the weights are approximately balanced (namely $\|\Delta_{j,j+1}\|_F$ are small) throughout the iterations of gradient descent under approximate balancedness at initialization and small learning rate. Notice that approximate balancedness holds for small initialization near origin (small variance for Gaussian initialization).

Specifically, under gradient flow the balanced condition (defined as $\|\Delta_{j,j+1}\|_F \equiv 0$ or equivalently $\Delta_{j,j+1} \equiv O, \forall j \in [1,N-1] \cap \mathbb{N}^*$) holds strictly at arbitrary time under balanced initialization, which is defined as $\Delta_{j,j+1}(t=0) \equiv O, \forall j \in [1,N-1] \cap \mathbb{N}^*$.

Remark 2. As previously discussed, balance condition holds approximately under small initialization, so such regularization's affect on the training process is relatively weak, especially when weight matrices grow larger and be away from origin.

4 Training Dynamics under Balanced Gaussian Initialization

To exhibit the convergence dynamics clearly, we present the global convergence under the simplified scenario of balanced Gaussian initialization (formally defined in Section 4.1) and gradient flow. Notice that the adjacent matrices remain balanced due to the non-increasing property of regularization term (Lemma 25).

Theorem 2. (Informal) Global convergence bound under balanced Gaussian initialization, gradient flow. For four-layer matrix factorization under gradient flow, balanced Gaussian initialization with scaling factor $\epsilon \leq \sigma_1^{1/4}(\Sigma)/poly(c_1, c_2, d)$, then for target matrix with identical singular values,

1. For $\mathbb{F} = \mathbb{R}$, with probability at least $\frac{1}{2}$ the loss does not converge to zero.

2. For $\mathbb{F} = \mathbb{C}$ with high probability and for $\mathbb{F} = \mathbb{R}$ with probability close to $\frac{1}{2}$, there exists $T(\delta) = \sigma_1^{-3/2}(\Sigma) \cdot poly\left(c_2, d, \sigma_1^{1/4}(\Sigma)/\epsilon, \ln\left(\frac{d\sigma_1^2(\Sigma)}{\delta}\right)\right)$, such that for any $\delta > 0$, when $t > T(\delta)$, $\mathcal{L}(t) < \delta$.

The formal version is stated in Theorem 36 in the Appendix.

4.1 BALANCED GAUSSIAN INITIALIZATION

Generally, random Gaussian initialization does not satisfy strict balancedness. To adapt the random Gaussian initialization to ensure balanced condition, we introduce a *balanced Gaussian initialization* scheme for the analysis below. The procedure is defined as follows:

- (1) Sample G with entries $G_{ij} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)_{\mathbb{F}}, Q_{k,k+1;k\in[0,N]\cap\mathbb{N}} \overset{\text{i.i.d.}}{\sim} Haar$ on $U(d,\mathbb{C})$ for $\mathbb{F} = \mathbb{C}$ (or $O(d,\mathbb{R})$ for $\mathbb{F} = \mathbb{R}$). $s_{j,j\in[1,N]\cap\mathbb{N}^*} \in \mathbb{F}$ are arbitrary constants with modulus/absolute value 1.
- (2) For scaling factor $\epsilon \in \mathbb{R}^+$, which is a small positive constant, set the weight matrices by:

$$W_{j} = \begin{cases} s_{j} \in Q_{j,j+1} G Q_{j-1,j}^{H} & ,2 \nmid j \\ s_{j} \in Q_{j,j+1} G^{H} Q_{j-1,j}^{H} & ,2 \mid j \end{cases}$$
 (6)

Intuitively, $Q_{k,k+1;k\in[0,N]\cap\mathbb{N}}$ are i.i.d. uniformly distributed unitary/orthogonal matrices. By Corollary 15 in the Appendix, each matrix is a ϵ -scaled Gaussian random matrix ensemble (but not independent of the others), while satisfying balanced condition $\Delta_{j,j+1}(0) = O, \forall j \in [1,N-1] \cap \mathbb{N}^*$.

Theorem 3. Under ϵ -scaled balanced Gaussian initialization with even number of depth $2 \mid N$, suppose W is $W = U \Sigma_w^N V^H$, where U, V are unitary/orthogonal matrices, Σ_w is positive semi-definite and diagonal, denote $s := \prod_{i=1}^N s_i$, then:

1. If $\mathbb{F} = \mathbb{C}$, with probability $1 - \delta$ such that

$$\|\Sigma_{w}\|_{op} \le c_{1}(\delta)\sqrt{d}\epsilon, \|(U-V)\Sigma_{w}\|_{F}|_{t=0} \le 2c_{1}(\delta)d\epsilon$$

$$\sigma_{min}((U+V)\Sigma_{w})|_{t=0} \ge c_{2}(\delta)^{-1}d^{-3/2}\epsilon.$$
(7)

2. If $\mathbb{F} = \mathbb{R}$, $\Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = 1) = \Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = -1) = \frac{1}{2}$. Under $\Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = -1)$, $\sigma_{min}((U+V)\Sigma_w)|_{t=0}$; under $\Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = 1)$, with probability $1 - \delta$ such that

$$\|\Sigma_{w}\|_{op} \le c_{1}(\delta)\sqrt{d}\epsilon, \|(U-V)\Sigma_{w}\|_{F}|_{t=0} \le 2c_{1}(\delta)d\epsilon$$

$$\sigma_{min}((U+V)\Sigma_{w})|_{t=0} \ge c_{2}(\delta)^{-1}d^{-3/2}\epsilon.$$
(8)

Proof is presented in Appendix B.3.

4.2 Non-increasing Skew-Hermitian Error

As presented in Lemma 24 in the Appendix, the product matrix can be factorized in to the form of $W(t) = U(t)\Sigma_w(t)^N V(t)^H$, where $\Sigma_w(t)$ is positive semi-definite and diagonal (consequently real-valued), U and V are unitary/orthogonal matrices, U, V and Σ_w are analytic. For simplicity, we denote $\sigma_{w,j}$ as the j^{th} diagonal entry of Σ_w , and u_j , v_j as the j^{th} column of U, V. Under this representation of product matrix, we obtain a non-increasing skew-hermitian/symmetric term:

Theorem 4. (Informal) Skew-hermitian error term is non-increasing.

Under balanced initialization with product matrix $W(t) = U(t)\Sigma_w(t)^N V(t)^H$, for depth $N \geq 2$, if $\mathbb{F} = \mathbb{R}$ or $2 \mid N$, and singular values of product matrix at initial W(0) are non-zero and distinct, then the following skew-hermitian error $\left\|\Sigma^{1/2}(U-V)\Sigma_w\right\|_F^2$ is non-increasing:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \Sigma^{1/2} (U - V) \Sigma_w \right\|_F^2 \le 0. \tag{9}$$

Proof sketch. Proof of the Theorem 4 involves technical and lengthy calculations. The two cases of $\mathbb{F} = \mathbb{R}$ and under $2 \mid N$ are provided separately in Theorem 32 and Theorem 33. For Theorem 32, the idea is to decompose the derivative of this term into the derivative of $\sigma_{w,j}$ and u_j, v_j , which have been characterized by Theorem 3 and Lemma 2 in Arora et al. (2019b) respectively. This method is hard to generalize into unbalanced setting. For Theorem 33, this term is directly derived from

 derivative of $W_N W_N^H$, $W_1^H W_1$ and W. This approach is straight forward and can be extended to unbalanced initialization, but encounters difficulty under odd depth $2 \nmid N$.

Remark 3. This result is under the reduction of target matrix. For general target matrix, suppose its SVD is $\Sigma = U_{\Sigma} \Sigma' V_{\Sigma}^H$, then Theorem 4 becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| {\Sigma'}^{1/2} (U_{\Sigma}^H U - V_{\Sigma}^H V) \Sigma_w \right\|_F^2 \le 0. \tag{10}$$

Explanation of the result. This theorem provides an intrinsic invariant (under initialization close to origin, this term is already small at initial) of the system. Though the result is accurately derived under strictly balanced initialization and gradient flow, one may expect similar property to hold under small initialization and gradient descent.

Moreover, this theorem characterizes when U and V become aligned. The product matrix can be expressed as $W = \sum_{i=1}^d \sigma_{w,j}^N u_j v_j^H$, while the error can be rewritten as $\sum_{j=1}^d \sigma_{w,j}^2 \left\| \sum^{1/2} (u_j - v_j) \right\|_F^2$. Each term $\sigma_{w,j}^N u_j v_j^H$ of the product matrix can be interpreted as a "feature" of the linear neural network, containing one "value" $\sigma_{w,j}^N$ and two "directions" u_j, v_j . When the loss converges, each feature converges to $\sigma_j u_{\Sigma,j} u_{\Sigma,j}^H$, where $\Sigma = \sum_{j=1}^d \sigma_j u_{\Sigma,j} u_{\Sigma,j}^H$ is a SVD of Σ . This shows that under initialization near origin, once a "value" of the j^{th} feature increases to a relatively large value (comparing to initialization), the directions of this feature automatically align with each other (i.e. $\langle u_j, v_j \rangle \approx 1$). Followed by Theoretical illustration part of Arora et al. (2019b), Section 3, generally the alignment of U, V leads to convergence.

As shown in the proof sketch, the analysis for odd N encounters difficulty when generalized to the unbalanced case, thus this intrinsic invariant becomes considerably more challenging to characterize. This is why we have developed the convergence proof for the four-layer case rather than the three-layer architecture.

4.3 Non-Decreasing Hermitian Main Term

This section shows the dynamics of the minimum singular value of hermitian main term $(U+V)\Sigma_w$.

The motivation of studying this specific term is that it provides a bound for $\sigma_k(\Sigma_w)$, $k \in [1, N-1] \cap \mathbb{N}^*$, especially a tight bound for $\sigma_{min}(\Sigma_w)$ (refer to Lemma 20):

$$\frac{1}{2}\sigma_{k}\left((U+V)\Sigma_{w}\right) \leq \sigma_{k}(\Sigma_{w}) \leq \frac{\sqrt{2}}{2}\sqrt{\sigma_{k}^{2}\left((U+V)\Sigma_{w}\right) + \left\|(U-V)\Sigma_{w}\right\|_{op}^{2}} \\
\frac{1}{2}\sigma_{min}\left((U+V)\Sigma_{w}\right) \leq \sigma_{min}(\Sigma_{w}) \leq \frac{1}{2}\sqrt{\sigma_{min}^{2}\left((U+V)\Sigma_{w}\right) + \left\|(U-V)\Sigma_{w}\right\|_{op}^{2}}.$$
(11)

Notice that the extra term in the upper bound is bounded by the skew-hermitian error term discussed in the previous section.

Although the evolution of $\sigma_k((U+V)\Sigma_w)$ is generally difficult to characterize, we find that in the special case of $\Sigma = \sigma_1(\Sigma)I$ and N=4, it exhibits a monotonically increasing pattern before local convergence:

Theorem 5. Dynamics of minimum singular value of hermitian term.

Under balanced initialization with product matrix $W(t) = U(t)\Sigma_w(t)^N V(t)^H$, for target matrix with identical singular values (reduces to $\Sigma = \sigma_1(\Sigma)I$) and depth N=4, the time derivative of the k^{th} singular value of the hermitian term $x_k := \frac{1}{2}\sigma_k((U+V)\Sigma_w)$ is bounded by:

$$\left(2\sigma_{1}(\Sigma)-x_{k}^{4}-\frac{1}{2}\|\Sigma_{w}\|_{op}^{2}\|((U-V)\Sigma_{w})|_{t=0}\|_{F}^{2}\right)x_{k}^{4}-\frac{1}{16}x_{k}^{2}\|\Sigma_{w}\|_{op}^{2}\|((U-V)\Sigma_{w})|_{t=0}\|_{F}^{4}$$

$$\leq \frac{\mathrm{d}}{\mathrm{d}t}x_{k}^{2}\leq \sigma_{1}(\Sigma)\left(2\|\Sigma_{w}\|_{op}^{2}+\|((U-V)\Sigma_{w})|_{t=0}\|_{F}^{2}\right)x_{k}^{2}.$$
(12)

Detailed proof is presented in D.2.

This theorem implies that under small initialization, if all singular values $\sigma_k((U+V)\Sigma_w)$ are initially non-zero, they increase monotonically to relatively large values, leading to subsequent local convergence. However, if any singular value is initialized to zero (which occurs with probability at least 1/2 for $\mathbb{F}=\mathbb{R}$, as shown in Theorem 3), it remains zero throughout the optimization (see Corollary 35), thereby explaining the 1/2 convergence probability in Theorem 2. Numerical simulations under the identity target setting are provided in Figure 1, with additional results and discussions for non-identity targets shown in Figure 2.

5 CONVERGENCE UNDER RANDOM GAUSSIAN INITIALIZATION

This section presents the proof sketch for Theorem 1, extending our analytical framework in the previous section to accommodate random Gaussian initialization.

For random Gaussian Initialization with balance regularization term, the balanced condition holds approximately. Following the methodology in balanced initialization scheme, Section 4, we then characterize the skew-hermitian error term and hermitian main term by $\|W_1 - W_2^{-1}W_3^HW_4^H\|_F^2$ and $\lambda_{min}\left(\left(W_1 + W_2^{-1}W_3^HW_4^H\right)^H\left(W_1 + W_2^{-1}W_3^HW_4^H\right)\right)$ respectively.

5.1 RANDOM GAUSSIAN INITIALIZATION

We consider the canonical setting of random Gaussian initialization near origin:

$$(W_{1,2,\cdots,N})_{ij} \stackrel{\text{i.i.d.}}{\sim} \epsilon \cdot \mathcal{N}(0,1)_{\mathbb{F}}.$$
 (13)

Specifically, we apply Gaussian distribution to generate $W_{1,2,\cdots,N} \in \mathbb{F}^{d\times d}$, $F=\mathbb{R}$ or \mathbb{C} elementwisely and independently. Then the initialization is scaled by a small positive constant $\epsilon \in \mathbb{R}^+$. The scale of ϵ is determined in the main convergence Theorem 1.

Theorem 6. For ϵ -scaled random Gaussian initialization on $W_{k,k=[1,N]\cap\mathbb{N}^*}$ over $\mathbb{F}=\mathbb{R}$ or \mathbb{C} , $N\in\mathbb{N}^*$, the initial product matrix $W=\prod_{k=N}^1W_k$ satisfy the following properties:

1. If $\mathbb{F} = \mathbb{C}$, with probability at least $1 - \delta$,

$$\max_{j,k} \sigma_k(W_j) \le c_1(\delta, N) \sqrt{d}\epsilon, \ \min_{j,k} \sigma_k(W_j) \le \frac{\epsilon}{c_1(\delta, N) \sqrt{d}}$$

$$\sigma_{min} \left(W + \left(W W^H \right)^{1/2} \right) \ge c_2(\delta, N)^{-1} \cdot d^{-(N/2+1)} \epsilon^N.$$
(14)

2. If $\mathbb{F} = \mathbb{R}$, the determinants $\det(W) > 0$ and $\det(W) < 0$ occur each with probability 1/2. If $\det(W) < 0$, then $\sigma_{min} \left(W + \left(WW^{\top} \right)^{1/2} \right) = 0$; if $\det(W) > 0$, then with conditional probability at least $1 - \delta$,

$$\max_{j,k} \sigma_k(W_j) \le c_1(\delta, N) \sqrt{d}\epsilon, \ \min_{j,k} \sigma_k(W_j) \le \frac{\epsilon}{c_1(\delta, N) \sqrt{d}}$$
$$\sigma_{min} \left(W + \left(W W^\top \right)^{1/2} \right) \ge c_2(\delta, N)^{-1} \cdot d^{-(N/2+1)} \epsilon^N, \tag{15}$$

where $c_1(\delta, N)$, $c_2(\delta, N)$ are positive constants depending on δ and N.

Proof is provided in Appendix B.2.

In the convergence proof below, we consider the initialization where (14) and (15) holds. We divide the training dynamics into three stages consisting of an alignment stage $t \in [0, T_1]$, a saddle-avoidance stage $t \in [T_1, T_1 + T_2]$, and a local convergence stage $t \in [T_2, +\infty)$, to an-

alyze the convergence process clearly. Here $T_1 = O\left(\frac{1}{\eta\sigma_1^{3/2}(\Sigma) \cdot poly\left(c_1, c_2, d, \epsilon/\sigma_1^{1/4}(\Sigma)\right)}\right)$, $T_2 = O\left(\frac{1}{\eta\sigma_1^{3/2}(\Sigma)} \cdot poly\left(c_1, c_2, d, \sigma_1^{1/4}/\epsilon\right)\right)$, refer to Theorem 49 and 53 respectively.

5.2 STAGE 1: ALIGNMENT STAGE

During this stage, the weight matrices align with each other under the convergence of the regularization term, while the hermitian main term stays away from origin at the end of this stage.

5.2.1 Convergence of Regularization term:

The convergence rate of the regularization term is related to the smallest singular value of weight matrices:

Theorem 7. (Informal) Convergence rate of the regularization term.

For four-layer matrix factorization, suppose the maximum and minimum singular values of the weight matrices are bounded by M and δ respectively, then the regularization term decays by

$$\mathcal{L}_{reg}(t+1) \le \left(1 - \Omega\left(\eta a \delta^4 M^{-2}\right)\right) \cdot \mathcal{L}_{reg}(t) + O(\eta^2 a^2). \tag{16}$$

The formal version can be found in Theorem 30. A N-layer version of this Theorem, along with a generalized loss function under gradient flow is provided in Theorem 26. This shows the importance of bounding the extreme singular values of W_j , otherwise the linear convergence of the regularization term (along with the balancedness) might not be guaranteed.

Theorem 8. (Informal) Under a small learning rate, the change in the maximum and minimum singular values is approximately independent of the regularization term:

$$\max_{j,k} \sigma_k^2(W_j(t+1)) - \max_{j,k} \sigma_k^2(W_j(t)) \le 2\eta \max_{j,k} \sigma_k(W_j(t)) \max_j \|\nabla_{W_j} \mathcal{L}_{ori}(t)\|_{op} + O(\eta^2 a^2)$$

$$\min_{j,k} \sigma_k^2(W_j(t+1)) - \min_{j,k} \sigma_k^2(W_j(t)) \ge -2\eta \min_{j,k} \sigma_k(W_j(t)) \max_j \|\nabla_{W_j} \mathcal{L}_{ori}(t)\|_{op} + O(\eta^2 a^2).$$
(17)

Here a is the coefficient of the regularization term.

This Theorem ensures the smooth change of the extreme singular values over short time intervals. Although the regularization term can induce significant fluctuations in individual singular values due to its potentially large coefficient, the largest and smallest singular values remain stable. This theoretical conclusion is corroborated by numerical simulations, as shown in Figure 3. The complete formal statement can be found in Theorem 31 (and Theorem 27 for the continuous-time case) in the Appendix.

5.2.2 THE LIMIT BEHAVIOR OF THE HERMITIAN MAIN TERM

Typically, the dynamics of the smallest singular value of the hermitian main term $W_1 + W_2^{-1}W_3^HW_4^H$ is involved and does not obtain a non-trivial lower bound during this stage. However its limit behavior after the convergence of regularization term can be characterized.

To simplify the analysis, ignore the original square loss \mathcal{L}_{ori} and consider gradient flow. For $t \to +\infty$, regularization term is exactly zero and thus the adjacent matrices are strictly balanced. Moreover, the product matrix does not change through the optimization: $W(+\infty) = W(0)$. Then under this scenario, the limit behavior of the hermitian main term is $(W_1 + W_2^{-1}W_3^HW_4^H)|_{t\to +\infty} = W(0)$.

$$\left. \left(W_2^{-1} W_3^{-1} W_4^{-1} \right) \right|_{t \to +\infty} \left(W(0) + \left(W(0) W(0)^H \right)^{1/2} \right).$$

This explains the reason of studying $\sigma_{min} \left(W(0) + \left(W(0)W(0)^H \right)^{1/2} \right)$ in the initialization section. Detailed analysis considering error terms is presented in Corollary 52.

Remark 4. Note that $\sigma_{\min} \left(W_1 + W_2^{-1} W_3^H W_4^H \right)$ is not necessarily lower-bounded by the above expression minus some error terms during the alignment stage. Instead, it may exhibit oscillations or a transient decrease, achieving stability only upon convergence of the regularization term. This behavior is illustrated in Figure 4 in the Appendix.

5.3 STAGE 2: SADDLE AVOIDANCE STAGE

Intuitively, this section focuses on generalizing Theorem 4 and 5 into unbalanced case by bounding the error terms introduced by unbalanceness.

The main technical challenge is to bound the operator norm of the inverse of W_2 below infinity, since both the skew-hermitian term and hermitian main term are characterized by W_2^{-1} and hence need to be well-defined. Under small balance error (equivalently small regularization term) which is guaranteed by the previous stage, W_2^{-1} , which is rigorously proved in Lemma 56.

Lemma 9. Skew-hermitian error in saddle avoidance stage, gradient descent. For $t \in [T_1, T_1 + T_2]$,

 $\|W_1 - W_2^{-1} W_3^H W_4^H\|_F \le 3c_1 d\epsilon. \tag{18}$

Lemma 10. The minimum eigenvalue of Hermitian term. For $t = T_1 + T_2$,

$$\sigma_{min}\left(W_1 + W_2^{-1}W_3^H W_4^H\right)|_{t=T_1+T_2} \ge 2^{3/4}\sigma_1^{1/4}(\Sigma). \tag{19}$$

Proofs are presented in H.2 in the Appendix.

5.4 STAGE 3: LOCAL CONVERGENCE STAGE

Since both the balanced error and skew-Hermitian error remain small, the minimal singular values of the weight matrices, after growing to the scale of the target matrix's, are prevented from decaying. This guarantees the local convergence.

Theorem 11. (Informal) Local convergence. After the second stage ($t \ge T_1 + T_2$),

$$\mathcal{L}(t) \leq \mathcal{L}_{ori}(T_1 + T_2) \exp\left(-\eta \sigma_1^{3/2}(\Sigma)(t - T_1 - T_2)\right)$$

$$\sigma_{min}\left(W_1(t) + W_2(t)^{-1}W_3(t)^H W_4(t)^H\right) \geq 2^{3/4} \sigma_1^{1/4}(\Sigma)$$

$$\|W_1(t) - W_2(t)^{-1}W_3(t)^H W_4(t)^H\|_F \leq 3c_1 d\epsilon.$$
(20)

Proof is presented in H.3 in the Appendix.

6 CONCLUSIONS, LIMITATIONS AND FUTURE WORK

In this work, we establish a polynomial-time global convergence guarantee for gradient descent applied to four-layer matrix decomposition, under the setting of a target matrix with identical singular values and small random Gaussian initialization beyond the NTK regime. For complex random Gaussian initialization, global convergence is ensured with high probability, whereas for real random Gaussian initialization, it is guaranteed with a probability close to $\frac{1}{2}$.

The analysis developed in this work reveals intrinsic properties of the training dynamics, such as the effective behavior of the regularization term, the monotonically increasing lower bound for the minimum singular value, and the non-increasing nature of the skew-Hermitian error. These findings might provide deeper insight into the training process of Deep Linear Networks.

We anticipate that this work will stimulate further research on global convergence proofs under general random initialization for matrix factorization with arbitrary depth and arbitrary - possibly low-rank - target matrices.

The observed divergence in convergence behavior between real and complex initializations also reveals a subtle disparity, suggesting that complex initializations may circumvent certain saddle points that real initializations cannot. This insight might motivate more detailed analysis of the performance gap between complex and real neural networks.

REPRODUCIBILITY STATEMENT

All theoretical results stated in this paper are proved in full detail in the Appendix , from Section A to H, including the proofs of all main-text theorems as well as intermediate lemmas and derivations, so that a reader can verify each step independently. The numerical illustration in Appendix I, where we specify the hyper-parameters in that section. Because the experiments are straightforward, we have not released an implementation.

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A REDUCTION TO DIAGONAL (IDENTICAL) TARGET

For arbitrary ground truth $\Sigma \in \mathbb{F}^{d \times d}$, $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , suppose its singular value decomposition is $\Sigma = U_{\Sigma} \Sigma' V_{\Sigma}^H$ (replace \cdot^H by \cdot^{\top} for the real case, same for the rest of the analysis), we apply the following transformation:

$$\begin{cases} W'_1 &= W_1 V_{\Sigma} \\ W'_j &= W_j, \ j \in [2, N-1] \cap \mathbb{N}^* \\ W'_N &= U_{\Sigma}^H W_N \end{cases}$$
 (21)

Then the balance error can be rewritten as

$$\Delta_{j,j+1} = \begin{cases} W_j' W_j'^H - W_{j+1}'^H W_{j+1}' &, j \in [1, N-1] \cap \mathbb{N}^* \\ O^{d \times d} &, j \in \{0, N\} \end{cases}$$
(22)

A.1 TRAINING DYNAMICS

For gradient flow, the dynamics becomes

$$\frac{\mathrm{d}W'_{j}}{\mathrm{d}t} = \left(\prod_{k=j+1}^{N} W'_{k}^{H}\right) \left(\Sigma' - \prod_{k=N}^{1} W'_{k}\right) \left(\prod_{k=1}^{j-1} W'_{k}^{H}\right) + aW'_{j}\Delta_{j-1,j} - a\Delta_{j,j+1}W'_{j}. \tag{23}$$

For gradient descent,

$$W'_{j}(t+1) = W'_{j}(t) + \eta \left(\prod_{k=j+1}^{N} W'_{k}(t)^{H} \right) \left(\Sigma' - \prod_{k=N}^{1} W'_{k}(t) \right) \left(\prod_{k=1}^{j-1} W'_{k}(t)^{H} \right)$$

$$+ \eta a W'_{j}(t) \Delta_{j-1,j}(t) - \eta a \Delta_{j,j+1}(t) W'_{j}(t).$$
(24)

Both share the same form as the original one (by replacing Σ with Σ').

A.2 INITIALIZATION

However, the distributions of W_1 and W_N at initialization change correspondingly. To address this issue, we introduce the following definition:

Definition 1. Input-Output Unitary(Orthogonal)-Invariant initialization.

For a N-layer complex (real) matrix factorization $W = \prod_{j=N}^1 W_j$, an initialization is input-output unitary-invariant (in the complex case) or orthogonal-invariant (in the real case) if the distribution of W_N is left unitarily (or orthogonally) invariant and the distribution of W_1 is right unitarily (or orthogonally) invariant. That is, for all $U, V \in U(d, \mathbb{C})$ (or $O(d, \mathbb{R})$ in the real case),

$$W_N \stackrel{dist}{=} UW_N, W_1 \stackrel{dist}{=} W_1 V. \tag{25}$$

Remark 5. The distribution of $W_{j,j\in[1,N]\cap\mathbb{N}^*}$ does not change under transformation 21 if the initialization is Input-Output Unitary(Orthogonal)-Invariant.

Throughout this work, the initialization schemes discussed (including random Gaussian initialization and balanced Gaussian initialization) are Input-Output Unitary(Orthogonal)-Invariant. This is from the left and right invariance under multiplication of unitary/orthogonal matrices.

Thus without loss of generality, the target matrix can be reduced to positive semi-definite diagonal matrix. Under Input-Output Unitary(Orthogonal)-Invariant initialization discussed in Definition 1, the initialization on W_1 and W_N is not affected by this reduction.

Moreover, if all singular values of Σ are the same (to rephrase, a unitary/orthogonal matrix scaled by a constant), the convergence analysis can be reduced to $\Sigma' = \sigma_1(\Sigma)I$.

B INITIALIZATION

First and foremost, we introduce the concept of Circular ensembles (Dyson, 1962) along with some properties.

B.1 Lemmas for Gaussian random matrix ensemble and Haar measure on $U(d,\mathbb{C})$ and $O(d,\mathbb{R})$

In the following derivations, we denote $O(d, \mathbb{R})$ as the d-dimensional orthogonal group on real number, and $U(d, \mathbb{C})$ as the d-dimensional unitary group on complex number.

We list the classical conclusions in Linear Algebra without proof:

Lemma 12. The eigenvalues of Orthogonal/Unitary Matrices.

- 1. Unitary matrices. $\forall U \in U(d, \mathbb{C}), d \in \mathbb{N}^*$, the eigenvalues of U are $e^{i\theta_{1,2,\cdots,d}}$, where $\theta_i \in [0,2\pi)$.
- 2. Orthogonal matrices. $\forall O \in O(d, \mathbb{R}), d \in \mathbb{N}^*$, the eigenvalues of O are:

$$\begin{cases}
1, e^{\pm i\theta_{1,2,\dots,m}} &, d = 2m+1, \det(O) = 1 \\
-1, e^{\pm i\theta_{1,2,\dots,m}} &, d = 2m+1, \det(O) = -1 \\
e^{\pm i\theta_{1,2,\dots,m}} &, d = 2m, \det(O) = 1 \\
1, -1, e^{\pm i\theta_{1,2,\dots,m-1}} &, d = 2m, \det(O) = -1
\end{cases}$$
(26)

Following the conventions, we call the argument of the eigenvalues as eigenangles.

Definition 2. Circular ensembles. (refer to Dyson (1962), Forrester (2010))

The circular ensembles are measures on spaces of unitary(or orthogonal, when generalizing from complex number to real number) matrices.

- 1. Unitary circular ensemble. The distribution of the unitary circular ensemble (CUE) is the Haar measure on d-dimensional (complex) unitary group $U(d, \mathbb{C})$.
- 2. Circular real ensemble. The distribution of the circular real ensemble (CRE) is the Haar measure on d-dimensional real orthogonal group $O(d, \mathbb{R})$.

Lemma 13. 1-point correlation function of CUE(d) and CRE(d).

1. CUE. The 1-point correlation function of CUE(d) is

$$\rho_{(1),CUE}(\theta) = \frac{d}{2\pi}.\tag{27}$$

2. CRE, determinant 1. The 1-point correlation function of CRE(d) under determinant 1 is

$$\rho_{(1),CRE,det=1}(\theta) = \frac{1}{2\pi} \left(d - 1 + (-1)^d \frac{\sin(d-1)|\theta|}{\sin|\theta|} \right), \ \theta \in (-\pi, \pi].$$
 (28)

Remark 6. 1-point correlation function $\rho_{(1)}(\theta)$ can be interpreted as the density of eigenangles at θ (despite probably existed fixed eigenangles, e.g. $0, \pi$).

Proof. Part 1. CUE.

From (146) of Dyson (1962) and Forrester (2010), the joint probability density function of eigenangles is

$$p_{CUE}(\theta_{k,k \in [1,d] \cap \mathbb{N}^*}) \propto \prod_{1 \le k < j \le d} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 = \prod_{1 \le k < j \le d} \left| e^{i(\theta_j - \theta_k)} - 1 \right|^2.$$
 (29)

Notice that it is rotation invariant, that is $\forall \Delta \theta \in [0, 2\pi]$, $p_{CUE}(\theta_{k,k \in [1,d] \cap \mathbb{N}^*}) = p_{CUE}((\theta_k + \Delta \theta)_{k \in [1,d] \cap \mathbb{N}^*})$. Thus the 1-point correlation function (density of eigenangles at θ) is uniform, which is $\frac{d}{2\pi}$.

Part 2. CRE.

Below we define
$$x_i = \cos \theta_i$$
, then $\rho_{(1)}(\theta) = \sin \theta \cdot \rho_{(1)}(x)$, $p(x_{k,k \in [1,N] \cap \mathbb{N}^*}) = \left(\prod_{k=1}^N \frac{1}{\sqrt{1-x^2}}\right) p(\theta_{k,k \in [1,N] \cap \mathbb{N}^*})$.

By combining Proposition 5.1.1 and 5.1.2 in Forrester (2010) together, suppose with $p_k(x)$ a polynomial of degree k which is further more monic (i.e. the coefficient of x^k is unity), $\{p_k(x)\}_{k\in\mathbb{N}}$ is the orthogonal polynomials associated with the weight function $w_2(x)$,

$$\int_{-\infty}^{+\infty} p_j(x) p_k(x) w_2(x) dx =: \langle p_j, p_k \rangle_2 = \langle p_j, p_j \rangle_2 \delta_{j,k}.$$
 (30)

and the joint probability density function satisfies

$$p(x_{k,k\in[1,N]\cap\mathbb{N}^*}) \propto \prod_{1\leq k\leq j\leq N} (x_j - x_k)^2 \prod_{l=1}^N w_2(x).$$
 (31)

the 1-point correlation function is

$$\rho_{(1)}(x) = w_2(x) \sum_{\nu=0}^{N-1} \frac{p_{\nu}^2(x)}{\langle p_{\nu}, p_{\nu} \rangle_2}.$$
 (32)

Note that the restriction of monic can be ommitted since there is a normalization coefficient on the denominator.

2.1. CRE, determinant 1, d=2N. From (135) of Dyson (1962), Section 2.9 of Forrester (2010) and Girko (1985),

$$p_{CRE,even,det=1}(\theta_{k,k\in[1,N]\cap\mathbb{N}^*}) \propto \prod_{1\leq k < j\leq N} \left|\cos\theta_j - \cos\theta_k\right|^2, \ \theta_{k,k\in[1,N]\cap\mathbb{N}^*} \in [0,\pi]. \tag{33}$$

By the change of variables,

$$p_{CRE,even,det=1}(x_{k,k\in[1,N]\cap\mathbb{N}^*}) \propto \prod_{1\leq k < j\leq N} (x_j - x_k)^2 \prod_{l=1}^N \frac{1}{\sqrt{1 - x_l^2}}.$$
 (34)

Here $w_2(x) = \frac{1}{\sqrt{1-x^2}}$. From knowledge of orthogonal polynomials ((1.12.3), (4.1.7), Szegő (1939)), Chebyshev polynomials of the first kind $T_n(x) = \cos(n\arccos x)$ associates with $w_2(x) = \frac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_j(x) T_k(x) w_2(x) dx = \begin{cases} \pi, & j = k = 0\\ \frac{\pi}{2}, & j = k \ge 1\\ 0, & j \ne k \end{cases}$$
 (35)

By (32),

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1. $\mathbb{F} = \mathbb{C}$. $\Pr(\sigma_{min}(I+Q) \geq \pi \delta d^{-1}) \geq 1 - \delta$.

 $\rho_{(1),CRE,even,det=1}(x) = \frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{i=1}^{N-1} \cos^2 \nu \theta\right)$ (36) $= \frac{1}{2\pi \sin \theta} \left[2N - 1 + \frac{\sin(2N-1)\theta}{\sin \theta} \right].$

 $\rho_{(1),CRE,even,det=1}(\theta) = \frac{1}{2\pi} \left[d - 1 + \frac{\sin(d-1)\theta}{\sin\theta} \right], \ \theta \in [0,\pi].$ (37)

From symmetry, $\rho_{(1),CRE,odd,det=1}(-\theta) = \rho_{(1),CRE,odd,det=1}(\theta)$.

2.2. CRE, determinant 1, d = 2N + 1. From (137) of Dyson (1962), Section 2.9 of Forrester (2010) and Girko (1985),

 $p_{CRE,odd,det=1}(\theta_{k,k\in[1,N]\cap\mathbb{N}^*}) \propto \prod_{1\leq k < j \leq N} \left|\cos\theta_j - \cos\theta_k\right|^2 \prod_{l=1}^N (1-\cos\theta_l), \ \theta_{k,k\in[1,N]\cap\mathbb{N}^*} \in [0,\pi].$ (38)

By the change of variables,

$$p_{CRE,odd,det=1}(x_{k,k \in [1,N] \cap \mathbb{N}^*}) \propto \prod_{1 \le k \le j \le N} (x_j - x_k)^2 \prod_{l=1}^N \sqrt{\frac{1 - x_l}{1 + x_l}}.$$
 (39)

Here $w_2(x) = \sqrt{\frac{1-x}{1+x}}$. From knowledge of orthogonal polynomials ((1.12.3), (4.1.7), Szegő (1939)), Chebyshev polynomials of the fourth kind $W_n(x) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{\theta}{n})}$, $\theta = \arccos x$ associates with $w_2(x) = \sqrt{\frac{1-x}{1+x}}$:

$$\int_{-1}^{1} W_j(x) W_k(x) w_2(x) dx = \begin{cases} \pi, & j = k \ge 0 \\ 0, & j \ne k \end{cases}$$
 (40)

By (32),

 $\rho_{(1),CRE,odd,det=1}(x) = \sqrt{\frac{1-x}{1+x}} \cdot \left(\frac{1}{\pi} \sum_{i=0}^{N-1} \left(\frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^{2}\right)$ (41) $= \frac{1}{2\pi \sin{(\theta)}} \left[2N - \frac{\sin(2N\theta)}{\sin{\theta}} \right].$

 $\rho_{(1),CRE,odd,det=1}(\theta) = \frac{1}{2\pi} \left[d - 1 - \frac{\sin(d-1)\theta}{\sin\theta} \right], \ \theta \in [0,\pi].$ (42)

From symmetry, $\rho_{(1),CRE,odd,det=1}(-\theta) = \rho_{(1),CRE,odd,det=1}(\theta)$.

This completes the proof.

Theorem 14. For Q sampled from Haar measure on $U(d,\mathbb{C})$ (or $O(d,\mathbb{R})$ if $\mathbb{F}=\mathbb{R}$),

2. $\mathbb{F} = \mathbb{R}$. If $d \geq 2$, $\Pr\left(\sigma_{min}(I+Q) \geq \frac{\pi\delta}{2}(d-1)^{-1} | \det(Q) = 1\right) \geq 1 - \delta$.

Remark 7. For $\mathbb{F} = \mathbb{R}$, d = 1, the eigenvalue of Q is $\det(Q)$, and thus $\Pr(\sigma_{min}(I + Q) \geq 2 - \Delta | \det(Q) = 1) = 1$, $\forall \Delta \in (0, 2)$.

Remark 8. For $\mathbb{F} = \mathbb{R}$, $\Pr(\det(Q) = 1) = \Pr(\det(Q) = -1) = \frac{1}{2}$. If $\det(Q) = -1$, Q has an eigenvalue of -1, causing $\Pr(\sigma_{min}(I+Q)) = 0$.

Proof. Consider $\theta_k \in (-\pi, \pi]$,

$$\sigma_k(I+Q) = \sqrt{\lambda_k(2I+Q+Q^H)} = \sqrt{2 + e^{i\theta_k} + 1/e^{i\theta_k}} = 2\cos\left(\frac{\theta_k}{2}\right)$$

$$\sigma_{min}(I+Q) = \min_k \cos\left(\frac{\theta_k}{2}\right).$$
(43)

The second step is from the fact that $Q^H = Q^{-1}$ shares the same eigenvectors with Q, and corresponding eigenvalues are the reciprocal of the original eigenvalues.

Denote $N(\delta\theta)$ to be number of eigenvectors in $(-\pi, -\pi + \delta\theta] \cup [\pi - \delta\theta, \pi]$, $\delta\theta \in (0, \pi)$. From Markov inequality,

$$\Pr\left(\sigma_{min}(I+Q) \ge \delta\theta\right) \ge \Pr\left(\sigma_{min}(I+Q) \ge 2\sin\frac{\delta\theta}{2}\right)$$

$$= 1 - \Pr(N(\delta\theta) \ge 1)$$

$$\ge 1 - \mathbb{E}(N(\delta\theta)) = 1 - \int_{\theta \in (-\pi, -\pi + \delta\theta) \cup [\pi - \delta\theta, \pi]} \rho_{(1)}(\theta) d\theta.$$
(44)

By invoking Lemma 13,

1. For $\mathbb{F} = \mathbb{C}$,

$$\mathbb{E}(N(\delta\theta)) = \frac{d}{2\pi} \cdot 2\delta\theta. \tag{45}$$

By setting $\delta\theta = \pi\delta d^{-1}$, $\Pr\left(\sigma_{min}(I+Q) \geq \delta\theta\right) \geq 1-\delta$.

2. For
$$\mathbb{F} = \mathbb{R}$$
 under determinant 1, for $\theta' \in [0, \pi]$, $\rho_{(1)}(\pi - \theta') = \frac{1}{2\pi} \left(d - 1 + \frac{\sin(d-1)\theta'}{\sin \theta'} \right)$.

If d=1, $\rho_{(1)}(\theta)\equiv 0$ and thus $\mathbb{E}(N(\delta\theta))=0$. For $d\geq 2$:

From $\frac{\sin(d-1)\theta}{\sin\theta} \le d-1$,

$$\mathbb{E}(N(\delta\theta)) = 2\int_0^{\delta\theta} \rho_{(1)}(\pi - \theta') d\theta' \le 2\int_0^{\delta\theta} \frac{1}{2\pi} \cdot 2(d-1) d\theta' = \frac{2(d-1)}{\pi} \delta\theta. \tag{46}$$

By setting $\delta\theta=\frac{\pi\delta}{2}(d-1)^{-1}$, $\Pr\left(\sigma_{min}(I+Q)\geq\delta\theta|\det(Q)=1\right)\geq 1-\delta$.

This completes the proof.

B.2 RANDOM GAUSSIAN INITIALIZATION

In the following, we present the proof for Theorem 6.

Proof. The upper and lower bound for singular values of W_k follow by Corollary 2.3.5 and Theorem 2.7.5 of Tao immediately. The main challenge is the minimum singular value of $W + (WW^H)^{1/2}$.

At the beginning, we define a modification of Gaussian random matrix ensemble for simplification:

W is sampled from (complex or real) Gaussian random matrix ensemble, and if rank(W) is not full, sample W from Gaussian random matrix ensemble again until it is full rank.

Since the set of rank(W) not being full is zero measure, the distribution of W shares the same with the one before modification almost surely, and thus changing Gaussian random matrix ensemble to modified version *does not affect* the analysis below essentially.

This modification is for better expression on definition of left and right unitary (orthogonal) matrix of SVD. For full rank square matrix $W = U\Sigma V^H$, U and V are not unique, but VU^H is (even if the singular values are non-distinct, or changing the order of diagonal elements of Σ . This is due to the uniqueness of polar decomposition W = SQ under full rank, where $Q = UV^H$, $S = (WW^H)^{1/2}$.) and thus well-defined.

Without changing the result, we analysis the initialization scheme of modified Gaussian random matrix ensemble instead. Then W is full rank and thus polar decomposition is unique.

Generally, suppose the right polar decomposition of W is $W = \left(WW^H\right)^{1/2}Q$, then

$$W + (WW^H)^{1/2} = (WW^H)^{1/2} (I + Q).$$
(47)

If $\mathbb{F} = \mathbb{R}$, $\Pr(\det(W) > 0) = \Pr(\det(W) < 0) = \frac{1}{2}$ due to the symmetry of Gaussian random matrix ensemble. If $\det(W) = \det\left(\left(WW^H\right)^{1/2}\right) \det\left(Q\right) < 0$, $\det\left(Q\right) = -1$, then $\sigma_{min}(I + Q) = 0$ and further $\sigma_{min}\left(W + \left(WW^\top\right)^{1/2}\right) = 0$.

Consider both $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, $\det(W) > 0$ (which indicates $\det(Q) = 1$):

$$\sigma_{min}\left(W + \left(WW^{H}\right)^{1/2}\right) \geq \sigma_{min}\left(\left(WW^{H}\right)^{1/2}\right)\sigma_{min}\left(I + Q\right)$$

$$= \sigma_{min}(W)\sigma_{min}\left(I + Q\right)$$

$$\geq \left[\prod_{k=1}^{N}\sigma_{min}(W_{k})\right]\sigma_{min}\left(I + Q\right).$$
(48)

From Theorem 2.7.5 of Tao, by applying union bound, $\sigma_{min}(W_{k,k\in[1,N]\cap\mathbb{N}^*}) > c_1^{-1}(\delta,N)d^{-1/2}\epsilon$ with high probability $1-\delta/2$. Then $\left[\prod_{k=1}^N \sigma_{min}(W_k)\right] \geq \left(c_1^{-1}(\delta,N)d^{-1/2}\epsilon\right)^N$, and it remains to find lower bound for $\sigma_{min}(I+Q)$.

To apply results in Theorem 14, it is sufficient to show that Q follows Haar measure on $U(d, \mathbb{C})$ (or $O(d, \mathbb{R})$).

Due to the property of invariance under left and right multiplication of unitary (orthogonal) matrix for Gaussian random matrix ensemble (Section 2.6.2, (2.131), Tao), \forall fixed $Q_0 \in U(d,\mathbb{C})$ (or $O(d,\mathbb{R})$ if $\mathbb{F}=\mathbb{R}$), $W_1Q_0^H$ follows the same distribution as W_1 while still independent of $W_{k,k\in[2,N]\cap\mathbb{N}^*}$, resulting that WQ_0^H follows the same distribution as W. Since the right polar decomposition of WQ_0^H is $WQ_0^H = (WQ_0^HQ_0W^H)^{1/2}QQ_0^H = (WW^H)^{1/2}(QQ_0^H)$, we have

$$Q_0Q\stackrel{\text{dist}}{=} Q, \ \forall \text{ fixed } Q_0 \in U(d,\mathbb{C}) \text{ (or } O(d,\mathbb{R}) \text{ if } \mathbb{F} = \mathbb{R}).$$
 (49)

Likewise

$$QQ_0 \stackrel{\text{dist}}{=} Q, \ \forall \text{ fixed } Q_0 \in U(d, \mathbb{C}) \text{ (or } O(d, \mathbb{R}) \text{ if } \mathbb{F} = \mathbb{R}).$$
 (50)

From the fact that the only measure invariant under left (or right) multiplication of arbitrary element of a compact lie group is Haar measure, Q follows Haar measure on $U(d,\mathbb{C})$ (or $O(d,\mathbb{R})$), and the proof is completed.

By Theorem 6, for depth N=4, if $\mathbb{F}=\mathbb{C}$ then with high probability $1-\delta$ (if $\mathbb{F}=\mathbb{R}$ then with probability 1/2, $\sigma_{min}\left(W(0)+\left(W(0)W(0)^{\top}\right)^{1/2}\right)=0$, and with probability $(1-\delta)/2$ the following holds), $\exists c_1(\delta), c_2(\delta)$ such that

$$\max_{j,k} \sigma_k(W_j(0)) \le c_1(\delta)\sqrt{d}\epsilon$$

$$\min_{j,k} \sigma_k(W_j(0)) \le \frac{1}{c_1(\delta)\sqrt{d}} \cdot \epsilon$$

$$\sigma_{min}\left(W(0) + \left(W(0)W(0)^H\right)^{1/2}\right) \ge \frac{1}{c_2(\delta)d^3} \cdot \epsilon^4.$$
(51)

Consequently,

$$e_{\Delta}(0) := \sqrt{\sum_{i=1}^{3} \|\Delta_{i,i+1}\|_F^2} \bigg|_{t=0} \le \sqrt{3} \cdot 2\sqrt{d} \cdot \max_{j,k} \sigma_k^2(W_j(0)) = 2\sqrt{3}c_1^2(\delta)d^{3/2}\epsilon^2.$$
 (52)

B.3 BALANCED GAUSSIAN INITIALIZATION

This section analyzes the balanced Gaussian initialization scheme.

Corollary 15. Under balanced Gaussian initialization scheme (6), each matrix $W_{k,k\in[1,N]\cap\mathbb{N}^*}$ is a Gaussian random matrix ensemble scaled by ϵ .

Proof. This is immediately from the property of invariance under left and right multiplication of unitary (orthogonal) matrix for Gaussian random matrix ensemble (Section 2.6.2, (2.131), Tao).

Due to Corollary 24, the product matrix can be expressed as $U\Sigma_w^N V^H$. Then we present the proof of Theorem 3.

Proof. From (6),
$$W(t=0) = s\epsilon^N Q_{N,N+1} (G^H G)^{N/2} Q_{01}^H$$
.

Naturally $\|\Sigma_w\|_{op} = \epsilon \|(G^H G)^{1/2}\|_{op} = \epsilon \|G\|_{op} \le c_1(\delta) \sqrt{d}\epsilon$. Last step is from Corollary 2.3.5 of Tao directly.

For the other two terms,

$$\sigma_{min} ((U+V)\Sigma_{w})|_{t=0}$$

$$= \sqrt{\lambda_{min} ((U+V)\Sigma_{w}^{2}(U+V)^{H})}\Big|_{t=0}$$

$$= \sqrt{\lambda_{min} \left((WW^{H})^{\frac{1}{N}} + (W^{H}W)^{\frac{1}{N}} + (WW^{H})^{-\frac{N-2}{2N}} W + (W^{H}W)^{-\frac{N-2}{2N}} W^{H} \right)}\Big|_{t=0}$$

$$= \epsilon \sqrt{\lambda_{min} \left((Q_{01} + sQ_{N,N+1}) (G^{H}G) (Q_{01} + sQ_{N,N+1})^{H} \right)}$$

$$\in \left[\epsilon \sigma_{min} (I + sQ_{01}^{H}Q_{N,N+1}) \sigma_{min}(G), \epsilon \sigma_{min} (I + sQ_{01}^{H}Q_{N,N+1}) \sigma_{max}(G) \right].$$
(53)

And

$$\|(U-V)\Sigma_w\|_F|_{t=0} \le 2\sqrt{d}\epsilon \|G\|_{op}.$$
 (54)

- Since $Q_{N,N+1}$ and Q_{01} are independent and both sampled from Haar measure, then $Q_{01}^HQ_{N,N+1}\stackrel{\text{dist}}{\sim} Haar$ on $U(d,\mathbb{C})$ (or $O(d,\mathbb{R})$ if $\mathbb{F}=\mathbb{R}$) as well.
- For $\mathbb{F} = \mathbb{R}$, since s is independent of $Q_{j,j \in [0,N] \cap \mathbb{N}}$, $\Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = 1) = \Pr(s \det(Q_{N,N+1}) \det(Q_{01}) = -1) = \frac{1}{2}$ is directly from symmetry of Haar measure.
 - Then by combining Theorem 14 and Corollary 2.3.5, Theorem 2.7.5 of Tao (with high probability $1 \delta'$, $\max(\|G\|_{op}, \|G^{-1}\|_{op}) \le c(\delta')\sqrt{d}$), the proof is completed.

B.4 GENERAL BALANCED INITIALIZATION

 This section introduces a property for general balanced and input-output orthogonal-invariant initialization (refer to Definition 1) under real field.

Theorem 16. For any real matrix factorization, if the initialization is balanced and input-output orthogonal-invariant, then the minimum singular value of $W + (WW^{\top})^{1/2}$ at t = 0 is exactly 0 with at least probability 1/2:

$$\Pr\left(\sigma_{min}\left(W + \left(WW^{\top}\right)^{1/2}\right) = 0\right) \ge 1/2. \tag{55}$$

Proof. As a direct consequence of Definition 1, W is left and right orthogonal invariant:

$$W \stackrel{\text{dist}}{=} UWV, \forall U, V \in O(d, \mathbb{R}). \tag{56}$$

Suppose the right polar decomposition of W is $W = WW^{\top}Q$, following the same arguments in the proof (B.2) of Theorem 6,

$$W + (WW^{\top})^{1/2} = (WW^{\top})^{1/2} (I + Q), Q \stackrel{\text{dist}}{\sim} Haar.$$
 (57)

From Theorem 14, $\Pr(\sigma_{min}(I+Q)=0)=\frac{1}{2}$, resulting

$$\Pr\left(\sigma_{min}\left(W + \left(WW^{\top}\right)^{1/2}\right) = 0\right) \ge \Pr(\sigma_{min}(I + Q) = 0) = \frac{1}{2}.$$
 (58)

This completes the proof.

C BASIC LEMMAS

C.1 CLASSIC MATRIX ANALYSIS CONCLUSIONS

Lemma 17. Let $R \in \mathbb{F}^{d \times d}$, where $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . Then:

- 1. $I RR^H$ and $I R^HR$ (or $I RR^T$ and $I R^TR$ if $\mathbb{F} = \mathbb{R}$) share the same set of eigenvalues.
- 2. These eigenvalues are real-valued.

Proof. We prove the complex case, and the real case follows. Suppose the singular value decomposition of R is $U_R \Sigma_R V_R^H$, then

$$I - RR^{H} = I - U_{R} \Sigma_{R}^{2} U_{R}^{H} = U_{R} \left(I - \Sigma_{R}^{2} \right) U_{R}^{H}$$

$$I - R^{H} R = I - V_{R} \Sigma_{R}^{2} V_{R}^{H} = V_{R} \left(I - \Sigma_{R}^{2} \right) V_{R}^{H}.$$
(59)

Thus both $I - RR^H$ and $I - R^HR$ are unitarily similar to $I - \Sigma_R^2$, which completes the proof. \square

Lemma 18. Given symmetric matrices $X, \Delta \in \mathbb{F}^{d \times d}$, where $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , suppose $X \succ \|\Delta\|_{op} I \succ O$, then

$$\left\| X^{1/2} - (X + \Delta)^{1/2} \right\|_{op} \le \frac{\|\Delta\|_{op}}{2(\lambda_{min}(X) - \|\Delta\|_{op})^{1/2}}.$$
 (60)

Proof. Directly by Theorem X.3.8 and inequality (X.46) in Bhatia (1996).

Lemma 19. $\forall X, \Delta \in \mathbb{F}^{d \times d}$, where $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , if X and $X + \Delta$ are both invertible, then

$$(X + \Delta)^{-1} - (X^{-1} - X^{-1}\Delta X^{-1}) = X^{-1}\Delta X^{-1}\Delta (X + \Delta)^{-1}.$$
 (61)

Proof

$$(X + \Delta)^{-1} - (X^{-1} - X^{-1}\Delta X^{-1}) = X^{-1} [X - (X - \Delta)X^{-1}(X + \Delta)] (X + \Delta)^{-1}$$

= $X^{-1}\Delta X^{-1}\Delta (X + \Delta)^{-1}$. (62)

Lemma 20. Bound of eigenvalues under perturbation.

For unitary (or orthogonal, for real field) d-dimensional matrices U, V, positive semi-definite matrix S, denote $P := \left(\frac{U+V}{2}\right) S\left(\frac{U+V}{2}\right)^H$, then the eigenvalues of S are bounded by

$$\lambda_{k}(P) \leq \lambda_{k}(S) \leq \begin{cases} 2\left[\lambda_{k}(P) + \left\|\left(\frac{U-V}{2}\right)S\left(\frac{U-V}{2}\right)^{H}\right\|_{op}\right] &, 1 \leq k \leq d-1\\ \lambda_{k}(P) + \left\|\left(\frac{U-V}{2}\right)S\left(\frac{U-V}{2}\right)^{H}\right\|_{op} &, k = d \end{cases}$$
(63)

Proof. Let $Q = U^H V$.

Due to Courant-Fischer min-max Theorem, $A \succeq B$ indicates $\lambda_k(A) \geq \lambda_k(B)$. Then the lower bound is straight forward:

$$\lambda_{k} \left(\left(\frac{U+V}{2} \right) S \left(\frac{U+V}{2} \right)^{H} \right) = \lambda_{k} \left(S^{1/2} \left(\frac{U+V}{2} \right) \left(\frac{U+V}{2} \right)^{H} S^{1/2} \right)
\leq \lambda_{k} \left(S^{1/2} \left(\left\| \frac{U+V}{2} \right\|_{op}^{2} I \right) S^{1/2} \right)
\leq \lambda_{k} \left(S^{1/2} \left(\left(\frac{\|U\|_{op} + \|V\|_{op}}{2} \right)^{2} I \right) S^{1/2} \right) = \lambda_{k} \left(S \right).$$
(64)

For upper bound, by applying Wely inequality,

$$\lambda_{k} \left(\left(\frac{U+V}{2} \right) S \left(\frac{U+V}{2} \right)^{H} \right) = \lambda_{k} \left(\left(\frac{I+Q}{2} \right) S \left(\frac{I+Q^{H}}{2} \right) \right) \\
\geq \lambda_{k} \left(\left(\frac{I+Q}{2} \right) S \left(\frac{I+Q^{H}}{2} \right) + \left(\frac{I-Q}{2} \right) S \left(\frac{I-Q^{H}}{2} \right) \right) - \left\| \left(\frac{I-Q}{2} \right) S \left(\frac{I-Q^{H}}{2} \right) \right\|_{op} \\
= \frac{1}{2} \lambda_{k} \left(S + QSQ^{H} \right) - \left\| \left(\frac{U-V}{2} \right) S \left(\frac{U-V}{2} \right)^{H} \right\|_{op} .$$
(65)

For arbitrary k, $\lambda_k \left(S + QSQ^H\right) \geq \lambda_k \left(S\right)$; for k = d, $\lambda_d \left(S + QSQ^H\right) \geq 2\lambda_d \left(S\right)$. This completes the proof.

C.2 Lemmas on Analytic Singular Value Decomposition of Product Matrix under Balanced Initialization and Gradient Flow

Lemma 21. Existence of analytic singular value decomposition (ASVD).

Under Section 3 with gradient flow and balanced initialization, for $t \in \mathbb{R}^+ \cup \{0\}$, there exists analytical singular value decompositions for $W_{j,j\in[1,N]\cap\mathbb{N}^*}(t)$ and W(t).

Proof. For $\mathbb{F}=\mathbb{R}$, the proof is exactly the same as Lemma 1 in Arora et al. (2019b): real analytic matrices have ASVD (Theorem 1 in Bunse-Gerstner et al. (1991/92)), and $W_j(t)$ are analytic then so does W(t). For complex case, Theorem 1 and 3 in De Moor & Boyd (1989) gives that complex analytic matrices (of a real parameter) have ASVD, then the rest of proof follows.

Remark 9. For complex field here, the "analytic" here has **no relation with the standard definition of "complex analytic function"**, who has complex parameters and consequently more restrictions on definition of derivatives.

Throughout the proof for gradient flow (continuous time), we only deal with real-valued parameter $t \in \mathbb{R}^+ \cup \{0\}$, so any "analytic" means real-analytic (for $\mathbb{F} = \mathbb{C}$, it means the real and imaginary part are both real-analytic), not complex-analytic.

Lemma 22. Suppose the analytic singular value decomposition of M(t) exists and is $U(t)\Sigma_M(t)V^H(t)$, $M(t)\in\mathbb{F}^{d\times d}$, where $\mathbb{F}=\mathbb{C}$ or \mathbb{R} , then the derivative of the k^{th} singular value is

$$\frac{\mathrm{d}\sigma_k(M)}{\mathrm{d}t} = \Re\left(u_k^H \frac{\mathrm{d}M}{\mathrm{d}t} v_k\right),\tag{66}$$

where u_k , v_k are the k^{th} column vectors of left and right unitary (or orthogonal if $\mathbb{F} = \mathbb{R}$) matrices respectively.

Proof. We prove the case when $\mathbb{F} = \mathbb{C}$. For $\mathbb{F} = \mathbb{R}$, replace \cdot^H by \cdot^\top .

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t} \Sigma_M V^H + U \frac{\mathrm{d}\Sigma_M}{\mathrm{d}t} V^H + U \Sigma_M \frac{\mathrm{d}V}{\mathrm{d}t}^H. \tag{67}$$

Then

 $\Re\left(u_k^H \frac{\mathrm{d}M}{\mathrm{d}t} v_k\right) = \Re\left(u_k^H \frac{\mathrm{d}U}{\mathrm{d}t} \Sigma_M V^H v_k + u_k^H U \frac{\mathrm{d}\Sigma_M}{\mathrm{d}t} V^H v_k + u_k^H U \Sigma_M \frac{\mathrm{d}V}{\mathrm{d}t}^H v_k\right) \\
= \frac{\mathrm{d}\sigma_k(M)}{\mathrm{d}t} + \sigma_k(M) \left(\Re\left(u_k^H \frac{\mathrm{d}u_k}{\mathrm{d}t}\right) + \Re\left(\frac{\mathrm{d}v_k^H}{\mathrm{d}t} v_k\right)\right).$ (68)

From $\Re\left(u_k^H \frac{\mathrm{d}u_k}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_k\|^2\right) = 0, \Re\left(\frac{\mathrm{d}v_k^H}{\mathrm{d}t}v_k\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|v_k\|^2\right) = 0$, the proof is done.

Remark 10. If M is hermitian, then the \Re can be omitted.

Remark 11. This generalizes Lemma 2 in Arora et al. (2019b) from real field into complex field by adding a \Re on the right side:

$$\frac{\mathrm{d}\sigma_r(S)}{\mathrm{d}t} = -N(\sigma_r^2(S))^{1-1/N} \cdot \Re\left(\left\langle \nabla_W \mathcal{L}(W), u_r v_r^H \right\rangle\right). \tag{69}$$

Lemma 23. Under Section 3 with gradient flow, \mathcal{L}_{ori} is non-increasing.

For $t \in [0, +\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{ori} \le -2N \min_{i,k} |\sigma_k(W_j)|^{2(N-1)} \mathcal{L}_{ori}. \tag{70}$$

Proof. Naturally we have the derivative of product matrix W(t):

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_{j=1}^{N} W_{\prod_{L},j+1} \left[W_{\prod_{L},j+1}^{H} \left(\Sigma - W \right) W_{\prod_{R},j-1}^{H} + a \left(W_{j} \Delta_{j-1,j} - \Delta_{j,j+1} W_{j} \right) \right] W_{\prod_{R},j-1}$$

$$= \sum_{j=1}^{N} W_{\prod_{L},j+1} W_{\prod_{L},j+1}^{H} \left(\Sigma - W \right) W_{\prod_{R},j-1}^{H} W_{\prod_{R},j-1}$$

$$+ a \sum_{j=1}^{N} W_{\prod_{L},j} \Delta_{j-1,j} W_{\prod_{R},j-1} - a \sum_{j=1}^{N} W_{\prod_{L},j+1} \Delta_{j,j+1} W_{\prod_{R},j}$$

$$= \sum_{j=1}^{N} W_{\prod_{L},j+1} W_{\prod_{L},j+1}^{H} \left(\Sigma - W \right) W_{\prod_{R},j-1}^{H} W_{\prod_{R},j-1} + a \left(W \Delta_{0,1} - \Delta_{N,N+1} W \right)$$

$$= \sum_{j=1}^{N} W_{\prod_{L},j+1} W_{\prod_{L},j+1}^{H} \left(\Sigma - W \right) W_{\prod_{R},j-1}^{H} W_{\prod_{R},j-1}.$$
(71)

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{ori} = -\Re\left(\left\langle \Sigma - W, \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle\right)$$

$$= -\Re\left(\left\langle \Sigma - W, \sum_{j=1}^{N} W_{\prod_{L}, j+1} W_{\prod_{L}, j+1}^{H} \left(\Sigma - W\right) W_{\prod_{R}, j-1}^{H} W_{\prod_{R}, j-1} \right\rangle\right)$$

$$= -\sum_{j=1}^{N} \Re\left(\left\langle \Sigma - W, W_{\prod_{L}, j+1} W_{\prod_{L}, j+1}^{H} \left(\Sigma - W\right) W_{\prod_{R}, j-1}^{H} W_{\prod_{R}, j-1} \right\rangle\right)$$

$$= -\sum_{j=1}^{N} \Re\left(\left\langle W_{\prod_{L}, j+1}^{H} \left(\Sigma - W\right) W_{\prod_{R}, j-1}^{H}, W_{\prod_{L}, j+1}^{H} \left(\Sigma - W\right) W_{\prod_{R}, j-1}^{H} \right\rangle\right)$$

$$= -\sum_{j=1}^{N} \left\|W_{\prod_{L}, j+1}^{H} \left(\Sigma - W\right) W_{\prod_{R}, j-1}^{H} \right\|_{F}^{2}.$$
(72)

From $\|LXR\|_F \geq \sigma_{min}(L)\sigma_{min}(R)\|X\|_F$, $\sigma_{min}\left(W^H_{\prod_L,j+1}\right) \geq \min_{j,k}|\sigma_k(W_j)|^{N-j}$ and $\sigma_{min}\left(W^H_{\prod_R,j-1}\right) \geq \min_{j,k}|\sigma_k(W_j)|^{j-1}$, the proof is completed.

Lemma 24. Analytic singular value decomposition of product matrix with positive semi-definite diagonal matrix.

 Under Section 3 with gradient flow and any bounded (i.e. $W_{j,j\in[1,N]\cap\mathbb{N}^*}(t=0)$ is bounded) balanced initialization, $\forall N\in[2,+\infty)\cap\mathbb{N}^*$, the product matrix W(t) can be expressed as:

$$W(t) = U(t)S(t)V(t)^{H},$$
(73)

where: $U(t) \in \mathbb{F}^{d \times d}$, $S(t) \in \mathbb{R}^{d \times d}$ and $V(t) \in \mathbb{F}^{d \times d}$ are analytic functions of t, U(t) and V(t) are orthogonal matrices, S(t) is diagonal and positive semi-definite (elements on its diagonal may appear in any order), $\Sigma_w(t) := S(t)^{1/N}$ is well-defined (meaning the real-valued operation $S_{ii} \mapsto (S_{ii})^{1/N}$ is applied to each diagonal element of S(t), resulting in another semi-positive diagonal matrix) and analytic.

Moreover, if the singular values of product matrix W are non-zero, then throughout the optimization W remains full rank in finite time.

Proof. From Lemma 21, it is left to construct a new ASVD (analytic singular value decomposition) of W(t) using existed ASVD $W(t) = U(t)S(t)V(t)^H$ (S(t) is not guaranteed to be positive semi-definite).

By Lemma 23, $\|\Sigma - W\|_F \le \|\Sigma - W(t=0)\|_F$. Then the following term is bounded by a constant for all $t \in \mathbb{R}^+ \cup \{0\}$:

$$\left|\left\langle \nabla l(W(t)), u_r(t)v_r(t)^H \right\rangle \right| \le \left\| \nabla l(W(t)) \right\|_{op} = \left\| \Sigma - W \right\|_{op}$$

$$\le \left\| \Sigma - W \right\|_F \le \left\| \Sigma - W(t=0) \right\|_F.$$
(74)

By invoking Theorem 3 in Arora et al. (2019b) (for complex case, add \Re), the absolute value of time derivative of $\sigma_r(t)$ is bounded by:

$$\left| \frac{\mathrm{d}\sigma_r(t)}{\mathrm{d}t} \right| \le \left\| \Sigma - W(t=0) \right\|_F \cdot N \left(\sigma_r^2(t) \right)^{1-1/N}. \tag{75}$$

Thus all $\sigma_r(t)$ do not change sign for $t \in \mathbb{R}^+ \cup \{0\}$. Moreover, if $|\sigma_r(t=0)| > 0$, the it never decrease to 0 in finite time.

Then we construct $S_{new}(t)$ by flipping the sign of negative diagonal terms, and $U_{new}(t)$ by changing the sign of corresponding columns of U(t). Now $W(t) = U_{new}(t)S_{new}(t)V(t)^H$ is also an ASVD of W(t), $U_{new}(t)$ is analytic and unitary (orthogonal), $S_{new}(t)$ is analytic, diagonal and positive semi-definite.

Specially, if for some r, $\sigma_r(t)=0$ at time t, then it remains zero. Thus, from $S_{new}(t)$ is analytic, so is $\Sigma_w(t)$. This completes the proof.

C.3 LEMMAS ON REGULARIZATION, GRADIENT FLOW

Lemma 25. Consider optimizing a generalized loss function coupled with a generalized regularization term using gradient flow:

$$\mathcal{L}(W_1, \cdots, W_N) := \mathcal{L}_{ori} \left(\prod_{j=N}^1 W_j \right) + \frac{1}{4} \sum_{j=1}^{N-1} a_{j,j+1} \|\Delta_{j,j+1}\|_F^2, \ a_{j,j+1} \in \mathbb{R}^+ \cup \{0\}.$$
 (76)

Then the regularization terms decays by:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=1}^{N-1} a_{j,j+1} \| \Delta_{j,j+1} \|_F^2 \right) = -4 \sum_{j=1}^N \| a_{j,j+1} \Delta_{j,j+1} W_j - a_{j-1,j} W_j \Delta_{j-1,j} \|_F^2. \tag{77}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}W_{j}W_{j}^{H} = -\left[\left(\nabla_{W_{j}}\mathcal{L}_{ori}\right)W_{j}^{H} + W_{j}\left(\nabla_{W_{j}}\mathcal{L}_{ori}\right)^{H} - 2a_{j-1,j}W_{j}\Delta_{j-1,j}W_{j}^{H} + a_{j,j+1}\left(\Delta_{j,j+1}W_{j}W_{j}^{H} + W_{j}W_{j}^{H}\Delta_{j,j+1}\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}W_{j+1}^{H}W_{j+1} = -\left[\left(\nabla_{W_{j+1}}\mathcal{L}_{ori}\right)^{H}W_{j+1} + W_{j+1}^{H}\left(\nabla_{W_{j+1}}\mathcal{L}_{ori}\right) + 2a_{j+1,j+2}W_{j+1}^{H}\Delta_{j+1,j+2}W_{j+1}\right]$$

$$- a_{j,j+1}\left(\Delta_{j,j+1}W_{j+1}^{H}W_{j+1} + W_{j+1}^{H}W_{j+1}\Delta_{j,j+1}\right)\right].$$
(78)

Denote $W_{\prod_L,j} := \prod_{k=N}^j W_k$, $W_{\prod_R,j} := \prod_{k=j}^1 W_k$, $W := \prod_{k=N}^1 W_k = W_{\prod_L,1} = W_{\prod_R,N}$. From property of the loss \mathcal{L}_{ori} ,

$$\left(\nabla_{W_{j}}\mathcal{L}_{ori}\right)W_{j}^{H} = W_{\prod_{L},j+1}^{H}\left(\nabla_{W}\mathcal{L}_{ori}(W)\right)W_{\prod_{R},j} = W_{j+1}^{H}\left(\nabla_{W_{j+1}}\mathcal{L}_{ori}\right), \forall j \in [1, N-1] \cap \mathbb{N}^{*}.$$
(79)

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{j,j+1} = 2a_{j-1,j}W_{j}\Delta_{j-1,j}W_{j}^{H} + 2a_{j+1,j+2}W_{j+1}^{H}\Delta_{j+1,j+2}W_{j+1}
- a_{j,j+1}\left(\Delta_{j,j+1}\left(W_{j}W_{j}^{H} + W_{j+1}^{H}W_{j+1}\right) + \left(W_{j}W_{j}^{H} + W_{j+1}^{H}W_{j+1}\right)\Delta_{j,j+1}\right),$$
(80)

$$\frac{\mathrm{d}\|\Delta_{j,j+1}\|_{F}^{2}}{\mathrm{d}t} = 4a_{j-1,j}\mathrm{tr}\left(W_{j}\Delta_{j-1,j}W_{j}^{H}\Delta_{j,j+1}\right)
+ 4a_{j+1,j+2}\mathrm{tr}\left(W_{j+1}\Delta_{j,j+1}W_{j+1}^{H}\Delta_{j+1,j+2}\right)
- 4a_{j,j+1}\mathrm{tr}\left((W_{j}W_{j}^{H} + W_{j+1}^{H}W_{j+1})\Delta_{j,j+1}^{2}\right)
= -\frac{2}{a_{j,j+1}} \left[\|a_{j,j+1}\Delta_{j,j+1}W_{j} - a_{j-1,j}W_{j}\Delta_{j-1,j}\|_{F}^{2} \right]
+ \|a_{j+1,j+2}\Delta_{j+1,j+2}W_{j+1} - a_{j,j+1}W_{j+1}\Delta_{j,j+1}\|_{F}^{2}
+ a_{j,j+1}^{2}\left(\|\Delta_{j,j+1}W_{j}\|_{F}^{2} + \|W_{j+1}\Delta_{j,j+1}\|_{F}^{2}\right)
- a_{j-1,j}^{2}\|W_{j}\Delta_{j-1,j}\|_{F}^{2} - a_{j+1,j+2}^{2}\|\Delta_{j+1,j+2}W_{j+1}\|_{F}^{2} \right].$$
(81)

By taking weighted sum,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=1}^{N-1} a_{j,j+1} \| \Delta_{j,j+1} \|_F^2 \right) = -4 \sum_{j=1}^N \| a_{j,j+1} \Delta_{j,j+1} W_j - a_{j-1,j} W_j \Delta_{j-1,j} \|_F^2. \tag{82}$$

Below we back to $a_{i,j+1} \equiv a \in \mathbb{R}^+ \cup \{0\}, \forall j \in [1, N-1] \cap \mathbb{N}^*$. Then 76 becomes

$$\mathcal{L}(W_1, \cdots, W_N) := \mathcal{L}_{ori} \left(\prod_{j=N}^1 W_j \right) + \frac{1}{4} \sum_{j=1}^{N-1} a \|\Delta_{j,j+1}\|_F^2, \ a \in \mathbb{R}^+ \cup \{0\}.$$
 (83)

Theorem 26. Suppose for all $j \in [1, N] \cap \mathbb{N}^*$, $\sigma_{min}(W_j) \geq \delta$, $\sigma_{max}(W_j) \leq M$. Consider optimizing 83 using gradient flow, then the convergence rate of the regularization term is lower bounded:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=1}^{N-1} \|\Delta_{j,j+1}\|_F^2 \right) \le -4a \cdot \frac{2}{N-1} \frac{M^2 - \delta^2}{\left(\frac{M}{\delta}\right)^{2\lfloor N/2 \rfloor} - 1} \cdot \left(\sum_{j=1}^{N-1} \|\Delta_{j,j+1}\|_F^2 \right). \tag{84}$$

Proof. Denote $D_j = \Delta_{j,j+1}W_j - W_j\Delta_{j-1,j}$. Then

$$\Delta_{j,j+1} = (D_j + W_j \Delta_{j-1,j}) W_j^{-1}. \tag{85}$$

Deducing

$$\|\Delta_{j,j+1}\|_{F} \le \|W_{j}^{-1}\|_{op} (\|D_{j}\|_{F} + \|\Delta_{j-1,j}\|_{F} \|W_{j}\|_{op}) \le \frac{1}{\delta} \|D_{j}\|_{F} + \frac{M}{\delta} \|\Delta_{j-1,j}\|_{F}.$$
 (86)

From $\Delta_{0,1} = O$, inductively we have

$$\|\Delta_{j,j+1}\|_{F}^{2} \leq \frac{1}{\delta^{2}} \left(\sum_{k=1}^{j} \left(\frac{M}{\delta} \right)^{j-k} \|D_{k}\|_{F} \right)^{2} \leq \frac{1}{\delta^{2}} \left(\sum_{k=1}^{j} \left(\frac{M}{\delta} \right)^{2(j-k)} \right) \left(\sum_{k=1}^{j} \|D_{k}\|_{F}^{2} \right)$$

$$= \frac{1}{\delta^{2}} \frac{\left(\frac{M}{\delta} \right)^{2j} - 1}{\left(\frac{M}{\delta} \right)^{2} - 1} \sum_{k=1}^{j} \|D_{k}\|_{F}^{2}.$$
(87)

The last two step use Cauchy-Schwarz inequality.

From $\Delta_{N,N+1} = O$, following the same procedure we have

$$\|\Delta_{N-j,N-j+1}\|_F^2 \le \frac{1}{\delta^2} \frac{\left(\frac{M}{\delta}\right)^{2j} - 1}{\left(\frac{M}{\delta}\right)^2 - 1} \sum_{k=N-j+1}^N \|D_k\|_F^2. \tag{88}$$

Summing all terms up, for odd N we have

$$\sum_{j=1}^{N-1} \|\Delta_{j,j+1}\|_{F}^{2} = \sum_{j=1}^{(N-1)/2} \left(\|\Delta_{j,j+1}\|_{F}^{2} + \|\Delta_{N-j,N-j+1}\|_{F}^{2} \right)
\leq \sum_{j=1}^{(N-1)/2} \left(\frac{1}{\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{2j} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \sum_{k=1}^{j} \left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right) \right)
= \sum_{k=1}^{(N-1)/2} \left(\left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right) \sum_{j=k}^{(N-1)/2} \left(\frac{1}{\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{2j} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \right) \right)
\leq \frac{N-1}{2} \frac{\left(\frac{M}{\delta}\right)^{N-1} - 1}{M^{2} - \delta^{2}} \left(\sum_{k=1}^{N} \|D_{k}\|^{2} \right).$$
(89)

For even N,

$$\sum_{j=1}^{N-1} \|\Delta_{j,j+1}\|_{F}^{2} = \sum_{j=1}^{N/2-1} \left(\|\Delta_{j,j+1}\|_{F}^{2} + \|\Delta_{N-j,N-j+1}\|_{F}^{2} \right) + \|\Delta_{N/2,N/2+1}\|_{F}^{2}
\leq \sum_{j=1}^{N/2-1} \left(\frac{1}{\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{2j} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \sum_{k=1}^{j} \left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right) \right)
+ \frac{1}{2\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{N} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \sum_{k=1}^{N/2} \left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right)
= \sum_{k=1}^{N/2-1} \left(\left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right) \sum_{j=k}^{N/2-1} \left(\frac{1}{\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{2j} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \right) \right)
+ \frac{1}{2\delta^{2}} \frac{\left(\frac{M}{\delta}\right)^{N} - 1}{\left(\frac{M}{\delta}\right)^{2} - 1} \sum_{k=1}^{N/2} \left(\|D_{k}\|_{F}^{2} + \|D_{N+1-k}\|_{F}^{2} \right)
\leq \frac{N-1}{2} \frac{\left(\frac{M}{\delta}\right)^{N} - 1}{M^{2} - \delta^{2}} \left(\sum_{k=1}^{N} \|D_{k}\|^{2} \right).$$
(90)

Thus

$$\sum_{j=1}^{N} \|D_j\|^2 \ge \frac{2}{N-1} \frac{M^2 - \delta^2}{\left(\frac{M}{\delta}\right)^{2\lfloor N/2 \rfloor} - 1} \sum_{i=1}^{N-1} \|\Delta_{i,i+1}\|_F^2. \tag{91}$$

Combine with Lemma 25, then the proof is done.

Remark 12. For N = 4, Theorem 26 reduces to

 $\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=1}^{3} \|\Delta_{j,j+1}\|_F^2 \right) \le -\frac{8a}{3} \frac{\delta^4}{M^2 + \delta^2} \cdot \left(\sum_{j=1}^{3} \|\Delta_{j,j+1}\|_F^2 \right). \tag{92}$

Theorem 27. Under problem settings in section 3 with gradient flow, the change of maximum and minimum singular values of W_i s have bounds that are irrelevant of the regularization term:

$$\frac{\operatorname{d}\max_{j,k}\sigma_{k}^{2}(W_{j})}{\operatorname{d}t} \leq 2\max_{j,k}|\sigma_{k}(W_{j})|\max_{j}\left\|\nabla_{W_{j}}\mathcal{L}_{ori}\right\|_{op}$$

$$\frac{\operatorname{d}\min_{j,k}\sigma_{k}^{2}(W_{j})}{\operatorname{d}t} \geq -2\min_{j,k}|\sigma_{k}(W_{j})|\max_{j}\left\|\nabla_{W_{j}}\mathcal{L}_{ori}\right\|_{op}.$$
(93)

Remark 13. If $\arg \max_{(j,k)} |\sigma_k(W_j)|$, $\arg \min_{(j,k)} |\sigma_k(W_j)|$ are not unique, the derivatives are not well-defined. In these cases, the inequalities become:

$$\frac{\mathrm{d}\sigma_{k'}^{2}(W_{j'})}{\mathrm{d}t} \leq 2 \max_{j,k} |\sigma_{k}(W_{j})| \max_{j} \left\| \nabla_{W_{j}} \mathcal{L}_{ori} \right\|_{op}, \quad (j',k') \in \arg \max_{(j,k)} |\sigma_{k}(W_{j})|
\frac{\mathrm{d}\sigma_{k'}^{2}(W_{j'})}{\mathrm{d}t} \geq -2 \min_{j,k} |\sigma_{k}(W_{j})| \max_{j} \left\| \nabla_{W_{j}} \mathcal{L}_{ori} \right\|_{op}, \quad (j',k') \in \arg \min_{(j,k)} |\sigma_{k}(W_{j})|.$$
(94)

Proof. For simplicity, set $W_0 \equiv W_1, W_5 \equiv W_4$.

Denote the analytic singular value decomposition of $W_j(t)$ to be $U^{(j)}\Sigma_w^{(j)}V^{(j)H}$, then from Lemma 22, we have

1459
1460
1461
$$\frac{\mathrm{d}\sigma_{k}(W_{j})}{\mathrm{d}t} = \Re\left(u_{k}^{(j)H}\left(-\nabla_{W_{j}}\mathcal{L}_{ori} + aW_{j}\Delta_{j-1,j} - a\Delta_{j,j+1}W_{j}\right)v_{k}^{(j)}\right)$$
1462
$$= \Re\left(u_{k}^{(j)H}\left(-\nabla_{W_{j}}\mathcal{L}_{ori}\right)v_{k}^{(j)}\right)$$
1463
1464
$$+ au_{k}^{(j)H}\left(W_{j}W_{j-1}W_{j-1}^{H} + W_{j+1}^{H}W_{j+1}W_{j} - 2W_{j}W_{j}^{H}W_{j}\right)v_{k}^{(j)}$$
1465
1466
$$= \Re\left(u_{k}^{(j)H}\left(-\nabla_{W_{j}}\mathcal{L}_{ori}\right)v_{k}^{(j)}\right)$$
1467
1468
$$+ a\left[\left(u_{k}^{(j)H}W_{j+1}^{H}W_{j+1}u_{k}^{(j)} + v_{k}^{(j)H}W_{j-1}W_{j-1}^{H}v_{k}^{(j)}\right)\sigma_{k}(W_{j}) - 2\sigma_{k}(W_{j})^{3}\right].$$

From $u_k^{(j)H}W_{j+1}^HW_{j+1}u_k^{(j)}, v_k^{(j)H}W_{j-1}W_{j-1}^Hv_k^{(j)} \in [\min_{j,k}\sigma_k^2(W_j), \max_{j,k}\sigma_k^2(W_j)]$, the proof is completed.

1474 Note:

$$\max_{j} \left\| \nabla_{W_{j}} \mathcal{L}_{ori} \right\|_{op} \le \max_{j,k} \left| \sigma_{k}(W_{j}) \right|^{N-1} \left(\sigma_{1}(\Sigma) + \max_{j,k} \left| \sigma_{k}(W_{j}) \right|^{N} \right). \tag{96}$$

C.4 Lemmas on Eigenvalue Change under Discrete Time

Lemma 28. Suppose $\Sigma, S \in \mathbb{F}^{d \times d}$ are positive semi-definite matrices, $0 \le \alpha \le \frac{1}{6} ||S||_{op}^{-1}$, $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . Consider $S' = (I + \alpha(\Sigma - S))S(I + \alpha(\Sigma - S))$,

$$\lambda_{min}\left(S'\right) \ge \lambda_{min}(S)\left(1 + \alpha(\lambda_{min}(\Sigma) - \lambda_{min}(S))\right)^{2} + O\left(\alpha^{2}\left(\|\Sigma\|_{op}^{2} + \|S\|_{op}^{2}\right)\|S\|_{op}\right)$$

$$\lambda_{max}\left(S'\right) \le \lambda_{max}(S)\left(1 + \alpha(\lambda_{max}(\Sigma) - \lambda_{max}(S))\right)^{2}.$$
(97)

This generalizes Lemma 3.2 in Ye & Du (2021).

Proof. Following the derivations in Ye & Du (2021), $\forall \beta \in (0,1)$, rewrite the terms by the following:

$$S' = \beta \left(I - \frac{\alpha}{\beta} S \right) S \left(I - \frac{\alpha}{\beta} S \right) + (1 - \beta) \left(I + \frac{\alpha}{1 - \beta} \Sigma \right) S \left(I + \frac{\alpha}{1 - \beta} \Sigma \right)$$

$$- \frac{\alpha^2}{\beta (1 - \beta)} \left[(1 - \beta) S + \beta \Sigma \right] S \left[(1 - \beta) S + \beta \Sigma \right].$$
(98)

The first term has eigenvalues $\lambda_{i'}(S') = \beta \left(1 - \frac{\alpha}{\beta}\lambda_i(S)\right)^2 \lambda_i(S)$ (note that $f(x) = (1-x)^2 x$ is non-decreasing in $\left[0, \frac{1}{3}\right]$, so $\lambda_{i'}(S')$ is exactly the i^{th} eigenvalue of the first term when $\beta \geq \frac{1}{2}$), while the second term is bounded by

$$(1-\beta)\left(I + \frac{\alpha}{1-\beta}\lambda_{min}(\Sigma)\right)^{2}\lambda_{min}(S) \leq term2 \leq (1-\beta)\left(I + \frac{\alpha}{1-\beta}\lambda_{max}(\Sigma)\right)^{2}\lambda_{max}(S). \tag{99}$$

By treating the third term as error term and taking $\beta = \frac{1}{2}$, the proof is completed.

Lemma 29. Suppose $D, S \in \mathbb{F}^{d \times d}$ are positive semi-definite matrices, $E \in \mathbb{F}^{d \times d}$, $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . Denote M = S + D. Consider $S' = \left(I + \eta \left(aM - M^3 + E\right)\right)S\left(I + \eta \left(aM - M^3 + E\right)\right)$, under $\eta < \frac{1}{16\left(\|M\|_{op}^3 + \|E\|_{op}\right)}$,

1513
$$\lambda_{min}(S') \ge \lambda_{min}(S) + 2\eta \left(a - 2\|D\|_{op}\|M\|_{op} - \|M\|_{op}\lambda_{min}(S)\right) \lambda_{min}^{2}(S)$$
1515
$$-2\eta \left(\|E\|_{op} + \|D\|_{op}^{2}\|M\|_{op}\right) \lambda_{min}(S)$$
1516
$$+O\left(\left(a^{2}\|M\|_{op}^{2} + \|M\|_{op}^{6} + \|E\|_{op}^{2}\right) \|S\|_{op}\right).$$
(100)

Proof. Expand the expression of S':

$$S' = S + \eta (aM + E - DMD) S + \eta S (aM + E - DMD)$$

$$- \eta (DMS^{2} + S^{2}MD) - \eta S (MD + DM) S - \eta (SMS^{2} + S^{2}MS) + \eta^{2} M_{error}$$

$$= \frac{1}{4} (I + 4\eta (aM + E - DMD)) S (I + 4\eta (aM + E - DMD))$$

$$+ \frac{1}{4s} (I - 4\eta sDM) S^{2} (I - 4\eta sMD) + \frac{1}{4s} S (I - 4\eta s (MD + DM)) S$$

$$+ \frac{1}{4s^{2}} S (I - 4\eta s^{2}M) S (I - 4\eta s^{2}M) S + \left(\frac{3}{4}S - \frac{1}{2s}S^{2} - \frac{1}{4s^{2}}S^{3}\right) + \eta^{2} M'_{error}.$$
(101)

where $||M'_{error}||_{op} = O\left(\left(a^2||M||_{op}^2 + ||M||_{op}^6 + ||E||_{op}^2\right)||S||_{op}\right).$

Notice that $\frac{3}{4}S - \frac{1}{2s}S^2 - \frac{1}{4s^2}S^3$ has eigenvalues $\lambda_{i'}(S') = \frac{3}{4}\lambda_i(S) - \frac{1}{2s}\lambda_i^2(S) - \frac{1}{4s^2}\lambda_i^3(S)$, so by taking $s = 2\|S\|_{op}, \lambda_{i'}(S')$ is exactly the i^{th} eigenvalue of S'.

This further gives

$$\lambda_{min}(S') \geq \frac{1}{4} \left(1 + 4\eta \left(a\lambda_{min}(M) - \|E\|_{op} - \|D\|_{op}^{2} \|M\|_{op} \right) \right)^{2} \lambda_{min}(S)$$

$$+ \frac{1}{4s} \left(1 - 4\eta s \|D\|_{op} \|M\|_{op} \right)^{2} \lambda_{min}^{2}(S) + \frac{1}{4s} \left(1 - 8\eta s \|M\|_{op} \|D\|_{op} \right) \lambda_{min}^{2}(S)$$

$$+ \frac{1}{4s^{2}} \left(1 - 4\eta s^{2} \|M\|_{op} \right)^{2} \lambda_{min}^{3}(S) + \left(\frac{3}{4} \lambda_{min}(S) - \frac{1}{2s} \lambda_{min}^{2}(S) - \frac{1}{4s^{2}} \lambda_{min}^{3}(S) \right)$$

$$+ \eta^{2} \|M'_{error}\|_{op}$$

$$\geq \lambda_{min}(S) + 2\eta \left(a\lambda_{min}(M) - 2\|D\|_{op} \|M\|_{op} \lambda_{min}(S) - \|M\|_{op} \lambda_{min}^{2}(S) \right) \lambda_{min}(S)$$

$$- 2\eta \left(\|E\|_{op} + \|D\|_{op}^{2} \|M\|_{op} \right) \lambda_{min}(S) + \eta^{2} \|M'_{error}\|_{op}.$$
(102)

From $\lambda_{min}(M) \geq \lambda_{min}(S)$, the proof is completed.

C.5 LEMMAS ON REGULARIZATION, GRADIENT DESCENT

Theorem 30. Suppose for all $j \in [1,4] \cap \mathbb{N}^*$, $\sigma_{min}(W_j(t)) \geq \delta$, $\sigma_{max}(W_j(t)) \leq M$, then the convergence rate of the regularization term is lower bounded by:

$$\mathcal{L}_{reg}(t+1) \leq \left(1 - \frac{8}{3} \frac{\eta a \delta^4}{M^2 + \delta^2}\right) \cdot \mathcal{L}_{reg}(t)$$

$$+ \eta^2 O\left(a^2 M^4 \mathcal{L}_{reg}(t) + \sqrt{a \mathcal{L}_{reg}(t)} M^6 \mathcal{L}_{ori}(t)\right)$$

$$+ \eta^4 O\left(a M^{12} \mathcal{L}_{ori}(t)^2 + a^3 M^4 \mathcal{L}_{reg}(t)^2\right).$$

$$(103)$$

Proof.

$$\Delta_{j,j+1}(t+1) - \Delta_{j,j+1}(t) = 2\eta a W_{j}(t) \Delta_{j-1,j}(t) W_{j}(t)^{H} + 2\eta a W_{j+1}(t)^{H} \Delta_{j+1,j+2}(t) W_{j+1}(t) - \eta a \Delta_{j,j+1}(t) \left(W_{j}(t) W_{j}(t)^{H} + W_{j+1}(t)^{H} W_{j+1}(t) \right) - \eta a \left(W_{j}(t) W_{j}(t)^{H} + W_{j+1}(t)^{H} W_{j+1}(t) \right) \Delta_{j,j+1}(t) + \eta^{2} \left[\nabla_{W_{j}} \mathcal{L}(t) \nabla_{W_{j}} \mathcal{L}(t)^{H} - \nabla_{W_{j+1}} \mathcal{L}(t)^{H} \nabla_{W_{j+1}} \mathcal{L}(t) \right].$$
(104)

From

$$\|\nabla_{W_{j}}\mathcal{L}(t)\|_{F} \leq \|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\|_{F} + \|\nabla_{W_{j}}\mathcal{L}_{reg}(t)\|_{F}$$

$$= O\left(M^{3}\sqrt{\mathcal{L}_{ori}(t)} + M\sqrt{a\mathcal{L}_{reg}(t)}\right)$$

$$\|\Delta_{j,j+1}(t+1) - \Delta_{j,j+1}(t)\|_{F} = O\left(\eta M^{2}\sqrt{a\mathcal{L}_{reg}(t)} + \eta^{2} \|\nabla_{W_{j}}\mathcal{L}(t)\|_{F}^{2}\right)$$

$$= O\left(\eta M^{2}\sqrt{a\mathcal{L}_{reg}(t)} + \eta^{2} M^{6}\mathcal{L}_{ori}(t) + \eta^{2} a M^{2}\mathcal{L}_{reg}(t)\right).$$
(105)

We have

$$\mathcal{L}_{reg}(t+1) - \mathcal{L}_{reg}(t) = 2a \sum_{j=1}^{3} \langle \Delta_{j,j+1}(t+1) - \Delta_{j,j+1}(t), \Delta_{j,j+1}(t) \rangle
+ a \sum_{j=1}^{3} \|\Delta_{j,j+1}(t+1) - \Delta_{j,j+1}(t)\|_{F}^{2}
= -4\eta a^{2} \sum_{j=1}^{4} \|\Delta_{j,j+1}(t)W_{j}(t) - W_{j}(t)\Delta_{j-1,j}(t)\|_{F}^{2}
+ O\left(\eta^{2}\sqrt{a\mathcal{L}_{reg}(t)}\left(aM^{2}\mathcal{L}_{reg}(t) + M^{6}\mathcal{L}_{ori}(t)\right)\right)
+ O\left(\eta^{2}a^{2}M^{4}\mathcal{L}_{reg}(t) + \eta^{4}aM^{12}\mathcal{L}_{ori}(t)^{2} + \eta^{4}a^{3}M^{4}\mathcal{L}_{reg}(t)^{2}\right)
= -4\eta a^{2} \sum_{j=1}^{4} \|\Delta_{j,j+1}(t)W_{j}(t) - W_{j}(t)\Delta_{j-1,j}(t)\|_{F}^{2}
+ \eta^{2}O\left(a^{2}M^{4}\mathcal{L}_{reg}(t) + \sqrt{a\mathcal{L}_{reg}(t)}M^{6}\mathcal{L}_{ori}(t)\right)
+ \eta^{4}O\left(aM^{12}\mathcal{L}_{ori}(t)^{2} + a^{3}M^{4}\mathcal{L}_{reg}(t)^{2}\right).$$
(106)

Follow previous analysis in continuous case,

$$\sum_{j=1}^{4} \|\Delta_{j,j+1}(t)W_j(t) - W_j(t)\Delta_{j-1,j}(t)\|^2 \ge \frac{2}{3} \frac{\delta^4}{M^2 + \delta^2} \sum_{i=1}^{3} \|\Delta_{i,i+1}(t)\|_F^2.$$
 (107)

Then the proof is done.

Theorem 31. The maximum and minimum singular values of W_j s are irrelevant of the regularization term.

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Under
$$\eta \leq \min\left(\frac{1}{18a \max_{j,k} \sigma_k^2(W_j(t))}, \frac{\min_{j,k} \sigma_k(W_j(t))}{3 \max_j \left\|\nabla_{W_j} \mathcal{L}_{ori}(t)\right\|_{op}}\right)$$
,
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1623

$$\max_{j,k} \sigma_{k}^{2}(W_{j}(t+1)) - \max_{j,k} \sigma_{k}^{2}(W_{j}(t)) \leq 2\eta \max_{j,k} \sigma_{k}(W_{j}(t)) \max_{j} \left\| \nabla_{W_{j}} \mathcal{L}_{ori}(t) \right\|_{op}
+ \eta^{2} O\left(\left\| \nabla_{W_{j}} \mathcal{L}_{ori}(t) \right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t)) \right)
\min_{j,k} \sigma_{k}^{2}(W_{j}(t+1)) - \min_{j,k} \sigma_{k}^{2}(W_{j}(t)) \geq -2\eta \min_{j,k} \sigma_{k}(W_{j}(t)) \max_{j} \left\| \nabla_{W_{j}} \mathcal{L}_{ori}(t) \right\|_{op}
+ \eta^{2} O\left(\left\| \nabla_{W_{j}} \mathcal{L}_{ori}(t) \right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t)) \right).$$
(108)

Proof. For simplicity, set $W_0 \equiv W_1$, $W_5 \equiv W_4$. Generally,

1651
$$W_{j}(t+1)W_{j}(t+1)^{H} = W_{j}(t)W_{j}(t)^{H} - \eta W_{j}(t)\nabla_{W_{j}}\mathcal{L}(t)^{H} - \eta \nabla_{W_{j}}\mathcal{L}(t)W_{j}(t)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}(t)^{H} = W_{j}(t)W_{j}(t)^{H} - \eta W_{j}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} - \eta \nabla_{W_{j}}\mathcal{L}_{ori}(t)W_{j}(t)^{H} = W_{j}(t)W_{j}(t)^{H} - \eta W_{j}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} - \eta \nabla_{W_{j}}\mathcal{L}_{ori}(t)W_{j}(t)^{H} + 2\eta aW_{j}(t)\Delta_{j-1,j}(t)W_{j}(t)^{H} - \eta aW_{j}(t)W_{j}(t)^{H}\Delta_{j,j+1}(t) - \eta a\Delta_{j,j+1}(t)W_{j}(t)W_{j}(t)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}(t)^{H} = \frac{1}{3}W_{j}(t)\left(I + 3\eta a\Delta_{j-1,j}(t)\right)^{2}W_{j}(t)^{H} = \frac{1}{3}\left(I - 3\eta a\Delta_{j,j+1}(t)\right)W_{j}(t)W_{j}(t)^{H}\left(I - 3\eta a\Delta_{j,j+1}(t)\right) + \frac{1}{3}\left(W_{j}(t) - 3\eta\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right)\left(W_{j}(t) - 3\eta\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} + \eta^{2}\nabla_{W_{j}}\mathcal{L}_{ori}(t)\nabla_{W_{j}}\mathcal{L}_{ori}(t)^{H} - 3\eta^{2}a^{2}W_{j}(t)\Delta_{j-1,j}(t)^{2}W_{j}(t)^{H} - 3\eta^{2}a^{2}\Delta_{j,j+1}(t)W_{j}(t)W_{j}(t)^{H}\Delta_{j,j+1}(t).$$

Notice that $W_j(t) \left(I + 3\eta a \Delta_{j-1,j}(t)\right)^2 W_j(t)^H$ and $\left(I + 3\eta a \Delta_{j-1,j}(t)\right) W_j(t)^H W_j(t) \left(I + 3\eta a \Delta_{j-1,j}(t)\right)$ shares the same eigenvalues. Then from Lemma 28, the maximum and minimum singular values of $W_j(t+1)$ satisfy

$$\begin{split} \sigma_{max}^{2}(W_{j}(t+1)) &\leq \frac{1}{3}\sigma_{max}^{2}(W_{j}(t)) \left[1 + 3\eta a \left(\sigma_{max}^{2}(W_{j-1}(t)) - \sigma_{max}^{2}(W_{j}(t))\right)\right]^{2} \\ &+ \frac{1}{3}\sigma_{max}^{2}(W_{j}(t)) \left[1 + 3\eta a \left(\sigma_{max}^{2}(W_{j+1}(t)) - \sigma_{max}^{2}(W_{j}(t))\right)\right]^{2} \\ &+ \frac{1}{3} \left[\sigma_{max}(W_{j}(t)) + 3\eta \left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}\right]^{2} \\ &+ \eta^{2}O\left(\left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t))\right) \\ &= \sigma_{max}^{2}(W_{j}(t)) \left[1 + 3\eta a \left(\sigma_{max}^{2}(W_{j+1}(t)) + \sigma_{max}^{2}(W_{j-1}(t)) - 2\sigma_{max}^{2}(W_{j}(t))\right)\right] \\ &+ 2\eta \sigma_{max}(W_{j}(t)) \left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op} + \eta^{2}O\left(\left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t))\right) \\ \sigma_{min}^{2}(W_{j}(t+1)) &\geq \frac{1}{3}\sigma_{min}^{2}(W_{j}(t)) \left[1 + 3\eta a \left(\sigma_{min}^{2}(W_{j-1}(t)) - \sigma_{min}^{2}(W_{j}(t))\right)\right]^{2} \\ &+ \frac{1}{3}\left[\sigma_{min}(W_{j}(t)) - 3\eta \left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}\right]^{2} \\ &+ \eta^{2}O\left(\left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t))\right) \\ &= \sigma_{min}^{2}(W_{j}(t)) \left[1 + 3\eta a \left(\sigma_{min}^{2}(W_{j+1}(t)) + \sigma_{min}^{2}(W_{j-1}(t)) - 2\sigma_{min}^{2}(W_{j}(t))\right)\right] \\ &- 2\eta \sigma_{min}(W_{j}(t)) \left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}^{2} + \eta^{2}O\left(\left\|\nabla_{W_{j}}\mathcal{L}_{ori}(t)\right\|_{op}^{2} + a^{2} \max_{j,k} \sigma_{k}^{6}(W_{j}(t))\right). \end{split}$$

By taking maximum and minimum over $j \in [1,4] \cap \mathbb{N}^*$ (for $\eta \leq \frac{1}{6a\max_{j,k} \sigma_k^2(W_j(t))}$, the first term of R.H.S can be upper bounded by $\max_{j,k} \sigma_k^2(W_j(t))$ or lower bounded by $\min_{j,k} \sigma_k^2(W_j(t))$ respectively), the proof is completed.

D DYNAMICS UNDER BALANCED INITIALIZATION

This section analyzes the training dynamics under balanced initialization.

At the beginning, We derive some properties from Lemma 24. Under balanced condition,

$$W_{\prod_{L},j}W_{\prod_{L},j}^{H} = \left(\prod_{k=N}^{j}W_{k}\right)\left(\prod_{k=N}^{j}W_{k}\right)^{H} = \left(W_{N}W_{N}^{H}\right)^{N-j+1}$$

$$W_{\prod_{R},j}^{H}W_{\prod_{R},j} = \left(\prod_{k=j}^{1}W_{k}\right)^{H}\left(\prod_{k=j}^{1}W_{k}\right) = \left(W_{1}^{H}W_{1}\right)^{N-j+1}.$$
(111)

Consider j = 1 and j = N, then

$$W_N W_N^H = (WW^H)^{1/N} = U\Sigma_w^2 U^H W_1 W_1^H = (W^H W)^{1/N} = V\Sigma_w^2 V^H.$$
(112)

Suppose the non-negative ASVD of product matrix is $W = U \sum_{w}^{N} V^{H}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_w^2 U^H \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(W_N W_N^H \right) = \Sigma V \Sigma_w^N U^H + U \Sigma_w^N V^H \Sigma^H - 2U \Sigma_w^{2N} U^H
\frac{\mathrm{d}}{\mathrm{d}t} \left(V \Sigma_w^2 V^H \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(W_1^H W_1 \right) = V \Sigma_w^N U^H \Sigma + \Sigma^H U \Sigma_w^N V^H - 2V \Sigma_w^{2N} V^H
\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_{j=1}^N U \Sigma_w^{2(j-1)} U^H \Sigma V \Sigma_w^{2(N-j)} V^H - N U \Sigma_w^{3N-2} V^H.$$
(113)

The dynamics of $\sigma_r := \sigma_{w,r}^N$ is presented in (69).

D.1 SKEW-HERMITIAN ERROR

 In this section, we prove Theorem below:

Theorem 32. The skew-symmetric error is non-increasing.

For $\mathbb{F} = \mathbb{R}$, under balanced Gaussian initialization, suppose the ASVD of product matrix is $W(t) = U(t)\Sigma_w(t)^N V(t)^\top$, furthermore assume that the singular values of the product matrix W(0) are distinct and different from zero at initialization (refer to Lemma 2 in Arora et al. (2019b)).

Denote
$$\sigma_{w,j}=(\Sigma_w)_{jj}$$
, $U'=\Sigma^{1/2}U$, $V'=\Sigma^{1/2}V$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \Sigma^{1/2} (U - V) \Sigma_w \|_F^2 = -2 \sum_j \sigma_{w,j}^N \cdot \| \Sigma^{1/2} \left(u_j' - v_j' \right) \|^2 - 2 \sum_j \sigma_{w,j}^{2N} \cdot \| u_j' - v_j' \|^2
- \sum_{k \neq j} \frac{\sigma_{w,j}^2 \sigma_{w,k}^2 \left(\sigma_{w,j}^{N-2} - \sigma_{w,k}^{N-2} \right)}{\sigma_{w,j}^2 - \sigma_{w,k}^2} \left| u_j'^\top v_k' - v_j'^\top u_k' \right|^2
< 0.$$
(114)

Proof. By (69)

$$\frac{\mathrm{d}\sigma_{w,j}}{\mathrm{d}t} = \sigma_{w,j}^{N-1} \left(\frac{\langle u_j', v_j' \rangle + \langle v_j', u_j' \rangle}{2} - \sigma_{w,j}^N \right). \tag{115}$$

By Lemma 2 in Arora et al. (2019b),

$$\begin{cases}
\frac{\mathrm{d}U}{\mathrm{d}t} = U(F \odot M_U), & (M_U)_{jk} = \left\langle v_k', u_j' \right\rangle \sigma_{w,k}^N + \left\langle u_k', v_j' \right\rangle \sigma_{w,j}^N - 2\sigma_{w,j}^{2N} \delta_{jk} \\
\frac{\mathrm{d}V}{\mathrm{d}t} = V(F \odot M_V), & (M_V)_{jk} = \left\langle u_k', v_j' \right\rangle \sigma_{w,k}^N + \left\langle v_k', u_j' \right\rangle \sigma_{w,j}^N - 2\sigma_{w,j}^{2N} \delta_{jk}
\end{cases}$$
(116)

Here $\langle a,b\rangle := b^{\top}a$ follows the standard definition of (complex) inner product. Then

$$\frac{\mathrm{d}U'^{H}V'}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t}^{\top} \Sigma V + U^{\top} \Sigma \frac{\mathrm{d}V}{\mathrm{d}t} = (F^{\top} \odot M_{U}^{\top}) U^{\top} \Sigma V + U^{\top} \Sigma V (F \odot M_{V})$$

$$\frac{\mathrm{d}U'^{H}U'}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t}^{\top} \Sigma U + U^{\top} \Sigma \frac{\mathrm{d}U}{\mathrm{d}t} = (F^{\top} \odot M_{U}^{\top}) U^{\top} \Sigma U + U^{\top} \Sigma U (F \odot M_{U})$$

$$\frac{\mathrm{d}V'^{H}V'}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}^{\top} \Sigma V + V^{\top} \Sigma \frac{\mathrm{d}V}{\mathrm{d}t} = (F^{\top} \odot M_{V}^{\top}) V^{\top} \Sigma V + V^{\top} \Sigma V (F \odot M_{V}).$$
(117)

For each diagonal entry,

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$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle u_j', v_j' \right\rangle = \left(\frac{\mathrm{d}U'^H V'}{\mathrm{d}t}\right)_{jj}$$
1786
1787
$$= \sum_{k \neq j} \frac{1}{\sigma_{w,j}^2 - \sigma_{w,k}^2} \left[\left(\left\langle u_j', v_k' \right\rangle^2 + \left\langle u_k', v_j' \right\rangle^2 \right) \sigma_{w,j}^N + 2 \left\langle v_k', u_j' \right\rangle \left\langle v_j', u_k' \right\rangle \sigma_{w,k}^N \right]$$
1788
$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle u_j', u_j' \right\rangle = \left(\frac{\mathrm{d}U'^H U'}{\mathrm{d}t}\right)_{jj}$$
1790
1791
$$= \sum_{k \neq j} \frac{1}{\sigma_{w,j}^2 - \sigma_{w,k}^2} \left[\left\langle u_k', v_j' \right\rangle \left\langle u_j', u_k' \right\rangle \sigma_{w,j}^N + \left\langle v_k', u_j' \right\rangle \left\langle u_j', u_k' \right\rangle \sigma_{w,k}^N \right]$$
1793
1794
$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle v_j', v_j' \right\rangle = \left(\frac{\mathrm{d}V'^H V'}{\mathrm{d}t}\right)_{jj}$$
1795
1796
$$= \sum_{k \neq j} \frac{1}{\sigma_{w,j}^2 - \sigma_{w,k}^2} \left[\left\langle v_k', u_j' \right\rangle \left\langle v_j', v_k' \right\rangle \sigma_{w,j}^N + \left\langle u_k', v_j' \right\rangle \left\langle v_j', v_k' \right\rangle \sigma_{w,k}^N \right],$$
1798

which further gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u'_{j} - v'_{j}\|^{2}$$

$$= \sum_{k \neq j} \frac{\sigma_{w,j}^{N}}{\sigma_{w,j}^{2} - \sigma_{w,k}^{2}} \cdot \left[-2 \left(\left| \left\langle u'_{j}, v'_{k} \right\rangle \right|^{2} + \left| \left\langle u'_{k}, v'_{j} \right\rangle \right|^{2} \right) + \left(\left\langle u'_{k}, v'_{j} \right\rangle \left\langle u'_{j}, u'_{k} \right\rangle + \left\langle v'_{k}, u'_{j} \right\rangle \left\langle v'_{j}, v'_{k} \right\rangle \right) \right] + \sum_{k \neq j} \frac{\sigma_{w,k}^{N}}{\sigma_{w,j}^{2} - \sigma_{w,k}^{2}} \cdot \left[-4 \left\langle v'_{k}, u'_{j} \right\rangle \left\langle v'_{j}, u'_{k} \right\rangle + \left(\left\langle v'_{k}, u'_{j} \right\rangle \left\langle u'_{j}, u'_{k} \right\rangle \right) + \left(\left\langle u'_{k}, v'_{j} \right\rangle \left\langle v'_{j}, v'_{k} \right\rangle \right) \right].$$
(119)

For the L.H.S. of (114),

$$\frac{\mathrm{d}}{\mathrm{d}t} \| (U' - V') \Sigma_{w} \|_{F}^{2} = \sum_{j} \| u'_{j} - v'_{j} \|^{2} \frac{\mathrm{d}}{\mathrm{d}t} \sigma_{w,j}^{2} + \sum_{j} \sigma_{w,j}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{j} - v'_{j} \|^{2}$$

$$= \sum_{j} \sigma_{w,j}^{N} \left(\langle u'_{j}, v'_{j} \rangle + \langle v'_{j}, u'_{j} \rangle - 2\sigma_{w,j}^{N} \right) \| u'_{j} - v'_{j} \|^{2}$$

$$+ \frac{1}{2} \left(\sum_{j} \sigma_{w,j}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{j} - v'_{j} \|^{2} + \sum_{k} \sigma_{w,k}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{k} - v'_{k} \|^{2} \right). \tag{120}$$

The first term can be written by

$$\sum_{j} \sigma_{w,j}^{N} \left(\langle u'_{j}, v'_{j} \rangle + \langle v'_{j}, u'_{j} \rangle - 2\sigma_{w,j}^{N} \right) \|u'_{j} - v'_{j}\|^{2}$$

$$= \sum_{j} \sigma_{w,j}^{N} \left(u'_{j}^{\top} u'_{j} u'_{j}^{\top} v'_{j} + v'_{j}^{\top} u'_{j} u'_{j}^{\top} u'_{j} + u'_{j}^{\top} v'_{j} v'_{j}^{\top} v'_{j} + v'_{j}^{\top} v'_{j} v'_{j}^{\top} u'_{j} \right)$$

$$- \sum_{j} \sigma_{w,j}^{N} \left(u'_{j}^{\top} v'_{j} + v'_{j}^{\top} u'_{j} \right)^{2} - 2 \sum_{j} \sigma_{w,j}^{2N} \cdot \|u'_{j} - v'_{j}\|^{2}.$$
(121)

For the second term,

$$\frac{1}{2} \left(\sum_{j} \sigma_{w,j}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{j} - v'_{j} \|^{2} + \sum_{k} \sigma_{w,k}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{k} - v'_{k} \|^{2} \right)
= - \sum_{j,k,j\neq k} \frac{\sigma_{w,j}^{2} \sigma_{w,k}^{2} \left(\sigma_{w,j}^{N-2} - \sigma_{w,k}^{N-2} \right)}{\sigma_{w,j}^{2} - \sigma_{w,k}^{2}} \left| \left\langle v'_{k}, u'_{j} \right\rangle - \left\langle u'_{k}, v'_{j} \right\rangle \right|^{2}
- 2 \sum_{j,k,j\neq k} \sigma_{w,j}^{N} \cdot \left(\left| \left\langle u'_{j}, v'_{k} \right\rangle \right|^{2} + \left| \left\langle u'_{k}, v'_{j} \right\rangle \right|^{2} \right)
+ 2 \sum_{j,k,j\neq k} \sigma_{w,j}^{N} \cdot \left(\left\langle u'_{k}, v'_{j} \right\rangle \left\langle u'_{j}, u'_{k} \right\rangle + \left\langle u'_{j}, v'_{k} \right\rangle \left\langle v'_{k}, v'_{j} \right\rangle \right).$$
(122)

Notice that

$$-\sum_{j} \sigma_{w,j}^{N} \left(u_{j}^{\prime \top} v_{j}^{\prime} + v_{j}^{\prime \top} u_{j}^{\prime} \right)^{2} - 2 \sum_{j,k,j \neq k} \sigma_{w,j}^{N} \left(\left| \left\langle u_{j}^{\prime}, v_{k}^{\prime} \right\rangle \right|^{2} + \left| \left\langle u_{k}^{\prime}, v_{j}^{\prime} \right\rangle \right|^{2} \right)$$

$$= -2 \sum_{j} \sigma_{w,j}^{N} \cdot \left(u_{j}^{\prime \top} V^{\prime} V^{\prime \top} u_{j}^{\prime} + v_{j}^{\prime \top} U^{\prime} U^{\prime \top} v_{j}^{\prime} \right)$$

$$= -2 \sum_{j} \sigma_{w,j}^{N} \cdot \left(u_{j}^{\prime \top} \Sigma u_{j}^{\prime} + v_{j}^{\prime \top} \Sigma v_{j}^{\prime} \right),$$

$$(123)$$

and

$$\sum_{j} \sigma_{w,j}^{N} \left(u_{j}^{'} u_{j}^{'} u_{j}^{'} u_{j}^{'} + v_{j}^{'} u_{j}^{'} u_{j}^{'} u_{j}^{'} + u_{j}^{'} v_{j}^{'} v_{j}^{'} v_{j}^{'} v_{j}^{'} v_{j}^{'} v_{j}^{'} v_{j}^{'} u_{j}^{'} \right) \\
+ 2 \sum_{j,k,j \neq k} \sigma_{w,j}^{N} \cdot \left(\left\langle u_{k}^{\prime}, v_{j}^{\prime} \right\rangle \left\langle u_{j}^{\prime}, u_{k}^{\prime} \right\rangle + \left\langle u_{j}^{\prime}, v_{k}^{\prime} \right\rangle \left\langle v_{k}^{\prime}, v_{j}^{\prime} \right\rangle \right) \\
= 2 \sum_{j,k} \sigma_{w,j}^{N} \cdot \Re \left(u_{j}^{\prime} \left(U U^{\prime} + V V^{\prime} \right) v_{j} \right) \\
= 2 \sum_{j} \sigma_{w,j}^{N} \cdot \left(u_{j}^{\prime} \left(V U^{\prime} + V V^{\prime} \right) v_{j} \right) . \tag{124}$$

By combining the results above, L.H.S. = R.H.S. This completes the proof.

For even depth $2 \mid N$, we have a similar result written in matrix form:

Theorem 33. If $2 \mid N$, the singular values of the product matrix W(0) are different from zero at initialization, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \Sigma^{1/2} (U - V) \Sigma_w \right\|_F^2 = -2 \left\| \Sigma (U - V) \Sigma_w^{N/2} \right\|_F^2 - 2 \left\| \Sigma^{1/2} (U - V) \Sigma_w^N \right\|_F^2 - 2 \Re \left(\operatorname{tr} \left(\sum_{j=1}^{N/2-1} \Sigma U \Sigma_w^{2j} \left(U^H \Sigma V - V^H \Sigma U \right) \Sigma_w^{N-2j} V^H \right) \right) \right)$$

$$\leq 0 \tag{125}$$

We present another approach of proof which takes the inverse of some terms. This approach adapts to the skew-hermitian term in unbalanced initialization, where the proof of Theorem 32 in does not hold

To prove the theorem, we introduce the following lemma.

Lemma 34. If $2 \mid N$, Σ_w is full rank at initialization, then $\forall k = 0, 1, \dots, N/2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(U \pm V)\Sigma_{w}^{2k}(U \pm V)^{H}$$

$$= \sum_{j=1}^{k} \left[U\Sigma_{w}^{2(j-1)}U^{H}\Sigma V\Sigma_{w}^{N+2k-2j}U^{H} + U\Sigma_{w}^{N+2(j-1)}V^{H}\Sigma U\Sigma_{w}^{2(k-j)}U^{H} \right]$$

$$+ V\Sigma_{w}^{2(j-1)}V^{H}\Sigma U\Sigma_{w}^{N+2k-2j}V^{H} + V\Sigma_{w}^{N+2(j-1)}U^{H}\Sigma V\Sigma_{w}^{2(k-j)}V^{H} \right]$$

$$\pm \sum_{j=1}^{N/2+k} \left[U\Sigma_{w}^{2(j-1)}U^{H}\Sigma V\Sigma_{w}^{N+2k-2j}V^{H} + V\Sigma_{w}^{2(j-1)}V^{H}\Sigma U\Sigma_{w}^{N+2k-2j}U^{H} \right]$$

$$\mp \sum_{j=1}^{N/2-k} \left[U\Sigma_{w}^{2(j-1+k)}V^{H}\Sigma U\Sigma_{w}^{N-2j}V^{H} + V\Sigma_{w}^{2(j-1+k)}U^{H}\Sigma V\Sigma_{w}^{N-2j}U^{H} \right]$$

$$-2k(U \pm V)\Sigma_{w}^{2(N+k-1)}(U \pm V)^{H}.$$
(126)

Proof. $\forall l \in \mathbb{N}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{2l} U^{H} \right) = \sum_{j=1}^{l} U \Sigma_{w}^{2(j-1)} U^{H} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{2} U^{H} \right) \right) U \Sigma_{w}^{2(l-j)} U^{H}
= \sum_{j=1}^{l} U \Sigma_{w}^{2(j-1)} U^{H} \left(\Sigma V \Sigma_{w}^{N} U^{H} + U \Sigma_{w}^{N} V^{H} \Sigma^{H} - 2U \Sigma_{w}^{2N} U^{H} \right) U \Sigma_{w}^{2(l-j)} U^{H}.$$
(127)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(V \Sigma_{w}^{2l} V^{H} \right) = \sum_{j=1}^{l} V \Sigma_{w}^{2(j-1)} V^{H} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(V \Sigma_{w}^{2} V^{H} \right) \right) V \Sigma_{w}^{2(l-j)} V^{H}
= \sum_{j=1}^{l} V \Sigma_{w}^{2(j-1)} V^{H} \left(\Sigma U \Sigma_{w}^{N} V^{H} + V \Sigma_{w}^{N} U^{H} \Sigma^{H} - 2V \Sigma_{w}^{2N} V^{H} \right) V \Sigma_{w}^{2(l-j)} V^{H}.$$
(128)

From Lemma 24, $U\Sigma_w^{N-2k}U^H$ is invertible at arbitrary time $t\in[0,+\infty)$, thus

$$\begin{split} \frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_w^{-(N-2k)} U^H \right) &= - \left(U \Sigma_w^{N-2k} U^H \right)^{-1} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_w^{N-2k} U^H \right) \right] \left(U \Sigma_w^{N-2k} U^H \right)^{-1} \\ &= - \left(U \Sigma_w^{-(N-2k)} U^H \right) \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_w^{N-2k} U^H \right) \right] \left(U \Sigma_w^{-(N-2k)} U^H \right), \end{split} \tag{129}$$

which further gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{2k} V^{H} \right)$$

$$= \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{-(N-2k)} U^{H} \right) \right] U \Sigma_{w}^{N} V^{H} + U \Sigma_{w}^{-(N-2k)} U^{H} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{N} V^{H} \right) \right]$$

$$= - \left(U \Sigma_{w}^{-(N-2k)} U^{H} \right) \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{N-2k} U^{H} \right) \right] \left(U \Sigma_{w}^{2k} V^{H} \right)$$

$$= - \left(U \Sigma_{w}^{-(N-2k)} U^{H} \right) \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{N-2k} U^{H} \right) \right]$$

$$+ U \Sigma_{w}^{-(N-2k)} U^{H} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(U \Sigma_{w}^{N} V^{H} \right) \right]$$

$$= \sum_{j=1}^{N/2+k} U \Sigma_{w}^{2(j-1)} U^{H} \Sigma V \Sigma_{w}^{N+2(k-j)} V^{H} + \sum_{j=1}^{N/2-k} U \Sigma_{w}^{2(k+j-1)} V^{H} \Sigma^{H} U \Sigma_{w}^{N-2j} V^{H}$$

$$-2k U \Sigma_{w}^{2(N+k-1)} V^{H}.$$

$$(130)$$

Combine (127), (128) and (130) together, then the proof is completed.

19621963 Now we present the proof of Theorem 33.

 Proof. Denote $Q = U^H \Sigma V$, calculate the L.H.S. of (125) by setting k = 1 in Lemma 34:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \Sigma^{1/2} (U - V) \Sigma_{w} \right\|_{F}^{2} \\
= \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr} \left(\Sigma (U - V) \Sigma_{w}^{2} (U - V)^{H} \right) \\
= -2 \operatorname{tr} \left(\Sigma^{2} (U - V) \Sigma_{w}^{N} (U - V)^{H} \right) - 2 \operatorname{tr} \left(\Sigma (U - V) \Sigma_{w}^{2N} (U - V)^{H} \right) \\
-2 \Re \left(\operatorname{tr} \left(\sum_{j=1}^{N/2-1} \Sigma U \Sigma_{w}^{2j} \left(U^{H} \Sigma V - V^{H} \Sigma U \right) \Sigma_{w}^{N-2j} V^{H} \right) \right) \\
= -2 \left\| \Sigma (U - V) \Sigma_{w}^{N/2} \right\|_{F}^{2} - 2 \left\| \Sigma^{1/2} (U - V) \Sigma_{w}^{N} \right\|_{F}^{2} \\
-2 \Re \left(\operatorname{tr} \left(\sum_{j=1}^{N/2-1} \Sigma_{w}^{2j} (Q - Q^{H}) \Sigma_{w}^{N-2j} Q^{H} \right) \right). \tag{131}$$

To analyze the last term,

$$\Re\left(\operatorname{tr}\left(\sum_{j=1}^{N/2-1} \Sigma_{w}^{2j}(Q - Q^{H}) \Sigma_{w}^{N-2j} Q^{H}\right)\right) \\
= \Re\left(\sum_{m,n} \left(\sum_{j=1}^{N/2-1} \sigma_{m}^{2j}(\Sigma_{w}) (Q_{mn} - \overline{Q_{nm}}) \sigma_{n}^{N-2j}(\Sigma_{w}) \overline{Q_{mn}}\right)\right) \\
= \frac{1}{2} \sum_{m,n} \left(\sum_{j=1}^{N/2-1} \sigma_{m}^{2j}(\Sigma_{w}) \sigma_{n}^{N-2j}(\Sigma_{w}) (|Q_{mn}|^{2} + |Q_{nm}|^{2} - 2\Re(Q_{mn}Q_{nm}))\right) \\
= \frac{1}{2} \sum_{m,n} |Q_{mn} - \overline{Q_{nm}}|^{2} \left(\sum_{j=1}^{N/2-1} \sigma_{m}^{2j}(\Sigma_{w}) \sigma_{n}^{N-2j}(\Sigma_{w})\right) \geq 0.$$
(132)

Thus for arbitrary $\Sigma \succ O$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| \Sigma^{1/2} (U - V) \Sigma_w \right\|_F^2 = -2 \left\| \Sigma (U - V) \Sigma_w^{N/2} \right\|_F^2 - 2 \left\| \Sigma^{1/2} (U - V) \Sigma_w^N \right\|_F^2 \\
- \sum_{m,n} \left| Q_{mn} - \overline{Q_{nm}} \right|^2 \left(\sum_{j=1}^{N/2-1} \sigma_m^{2j} (\Sigma_w) \sigma_n^{N-2j} (\Sigma_w) \right) \\
\leq 0.$$
(133)

which completes the proof.

D.2 HERMITIAN MAIN TERM

This section proves Theorem 5.

Proof. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}(U+V)\Sigma_{w}^{2}(U+V)^{H}
= \Sigma(U+V)\Sigma_{w}^{N}(U+V)^{H} + (U+V)\Sigma_{w}^{N}(U+V)^{H}\Sigma - 2(U+V)\Sigma_{w}^{2N}(U+V)^{H}
+ \sum_{j=1}^{N/2-1} \left[U\Sigma_{w}^{2j}\left(U^{H}\Sigma V - V^{H}\Sigma U\right)\Sigma_{w}^{N-2j}V^{H} + V\Sigma_{w}^{2j}\left(V^{H}\Sigma U - U^{H}\Sigma V\right)\Sigma_{w}^{N-2j}U^{H}\right].$$
(134)

Denote
$$P=\frac{(U+V)\Sigma_w}{2},$$
 $Q=\frac{(U-V)\Sigma_w}{2}.$ Then $P^HQ=-Q^HP,$ $\Sigma_w^2=P^HP+Q^HQ.$

From $ABC^H - CBA^H = \frac{1}{2}\left[(A-C)B(A+C)^H - (A+C)B(A-C)^H\right]$ for arbitrary A,B,C we have

$$\frac{\mathrm{d}}{\mathrm{d}t}PP^{H} = \sum P \sum_{w}^{N-2} P^{H} + P \sum_{w}^{N-2} P^{H} \sum_{w} - 2P \sum_{w}^{2N-2} P^{H}
+ \sum_{j=1}^{N/2-1} \left[Q \sum_{w}^{2j-2} \left(Q^{H} \sum_{w} P - P^{H} \sum_{w} Q \right) \sum_{w}^{N-2j-2} P^{H}
- P \sum_{w}^{2j-2} \left(Q^{H} \sum_{w} P - P^{H} \sum_{w} Q \right) \sum_{w}^{N-2j-2} Q^{H} \right].$$
(135)

Suppose the k^{th} eigenvalue and eigenvector of PP^H are x_k^2 and ξ_k respectively, $P^H\xi_k=x_k\eta_k$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{k}^{2} = \xi_{k}^{H} \left(\frac{\mathrm{d}}{\mathrm{d}t}PP^{H}\right)\xi_{k}$$

$$= 2\xi_{k}^{H}\Sigma P\Sigma_{w}^{N-2}P^{H}\xi_{k} - 2\xi_{k}^{H}P\Sigma_{w}^{2N-2}P^{H}\xi_{k}$$

$$+ 2\xi_{k}^{H} \left[\sum_{j=1}^{N/2-1} Q\Sigma_{w}^{2j-2} \left(Q^{H}\Sigma P - P^{H}\Sigma Q\right)\Sigma_{w}^{N-2j-2}P^{\top}\right]\xi_{k}.$$
(136)

We focus on $N=4, \Sigma=\sigma_1(\Sigma)I$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{k}^{2} = 2\sigma_{1}(\Sigma)\xi_{k}^{H}P\Sigma_{w}^{2}P^{H}\xi_{k} - 2\xi_{k}^{H}P\Sigma_{w}^{6}P^{H}\xi_{k} + 4\sigma_{1}(\Sigma)\xi_{k}^{H}QQ^{H}PP^{H}\xi_{k}
= 2\sigma_{1}(\Sigma)\xi_{k}^{H}P\Sigma_{w}^{2}P^{H}\xi_{k} - 2\xi_{k}^{H}P\Sigma_{w}^{6}P^{H}\xi_{k} + 4\sigma_{1}(\Sigma)x_{k}^{2}\xi_{k}^{H}QQ^{H}\xi_{k}.$$
(137)

For the second term:

$$\xi_{k}^{H} P \Sigma_{w}^{6} P^{H} \xi_{k} = \xi_{k}^{H} P \left(P^{H} P + Q^{H} Q \right) \Sigma_{w}^{2} \left(P^{H} P + Q^{H} Q \right) P^{H} \xi_{k}
= x_{k}^{4} \xi_{k}^{H} P \Sigma_{w}^{2} P^{H} \xi_{k} + 2x_{k}^{2} \xi_{k}^{H} P \Sigma_{w}^{2} Q^{H} Q P^{H} \xi_{k} + \xi_{k}^{H} P Q^{H} Q \Sigma_{w}^{2} Q^{H} Q P^{H} \xi_{k}
\leq x_{k}^{4} \xi_{k}^{H} P \Sigma_{w}^{2} P^{H} \xi_{k} + 2x_{k}^{4} \|Q\|_{op}^{2} \|\Sigma_{w}\|_{op}^{2} + x_{k}^{2} \|Q\|_{op}^{4} \|\Sigma_{w}\|_{op}^{2}.$$
(138)

From Theorem 33, $||Q||_{op} \leq ||Q||_F \leq ||Q(t=0)||_F$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{k}^{2} \geq \left(2\sigma_{1}(\Sigma) - x_{k}^{4}\right)\xi_{k}^{H}P\Sigma_{w}^{2}P^{H}\xi_{k} - 2x_{k}^{4}\|Q\|_{op}^{2}\|\Sigma_{w}\|_{op}^{2} - x_{k}^{2}\|Q\|_{op}^{4}\|\Sigma_{w}\|_{op}^{2}$$

$$\geq \left(2\sigma_{1}(\Sigma) - x_{k}^{4} - \frac{1}{2}\|\Sigma_{w}\|_{op}^{2}\|((U - V)\Sigma_{w})|_{t=0}\|_{F}^{2}\right)x_{k}^{4} - \frac{1}{16}x_{k}^{2}\|\Sigma_{w}\|_{op}^{2}\|((U - V)\Sigma_{w})|_{t=0}\|_{F}^{4}.$$
(139)

The lower bound is proved.

For the upper bound,

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{k}^{2} \leq 2\sigma_{1}(\Sigma)x_{k}^{2}\|\Sigma_{w}\|_{op}^{2} + 4\sigma_{1}(\Sigma)x_{k}^{2}\|Q\|_{op}^{2}.$$
(140)

This completes the proof.

Corollary 35. If for some k, $\sigma_k((U+V)\Sigma_w)|_{t=0}=0$, then $\sigma_k((U+V)\Sigma_w)\equiv 0$ for finite time $t\in [0,+\infty)$.

Proof. Denote $x_k \equiv \frac{1}{2}\sigma_k((U+V)\Sigma_w)$. By Lemma 23, $\|\Sigma - W\|_F \leq \|\Sigma - W(0)\|_F$. Then $\|\Sigma_w\|_{op}$ is bounded:

$$\|\Sigma_{w}\|_{op} = \|W\|_{op}^{1/N} \le (\|\Sigma\|_{op} + \|\Sigma - W\|_{op})^{1/N} \le (\|\Sigma\|_{op} + \|\Sigma - W\|_{F})^{1/N}$$

$$\le (\|\Sigma\|_{op} + \|\Sigma - W(0)\|_{F})^{1/N}.$$
(141)

Then from Theorem 5, there exists some $C \in (0, +\infty)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k^2 \le \sigma_1(\Sigma) \left(2\|\Sigma_w\|_{op}^2 + \|((U - V)\Sigma_w)|_{t=0}\|_F^2 \right) x_k^2 \le Cx_k^2. \tag{142}$$

Giving

$$x_k^2(t) \le x_k^2(0)e^{Ct} = 0. (143)$$

This completes the proof.

D.3 CONVERGENCE PROOF

This section states the global convergence guarantee under balanced Gaussian initialization, with gradient flow.

Theorem 36. Global convergence bound under balanced Gaussian initialization, gradient flow.

For four-layer matrix factorization under gradient flow, balanced Gaussian initialization with scaling factor $\epsilon \leq \frac{\sigma_1^{1/4}(\Sigma)}{4c_1^2c_2d^{29/8}}$, then for target matrix with identical singular values,

1. For $\mathbb{F} = \mathbb{R}$, with probability at least $\frac{1}{2}$ the loss does not converge to zero. Specifically,

2109
$$\mathcal{L}(t) \geq \frac{1}{2}\sigma_1^2(\Sigma), \, \forall t \in [0, +\infty).$$

2. For $\mathbb{F} = \mathbb{C}$ with high probability and for $\mathbb{F} = \mathbb{R}$ with probability close to $\frac{1}{2}$, there exists $T(\delta) = \frac{16c_2^2d^3}{\sigma_1(\Sigma)\epsilon^2} + \frac{1}{8\sigma_1^{3/2}(\Sigma)}\ln\left(\frac{d\sigma_1^2(\Sigma)}{\delta}\right)$, such that for any $\delta > 0$, when $t > T(\delta)$, $\mathcal{L}(t) < \delta$.

Remark 14. The first part of this Theorem can be generalized to general (bounded) balanced initialization.

Proof. For the first conclusion, by Theorem 3 and Corollary 35, for $\mathbb{F} = \mathbb{R}$, $\sigma_{min}((U+V)\Sigma_w) \equiv 0$ with probability at least $\frac{1}{2}$. Consequently $\sigma_{min}((U+V)\Sigma_w^N) \equiv 0$.

Suppose at time t, for some unit vector y, $(U+V)\sum_{w}^{N}y(t)=0$. Then

$$\|\Sigma - W\|_{F} = \|\sigma_{1}(\Sigma)I - U\Sigma_{w}^{N}V^{\top}\|_{F} = \|\sigma_{1}(\Sigma)V - U\Sigma_{w}^{N}\|_{F}$$

$$\geq \|\sigma_{1}(\Sigma)V - U\Sigma_{w}^{N}\|_{op} \geq \|(\sigma_{1}(\Sigma)V - U\Sigma_{w}^{N})y\|$$

$$= \|(\sigma_{1}(\Sigma)V + V\Sigma_{w}^{N})y\| = \|(\sigma_{1}(\Sigma) + \Sigma_{w}^{N})y\| \geq \sigma_{1}(\Sigma).$$
(145)

(144)

For the second part:

From Lemma 23, $\|\Sigma - W\|_F \le \|\Sigma - W(0)\|_F < 2\sqrt{d}\sigma_1(\Sigma)$. Thus for any time t,

$$\|\Sigma_{w}\|_{op} = \|W\|_{op}^{1/4} \le (\|\Sigma\|_{op} + \|\Sigma - W\|_{op})^{1/4} \le (\|\Sigma\|_{op} + \|\Sigma - W\|_{F})^{1/N}$$

$$\le (\|\Sigma\|_{op} + \|\Sigma - W(0)\|_{F})^{1/4} \le \sqrt{2}d^{1/8}\sigma_{1}^{1/4}(\Sigma).$$
(146)

From Theorem 3, for $\mathbb{F}=\mathbb{C}$ with high probability (while for $\mathbb{F}=\mathbb{R}$ with probability close to $\frac{1}{2}$), $x_k(t=0)\geq \frac{\epsilon}{2c_2d^{3/2}}, \|(U-V)\Sigma_w\|_F|_{t=0}\leq 2c_1d\epsilon$. Thus by taking $\epsilon\leq \frac{\sigma_1^{1/4}(\Sigma)}{4c_1^2c_2d^{29/8}}$, for t such that $x_k(t)\geq x_k(0)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k^2 \ge \left(2\sigma_1(\Sigma) - \left(4c_1^2d^{9/4} + 8c_1^4c_2^2d^{29/4}\right)\epsilon^2\sigma_1^{1/2}(\Sigma) - x_k^4\right)x_k^4 \ge \left(\frac{5}{4}\sigma_1(\Sigma) - x_k^4\right)x_k^4. \tag{147}$$

This indicates that all x_k monotonically increase to $\sigma_1^{1/4}(\Sigma)$ in $T_1 = \frac{4}{\sigma_1(\Sigma)} \cdot x_k(0)^{-2} = \frac{16c_2^2d^3}{\sigma_1(\Sigma)\epsilon^2}$, and never decrease to below $\sigma_1^{1/4}(\Sigma)$ for $t > T_1$.

By Theorem 20, $\sigma_{min}(\Sigma_w) \geq x_k$. Then combine with Lemma 23,

$$\mathcal{L}_{ori}(t) \le \mathcal{L}_{ori}(0)e^{-8\sigma_{min}^6(\Sigma_w(T_1))(t-T_1)} \le d\sigma_1^2(\Sigma)e^{-8\sigma_1^{3/2}(\Sigma)(t-T_1)}.$$
 (148)

Thus it takes at most $t=T_1+\frac{1}{8\sigma_1^{3/2}(\Sigma)}\ln\left(\frac{d\sigma_1^2(\Sigma)}{\delta}\right)$ to reach δ -convergence.

NOTATIONS AND PRELIMINARIES UNDER THE DEPTH OF FOUR, UNBALANCED

To tackle the unbalanced initialization with depth N=4, we make the following notations and derive some basic properties.

Below we denote $R = W_2^{-1}W_3^H$, $W_1' = RW_4^H$, $W = W_4W_3W_2W_1$, $M_2 = W_2^HW_2$, $M_1 = W_1W_1^H$, $M_{\Delta 1234} = W_2W_1W_1^HW_2^H - W_3^HW_4^HW_4W_3$ $M_1' = W_1'W_1'^H$, $e_{\Delta} = \sqrt{\sum_{i=1}^3 \|\Delta_{i,i+1}\|_F^2}$. Then:

$$W = W_1^{\prime H} M_2 W_1, \tag{149}$$

$$RR^{H} = W_{2}^{-1}W_{3}^{H}W_{3}W_{2}^{H-1} = I - W_{2}^{-1}\Delta_{23}W_{2}^{H-1},$$
(150)

$$R^{-1}R^{H-1} = W_3^{H-1}W_2W_2^HW_3^{-1} = I + W_3^{H-1}\Delta_{23}W_3^{-1},$$
(151)

$$M_{\Delta 1234} = \left(\left(W_2^H W_2 \right)^2 - \left(W_3 W_3^H \right)^2 \right) + W_3^H \Delta_{34} W_3 + W_2 \Delta_{12} W_2^H$$

$$= \frac{1}{2} \left(\Delta_{23} \left(W_3^H W_3 + W_2 W_2^H \right) + \left(W_3^H W_3 + W_2 W_2^H \right) \Delta_{23} \right)$$

$$+ W_3^H \Delta_{34} W_3 + W_2 \Delta_{12} W_2^H,$$
(152)

$$M_1' - M_1 = W_2^{-1} M_{\Delta 1234} W_2^{H-1}. {153}$$

Deducing that

$$||R||_{op} \le \sqrt{1 + \frac{1}{\sigma_{min}^2(W_2)} \cdot ||\Delta_{23}||_{op}} \le \sqrt{1 + \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta}},$$
 (154)

$$\|R^{-1}\|_{op} \le \sqrt{1 + \frac{1}{\sigma_{min}^2(W_3)} \cdot \|\Delta_{23}\|_{op}} \le \sqrt{1 + \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta}},$$
 (155)

$$||I - RR^H||_{op} \le \frac{1}{\sigma_{min}^2(W_2)} \cdot ||\Delta_{23}||_{op} \le \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta},$$
 (156)

$$||I - R^{-1}R^{H-1}||_{op} \le \frac{1}{\sigma_{min}^2(W_3)} \cdot ||\Delta_{23}||_{op} \le \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta}, \tag{157}$$

$$||M_{\Delta_{1234}}||_{op} \le (||W_{2}||_{op}^{2} + ||W_{3}||_{op}^{2}) ||\Delta_{23}||_{op} + ||W_{3}||_{op}^{2} ||\Delta_{34}||_{op} + ||W_{2}||_{op}^{2} ||\Delta_{12}||_{op} \le \sqrt{6} \max_{i,k} \sigma_{k}^{2}(W_{j}) e_{\Delta},$$
(158)

$$||M_1' - M_1||_{op} \le \sqrt{6} \cdot \frac{\max_{j,k} \sigma_k^2(W_j)}{\sigma_{\min}^2(W_2)} e_{\Delta} \le \sqrt{6} \cdot \frac{\max_{j,k} \sigma_k^2(W_j)}{\min_{j,k} \sigma_k^2(W_j)} e_{\Delta}.$$
(159)

Applying Lemma 17,

$$||I - R^H R||_{op} \le \frac{1}{\sigma_{min}^2(W_2)} \cdot ||\Delta_{23}||_{op} \le \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta},$$
 (160)

$$||I - R^{H-1}R^{-1}||_{op} \le \frac{1}{\sigma_{min}^2(W_3)} \cdot ||\Delta_{23}||_{op} \le \frac{1}{\min_{j,k} \sigma_k^2(W_j)} \cdot e_{\Delta}.$$
 (161)

F SKEW-HERMITIAN ERROR TERM AND HERMITIAN MAIN TERM FOR FOUR-LAYER MATRIX DECOMPOSITION

In this section, we construct skew-hermitian error term and hermitian main term to prepare for the convergence proof, under four-layer setting with scaled identical target matrix $\Sigma = \sigma_1(\Sigma)I$.

F.1 SKEW-HERMITIAN ERROR TERM

The skew-hermitian error term is defined by $\|W_1 - W_1'\|_F^2$. To address the dynamics:

F.1.1 GRADIENT FLOW

 Consider $\Sigma = \sigma_1(\Sigma)I$. We study $\|W_1 - W_1'\|_F^2$. From the derivative of inverse,

$$\frac{\mathrm{d}W_2^{-1}}{\mathrm{d}t} = -W_2^{-1} \frac{\mathrm{d}W_2}{\mathrm{d}t} W_2^{-1} = -W_1'(\Sigma - W) W_1^H W_2^{-1} - a\Delta_{12} W_2^{-1} + aW_2^{-1} \Delta_{23}, \tag{162}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}W_{2}^{-1}}{\mathrm{d}t}W_{3}^{H} + W_{2}^{-1}\frac{\mathrm{d}W_{3}^{H}}{\mathrm{d}t}
= -RW_{4}^{H}(\Sigma - W)W_{1}^{H}R + W_{1}(\Sigma - W^{H})W_{4}
- a\Delta_{12}R + 2aW_{2}^{-1}\Delta_{23}W_{3}^{H} - aR\Delta_{34},$$
(163)

$$\frac{dW_{1}'}{dt} = \frac{dW_{2}^{-1}}{dt} W_{3}^{H} W_{4}^{H} + W_{2}^{-1} \frac{dW_{3}^{H}}{dt} W_{4}^{H} + W_{2}^{-1} W_{3}^{H} \frac{dW_{4}^{H}}{dt}
= -W_{1}'(\Sigma - W) W_{1}^{H} W_{1}' + W_{1} (\Sigma - W^{H}) W_{1}'^{H} R^{H-1} R^{-1} W_{1}'
+ RR^{H} W_{2}^{H} W_{2} W_{1} (\Sigma - W^{H}) - a \Delta_{12} W_{1}' + 2a W_{2}^{-1} \Delta_{23} W_{2} W_{1}'.$$
(164)

From $\Re(\operatorname{tr}(PQ))=0$ if $P=P^H$ and $Q=-Q^H$, we have

$$\Re\left(\operatorname{tr}\left(\left(W_{1}'W_{1}^{H}-W_{1}W_{1}'^{H}\right)W_{1}'\left(W_{1}-W_{1}'\right)^{H}\right)\right)$$

$$=-\frac{1}{2}\operatorname{tr}\left(\left(W_{1}'W_{1}^{H}-W_{1}W_{1}'^{H}\right)\left(W_{1}'W_{1}^{H}-W_{1}W_{1}'^{H}\right)^{H}\right).$$
(165)

Thus

$$\frac{d}{dt} \|W_1 - W_1'\|_F^2 = 2\Re \left(\operatorname{tr} \left(\frac{d(W_1 - W_1')}{dt} (W_1 - W_1')^H \right) \right)$$

$$= 2\Re \left(\operatorname{tr} \left(\left[M_2 W_1' (\Sigma - W) + W_1' (\Sigma - W) W_1^H W_1' \right] \right)$$

$$= 2\Re \left(\operatorname{tr} \left(\left[M_2 W_1' (\Sigma - W) + W_1' (\Sigma - W) W_1^H W_1' \right] \right)$$

$$- W_1 (\Sigma - W^H) W_1'^H R^{H-1} R^{-1} W_1' - R R^H M_2 W_1 (\Sigma - W^H) \right)$$

$$- a\Delta_{12} (W_1 - W_1') - 2a W_2^{-1} \Delta_{23} W_2 W_1' \right] (W_1 - W_1')^H \right)$$

$$= -2\sigma_1(\Sigma) \operatorname{tr} \left((W_1 - W_1')^H M_2 (W_1 - W_1') \right)$$

$$- \sigma_1(\Sigma) \operatorname{tr} \left((W_1' W_1^H - W_1 W_1'^H) (W_1' W_1^H - W_1 W_1'^H)^H \right)$$

$$- \operatorname{tr} \left(M_2 (M_1' + M_1) M_2 (W_1 - W_1') (W_1 - W_1')^H \right)$$

$$- \operatorname{tr} \left(M_2 (M_1' - M_1) M_2 (W_1' + W_1) (W_1 - W_1')^H \right)$$

$$- \operatorname{tr} \left(M_2 (M_1' - M_1) M_2 (W_1' + W_1) (W_1 - W_1')^H \right)$$

$$+ 2\operatorname{tr} \left(\left[-M_1' M_2 M_1 + M_1 M_2 M_1' W_1' (W_1 - W_1')^H \right) \right)$$

$$+ 2\Re \left(\operatorname{tr} \left(\left[W_1 (\Sigma - W^H) W_4 (R^H R - I) W_4^H (W_1 - W_1')^H \right) \right)$$

$$- 2a\Re \left(\operatorname{tr} \left(\Delta_{12} (W_1 - W_1') (W_1 - W_1')^H \right) \right)$$

$$- 4a\Re \left(\operatorname{tr} \left(W_2^{-1} \Delta_{23} W_2 W_1' (W_1 - W_1')^H \right) \right)$$

Note: $-M_1'M_2M_1 + M_1M_2M_1' = \frac{1}{2}[(M_1 - M_1')M_2(M_1 + M_1') - (M_1 + M_1')M_2(M_1 - M_1')].$

F.1.2 GRADIENT DESCENT

 From Lemma 19,

$$\begin{aligned} & \left\| W_{2}(t+1)^{-1} - W_{2}(t)^{-1} - \eta \left[-W'_{1}(t)(\Sigma - W(t))W_{1}(t)^{H}W_{2}(t)^{-1} - a\Delta_{12}(t)W_{2}(t)^{-1} + aW_{2}(t)^{-1}\Delta_{23}(t) \right] \right\|_{F} \\ & \leq \eta^{2} \left[\left(1 + e_{\Delta}(t) \left\| W_{2}(t)^{-1} \right\|_{op}^{2} \right) \left\| W_{1}(t) \right\|_{op} \left\| \Sigma - W(t) \right\|_{F} + \sqrt{2}ae_{\Delta}(t) \left\| W_{2}(t)^{-1} \right\|_{op} \right] \\ & \cdot \left\| W_{2}(t+1)^{-1} \right\|_{op} \left\| \nabla_{W_{2}}\mathcal{L}(t) \right\|_{F}. \end{aligned}$$
(167)

Under
$$||W_j(t+1)||_{op} = O(||W_j(t)||_{op}), e_{\Delta}(t) ||W_2(t)^{-1}||_{op}^2 = O(1),$$

 $-\eta \left[-W_1'(t)(\Sigma - W(t))W_1(t)^H W_1'(t)\right]$

 $+W_1(t) (\Sigma - W(t)^H) W_1'(t)^H R(t)^{H-1} R(t)^{-1} W_1'(t)$

 $-a\Delta_{12}(t)W_1'(t)+2aW_2^{-1}(t)\Delta_{23}(t)W_2(t)W_1'(t)\Big]\Big\|_{F}$

 $+R(t)R(t)^{H}W_{2}(t)^{H}W_{2}(t)W_{1}(t)\left(\Sigma-W(t)^{H}\right)$

 $||W_1'(t+1) - W_1'(t)||$

Finally giving

 $= \eta^{2} O\left(\left[\max_{j \in [1,4] \cap \mathbb{N}^{*}} \|W_{j}(t)\|_{op} \|\Sigma - W(t)\|_{F} + ae_{\Delta}(t) \|W_{2}(t)^{-1}\|_{op} \right] \right)$

 $\max_{j \in [1,4] \cap \mathbb{N}^*} \|W_j(t)\|_{op}^2 \cdot \|W_2(t+1)^{-1}\|_{op} \cdot \max_{j \in [1,4] \cap \mathbb{N}^*} \|\nabla_{W_j} \mathcal{L}(t)\|_F \right).$

(168)

```
2323
2325
                     \|W_1(t+1) - W_1'(t+1)\|_F^2 - \|W_1(t) - W_1'(t)\|_F^2
2326
                  =\Re\left(\operatorname{tr}\left(\left[\left(W_{1}(t+1)-W_{1}'(t+1)\right)+\left(W_{1}(t)-W_{1}'(t)\right)\right]\right)
2328
                   \cdot \left[ \left( W_1(t+1) - W_1'(t+1) \right) - \left( W_1(t) - W_1'(t) \right) \right]^H \right)
2329
2330
                 = -2\eta \sigma_1(\Sigma) \operatorname{tr} \left( (W_1(t) - W_1'(t))^H M_2(t) (W_1(t) - W_1'(t)) \right)
2331
                 -\eta \sigma_1(\Sigma) \operatorname{tr} \left( \left( W_1'(t) W_1(t)^H - W_1(t) W_1'(t)^H \right) \left( W_1'(t) W_1(t)^H - W_1(t) W_1'(t)^H \right)^H \right)
2332
2333
                 -\eta \text{tr} \left( M_2(t) \left( M_1'(t) + M_1(t) \right) M_2(t) \left( W_1(t) - W_1'(t) \right) \left( W_1(t) - W_1'(t) \right)^H \right)
2334
2335
                 -\eta \text{tr} \left( M_2(t) \left( M_1'(t) - M_1(t) \right) M_2(t) \left( W_1'(t) + W_1(t) \right) \left( W_1(t) - W_1'(t) \right)^H \right)
2336
2337
                 +2\eta \operatorname{tr}\left(\left[-M_{1}'(t)M_{2}(t)M_{1}(t)+M_{1}(t)M_{2}(t)M_{1}'(t)\right]W_{1}'(t)\left(W_{1}(t)-W_{1}'(t)\right)^{H}\right)
                                                                                                                                                                                       (169)
2338
                 +2\eta\Re\left(\operatorname{tr}\left(\left[W_{1}(t)(\Sigma-W(t)^{H})W_{4}(t)\left(R(t)^{H}R(t)-I\right)W_{4}(t)^{H}\right]\left(W_{1}(t)-W_{1}^{\prime}(t)\right)^{H}\right)\right)
2340
                 +2\eta\Re\left(\operatorname{tr}\left(\left[\left(I-R(t)R(t)^{H}\right)W_{2}(t)^{H}W_{2}(t)W_{1}(t)(\Sigma-W(t)^{H})\right]\left(W_{1}(t)-W_{1}'(t)\right)^{H}\right)\right)
2342
                 -2\eta a\Re\left(\operatorname{tr}\left(\Delta_{12}(t)\left(W_{1}(t)-W_{1}'(t)\right)\left(W_{1}(t)-W_{1}'(t)\right)^{H}\right)\right)
2344
                 -4\eta a\Re\left(\operatorname{tr}\left(W_{2}^{-1}(t)\Delta_{23}(t)W_{2}(t)W_{1}'(t)\left(W_{1}(t)-W_{1}'(t)\right)^{H}\right)\right)
2345
2346
                 +\eta^{2}O\left(\left[\max_{j\in[1,4]\cap\mathbb{N}^{*}}\|W_{j}(t)\|_{op}\|\Sigma-W(t)\|_{F}+ae_{\Delta}(t)\|W_{2}(t)^{-1}\|_{op}\right]^{2}
2347
2348
2349
                   \cdot \max_{i \in [1,4] \cap \mathbb{N}^*} \|W_j(t)\|_{op}^5 \cdot \|W_2(t+1)^{-1}\|_{op} \right).
2350
2351
```

F.2 SKEW-HERMITIAN ERROR TERM

F.2.1 GRADIENT FLOW

For gradient flow, we study the k^{th} singular value of $W_1 + W_1'$, or equivalently $\lambda_k \left((W_1 + W_1')^H (W_1 + W_1') \right) = \sigma_k^2 (W_1 + W_1')$. To address the dynamics:

Suppose the left and right singular vector of $W_1 + W_1'$ corresponding to $\sigma_k(t) = \sigma_k (W_1 + W_1') (t)$ are $\eta_k(t)$ and $\chi_k(t)$ respectively, $(W_1 + W_1') \chi_k = \sigma_k \eta_k$, $\eta_k^H (W_1 + W_1') = \sigma_k \chi_k$, $\|\chi_k\| = \|\eta_k\| = 1$. Then from Lemma 22,

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{k}\left(\left(W_{1}+W_{1}'\right)^{H}\left(W_{1}+W_{1}'\right)\right) = \chi_{k}^{H}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(W_{1}+W_{1}'\right)^{H}\left(W_{1}+W_{1}'\right)\right)\chi_{k}$$

$$= 2\Re\left(\chi_{k}^{H}\left(W_{1}+W_{1}'\right)^{H}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(W_{1}+W_{1}'\right)\right)\chi_{k}\right), \tag{170}$$

where

2377
2378
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(W_{1}+W_{1}'\right)=M_{2}W_{1}'(\Sigma-W)-W_{1}'(\Sigma-W)W_{1}^{H}W_{1}'$$

$$+W_{1}(\Sigma-W^{H})W_{1}'^{H}R^{H-1}R^{-1}W_{1}'+RR^{H}M_{2}W_{1}(\Sigma-W^{H})$$

$$-a\Delta_{12}(W_{1}+W_{1}')+2aW_{2}^{-1}\Delta_{23}W_{2}W_{1}'$$
2382
$$=M_{2}\left(W_{1}+W_{1}'\right)\Sigma+\left(W_{1}\Sigma W_{1}'^{H}-W_{1}'\Sigma W_{1}^{H}\right)W_{1}'$$
2383
$$-M_{2}\left(\frac{M_{1}+M_{1}'}{2}M_{2}\left(W_{1}+W_{1}'\right)+\frac{M_{1}-M_{1}'}{2}M_{2}\left(W_{1}-W_{1}'\right)\right)$$

$$+\left(M_{1}'M_{2}M_{1}-M_{1}M_{2}M_{1}'\right)W_{1}'$$
2386
$$-W_{1}(\Sigma-W^{H})W_{1}'^{H}\left(I-R^{H-1}R^{-1}\right)W_{1}'$$
2388
2389
$$-a\Delta_{12}(W_{1}+W_{1}')+2aW_{2}^{-1}\Delta_{23}W_{2}W_{1}'.$$

Consider arbitrary $\chi \in \mathbb{F}^d$. Notice that $(W_1 \Sigma W_1'^H - W_1' \Sigma W_1^H)$ is a skew-hermitian matrix:

$$\Re \left(2\chi^{H}(W_{1}+W_{1}')^{H}\left(W_{1}\Sigma W_{1}'^{H}-W_{1}'\Sigma W_{1}^{H}\right)W_{1}'\chi\right)
=\Re \left(\chi^{H}(W_{1}+W_{1}')^{H}\left(W_{1}\Sigma W_{1}'^{H}-W_{1}'\Sigma W_{1}^{H}\right)W_{1}'\chi\right)
-\Re \left(\chi^{H}W_{1}'^{H}\left(W_{1}\Sigma W_{1}'^{H}-W_{1}'\Sigma W_{1}^{H}\right)W_{1}\chi\right)
-\Re \left(\chi^{H}W_{1}^{H}\left(W_{1}\Sigma W_{1}'^{H}-W_{1}'\Sigma W_{1}^{H}\right)W_{1}\chi\right)
=\Re \left(\chi^{H}(W_{1}+W_{1}')^{H}\left(-W_{1}\Sigma W_{1}'^{H}+W_{1}'\Sigma W_{1}^{H}\right)(W_{1}-W_{1}')\chi\right).$$
(172)

From $\Sigma = \sigma_1(\Sigma)I$,

$$-W_1 \Sigma W_1'^H + W_1' \Sigma W_1^H = \sigma_1(\Sigma) (W_1 + W_1') (W_1 - W_1')^H + \sigma_1(\Sigma) (M_1' - M_1). \tag{173}$$

Likewise,

$$\Re \left(2\chi^{H}(W_{1}+W_{1}')^{H}\left(M_{1}'M_{2}M_{1}-M_{1}M_{2}M_{1}'\right)W_{1}'\chi\right) =\Re \left(\chi^{H}(W_{1}+W_{1}')^{H}\left(M_{1}'M_{2}M_{1}-M_{1}M_{2}M_{1}'\right)\left(W_{1}'-W_{1}\right)\chi\right).$$
(174)

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{k}^{2} = 2\sigma_{1}(\Sigma)\sigma_{k}^{2}\eta_{k}^{H}M_{2}\eta_{k} + \sigma_{1}(\Sigma)\sigma_{k}^{2}\chi_{k}^{H}(W_{1} - W_{1}')^{H}(W_{1} - W_{1}')\chi_{k}
+ \sigma_{1}(\Sigma)\sigma_{k}\Re\left(\eta_{k}^{H}(M_{1}' - M_{1})(W_{1} - W_{1}')\chi_{k}\right)
- \sigma_{k}^{2}\eta_{k}^{H}M_{2}(M_{1} + M_{1}')M_{2}\eta_{k} - \sigma_{k}\Re\left(\eta_{k}^{H}M_{2}(M_{1} - M_{1}')M_{2}(W_{1} - W_{1}')\chi_{k}\right)
+ \sigma_{k}\Re\left(\eta_{k}^{H}(M_{1}'M_{2}M_{1} - M_{1}M_{2}M_{1}')(W_{1}' - W_{1})\chi_{k}\right)
- 2\sigma_{k}\Re\left(\eta_{k}^{H}W_{1}(\Sigma - W^{H})W_{4}(R^{H}R - I)W_{4}^{H}\chi_{k}\right)
- 2\sigma_{k}\Re\left(\eta_{k}^{H}(I - RR^{H})M_{2}W_{1}(\Sigma - W^{H})\chi_{k}\right)
- 2\sigma_{k}^{2}\Re\left(\eta_{k}^{H}\Delta_{12}\eta_{k}\right) + 4\sigma_{k}\Re\left(\eta_{k}^{H}W_{2}^{-1}\Delta_{23}W_{2}W_{1}'\chi_{k}\right).$$
(175)

F.2.2 GRADIENT DESCENT

For gradient descent, we study $\lambda_{min} \left(\left(W_1 + W_1' \right)^H \left(W_1 + W_1' \right) \right) = \sigma_{min}^2 \left(W_1 + W_1' \right)$. To address the dynamics:

2430
2431
$$(W_{1}(t+1) + W'_{1}(t+1))$$
2432
$$= W_{1}(t) + W'_{1}(t)$$
2433
2434
$$+ \eta \left[\sigma_{1}(\Sigma)M_{2}(t) - M_{2}(t) \frac{M_{1}(t) + M'_{1}(t)}{2} M_{2}(t) \right] (W_{1}(t) + W'_{1}(t))$$
2436
$$+ \eta (M'_{1}(t)M_{2}(t)M_{1}(t) - M_{1}(t)M_{2}(t)M'_{1}(t)) W'_{1}(t)$$
2437
2438
$$+ \eta \sigma_{1}(\Sigma) \left(W_{1}(t)W'_{1}(t)^{H} - W'_{1}(t)W_{1}(t)^{H} \right) W'_{1}(t) + \eta E_{1}(t),$$
(176)

where the error term is bounded by

$$||E_{1}(t)||_{op} \leq \frac{1}{2} \max_{j \in [1,4] \cap \mathbb{N}^{*}} ||W_{j}(t)||_{op}^{4} ||W_{1}(t) - W'_{1}(t)||_{op} ||M_{1}(t) - M'_{1}(t)||_{op}$$

$$+ \left(||R(t)^{H}R(t) - I||_{op} + ||I - R(t)R(t)^{H}||_{op} \right) \max_{j \in [1,4] \cap \mathbb{N}^{*}} ||W_{j}(t)||_{op}^{3} ||\Sigma - W(t)||_{op}$$

$$+ ae_{\Delta}(t) \left(||W_{1}(t) + W'_{1}(t)||_{op} + 2 ||R(t)||_{op} ||W_{2}(t)^{-1}||_{op} \max_{j \in [1,4] \cap \mathbb{N}^{*}} ||W_{j}(t)||_{op}^{2} \right)$$

$$+ \eta O \left(\left[\max_{j \in [1,4] \cap \mathbb{N}^{*}} ||W_{j}(t)||_{op} ||\Sigma - W(t)||_{F} + ae_{\Delta}(t) ||W_{2}(t)^{-1}||_{op} \right]$$

$$\cdot \max_{j \in [1,4] \cap \mathbb{N}^{*}} ||W_{j}(t)||_{op}^{2} \cdot ||W_{2}(t+1)^{-1}||_{op} \cdot \max_{j \in [1,4] \cap \mathbb{N}^{*}} ||\nabla_{W_{j}}\mathcal{L}(t)||_{F} \right). \tag{177}$$

Follow the tricks in Lemma 28,

$$\lambda_{min} \left(\left(W_{1}(t+1) + W_{1}'(t+1) \right)^{H} \left(W_{1}(t+1) + W_{1}'(t+1) \right) \right)$$

$$\geq \lambda_{min} \left(\left(W_{1}(t) + W_{1}'(t) \right)^{H} \left(I + \eta \left[\sigma_{1}(\Sigma) M_{2}(t) - M_{2}(t) \frac{M_{1}(t) + M_{1}'(t)}{2} M_{2}(t) \right] \right)^{2} \left(W_{1}(t) + W_{1}'(t) \right) \right)$$

$$+ \eta \| E_{2}(t) \|_{op} + \eta^{2} O\left(\| \left(W_{1}(t+1) + W_{1}'(t+1) \right) - \left(W_{1}(t) + W_{1}'(t) \right) \|_{op}^{2} \right), \tag{178}$$

where

$$||E_{2}(t)||_{op} = \sigma_{min} \left(W_{1}(t+1) + W_{1}'(t+1) \right) \cdot \left[||E_{1}(t)||_{op} + ||W_{2}(t)||_{op}^{2} ||M_{1}(t) + M_{1}'(t)||_{op} ||M_{1}(t) - M_{1}'(t)||_{op} ||W_{1}(t) - W_{1}'(t)||_{op} \right].$$

$$(179)$$

G CONVERGENCE UNDER GRADIENT FLOW, STAGED ANALYSIS

In order to present the proof more clearly, we state the complete proof of convergence under Random Gaussian Initialization B.2 and gradient flow, before tackling gradient descent.

At the beginning we assume (51) holds. (For the complex case, it holds with high probability $1 - \delta$; for the real case, it holds with probability $\frac{1}{2}(1 - \delta)$.)

G.1 STAGE 1: ALIGNMENT STAGE

In this section, we set $\epsilon \leq \frac{\sigma_1^{1/4}(\Sigma)}{2c_1\sqrt{d}}$, $a \geq 2^5c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$, where $b \geq 2^4\ln(4c_1d) + \ln c_2$.

Without loss of generality, $c_1 \ge 2$, $c_2 \ge c_1^6$.

Theorem 37. At $T_1 = \frac{1}{32c_1^{14}c_2d^{10}\epsilon^2\sigma_1(\Sigma)}$, the following conclusions hold:

$$\sigma_{min}(W_1 + W_1')|_{t=T_1} \ge \frac{\epsilon}{2c_1^3 c_2 d^{9/2}}$$

$$e_{\Delta}(T_1) \le 2\sqrt{3}c_1^2 d^{3/2} \epsilon^2 \exp\left(-\frac{a}{32c_1^{20}c_2 d^{13}\sigma_1(\Sigma)}\right)$$

$$\max_{j,k} |\sigma_k(W_j(T_1))| \le (1 + 2^{-21})c_1\sqrt{d}\epsilon$$

$$\min_{j,k} |\sigma_k(W_j(T_1))| \ge (1 - 2^{-17})\frac{\epsilon}{c_1\sqrt{d}}.$$
(180)

This section proves the theorem above by following Lemmas and Corollaries.

Lemma 38. Maximum and minimum singular value bound of weight matrices in alignment stage.

For
$$t \in \left[0, \frac{1}{16c_1^4d^2\epsilon^2\sigma_1(\Sigma)}\right]$$
,

$$\min_{j,k} \sigma_k(W_j) \ge \frac{\epsilon}{c_1 \sqrt{d}} - 16c_1^3 d^{3/2} \epsilon^3 \sigma_1(\Sigma)t, \, \max_{j,k} \sigma_k(W_j) \le \frac{c_1 \sqrt{d\epsilon}}{\sqrt{1 - 4c_1^2 d\epsilon^2 \sigma_1(\Sigma)t}}. \tag{181}$$

Proof. For $t \geq 0$ such that $\max_{j,k} \sigma_k(W_j) \leq 2c_1 \sqrt{d\epsilon} \leq \sigma_1^{1/4}(\Sigma)$,

$$\max_{j} \left\| \nabla_{W_j} \mathcal{L}_{ori} \right\|_{op} \le \max_{j,k} |\sigma_k(W_j)|^3 \left(\sigma_1(\Sigma) + \max_{j,k} |\sigma_k(W_j)|^4 \right) \le 2\sigma_1(\Sigma) \max_{j,k} |\sigma_k(W_j)|^3.$$
(182)

By invoking Theorem 27,

$$\frac{\mathrm{d}\max_{j,k}\sigma_k^2(W_j)}{\mathrm{d}t} \le 4\max_{j,k}|\sigma_k(W_j)|^4\sigma_1(\Sigma)$$

$$\frac{\mathrm{d}\min_{j,k}\sigma_k^2(W_j)}{\mathrm{d}t} \ge -4\min_{j,k}|\sigma_k(W_j)|\max_{j,k}|\sigma_k(W_j)|^3\sigma_1(\Sigma).$$
(183)

By solving the differential inequality,

$$\max_{j,k} \sigma_k |W_j| \le \frac{\max_{j,k} \sigma_k |W_j(0)|}{\sqrt{1 - 4\sigma_1(\Sigma) \max_{j,k} \sigma_k |W_j(0)|^2 \cdot t}} \le \frac{c_1 \sqrt{d\epsilon}}{\sqrt{1 - 4c_1^2 d\epsilon^2 \sigma_1(\Sigma)t}}, t \in \left[0, \frac{3}{16c_1^2 d\epsilon^2 \sigma_1(\Sigma)}\right]. \tag{184}$$

$$\min_{j,k} |\sigma_k(W_j)| \ge \frac{\epsilon}{c_1 \sqrt{d}} - 16c_1^3 d^{3/2} \epsilon^3 \sigma_1(\Sigma) t, \ t \in \left[0, \frac{1}{16c_1^4 d^2 \epsilon^2 \sigma_1(\Sigma)}\right]. \tag{185}$$

This completes the proof.

Notice that

$$\max_{j,k} |\sigma_k(W_j(t \le T_1))| \le \frac{c_1 \sqrt{d\epsilon}}{\sqrt{1 - \frac{1}{8c_1^{12}c_2}}} \le (1 + 2^{-21})c_1 \sqrt{d\epsilon}
\min_{j,k} |\sigma_k(W_j(t \le T_1))| \ge \left(1 - \frac{1}{2c_1^{10}c_2}\right) \cdot \frac{\epsilon}{c_1 \sqrt{d}} \ge (1 - 2^{-17}) \frac{\epsilon}{c_1 \sqrt{d}}.$$
(186)

Corollary 39. Balanced term error in alignment stage.

For $t \in [0, T_1]$,

$$e_{\Delta}(t) \le 2\sqrt{3}c_1^2 d^{3/2} \epsilon^2 \exp\left(-\frac{a\epsilon^2}{c_1^6 d^3}t\right).$$
 (187)

Specially, at $t = T_1$,

$$e_{\Delta}(T_1) \le 2\sqrt{3}c_1^2d^{3/2}\epsilon^2 \exp\left(-\frac{a}{32c_1^{20}c_2d^{13}\sigma_1(\Sigma)}\right) \le \sqrt{3}\cdot 2^{-31}c_1^{-14}c_2^{-1}d^{-29/2}\epsilon^2.$$
 (188)

Proof. By simply combining Theorem 26 and Lemma 38.

Corollary 40. Main term at the end of alignment stage.

At $t = T_1$,

$$\sigma_{min} (W_1 + W_1')|_{t=T_1} \ge \frac{\epsilon}{2c_1^3 c_2 d^{9/2}}.$$
 (189)

Proof. For simplicity, denote $\Delta_X(t)=X(t)-X(0)$ for arbitrary X. Note: $\Delta_{X^H}=\Delta_X^H$. At $t=T_1$,

$$\|\Delta_{W}(T_{1})\|_{op} = \left\| \int_{0}^{T_{1}} \sum_{j=1}^{4} \left[W_{\prod_{L},j+1}(t')W_{\prod_{L},j+1}(t')^{H} \left(\Sigma - W(t') \right) W_{\prod_{R},j-1}(t')W_{\prod_{R},j-1}(t') \right] dt' \right\|_{op}$$

$$\leq \int_{0}^{T_{1}} \sum_{j=1}^{4} \left\| W_{\prod_{L},j+1}(t')W_{\prod_{L},j+1}(t')^{H} \left(\Sigma - W(t') \right) W_{\prod_{R},j-1}(t')W_{\prod_{R},j-1}(t') \right\|_{op} dt'$$

$$\leq \int_{0}^{T_{1}} \sum_{j=1}^{4} \left(\|\Sigma\|_{op} + \|W(t')\|_{op} \right) \left(\prod_{k \in [1,4] \cap \mathbb{N}^{*}, \ k \neq j} \|W_{i}(t')\|_{op}^{2} \right) dt'$$

$$\leq \int_{0}^{T_{1}} 4 \cdot 2\sigma_{1}(\Sigma) \cdot \left(\left(1 + 2^{-21} \right) c_{1} \sqrt{d} \epsilon \right)^{6} dt'$$

$$\leq 8 \left(1 + 2^{-18} \right) c_{1}^{6} d^{3} \epsilon^{6} \sigma_{1}(\Sigma) T_{1} = \left(1 + 2^{-18} \right) \cdot \frac{1}{4} c_{1}^{-8} c_{2}^{-1} d^{-7} \epsilon^{4}. \tag{190}$$

Thus

$$\|\Delta_{W^{H}W}(T_{1})\|_{op} = \left\| \frac{1}{2} \left[(W(T_{1}) + W(0))^{H} \Delta_{W}(T_{1}) + \Delta_{W}(T_{1})^{H} (W(T_{1}) + W(0)) \right] \right\|_{op}$$

$$\leq \left(\|W(T_{1})\|_{op} + \|W(0)\|_{op} \right) \|\Delta_{W}(T_{1})\|_{op}$$

$$\leq \left[1 + \left(1 + 2^{-21} \right)^{4} \right] c_{1}^{4} d^{2} \epsilon^{4} \cdot \|\Delta_{W}(T_{1})\|_{op} = (1 + 2^{-17}) \cdot \frac{1}{2} c_{1}^{-4} c_{2}^{-1} d^{-5} \epsilon^{8}.$$
(191)

From Corollary 39,

$$\left\| \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) \right)^{2} - W(T_{1})^{H} W(T_{1}) \right\|_{op}$$

$$= \left\| W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} M_{\Delta 1234}(T_{1}) W_{2}(T_{1}) W_{1}(T_{1}) \right\|_{op}$$

$$\leq \left\| W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} \right\|_{op} \left\| M_{\Delta 1234}(T_{1}) \right\|_{op} \left\| W_{2}(T_{1}) W_{1}(T_{1}) \right\|_{op}$$

$$\leq \left(\left(1 + 2^{-21} \right) c_{1} \sqrt{d} \epsilon \right)^{4} \cdot \sqrt{6} \left(\left(1 + 2^{-21} \right) c_{1} \sqrt{d} \epsilon \right)^{2} \cdot e_{\Delta}(T_{1})$$

$$\leq \sqrt{6} (1 + 2^{-18}) c_{1}^{6} d^{3} \epsilon^{6} e_{\Delta}(T_{1}) \leq 2^{-28} c_{1}^{-8} c_{2}^{-16} d^{-23/2} \epsilon^{8}.$$

$$(192)$$

Thus

$$\left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) \right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}$$

$$\leq \left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) \right)^{2} - W(T_{1})^{H}W(T_{1}) \right\|_{op} + \left\| \Delta_{W^{H}W}(T_{1}) \right\|_{op}$$

$$\leq (1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}.$$
(193)

From Lemma 18,

$$\left\| W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) - \left(W(T_{0})^{H}W(T_{0})\right)^{1/2} \right\|_{op} \\
\leq \frac{\left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1})\right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}}{2\sqrt{\lambda_{min}\left(W(T_{0})^{H}W(T_{0})\right) - \left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1})\right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}}} \\
\leq \frac{(1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}}{2\sqrt{\left(\frac{\epsilon}{c_{1}\sqrt{d}}\right)^{8} - (1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}}}} \leq 0.27c_{2}^{-1}d^{-3}\epsilon^{4}.$$
(194)

By (B.2),

$$\sigma_{min} \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right)$$

$$\geq \sigma_{min} \left(\left(W(T_{0})^{H} W(T_{0}) \right)^{1/2} + W(0)^{H} \right)$$

$$- \left\| W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) - \left(W(T_{0})^{H} W(T_{0}) \right)^{1/2} \right\|_{op} - \left\| \Delta_{W}(T_{1}) \right\|_{op} \qquad (195)$$

$$\geq c_{2}^{-1} d^{-3} \epsilon^{4} - 0.27 c_{2}^{-1} d^{-3} \epsilon^{4} - \left(1 + 2^{-18} \right) \cdot \frac{1}{4} c_{1}^{-8} c_{2}^{-1} d^{-7} \epsilon^{4}$$

$$\geq 0.72 c_{2}^{-1} d^{-3} \epsilon^{4},$$

which further gives

$$\sigma_{min} (W_{1} + W_{1}')|_{t=T_{1}}$$

$$= \sigma_{min} \left(\left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) \right)^{-1} \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right) \right)$$

$$\geq \left(\frac{1}{\max_{j,k} |\sigma_{k}(W_{j}(T_{1}))|} \right)^{3} \cdot \sigma_{min} \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right)$$

$$\geq \frac{\epsilon}{2c_{1}^{3} c_{2} d^{9/2}}.$$
(196)

G.2 STAGE 2: SADDLE AVOIDANCE STAGE

In this stage, we further assume $a \geq 32c_1^{20}c_2d^{13}\sigma_1(\Sigma)\left(5\ln\left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon}\right) + \frac{281}{8}\ln d + 23\ln(4c_1) + 7\ln c_2\right)$, while $\frac{\epsilon}{\sigma_1^{1/4}(\Sigma)} \leq \frac{1}{32c_1^5c_2d^{53/8}}$. From Lemma 25 and Theorem 37,

$$e_{\Delta}(t \in [T_1, +\infty)) \le e_{\Delta}(T_1) \le 2\sqrt{3}c_1^2 d^{3/2} \epsilon^2 \exp\left(-\frac{a}{32c_1^{20}c_2 d^{13}\sigma_1(\Sigma)}\right)$$

$$\le \sqrt{3} \cdot 2^{-45}c_1^{-21}c_2^{-7} d^{-269/8} \epsilon^7 \sigma_1^{-5/4}(\Sigma).$$
(197)

Moreover,
$$a \ge 32c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$$
, where $b - \ln b \ge 3\ln\left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon}\right) + \frac{303}{8}\ln d + 37\ln(2c_1) + 6\ln c_2$.

$$ae_{\Delta}(t \in [T_1, +\infty)) \le ae_{\Delta}(T_1) \le 2^6 \sqrt{3} c_1^{22} c_2 d^{29/2} \epsilon^2 \sigma_1(\Sigma) \exp(-(b - \ln b))$$

$$\le \sqrt{3} \cdot 2^{-31} c_1^{-15} c_2^{-5} d^{-187/8} \epsilon^5 \sigma_1^{1/4}(\Sigma).$$
(198)

Theorem 41. At $T_1 + T_2$, $T_2 = \frac{32c_1^6c_2^2d^9}{\sigma_1(\Sigma)\epsilon^2}$, the following conclusions hold:

$$||W_1(T_1 + T_2) - W_1'(T_1 + T_2)||_F \le 3c_1 d\epsilon$$

$$\sigma_{min}(W_1 + W_1')(T_1 + T_2) \ge 2^{3/4} \sigma_1^{1/4}(\Sigma).$$
(199)

Lemma 42. Bound of operator norms throughout the optimization process.

For
$$t \in [0, +\infty)$$
,

$$\|\Sigma - W(t)\|_{op} \le \|\Sigma - W(t)\|_{F} \le 1.01\sqrt{d}\sigma_{1}(\Sigma)$$

$$\|W\|_{op} \le \|W\|_{F} \le 3\sqrt{d}\sigma_{1}(\Sigma)$$

$$\max_{j} \|W_{j}\|_{op} \le \max_{j} \|W_{j}\|_{F} \le \sqrt{2}d^{1/8}\sigma_{1}^{1/4}(\Sigma).$$
(200)

Proof. For $t \in [0, T_1]$, the result is obvious from Theorem 37 and Lemma 38.

For $t \in (T_1, +\infty)$: from Lemma 23,

$$\|\Sigma - W(t)\|_{op} \le \|\Sigma - W(t)\|_F \le \|\Sigma - W(0)\|_F \le \|\Sigma\|_F + \|W(0)\|_F \le \sqrt{2d}\sigma_1(\Sigma). \tag{201}$$

Giving

$$||W(t)||_{op} \le ||W(t)||_F \le ||\Sigma - W(t)||_F + ||\Sigma||_F \le 3\sqrt{d}\sigma_1(\Sigma).$$
 (202)

For the last inequality, prove by contradiction.

Suppose $\max_j ||W_j||_{op} \ge \sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma)$, then by invoking Corollary 39,

$$e_{\Delta}(t) \le e_{\Delta}(T_1) \le \sqrt{3} \cdot 2^{-15} c_1^{-14} c_2^{-16} d^{-29/2} \epsilon^2 \le 2^{-15} \max_j \|W_j\|_{op}^2.$$
 (203)

Thus for $t > T_1$,

$$||W||_{op}^{2} = ||WW^{H}||_{op} = ||W_{4}W_{3}W_{2}W_{1}W_{1}^{H}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op}$$

$$\geq ||W_{4}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{2}\Delta_{12}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op}$$

$$- ||W_{4}W_{3}\Delta_{23}W_{2}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{2}W_{2}^{H}\Delta_{23}W_{3}^{H}W_{4}^{H}||_{op}$$

$$- ||W_{4}\Delta_{34}(W_{3}W_{3}^{H})^{2}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{3}^{H}\Delta_{34}W_{3}W_{3}^{H}W_{4}^{H}||_{op} - ||W_{4}(W_{3}W_{3}^{H})^{2}\Delta_{34}W_{4}^{H}||_{op}$$

$$\geq \left(\max_{j}||W_{j}||_{op}^{2} - 3e_{\Delta}\right)^{4} - 6e_{\Delta}\max_{j}||W_{j}||_{op}^{6} > 15\sqrt{d}\sigma_{1}(\Sigma).$$

$$(204)$$

which contradicts inequality (202). This completes the proof.

Lemma 43. Bound of $\|W_2^{-1}\|_{op}$, $\|W_3^{-1}\|_{op}$, and relevant terms.

For $t \in [T_1, T_1 + T_2]$,

$$\max\left(\left\|W_{2}^{-1}(t)\right\|_{op}, \left\|W_{3}^{-1}(t)\right\|_{op}\right) \le 128c_{1}^{6}c_{2}^{2}d^{77/8}\epsilon^{-2}\sigma_{1}^{1/4}(\Sigma),\tag{205}$$

$$\max\left(e_{\Delta}(t)\left\|W_{2}^{-1}(t)\right\|_{op}^{2},\ e_{\Delta}(t)\left\|W_{3}^{-1}(t)\right\|_{op}^{2}\right) \leq \sqrt{3}\cdot 2^{-31}c_{1}^{-9}c_{2}^{-3}d^{-115/8}\epsilon^{3}\sigma_{1}^{-3/4}(\Sigma). \tag{206}$$

Proof. We begin with the time derivative of W_2^{-1} and W_3^{-1} :

$$\frac{dW_2^{-1}}{dt} = -RW_4^H(\Sigma - W)W_1^HW_2^{-1} - a\Delta_{12}W_2^{-1} + aW_2^{-1}\Delta_{23}
\frac{dW_3^{-1}}{dt} = -W_3^{-1}W_4^H(\Sigma - W)W_1^HR^{H-1} - a\Delta_{23}W_3^{-1} + aW_3^{-1}\Delta_{34}.$$
(207)

From $\frac{d}{dt} \|M\|_{op} \le \left\| \frac{d}{dt} M \right\|_{op}$ (this in equality is from triangular inequality and standard calculus analysis),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_{2}^{-1}\|_{op} \leq \|R\|_{op} \|W_{4}\|_{op} \|\Sigma - W\|_{op} \|W_{1}^{H}W_{2}^{-1}\|_{op}
+ a \|\Delta_{12}\|_{op} \|W_{2}^{-1}\|_{op} + a \|W_{2}^{-1}\|_{op} \|\Delta_{23}\|_{op}
\frac{\mathrm{d}}{\mathrm{d}t} \|W_{3}^{-1}\|_{op} \leq \|W_{3}^{-1}W_{4}^{H}\|_{op} \|\Sigma - W\|_{op} \|W_{1}\|_{op} \|R\|_{op}
+ a \|\Delta_{23}\|_{op} \|W_{3}^{-1}\|_{op} + a \|W_{3}^{-1}\|_{op} \|\Delta_{34}\|_{op}.$$
(208)

From Lemma 42 and

$$||R||_{op} \leq \sqrt{1 + \frac{1}{\sigma_{min}^{2}(W_{2})} \cdot ||\Delta_{23}||_{op}}$$

$$||R^{-1}||_{op} \leq \sqrt{1 + \frac{1}{\sigma_{min}^{2}(W_{3})} \cdot ||\Delta_{23}||_{op}}$$

$$||W_{1}^{H}W_{2}^{-1}||_{op} = \sqrt{||W_{2}^{H-1}W_{1}W_{1}^{H}W_{2}^{-1}||_{op}} = \sqrt{||I + W_{2}^{H-1}\Delta_{12}W_{2}^{-1}||}$$

$$\leq \sqrt{1 + e_{\Delta} ||W_{2}^{-1}||_{op}^{2}}$$

$$||W_{3}^{-1}W_{4}^{H}||_{op} = \sqrt{||W_{3}^{-1}W_{4}^{H}W_{4}W_{3}^{H-1}||_{op}} = \sqrt{||I - W_{3}^{-1}\Delta_{34}W_{3}^{H-1}||}$$

$$\leq \sqrt{1 + e_{\Delta} ||W_{3}^{-1}||_{op}^{2}}.$$

$$(209)$$

Further we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_{2}^{-1}\|_{op} \leq 2\sqrt{2} \left(1 + e_{\Delta} \|W_{2}^{-1}\|_{op}^{2}\right) d^{5/8} \sigma_{1}^{5/4}(\Sigma) + \sqrt{2} a e_{\Delta} \|W_{2}^{-1}\|_{op}
\frac{\mathrm{d}}{\mathrm{d}t} \|W_{3}^{-1}\|_{op} \leq 2\sqrt{2} \left(1 + e_{\Delta} \|W_{3}^{-1}\|_{op}^{2}\right) d^{5/8} \sigma_{1}^{5/4}(\Sigma) + \sqrt{2} a e_{\Delta} \|W_{3}^{-1}\|_{op}.$$
(210)

Combine with (197) and (198), for $t \ge T_1$ such that (205) holds,

$$\max\left(\frac{\mathrm{d}}{\mathrm{d}t} \|W_{2}^{-1}\|_{op}, \frac{\mathrm{d}}{\mathrm{d}t} \|W_{3}^{-1}\|_{op}\right) \\
\leq 2\sqrt{2}(1+\sqrt{3}\cdot 2^{-31})d^{5/8}\sigma_{1}^{5/4}(\Sigma) + 2^{-22}c_{1}^{-9}c_{2}^{-3}d^{-55/4}\epsilon^{3}\sigma_{1}^{1/2}(\Sigma) \\
\leq 2\sqrt{2}(1+2^{-20})d^{5/8}\sigma_{1}^{5/4}(\Sigma). \tag{211}$$

From Theorem 37, $\max\left(\left\|W_2(T_1)^{-1}\right\|_{op}, \left\|W_3(T_1)^{-1}\right\|_{op}\right) \leq \frac{1}{\min_{j,k} |\sigma_k(W_j(T_1))|} \leq \frac{c_1\sqrt{d}}{(1-2^{-17})\epsilon}$, then the proof of the first inequality is completed via integration during the time interval $[T_1, T_1 + T_2]$. The second inequality follows immediately.

Remark 15. This Lemma verifies that $W_{2,3}^{-1}$ are bounded (consequently $W_{2,3}$ are full rank), then R is well defined throughout this stage. For $t > T_1 + T_2$, further analysis shows that the minimum singular values of W_2 and W_3 are lower bounded by $\Omega(\sigma_1^{1/4}(\Sigma))$.

Lemma 44. Skew-hermitian error.

For $t \in [T_1, T_1 + T_2]$,

$$||W_1 - W_1'||_F \le 3c_1 d\epsilon. \tag{212}$$

Proof. From section F.1.1,

$$\frac{d}{dt} \|W_{1} - W_{1}'\|_{F}^{2} = -2\sigma_{1}(\Sigma) \operatorname{tr} \left((W_{1} - W_{1}')^{H} M_{2} (W_{1} - W_{1}') \right) \\
- \sigma_{1}(\Sigma) \operatorname{tr} \left((W_{1}'W_{1}^{H} - W_{1}W_{1}'^{H}) (W_{1}'W_{1}^{H} - W_{1}W_{1}'^{H})^{H} \right) \\
- \operatorname{tr} \left(M_{2} (M_{1}' + M_{1}) M_{2} (W_{1} - W_{1}') (W_{1} - W_{1}')^{H} \right) \\
- \operatorname{tr} \left(M_{2} (M_{1}' - M_{1}) M_{2} (W_{1}' + W_{1}) (W_{1} - W_{1}')^{H} \right) \\
+ 2\operatorname{tr} \left([-M_{1}'M_{2}M_{1} + M_{1}M_{2}M_{1}'] W_{1}' (W_{1} - W_{1}')^{H} \right) \\
+ 2\Re \left(\operatorname{tr} \left([W_{1}(\Sigma - W^{H})W_{4} (R^{H}R - I) W_{4}^{H}] (W_{1} - W_{1}')^{H} \right) \right) \\
+ 2\Re \left(\operatorname{tr} \left([(I - RR^{H}) W_{2}^{H} W_{2} W_{1}(\Sigma - W^{H})] (W_{1} - W_{1}')^{H} \right) \right) \\
- 2a\Re \left(\operatorname{tr} \left(\Delta_{12} (W_{1} - W_{1}') (W_{1} - W_{1}')^{H} \right) \right) \\
- 4a\Re \left(\operatorname{tr} \left(W_{2}^{-1} \Delta_{23} W_{2} W_{1}' (W_{1} - W_{1}')^{H} \right) \right).$$

Note: $-M_1'M_2M_1 + M_1M_2M_1' = \frac{1}{2} \left[(M_1 - M_1') M_2 (M_1 + M_1') - (M_1 + M_1') M_2 (M_1 - M_1') \right].$ From Lemma 43, for $t \in [T_1, T_1 + T_2]$,

$$\max\left(\left\|R^{H}R - I\right\|_{op}, \left\|I - RR^{H}\right\|_{op}\right) \le e_{\Delta} \left\|W_{2}^{-1}\right\|_{op}^{2}$$

$$\le \sqrt{3} \cdot 2^{-31} c_{1}^{-9} c_{2}^{-3} d^{-115/8} \epsilon^{3} \sigma_{1}^{-3/4}(\Sigma),$$
(214)

$$||M_{1} - M_{1}'||_{op} \leq \sqrt{6} \cdot \frac{\max_{j,k} \sigma_{k}^{2}(W_{j})}{\sigma_{min}^{2}(W_{2})} e_{\Delta}$$

$$\leq 2^{-27} c_{1}^{-9} c_{2}^{-3} d^{-113/8} \epsilon^{3} \sigma_{1}^{-1/4}(\Sigma),$$
(215)

$$\left\| M_{2} - \frac{M_{1} + M_{1}'}{2} \right\|_{op} \leq \left\| \Delta_{12} \right\|_{op} + \frac{1}{2} \left\| M_{1} - M_{1}' \right\|_{op} \leq \left[1 + \frac{\sqrt{6}}{2} \cdot \frac{\max_{j,k} \sigma_{k}^{2}(W_{j})}{\sigma_{min}^{2}(W_{2})} \right] e_{\Delta}$$

$$\leq 2^{-28} c_{1}^{-9} c_{2}^{-3} d^{-113/8} \epsilon^{3} \sigma_{1}^{-1/4}(\Sigma).$$
(216)

Consequently:

$$||R||_{op} \le \sqrt{1 + e_{\Delta} ||W_2^{-1}||_{op}^2} \le 1 + \sqrt{3} \cdot 2^{-32} c_1^{-9} c_2^{-3} d^{-115/8} \epsilon^3 \sigma_1^{-3/4}(\Sigma), \tag{217}$$

$$\|W_1'\|_{op} \le \|W_1'\|_F \le \sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma) \|R\|_{op} \le \left(1 + 2^{-31}\right)\sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma), \tag{218}$$

$$\left\| \frac{M_1 + M_1'}{2} \right\|_{\text{op}} \le \|M_2\|_{op} + \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{\text{op}} \le \left(1 + 2^{-29} \right) 2d^{1/4} \sigma_1^{1/2}(\Sigma), \tag{219}$$

$$||M_1'M_2M_1 - M_1M_2M_1'||_{op} \le ||M_1 - M_1'|| ||M_2|| ||M_1 + M_1'||$$

$$\le (1 + 2^{-29}) 2^{-25} c_1^{-9} c_2^{-3} d^{-109/8} \epsilon^3 \sigma_1^{3/4}(\Sigma).$$
(220)

By combining all results above, for $t \in [T_1, T_1 + T_2]$ such that $||W_1 - W_1'||_F \leq 3c_1 d\epsilon$ holds,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_{1} - W_{1}'\|_{F}^{2} \leq -0 - 0 - 0 \\
+ \|M_{2}\|_{F} \|M_{1}' - M_{1}\|_{op} \|M_{2}\|_{op} \left(\|W_{1}'\|_{op} + \|W_{1}\|_{op}\right) \|W_{1} - W_{1}'\|_{F} \\
+ 2 \|-M_{1}'M_{2}M_{1} + M_{1}M_{2}M_{1}'\|_{op} \|W_{1}'\|_{F} \|W_{1} - W_{1}'\|_{F} \\
+ 2 \max_{j} \|W_{j}\|_{op}^{3} \|\Sigma - W\|_{F} \left(\|R^{H}R - I\|_{op} + \|I - RR^{H}\|_{op}\right) \|W_{1} - W_{1}'\|_{F} \\
+ 2ae_{\Delta} \|W_{1} - W_{1}'\|_{F}^{2} \\
+ 4ae_{\Delta} \|W_{2}^{-1}\|_{op} \|W_{2}\|_{F} \|W_{1}'\|_{op} \|W_{1} - W_{1}'\|_{F} \\
\leq 2^{-22}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma) \\
+ 2^{-21}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma) \\
+ 2^{-24}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma) \\
+ 2^{-26}c_{1}^{-13}c_{2}^{-5}d^{-171/8}\epsilon^{7}\sigma_{1}^{1/4}(\Sigma) \\
+ 2^{-18}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma) \\
\leq 2^{-17}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma). \tag{221}$$

From Theorem 37, at $t = T_1$,

$$||W_{1}(T_{1}) - W_{1}'(T_{1})||_{F} \leq ||W_{1}(T_{1})||_{F} + ||W_{1}'(T_{1})||_{F} \leq ||W_{1}(T_{1})||_{F} + ||W_{4}(T_{1})||_{F} ||R(T_{1})||_{op}$$

$$\leq (1 + 2^{-32}) 2\sqrt{d} \cdot (1 + 2^{-21}) c_{1}\sqrt{d}\epsilon \leq (1 + 2^{-20}) 2c_{1}d\epsilon.$$
(222)

Thus $\|W_1 - W_1'\|_F^2 \le \sqrt{\left[(1 + 2^{-20}) 2c_1 d\epsilon\right]^2 + 2^{-17} c_1^{-8} c_2^{-3} d^{-25/2} \epsilon^4 \sigma_1(\Sigma)(t - T_1)}$, when both $t \in [T_1, T_1 + T_2]$ and $\|W_1 - W_1'\|_F^2 \le 3c_1 d\epsilon$ hold. Then

$$||W_{1}(T_{1} + T_{2}) - W'_{1}(T_{1} + T_{2})||_{F}^{2}$$

$$\leq \sqrt{\left[\left(1 + 2^{-20}\right) 2c_{1}d\epsilon\right]^{2} + 2^{-17}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma)T_{2}}$$

$$\leq \sqrt{\left[\left(1 + 2^{-20}\right) 2c_{1}d\epsilon\right]^{2} + 2^{-12}c_{1}^{-2}c_{2}^{-1}d^{-7/2}\epsilon^{2}} < 3c_{1}d\epsilon.$$
(223)

which completes the proof.

Corollary 45. The minimum eigenvalue of Hermitian term.

For any $\sigma_k(W_1+W_1')(T_1)\geq \frac{\epsilon}{2c_1^3c_2d^{9/2}}$, it takes at most time T_2 to increase to $2^{3/4}\sigma_1^{1/4}(\Sigma)$.

Proof. We analyze the dynamics of $\lambda_k \left(\left(W_1 + W_1' \right)^H \left(W_1 + W_1' \right) \right) = \sigma_k^2$. The definition of $\eta_k(t)$ and $\chi_k(t)$ follows section F.2.1. The dynamics can be expressed as below:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{k}^{2} = 2\sigma_{1}(\Sigma)\sigma_{k}^{2}\eta_{k}^{H}M_{2}\eta_{k} + \sigma_{1}(\Sigma)\sigma_{k}^{2}\chi_{k}^{H}\left(W_{1} - W_{1}'\right)^{H}\left(W_{1} - W_{1}'\right)\chi_{k}
+ \sigma_{1}(\Sigma)\sigma_{k}\Re\left(\eta_{k}^{H}\left(M_{1}' - M_{1}\right)\left(W_{1} - W_{1}'\right)\chi_{k}\right)
- \sigma_{k}^{2}\eta_{k}^{H}M_{2}\left(M_{1} + M_{1}'\right)M_{2}\eta_{k} - \sigma_{k}\Re\left(\eta_{k}^{H}M_{2}\left(M_{1} - M_{1}'\right)M_{2}\left(W_{1} - W_{1}'\right)\chi_{k}\right)
+ \sigma_{k}\Re\left(\eta_{k}^{H}\left(M_{1}'M_{2}M_{1} - M_{1}M_{2}M_{1}'\right)\left(W_{1}' - W_{1}\right)\chi_{k}\right)
- 2\sigma_{k}\Re\left(\eta_{k}^{H}W_{1}(\Sigma - W^{H})W_{4}\left(R^{H}R - I\right)W_{4}^{H}\chi_{k}\right)
- 2\sigma_{k}\Re\left(\eta_{k}^{H}\left(I - RR^{H}\right)M_{2}W_{1}(\Sigma - W^{H})\chi_{k}\right)
- 2\sigma_{k}^{2}\Re\left(\eta_{k}^{H}\Delta_{12}\eta_{k}\right) + 4\sigma_{k}\Re\left(\eta_{k}^{H}W_{2}^{-1}\Delta_{23}W_{2}W_{1}'\chi_{k}\right).$$
(224)

$$\begin{array}{lll} \text{From} & \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & (1 + 2^{-29}) \ 2 d^{1/4} \sigma_1^{1/2}(\Sigma), \end{array}$$

$$\eta_k^H M_2 \eta_k \ge \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) \eta_k - \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{op} \\
\ge \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) \eta_k - 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4} (\Sigma) \\
\eta_k^H M_2 (M_1 + M_1') M_2 \eta_k \le \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) (M_1 + M_1') \left(\frac{M_1 + M_1'}{2}\right) \eta_k \\
+ 2 \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{op} \left\| \frac{M_1 + M_1'}{2} \right\|_{op} \left(\|M_2\|_{op} + \left\| \frac{M_1 + M_1'}{2} \right\|_{op} \right) \\
\le \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) (M_1 + M_1') \left(\frac{M_1 + M_1'}{2}\right) \eta_k \\
+ (1 + 2^{-28}) 2^{-24} c_1^{-9} c_2^{-3} d^{-109/8} \epsilon^3 \sigma_1^{3/4} (\Sigma). \tag{225}$$

By Lemma 44, $\|W_1 - W_1'\|_{op} \le \|W_1 - W_1'\|_F \le 3c_1 d\epsilon$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_k^2 \geq 2\sigma_1(\Sigma) \sigma_k^2 \eta_k^H M_2 \eta_k + 0$$

$$-\sigma_1(\Sigma) \sigma_k \| M_1' - M_1\|_{op} \| W_1 - W_1'\|_{op}$$

$$-\sigma_1(\Sigma) \sigma_k \| M_1' - M_1\|_{op} \| W_1 - W_1'\|_{op}$$

$$-\sigma_k^2 \eta_k^H M_2 (M_1 + M_1') M_2 \eta_k - \sigma_k \max_j \| W_j \|_{op}^4 \| M_1 - M_1'\|_{op} \| W_1 - W_1'\|_{op}$$

$$-\sigma_k \| M_1' M_2 M_1 - M_1 M_2 M_1'\|_{op} \| W_1' - W_1\|_{op}$$

$$-2\sigma_k \max_j \| W_j \|_{op}^3 \| \Sigma - W\|_{op} \left(\| R^H R - I\|_{op} + \| I - RR^H \|_{op} \right)$$

$$-2ae_\Delta \sigma_k^2 - 4ae_\Delta \sigma_k \| W_2^{-1} \|_{op} \max_j \| W_j \|_{op}^2 \| R\|_{op}$$

$$\geq 2\sigma_1(\Sigma) \sigma_k^2 \left(\eta_k^H \left(\frac{M_1 + M_1'}{2} \right) \eta_k - 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4} (\Sigma) \right)$$

$$-\sigma_k \| W_1 - W_1' \|_{op} \cdot 2^{-27} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{3/4} (\Sigma)$$

$$-\sigma_k^2 \left[\eta_k^H \left(\frac{M_1 + M_1'}{2} \right) (M_1 + M_1') \left(\frac{M_1 + M_1'}{2} \right) \eta_k + (1 + 2^{-28}) 2^{-24} c_1^{-9} c_2^{-3} d^{-109/8} \epsilon^3 \sigma_1^{3/4} (\Sigma) \right]$$

$$-\sigma_k \| W_1 - W_1' \|_{op} \cdot (1 + 2^{-29}) 2^{-25} c_1^{-9} c_2^{-3} d^{-109/8} \epsilon^3 \sigma_1^{3/4} (\Sigma)$$

$$-\sigma_k \| W_1 - W_1' \|_{op} \cdot (1 + 2^{-29}) 2^{-25} c_1^{-9} c_2^{-3} d^{-109/8} \epsilon^3 \sigma_1^{3/4} (\Sigma)$$

$$-\sigma_k \cdot 2^{-25} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot 2^{-25} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot 2^{-25} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

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$$-\sigma_k \cdot 2^{-25} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot 2^{-25} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot (1 + 2^{-1}) 2^{-22} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot (1 + 2^{-1}) 2^{-22} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot (1 + 2^{-1}) 2^{-22} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot (1 + 2^{-1}) 2^{-22} c_1^{-9} c_2^{-3} d^{-27/2} \epsilon^3 \sigma_1 (\Sigma)$$

$$-\sigma_k \cdot (1 + 2^{-1}) 2^{-22} c_1^{-9} c_2^{-3} d$$

under $\sigma_k \geq \frac{\epsilon}{2c_1^3c_2d^{9/2}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k^2 \ge 2\sigma_k^2 \eta_k^H \left[\sigma_1(\Sigma) \left(\frac{M_1 + M_1'}{2} \right) - \left(\frac{M_1 + M_1'}{2} \right)^3 \right] \eta_k - 2^{-18} \sigma_1(\Sigma) \sigma_k^4. \tag{227}$$

Denote $P=rac{W_1+W_1'}{2},\,Q=rac{W_1-W_1'}{2}.$ Notice that

$$PP^{H} + QQ^{H} = \frac{M_1 + M_1'}{2}, P^{H}\eta_k = \frac{1}{2}\sigma_k\chi_k,$$
 (228)

$$\eta_k^H \left(\frac{M_1 + M_1'}{2} \right) \eta_k = \eta_k^H \left(P P^H + Q Q^H \right) \eta_k \ge \frac{1}{4} \sigma_k^2,$$
(229)

$$\eta_k^H \left(\frac{M_1 + M_1'}{2}\right)^3 \eta_k = \eta_k^H \left(PP^H + QQ^H\right) \left(\frac{M_1 + M_1'}{2}\right) \left(PP^H + QQ^H\right) \eta_k
= \frac{1}{16} \sigma_k^4 \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) \eta_k + \eta_k^H QQ^H \left(\frac{M_1 + M_1'}{2}\right) QQ^H \eta_k
+ \frac{1}{4} \sigma_k^2 \eta_k^H \left[QQ^H \left(\frac{M_1 + M_1'}{2}\right) + \left(\frac{M_1 + M_1'}{2}\right) QQ^H\right] \eta_k
\leq \frac{1}{16} \sigma_k^4 \eta_k^H \left(\frac{M_1 + M_1'}{2}\right) \eta_k + \left\|\frac{M_1 + M_1'}{2}\right\|_{op} \left(\frac{1}{2} \sigma_k^2 \|Q\|_{op}^2 + \|Q\|_{op}^4\right).$$
(230)

Notice
$$\|Q\|_{op} = \frac{1}{2} \|W_1 - W_1'\|_F \le \frac{3}{2} c_1 d\epsilon \le \sigma_k \cdot 3c_1^4 c_2 d^{11/2}, \epsilon \le \frac{1}{32c_1^5 c_2 d^{53/8}} \sigma_1^{1/4}(\Sigma),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{k}^{2} \geq 2\sigma_{k}^{2} \left[\left(\sigma_{1}(\Sigma) - \frac{1}{16}\sigma_{k}^{4} \right) \eta_{k}^{H} \left(\frac{M_{1} + M_{1}'}{2} \right) \eta_{k} - \left\| \frac{M_{1} + M_{1}'}{2} \right\|_{op} \left(\frac{1}{2}\sigma_{k}^{2} \|Q\|_{op}^{2} + \|Q\|_{op}^{4} \right) \right] \\
- 2^{-18}\sigma_{1}(\Sigma)\sigma_{k}^{4} \\
\geq \frac{1}{2}\sigma_{k}^{4} \left(\sigma_{1}(\Sigma) - \frac{1}{16}\sigma_{k}^{4} \right) - 2\sigma_{k}^{2} \left\| \frac{M_{1} + M_{1}'}{2} \right\|_{op} \|Q\|_{op}^{2} \left(\frac{1}{2}\sigma_{k}^{2} + \|Q\|_{op}^{2} \right) - 2^{-18}\sigma_{1}(\Sigma)\sigma_{k}^{4} \\
\geq \frac{1}{2}\sigma_{k}^{4}\sigma_{1}(\Sigma) - \frac{1}{32}\sigma_{k}^{8} - 81\left(1 + 2^{-5} \right) c_{1}^{10}c_{2}^{2}d^{53/4}\epsilon^{2}\sigma_{1}^{1/2}(\Sigma)\sigma_{k}^{4} - 2^{-18}\sigma_{1}(\Sigma)\sigma_{k}^{4} \\
\geq \frac{3}{8}\sigma_{k}^{4}\sigma_{1}(\Sigma) - \frac{1}{32}\sigma_{k}^{8}. \tag{231}$$

This indicates that for $\sigma_k \in \left[\frac{\epsilon}{2c_1^3c_2d^{9/2}}, 2^{3/4}\sigma_1^{1/4}(\Sigma)\right]$, σ_k is monotonically increasing. By standard calculus, it takes at most time $\Delta t\left(\sigma_k \geq 2^{3/4}\sigma_1^{1/4}(\Sigma)\right) \leq T_2$ for σ_k to increase from at least $\frac{\epsilon}{2c_1^3c_2d^{9/2}}$ to $2^{3/4}\sigma_1^{1/4}(\Sigma)$:

$$\Delta t \left(\sigma_{k} \geq 2^{3/4} \sigma_{1}^{1/4}(\Sigma) \right) \leq \int_{\frac{2c_{1}^{3} c_{2} d^{9/2}}{2c_{1}^{3} c_{2} d^{9/2}}}^{2 \cdot \sqrt{\frac{\sigma_{1}(\Sigma)}{2}}} \left(\frac{3}{8} \sigma_{1}(\Sigma) \sigma_{k}^{4} - \frac{1}{32} \sigma_{k}^{8} \right)^{-1} d \left(\sigma_{k}^{2} \right)$$

$$= \int_{\frac{\epsilon}{4c_{1}^{6} c_{2}^{2} d^{9}}}^{4 \cdot \sqrt{\frac{\sigma_{1}(\Sigma)}{2}}} \left(\frac{3}{8} \sigma_{1}(\Sigma) \lambda_{k}^{2} - \frac{1}{32} \lambda_{k}^{4} \right)^{-1} d \lambda_{k}$$

$$\leq \int_{\frac{\epsilon}{4c_{1}^{6} c_{2}^{2} d^{9}}}^{4 \cdot \sqrt{\frac{\sigma_{1}(\Sigma)}{2}}} \left(\frac{3}{8} \sigma_{1}(\Sigma) \lambda_{k}^{2} - \frac{1}{4} \sigma_{1}(\Sigma) \lambda_{k}^{2} \right)^{-1} d \lambda_{k}$$

$$\leq 8 \left[\left(\frac{\epsilon}{4c_{1}^{6} c_{2}^{2} d^{9}} \right)^{-1} - \left(4 \cdot \sqrt{\frac{\sigma_{1}(\Sigma)}{2}} \right)^{-1} \right] \sigma_{1}^{-1}(\Sigma) \leq T_{2}.$$
(232)

And for $t \in \left[T_1 + \Delta t \left(\sigma_k \ge 2^{3/4} \sigma_1^{1/4}(\Sigma)\right), T_1 + T_2\right]$, σ_k does not decrease to less than $2^{3/4} \sigma_1^{1/4}(\Sigma)$ if $t \le T_1 + T_2$. This is from the continuity of σ_k and the time derivative of σ_k^2 at $\sigma_k = 2^{3/4} \sigma_1^{1/4}(\Sigma)$, $t \le T_1 + T_2$ is positive:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k^2\bigg|_{\sigma_k=2^{3/4}\sigma_*^{1/4}(\Sigma), t \le T_1 + T_2} \ge \frac{1}{8}\sigma_1(\Sigma) \cdot \left(2^{3/4}\sigma_1^{1/4}(\Sigma)\right)^4 > 0. \tag{233}$$

G.3 STAGE 3: LOCAL CONVERGENCE STAGE

In this stage, we analysis the time to reach δ -convergence, that is

$$T(\delta) = \inf_{t} \{ \mathcal{L}(t) \le \delta \}. \tag{234}$$

Lemma 46. $\sigma_{min}(W_1 + W_1')$ is lower bounded, while the skew-hermitian error is upper bounded. For $t \geq T_1 + T_2$,

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$$\sigma_{min} (W_1 + W_1') (t) \ge 2^{3/4} \sigma_1^{1/4} (\Sigma)$$

$$\|W_1 - W_1'\|_F \le 3c_1 d\epsilon.$$
(235)

Proof. (235) holds at $t=T_1+T_2$. Since both L.H.S. change continuously, it left to prove that the derivatives at the critical points (to be specific, $t' \geq T_2$ such that $\|W_1-W_1'\|_F|_{t=t'}=3c_1d\epsilon$ or $\sigma_k\left(W_1+W_1'\right)|_{t=t'}=2^{3/4}\sigma_1(\Sigma)$) are positive/negative. (If such time does not exist, then the proof is done.)

From

$$\sigma_{min}^{2}(W_{1}) + \sigma_{min}^{2}(W_{1}') \ge \frac{1}{2} \lambda_{min} \left((W_{1} + W_{1}')(W_{1} + W_{1}')^{H} + (W_{1} - W_{1}')(W_{1} - W_{1}')^{H} \right)$$

$$\ge \frac{1}{2} \sigma_{min}^{2} \left(W_{1} + W_{1}' \right),$$
(236)

and

$$\sigma_{min}(W_1') \le \sigma_{min}(W_1) + \|W_1 - W_1'\|_F. \tag{237}$$

For $t > T_1 + T_2$ such as (235) holds,

$$\sigma_{min}(W_2) \ge \sigma_{min}(W_1) - e_{\Delta} \ge \frac{1}{\sqrt{2}} \sigma_1^{1/4}(\Sigma). \tag{238}$$

Then by following almost the same arguments as Lemma 44 and 45,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_{1} - W_{1}'\|_{F}^{2} \leq -2\sigma_{1}(\Sigma) \mathrm{tr} \left((W_{1} - W_{1}')^{H} \sigma_{min}^{2}(W_{2}) (W_{1} - W_{1}') \right) - 0 - 0
+ 2^{-17} c_{1}^{-8} c_{2}^{-3} d^{-25/2} \epsilon^{4} \sigma_{1}(\Sigma)
\leq -\sigma_{1}^{3/2}(\Sigma) \|W_{1} - W_{1}'\|_{F}^{2} + 2^{-17} c_{1}^{-8} c_{2}^{-3} d^{-25/2} \epsilon^{4} \sigma_{1}(\Sigma),$$
(239)

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k^2(W_1 + W_1') \ge \frac{3}{8}\sigma_k^4(W_1 + W_1')\sigma_1(\Sigma) - \frac{1}{32}\sigma_k^8(W_1 + W_1'). \tag{240}$$

Suppose for some $t_1, t_2 \ge T_1 + T_2$ such that $||W_1 - W_1'||_F|_{t=t_1} = 3c_1 d\epsilon$, $\sigma_k (W_1 + W_1')|_{t=t_2} = 2^{3/4} \sigma_1(\Sigma)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_1 - W_1'\|_F^2 \bigg|_{t=t_1} \le 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_k^2 (W_1 + W_1') \bigg|_{t=t_2} \ge 0.$$
(241)

This completes the proof.

Theorem 47. Global convergence bound.

For four-layer matrix factorization under gradient flow, with random Gaussian initialization with scaling factor $\epsilon \leq \frac{\sigma_1^{1/4}(\Sigma)}{32c_1^5c_2d^{53/8}}$, regularization factor $a \geq 32c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$, where b satisfies

$$b \ge 5 \ln \left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon} \right) + \frac{281}{8} \ln d + 23 \ln(4c_1) + 7 \ln c_2$$

$$b - \ln b \ge 3 \ln \left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon} \right) + \frac{303}{8} \ln d + 37 \ln(2c_1) + 6 \ln c_2.$$
(242)

Then for target matrix with identical singular values, there exists following $T(\delta)$, such that for any $\delta>0$, (1) with high probability over the complex initialization (2) with probability close to $\frac{1}{2}$ over the real initialization, when $t > T(\delta)$, $\mathcal{L}(t) < \delta$.

$$T(\delta) \leq T_{1} + T_{2} + \sigma_{1}^{-3/2}(\Sigma) \ln\left(\frac{d\sigma_{1}^{2}(\Sigma)}{\delta}\right)$$

$$= \frac{1}{32c_{1}^{14}c_{2}d^{10}\epsilon^{2}\sigma_{1}(\Sigma)} + \frac{32c_{1}^{6}c_{2}^{2}d^{9}}{\sigma_{1}(\Sigma)\epsilon^{2}} + \sigma_{1}^{-3/2}(\Sigma) \ln\left(\frac{d\sigma_{1}^{2}(\Sigma)}{\delta}\right)$$

$$= O\left(\frac{c_{1}^{6}c_{2}^{2}d^{9}}{\sigma_{1}(\Sigma)\epsilon^{2}} + \frac{1}{\sigma_{1}^{3/2}(\Sigma)} \ln\left(\frac{d\sigma_{1}^{2}(\Sigma)}{\delta}\right)\right). \tag{243}$$

Proof. Following the derivations in Lemma 46,

$$\min_{j,k} \sigma_k(W_j)(t > T_1 + T_2) \ge \frac{1}{\sqrt{2}} \sigma_1^{1/4}(\Sigma). \tag{244}$$

By Lemma 23 and 42,

$$\mathcal{L}_{ori}(t) \leq \mathcal{L}_{ori}(T_1 + T_2) \exp\left(-8 \min_{j,k} |\sigma_k(W_j)(t > T_1 + T_2)|^6 (t - T_1 - T_2)\right)
\leq \mathcal{L}_{ori}(0) \exp\left(-8 \min_{j,k} |\sigma_k(W_j)(t > T_1 + T_2)|^6 (t - T_1 - T_2)\right)
\leq 0.52 d\sigma_1^2(\Sigma) \exp\left(-\sigma_1^{3/2}(\Sigma)(t - T_1 - T_2)\right).$$
(245)

For regularization term, by invoking Theorem 26, 37 and Lemma 42,

$$\mathcal{L}_{reg}(t) \leq \mathcal{L}_{reg}(T_1 + T_2) \exp\left(-\frac{4a}{3} \frac{\min_{j,k} |\sigma_k(W_j)(t > T_1 + T_2)|^4}{\max_{j,k} |\sigma_k(W_j)|^2} \cdot (t - T_1 - T_2)\right)
\leq \frac{a}{4} e_{\Delta}^2(T_1 + T_2) \exp\left(-\frac{4a}{3} \frac{\min_{j,k} |\sigma_k(W_j)(t > T_1 + T_2)|^4}{\max_{j,k} |\sigma_k(W_j)|^2} \cdot (t - T_1 - T_2)\right)
\leq \frac{a}{4} e_{\Delta}^2(T_1) \exp\left(-\frac{4a}{3} \frac{\min_{j,k} |\sigma_k(W_j)(t > T_1 + T_2)|^4}{\max_{j,k} |\sigma_k(W_j)|^2} \cdot (t - T_1 - T_2)\right)
\leq 2^{-76} c_1^{-36} c_2^{-12} d^{-57} \epsilon^{12} \sigma_1^{-1}(\Sigma) \exp\left(-16c_1^{20} c_2 d^{51/4} \sigma_1^{3/2}(\Sigma)(t - T_1 - T_2)\right).$$
(246)

By taking logarithm on the summation of these two inequalities, the proof is completed.

CONVERGENCE UNDER GRADIENT DESCENT, STAGED ANALYSIS

This section states the complete proof of convergence under Random Gaussian Initialization B.2.

At the beginning we still assume (51) holds. (For the complex case, it holds with high probability $1-\delta$; for the real case, it holds with probability $\frac{1}{2}(1-\delta)$.

 Theorem 48. Global convergence bound under random Gaussian initialization, gradient descent. For four-layer matrix factorization under gradient descent, random Gaussian initialization with scaling factor $\epsilon \leq \frac{\sigma_1^{1/4}(\Sigma)}{32c_1^5c_2d^{53/8}}$, regularization factor $a \geq 32c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$, where b satisfies

$$b \ge \max\left(5\ln\left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon}\right) + \frac{281}{8}\ln d + 23\ln(4c_1) + 7\ln c_2, \ 16\ln(2c_1c_2d)\right)$$

$$b - \ln b \ge 3\ln\left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon}\right) + \frac{303}{8}\ln d + 37\ln(2c_1) + 6\ln c_2.$$
(247)

Then for target matrix with identical singular values, there exists following learning rate η and convergence time $T(\delta, \eta)$, such that for any $\delta > 0$, (1) with high probability over the complex initialization (2) with probability close to $\frac{1}{2}$ over the real initialization, when $t > T(\delta)$, $\mathcal{L}(t) < \delta$.

$$\eta = O\left(\min\left(a^{-2}c_{1}^{-4}d^{-2}\epsilon^{-2}\sigma_{1}(\Sigma), ac_{1}^{-56}c_{2}^{-14}d^{-301/4}\epsilon^{8}\sigma_{1}^{-9/2}(\Sigma), a^{-1}c_{1}^{-44}c_{2}^{-10}d^{-219/4}\epsilon^{4}\sigma_{1}^{-3/2}(\Sigma), c_{1}^{-27}c_{2}^{-9}d^{-355/8}\epsilon^{9}\sigma_{1}^{-15/4}(\Sigma), a^{-1}c_{1}^{-21}c_{2}^{-7}d^{-273/8}\epsilon^{7}\sigma_{1}^{-9/4}(\Sigma)\right)\right)
T(\delta, \eta) \leq T_{1} + T_{2} + \eta^{-1}\sigma_{1}^{-3/2}(\Sigma)\ln\left(\frac{d\sigma_{1}^{2}(\Sigma)}{\delta}\right)
= O\left(\frac{c_{1}^{6}c_{2}^{2}d^{9}}{\eta\sigma_{1}(\Sigma)\epsilon^{2}} + \frac{1}{\eta\sigma_{1}^{3/2}(\Sigma)}\ln\left(\frac{d\sigma_{1}^{2}(\Sigma)}{\delta}\right)\right).$$
(248)

The following section completes the proof.

H.1 STAGE 1: ALIGNMENT STAGE

In this section, we set $\epsilon \leq \frac{\sigma_1^{1/4}(\Sigma)}{4c_1\sqrt{d}}$, $a \geq 2^5c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$, where $b \geq 2^4\ln(4c_1d) + \ln c_2$. $\eta = O\left(\frac{\sigma_1(\Sigma)}{a^2c_1^4d^2\epsilon^2}\right)$, with appropriate small constant. Without loss of generality, $c_1 \geq 2$, $c_2 \geq c_1^6$.

Theorem 49. At $T_1 = \frac{1}{32c_1^{14}c_2d^{10}\epsilon^2\sigma_1(\Sigma)n}$, the following conclusions hold:

$$\sigma_{min} (W_1 + W_1')|_{t=T_1} \ge \frac{\epsilon}{2c_1^3 c_2 d^{9/2}}$$

$$e_{\Delta}(T_1) \le 2\sqrt{3c_1^4 d^3 \epsilon^4 e^{-2b} + \eta O\left(a^{-1} c_1^{14} d^8 \epsilon^6 \sigma_1^2(\Sigma)\right)}$$

$$\max_{j,k} |\sigma_k(W_j(T_1))| \le (1 + 2^{-21}) c_1 \sqrt{d} \epsilon$$

$$\min_{j,k} |\sigma_k(W_j(T_1))| \ge (1 - 2^{-17}) \frac{\epsilon}{c_1 \sqrt{d}}.$$
(249)

This section proves the theorem above by following Lemmas and Corollaries.

Lemma 50. Maximum and minimum singular value bound of weight matrices in alignment stage.

For
$$t \in \left[0, \frac{1}{32c_1^4d^2\epsilon^2\sigma_1(\Sigma)\eta}\right]$$
,

$$\min_{j,k} \sigma_k(W_j) \ge \frac{\epsilon}{c_1 \sqrt{d}} - 16c_1^3 d^{3/2} \epsilon^3 \sigma_1(\Sigma)t, \, \max_{j,k} \sigma_k(W_j) \le \frac{c_1 \sqrt{d}\epsilon}{\sqrt{1 - 4c_1^2 d\epsilon^2 \sigma_1(\Sigma)t}}. \tag{250}$$

Proof. For $t \geq 0$ such that $\max_{j,k} \sigma_k(W_j) \leq 2c_1 \sqrt{d\epsilon} \leq \frac{1}{2} \sigma_1^{1/4}(\Sigma)$,

$$\max_{j} \left\| \nabla_{W_j} \mathcal{L}_{ori} \right\|_{op} \le \max_{j,k} |\sigma_k(W_j)|^3 \left(\sigma_1(\Sigma) + \max_{j,k} |\sigma_k(W_j)|^4 \right) \le \frac{3}{2} \max_{j,k} |\sigma_k(W_j)|^3 \sigma_1(\Sigma). \tag{251}$$

By invoking Corollary 31, for $t \geq 0$ such that $\min_{j,k} \sigma_k(W_j(t)) \geq \frac{\epsilon}{2c_1\sqrt{d}}$,

$$\max_{j,k} \sigma_{k}^{2}(W_{j}(t+1)) - \max_{j,k} \sigma_{k}^{2}(W_{j}(t)) \leq 3\eta \max_{j,k} |\sigma_{k}(W_{j}(t))|^{4} \sigma_{1}(\Sigma)
+ \eta^{2} O\left(a^{2}\left(\epsilon c_{1} \sqrt{d}\right)^{6}\right)
\leq 4\eta \max_{j,k} |\sigma_{k}(W_{j}(t))|^{4} \sigma_{1}(\Sigma)
\min_{j,k} \sigma_{k}^{2}(W_{j}(t+1)) - \min_{j,k} \sigma_{k}^{2}(W_{j}(t)) \geq -3\eta \min_{j,k} |\sigma_{k}(W_{j}(t))| \max_{j,k} |\sigma_{k}(W_{j}(t))|^{3} \sigma_{1}(\Sigma)
+ \eta^{2} O\left(a^{2}\left(\epsilon c_{1} \sqrt{d}\right)^{6}\right)
\geq -2\eta \left(\min_{j,k} |\sigma_{k}(W_{j}(t+1))| + \min_{j,k} |\sigma_{k}(W_{j}(t))|\right)
\cdot \max_{j,k} |\sigma_{k}(W_{j}(t))|^{3} \sigma_{1}(\Sigma).$$
(252)

By solving the differential inequality,

$$\max_{j,k} \sigma_k |W_j(t)| \le \frac{\max_{j,k} \sigma_k |W_j(0)|}{\sqrt{1 - 4\sigma_1(\Sigma) \max_{j,k} \sigma_k |W_j(0)|^2 \eta t}} \le \frac{c_1 \sqrt{d\epsilon}}{\sqrt{1 - 4c_1^2 d\epsilon^2 \sigma_1(\Sigma) \eta t}}, t \in \left[0, \frac{3}{16c_1^2 d\epsilon^2 \sigma_1(\Sigma) \eta}\right], \tag{253}$$

$$\min_{j,k} |\sigma_k(W_j(t))| \ge \frac{\epsilon}{c_1 \sqrt{d}} - 16c_1^3 d^{3/2} \epsilon^3 \sigma_1(\Sigma) \eta t, \ t \in \left[0, \frac{1}{32c_1^4 d^2 \epsilon^2 \sigma_1(\Sigma) \eta}\right]. \tag{254}$$

This completes the proof.

Notice that

$$\max_{j,k} |\sigma_k(W_j(t \le T_1))| \le \frac{c_1 \sqrt{d}\epsilon}{\sqrt{1 - \frac{1}{8c_1^{12}c_2}}} \le (1 + 2^{-21})c_1 \sqrt{d}\epsilon$$

$$\min_{j,k} |\sigma_k(W_j(t \le T_1))| \ge \left(1 - \frac{1}{2c_1^{10}c_2}\right) \cdot \frac{\epsilon}{c_1 \sqrt{d}} \ge (1 - 2^{-17}) \frac{\epsilon}{c_1 \sqrt{d}}.$$
(255)

Corollary 51. Balanced term error in alignment stage.

$$e_{\Delta}(T_1) \le \sqrt{3} \cdot 2^{-31} c_1^{-14} c_2^{-1} d^{-29/2} \epsilon^2.$$
 (256)

Proof. By simply combining Theorem 30 and Lemma 50, denote $M = \max_{i,k,t < T_1}(W_i(t))$,

$$\mathcal{L}_{reg}(t+1) \leq \left(1 - 2.509 \frac{\eta a \epsilon^{2}}{c_{1}^{6} d^{3}}\right) \cdot \mathcal{L}_{reg}(t) + \eta^{2} O\left(a^{2} M^{4} \mathcal{L}_{reg}(t) + \sqrt{a \mathcal{L}_{reg}(t)} M^{6} \mathcal{L}_{ori}(t)\right)
+ \eta^{4} O\left(a M^{12} \mathcal{L}_{ori}(t)^{2} + a^{3} M^{4} \mathcal{L}_{reg}(t)^{2}\right)
\leq \left(1 - \frac{2\eta a \epsilon^{2}}{c_{1}^{6} d^{3}}\right) \cdot \mathcal{L}_{reg}(t) + \eta^{2} O\left(a M^{8} \mathcal{L}_{ori}(t)\right)
\leq \left(1 - \frac{2\eta a \epsilon^{2}}{c_{1}^{6} d^{3}}\right) \cdot \mathcal{L}_{reg}(t) + \eta^{2} O\left(a c_{1}^{8} d^{5} \epsilon^{8} \sigma_{1}^{2}(\Sigma)\right),$$
(257)

giving

$$\mathcal{L}_{reg}(t) \le \mathcal{L}_{reg}(0)e^{-\frac{2\eta a\epsilon^2}{c_1^6 d^3}t} + \eta O\left(c_1^{14} d^8 \epsilon^6 \sigma_1^2(\Sigma)\right). \tag{258}$$

$$\mathcal{L}_{reg}(T_1) \le 3ac_1^4 d^3 \epsilon^4 e^{-2b} + \eta O\left(c_1^{14} d^8 \epsilon^6 \sigma_1^2(\Sigma)\right), \tag{259}$$

$$e_{\Delta}(T_1) = 2\sqrt{\frac{\mathcal{L}_{reg}(T_1)}{a}} \le \sqrt{3} \cdot 2^{-31} c_1^{-14} c_2^{-1} d^{-29/2} \epsilon^2.$$
 (260)

Corollary 52. Main term at the end of alignment stage.

At $t = T_1$,

$$\sigma_{min} (W_1 + W_1')|_{t=T_1} \ge \frac{\epsilon}{2c_1^3 c_2 d^{9/2}}.$$
 (261)

Proof. Denote $\Delta_X(t) = X(t) - X(0)$ for arbitrary X.

At $t = T_1$,

$$\|\Delta_{W}(T_{1})\|_{op} \leq \left\| \sum_{t'=0}^{4} \eta \left[\sum_{j=1}^{4} W_{\prod_{L},j+1}(t')W_{\prod_{L},j+1}(t')^{H} \left(\Sigma - W(t')\right) W_{\prod_{R},j-1}^{H}(t')W_{\prod_{R},j-1}(t') \right] \right\|_{op}$$

$$+ \eta^{2} \sum_{t'=0}^{T_{1}-1} O\left(\max_{j \in [1,4] \cap \mathbb{N}^{*}} \left\| \nabla_{W_{j}} \mathcal{L}(t') \right\|_{F}^{2} \cdot \max_{j \in [1,4] \cap \mathbb{N}^{*}} \left\| W_{j}(t') \right\|_{op}^{2} \right)$$

$$\leq \eta T_{1} \cdot 6\sigma_{1}(\Sigma) \cdot \left(\left(1 + 2^{-21}\right) c_{1} \sqrt{d} \epsilon \right)^{6} + \eta^{2} T_{1} O\left(a^{2} d\left(c_{1} \sqrt{d} \epsilon\right)^{8} \right)$$

$$\leq \eta T_{1} \cdot 8\sigma_{1}(\Sigma) \cdot \left(\left(1 + 2^{-21}\right) c_{1} \sqrt{d} \epsilon \right)^{6}$$

$$\leq \left(1 + 2^{-18}\right) \cdot \frac{1}{4} c_{1}^{-8} c_{2}^{-1} d^{-7} \epsilon^{4}. \tag{262}$$

Thus

$$\|\Delta_{W^{H}W}(T_{1})\|_{op} = \left\| \frac{1}{2} \left[\left(W(T_{1}) + W(0) \right)^{H} \Delta_{W}(T_{1}) + \Delta_{W}(T_{1})^{H} \left(W(T_{1}) + W(0) \right) \right] \right\|_{op}$$

$$\leq \left(1 + 2^{-17} \right) \cdot \frac{1}{2} c_{1}^{-4} c_{2}^{-1} d^{-5} \epsilon^{8}.$$
(263)

From Corollary 51,

$$\left\| \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) \right)^{2} - W(T_{1})^{H} W(T_{1}) \right\|_{op}$$

$$\leq \left\| W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} \right\|_{op} \left\| M_{\Delta 1234}(T_{1}) \right\|_{op} \left\| W_{2}(T_{1}) W_{1}(T_{1}) \right\|_{op}$$

$$\leq 2^{-12} c_{1}^{-8} c_{2}^{-16} d^{-23/2} \epsilon^{8}.$$

$$(264)$$

Thus

$$\left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) \right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}$$

$$\leq \left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) \right)^{2} - W(T_{1})^{H}W(T_{1}) \right\|_{op} + \left\| \Delta_{W^{H}W}(T_{1}) \right\|_{op}$$

$$\leq (1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}.$$
(265)

From Lemma 18,

$$\left\| W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1}) - \left(W(T_{0})^{H}W(T_{0})\right)^{1/2} \right\|_{op} \\
\leq \frac{\left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1})\right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}}{2\sqrt{\lambda_{min}\left(W(T_{0})^{H}W(T_{0})\right) - \left\| \left(W_{1}(T_{1})^{H}W_{2}(T_{1})^{H}W_{2}(T_{1})W_{1}(T_{1})\right)^{2} - W(T_{0})^{H}W(T_{0}) \right\|_{op}}} \\
\leq \frac{(1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}}{2\sqrt{\left(\frac{\epsilon}{c_{1}\sqrt{d}}\right)^{8} - (1 + 2^{-16}) \cdot \frac{1}{2}c_{1}^{-4}c_{2}^{-1}d^{-5}\epsilon^{8}}}} \leq 0.27c_{2}^{-1}d^{-3}\epsilon^{4}.$$
(266)

By (B.2),

$$\sigma_{min} \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right)$$

$$\geq \sigma_{min} \left(\left(W(T_{0})^{H} W(T_{0}) \right)^{1/2} + W(0)^{H} \right)$$

$$- \left\| W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) - \left(W(T_{0})^{H} W(T_{0}) \right)^{1/2} \right\|_{op} - \left\| \Delta_{W}(T_{1}) \right\|_{op}$$

$$\geq 0.72 c_{2}^{-1} d^{-3} \epsilon^{4}, \tag{267}$$

which further gives

$$\sigma_{min} (W_{1} + W_{1}')|_{t=T_{1}}$$

$$= \sigma_{min} \left((W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}))^{-1} (W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right)$$

$$\geq \left(\frac{1}{\max_{j,k} |\sigma_{k}(W_{j}(T_{1}))|} \right)^{3} \cdot \sigma_{min} \left(W_{1}(T_{1})^{H} W_{2}(T_{1})^{H} W_{2}(T_{1}) W_{1}(T_{1}) + W(T_{1})^{H} \right)$$

$$\geq \frac{\epsilon}{2c_{1}^{3} c_{2} d^{9/2}}.$$
(268)

H.2 STAGE 2: SADDLE AVOIDANCE STAGE

In this stage, we further assume
$$a \geq 32c_1^{20}c_2d^{13}\sigma_1(\Sigma)b$$
, where $b \geq \left(5\ln\left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon}\right) + \frac{281}{8}\ln d + 23\ln(4c_1) + 7\ln c_2\right)$. Meanwhile, $\frac{\epsilon}{\sigma_1^{1/4}(\Sigma)} \leq \frac{1}{32c_1^5c_2d^{53/8}}$.

From Theorem 49, for $\eta=O\left(ac_1^{-56}c_2^{-14}d^{-301/4}\epsilon^8\sigma_1^{-9/2}(\Sigma)\right)$ with appropriate small constant,

$$e_{\Delta}(T_1) \le 2\sqrt{3c_1^4 d^3 \epsilon^4 e^{-2b} + \eta O\left(a^{-1}c_1^{14} d^8 \epsilon^6 \sigma_1^2(\Sigma)\right)}$$

$$\le 2^{-44}c_1^{-21}c_2^{-7} d^{-269/8} \epsilon^7 \sigma_1^{-5/4}(\Sigma).$$
(269)

Moreover, $b - \ln b \geq 3 \ln \left(\frac{\sigma_1^{1/4}(\Sigma)}{\epsilon} \right) + \frac{303}{8} \ln d + 37 \ln(2c_1) + 6 \ln c_2$. Thus for $\eta = O\left(a^{-1}c_1^{-44}c_2^{-10}d^{-219/4}\epsilon^4\sigma_1^{-3/2}(\Sigma)\right)$ with appropriate small constant,

$$ae_{\Delta}(T_1) \le 2\sqrt{3 \cdot 2^{10} c_1^{44} c_2^2 d^{29} \epsilon^4 \sigma_1^2(\Sigma) \exp(-2(b - \ln b)) + \eta O\left(ac_1^{14} d^8 \epsilon^6 \sigma_1^2(\Sigma)\right)}$$

$$\le 2^{-30} c_1^{-15} c_2^{-5} d^{-187/8} \epsilon^5 \sigma_1^{1/4}(\Sigma).$$
(270)

Theorem 53. At $T_1 + T_2$, $T_2 = \frac{32c_1^6c_2^2d^9}{\eta\sigma_1(\Sigma)\epsilon^2}$, the following conclusions hold:

$$||W_1(T_1 + T_2) - W_1'(T_1 + T_2)||_F \le 3c_1 d\epsilon$$

$$\sigma_{min}(W_1 + W_1')(T_1 + T_2) \ge 2^{3/4} \sigma_1^{1/4}(\Sigma).$$
(271)

Lemma 54. \mathcal{L}_{ori} is approximately non-increasing.

For $t \in [0, +\infty)$, suppose $\|W_{j \in [1,N] \cap \mathbb{N}^*}(t)\|_{op} \leq M$, then

$$\mathcal{L}_{ori}(t+1) - \mathcal{L}_{ori}(t) \leq -2\eta N \min_{j,k} |\sigma_k(W_j(t))|^{2(N-1)} \mathcal{L}_{ori}(t)$$

$$+ \eta^2 O\left(M^8 \left(M^4 + \sqrt{\mathcal{L}_{ori}(t)}\right) \mathcal{L}_{ori}(t) + aM^4 \sqrt{\mathcal{L}_{ori}(t)} \mathcal{L}_{reg}(t)\right)$$

$$+ \eta^4 O\left(M^{16} \mathcal{L}_{ori}(t)^2 + a^2 M^8 \mathcal{L}_{reg}(t)^2\right).$$

$$(272)$$

Proof. Following the continuous case (71), the change of product matrix satisfy

$$\left\| W(t+1) - W(t) - \eta \sum_{j=1}^{N} W_{\prod_{L}, j+1}(t) W_{\prod_{L}, j+1}(t)^{H} \left(\Sigma - W(t) \right) W_{\prod_{R}, j-1}(t)^{H} W_{\prod_{R}, j-1}(t) \right\|_{F}$$

$$= \eta^{2} O\left(\max_{j \in [1, 4] \cap \mathbb{N}^{*}} \left\| \nabla_{W_{j}} \mathcal{L}(t) \right\|_{F}^{2} \cdot \max_{j \in [1, 4] \cap \mathbb{N}^{*}} \left\| W_{j}(t) \right\|_{op}^{2} \right). \tag{273}$$

Then

3457
3458
3459
$$\mathcal{L}_{ori}(t+1) - \mathcal{L}_{ori}(t) = -\Re\left(\left\langle \Sigma - \frac{W(t+1) + W(t)}{2}, W(t+1) - W(t) \right\rangle \right)$$
3460
3461
$$= -\eta \sum_{j=1}^{N} \left\| W_{\prod_{L}, j+1}(t)^{H} \left(\Sigma - W(t) \right) W_{\prod_{R}, j-1}(t)^{H} \right\|_{F}^{2}$$
3463
3464
$$+ \eta^{2} O\left(M^{2} \sqrt{\mathcal{L}_{ori}(t)} \cdot \max_{j \in [1, 4] \cap \mathbb{N}^{*}} \left\| \nabla_{W_{j}} \mathcal{L}(t) \right\|_{F}^{2} \right)$$
4665
3466
3467
$$+ \eta^{2} O\left(M^{6} \cdot \max_{j \in [1, 4] \cap \mathbb{N}^{*}} \left\| \nabla_{W_{j}} \mathcal{L}_{ori}(t) \right\|_{F}^{2} \right)$$
470
$$+ \eta^{4} O\left(M^{4} \cdot \max_{j \in [1, 4] \cap \mathbb{N}^{*}} \left\| \nabla_{W_{j}} \mathcal{L}(t) \right\|_{F}^{4} \right)$$
471
$$\leq -2\eta N \min_{j,k} |\sigma_{k}(W_{j}(t))|^{2(N-1)} \mathcal{L}_{ori}(t)$$
472
$$+ \eta^{2} O\left(M^{8} \left(M^{4} + \sqrt{\mathcal{L}_{ori}(t)} \right) \mathcal{L}_{ori}(t) + a M^{4} \sqrt{\mathcal{L}_{ori}(t)} \mathcal{L}_{reg}(t) \right)$$
474
3475

Below we further assume $\eta = O\left(\min\left(c_1^{-27}c_2^{-9}d^{-355/8}\epsilon^9\sigma_1^{-15/4}(\Sigma), a^{-1}c_1^{-21}c_2^{-7}d^{-273/8}\epsilon^7\sigma_1^{-9/4}(\Sigma)\right)\right)$ with appropriate small constant.

Lemma 55. Bound of operator norms.

For $t \in [T_1, T_1 + T_2]$,

$$\|\Sigma - W(t)\|_{F} \leq 1.01\sqrt{d}\sigma_{1}(\Sigma)$$

$$e_{\Delta}(t) \leq 1.01 \cdot 2^{-44}c_{1}^{-21}c_{2}^{-7}d^{-269/8}\epsilon^{7}\sigma_{1}^{-5/4}(\Sigma)$$

$$ae_{\Delta}(t) \leq 1.01 \cdot 2^{-30}c_{1}^{-15}c_{2}^{-5}d^{-187/8}\epsilon^{5}\sigma_{1}^{1/4}(\Sigma)$$

$$\|W\|_{op} \leq \|W\|_{F} \leq 3\sqrt{d}\sigma_{1}(\Sigma)$$

$$\max_{j} \|W_{j}\|_{op} \leq \max_{j} \|W_{j}\|_{F} \leq \sqrt{2}d^{1/8}\sigma_{1}^{1/4}(\Sigma).$$
(275)

Proof. We first prove that if the first three inequalities hold at some time t, then the rest follows. Then we prove the first three by mathematical induction.

1. For some t, it the first two hold, then

$$||W(t)||_{op} \le ||W(t)||_F \le ||\Sigma - W(t)||_F + ||\Sigma||_F \le 3\sqrt{d}\sigma_1(\Sigma).$$
 (276)

For the last inequality, prove by contradiction. (Omit t here)

Suppose $\max_{j} \|W_{j}\|_{op} \geq \sqrt{2}d^{1/8}\sigma_{1}^{1/4}(\Sigma)$, then

$$e_{\Delta}(t) \le 1.01 e_{\Delta}(T_1) \le 2^{-15} \max_{j} \|W_j\|_{op}^2.$$
 (277)

Thus for $t > T_1$,

3511
3512
$$||W||_{op}^{2} = ||W_{4}W_{3}W_{2}W_{1}W_{1}^{H}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op}$$

$$\geq ||W_{4}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{2}\Delta_{12}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op}$$

$$- ||W_{4}W_{3}\Delta_{23}W_{2}W_{2}^{H}W_{3}^{H}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{2}W_{2}^{H}\Delta_{23}W_{3}^{H}W_{4}^{H}||_{op}$$

$$- ||W_{4}\Delta_{34}(W_{3}W_{3}^{H})^{2}W_{4}^{H}||_{op} - ||W_{4}W_{3}W_{3}^{H}\Delta_{34}W_{3}^{H}W_{4}^{H}||_{op} - ||W_{4}(W_{3}W_{3}^{H})^{2}\Delta_{34}W_{4}^{H}||_{op}$$

$$\geq \left(\max_{j}||W_{j}||_{op}^{2} - 3e_{\Delta}\right)^{4} - 6e_{\Delta}\max_{j}||W_{j}||_{op}^{6} > 15\sqrt{d}\sigma_{1}(\Sigma),$$

$$(278)$$

which contradicts inequality (276).

2. Mathematical induction.

For $t = T_1$,

$$\|\Sigma - W(T_1)\|_F \le \|\Sigma\|_F + \|W(T_1)\|_F \le (1 + 2^{-39})\sqrt{d}\sigma_1(\Sigma). \tag{279}$$

Suppose for $t' \in [T_1, t]$ $(T_1 \le t < T_2)$, the first two properties hold. Denote $M = \max_j \|W_j(t' \in [T_1, t])\|_{op}$. By invoking Lemma 54 and 30, at t + 1,

$$\mathcal{L}_{ori}(t+1) = \mathcal{L}_{ori}(T_1) + \eta^2(t-T_1)O\left(M^8 \left(M^4 + \sqrt{\mathcal{L}_{ori}(T_1)}\right) \mathcal{L}_{ori}(T_1) + aM^4 \sqrt{\mathcal{L}_{ori}(T_1)} \mathcal{L}_{reg}(T_1)\right) + \eta^4(t-T_1)O\left(M^{16} \mathcal{L}_{ori}(T_1)^2 + a^2M^8 \mathcal{L}_{reg}(T_1)^2\right)$$

$$= \mathcal{L}_{ori}(T_1) + \eta^2 T_2 O\left(d^2 \sigma_1(\Sigma)^4 + d\sigma_1(\Sigma)^2 (ae_{\Delta}(T_1))^2\right) \le 1.01^2 \sqrt{d}\sigma_1(\Sigma).$$
(280)

Note that $\mathcal{L}_{ori} = \frac{a}{4}e_{\Delta}^2$. Under $\eta = O\left(\min\left(c_1^{-27}c_2^{-9}d^{-355/8}\epsilon^9\sigma_1^{-15/4}(\Sigma), a^{-1}c_1^{-21}c_2^{-7}d^{-273/8}\epsilon^7\sigma_1^{-9/4}(\Sigma)\right)\right)$ with appropriate small constant,

$$\mathcal{L}_{reg}(t+1) \leq \mathcal{L}_{reg}(T_{1}) + \eta^{2}(t-T_{1})O\left(a^{2}M^{4}\mathcal{L}_{reg}(t) + \sqrt{a\mathcal{L}_{reg}(t)}M^{6}\mathcal{L}_{ori}(t)\right)
+ \eta^{4}(t-T_{1})O\left(aM^{12}\mathcal{L}_{ori}(t)^{2} + a^{3}M^{4}\mathcal{L}_{reg}(t)^{2}\right)
\leq \mathcal{L}_{reg}(T_{1}) + \eta^{2}T_{2}O\left(\sqrt{a\mathcal{L}_{reg}(t)}M^{6}\mathcal{L}_{ori}(t)\right) + \eta^{4}T_{2}O\left(aM^{12}\mathcal{L}_{ori}(t)^{2}\right)
\leq \frac{1.01^{2}}{4}\min\left(a \cdot \left[2^{-44}c_{1}^{-21}c_{2}^{-7}d^{-269/8}\epsilon^{7}\sigma_{1}^{-5/4}(\Sigma)\right]^{2}, \frac{1}{a} \cdot \left[2^{-30}c_{1}^{-15}c_{2}^{-5}d^{-187/8}\epsilon^{5}\sigma_{1}^{1/4}(\Sigma)\right]^{2}\right).$$
(281)

This completes the proof.

Lemma 56. Bound of $\|W_2^{-1}\|_{op}$ and relevant term.

For $t \in [T_1, T_1 + T_2]$,

$$||W_2^{-1}(t)||_{op} \le 128c_1^6c_2^2d^{77/8}\epsilon^{-2}\sigma_1^{1/4}(\Sigma),$$
 (282)

$$e_{\Delta}(t) \|W_2^{-1}(t)\|_{\infty}^2 \le 1.01 \cdot 2^{-30} c_1^{-9} c_2^{-3} d^{-115/8} \epsilon^3 \sigma_1^{-3/4}(\Sigma).$$
 (283)

Proof. We begin with the update of W_2^{-1} . From Lemma 19,

$$\begin{aligned} & \left\| W_{2}^{-1}(t+1) - W_{2}^{-1}(t) \right. \\ & - \eta \left[-R(t)W_{4}(t)^{H}(\Sigma - W(t))W_{1}(t)^{H}W_{2}(t)^{-1} - a\Delta_{12}(t)W_{2}(t)^{-1} + aW_{2}(t)^{-1}\Delta_{23}(t) \right] \right\|_{op} \\ & \leq & \eta^{2} \|W_{2}(t)^{-1}\|_{op}^{2} \|W_{2}(t+1)^{-1}\|_{op} \|\nabla_{W_{2}}\mathcal{L}(t)\|_{op}^{2}. \end{aligned}$$

$$(284)$$

By triangular inequality,

$$\begin{aligned} \|W_{2}(t+1)^{-1}\|_{op} - \|W_{2}(t)^{-1}\|_{op} &\leq \eta \|R(t)\|_{op} \|W_{4}(t)\|_{op} \|\Sigma - W(t)\|_{op} \|W_{1}(t)^{H}W_{2}(t)^{-1}\|_{op} \\ &+ \eta a \|\Delta_{12}(t)\|_{op} \|W_{2}(t)^{-1}\|_{op} + \eta a \|W_{2}(t)^{-1}\|_{op} \|\Delta_{23}(t)\|_{op} \\ &+ \eta^{2} \|W_{2}(t)^{-1}\|_{op}^{2} \|W_{2}(t+1)^{-1}\|_{op} \|\nabla_{W_{2}}\mathcal{L}(t)\|_{op}^{2}. \end{aligned}$$

$$(285)$$

From

$$||R||_{op} \leq \sqrt{1 + \frac{1}{\sigma_{min}^{2}(W_{2})} \cdot ||\Delta_{23}||_{op}}$$

$$||W_{1}^{H}W_{2}^{-1}||_{op} = \sqrt{||W_{2}^{H-1}W_{1}W_{1}^{H}W_{2}^{-1}||_{op}} = \sqrt{||I + W_{2}^{H-1}\Delta_{12}W_{2}^{-1}||} \leq \sqrt{1 + e_{\Delta} ||W_{2}^{-1}||_{op}^{2}}.$$
(286)

Further we have

$$\begin{aligned} \|W_{2}(t+1)^{-1}\|_{op} - \|W_{2}(t)^{-1}\|_{op} &\leq 2\sqrt{2}\eta \left(1 + e_{\Delta}(t) \|W_{2}(t)^{-1}\|_{op}^{2}\right) d^{5/8}\sigma_{1}^{5/4}(\Sigma) \\ &+ \sqrt{2}\eta a e_{\Delta}(t) \|W_{2}(t)^{-1}\|_{op} \\ &+ \eta^{2}O\left(\|W_{2}(t)^{-1}\|_{op}^{2} \|W_{2}(t+1)^{-1}\|_{op} \|\nabla_{W_{2}}\mathcal{L}(t)\|_{op}^{2}\right). \end{aligned}$$

$$(287)$$

Combine with Lemma 55, for $t \ge T_1$ such that (282) holds,

$$\begin{aligned} & \|W_{2}(t+1)^{-1}\|_{op} - \|W_{2}(t)^{-1}\|_{op} \\ & \leq 2\sqrt{2}(1+1.01\cdot 2^{-30})\eta d^{5/8}\sigma_{1}^{5/4}(\Sigma) + 2^{-22}\eta c_{1}^{-9}c_{2}^{-3}d^{-55/4}\epsilon^{3}\sigma_{1}^{1/2}(\Sigma) \\ & + \eta^{2}O\left(c_{1}^{18}c_{2}^{6}d^{245/8}\epsilon^{-6}\sigma_{1}^{17/4}(\Sigma)\right) \\ & \leq 2\sqrt{2}(1+2^{-20})\eta d^{5/8}\sigma_{1}^{5/4}(\Sigma). \end{aligned} \tag{288}$$

From Theorem 49, $\max\left(\left\|W_2(T_1)^{-1}\right\|_{op}, \left\|W_3(T_1)^{-1}\right\|_{op}\right) \leq \frac{1}{\min_{j,k} |\sigma_k(W_j(T_1))|} \leq \frac{c_1\sqrt{d}}{(1-2^{-17})\epsilon}$, then the proof of the first inequality is completed via integration during the time interval $[T_1, T_1 + T_2]$. The second inequality follows immediately.

Remark 16. This Lemma verifies that $W_{2,3}^{-1}$ are bounded (consequently $W_{2,3}$ are full rank), then R is well defined throughout this stage. For $t > T_1 + T_2$, further analysis shows that the minimum singular values of W_2 and W_3 are lower bounded by $\Omega(\sigma_1^{1/4}(\Sigma))$.

Now we begin the proof of Lemma 9 and 10.

Proof for Lemma 9:

Proof. From Lemma 56, for $t \in [T_1, T_1 + T_2]$,

$$\max\left(\left\|R^{H}R - I\right\|_{op}, \left\|I - RR^{H}\right\|_{op}\right) \le e_{\Delta} \left\|W_{2}^{-1}\right\|_{op}^{2}$$

$$\le 1.01 \cdot 2^{-30} c_{1}^{-9} c_{2}^{-3} d^{-115/8} \epsilon^{3} \sigma_{1}^{-3/4}(\Sigma),$$
(289)

$$||M_{1} - M'_{1}||_{op} \leq \sqrt{6} \cdot \frac{\max_{j,k} \sigma_{k}^{2}(W_{j})}{\sigma_{min}^{2}(W_{2})} e_{\Delta}$$

$$\leq 2^{-27} c_{1}^{-9} c_{2}^{-3} d^{-113/8} \epsilon^{3} \sigma_{1}^{-1/4}(\Sigma),$$
(290)

$$\left\| M_{2} - \frac{M_{1} + M_{1}'}{2} \right\|_{op} \leq \left\| \Delta_{12} \right\|_{op} + \frac{1}{2} \left\| M_{1} - M_{1}' \right\|_{op} \leq \left[1 + \frac{\sqrt{6}}{2} \cdot \frac{\max_{j,k} \sigma_{k}^{2}(W_{j})}{\sigma_{min}^{2}(W_{2})} \right] e_{\Delta}$$

$$\leq 2^{-28} c_{1}^{-9} c_{2}^{-3} d^{-113/8} \epsilon^{3} \sigma_{1}^{-1/4}(\Sigma).$$
(291)

Consequently:

$$||R||_{op} \le \sqrt{1 + e_{\Delta} ||W_2^{-1}||_{op}^2} \le 1 + 1.01 \cdot 2^{-31} c_1^{-9} c_2^{-3} d^{-115/8} \epsilon^3 \sigma_1^{-3/4}(\Sigma), \tag{292}$$

$$\|W_1'\|_{op} \le \|W_1'\|_F \le \sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma) \|R\|_{op} \le (1 + 1.01 \cdot 2^{-31})\sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma), \tag{293}$$

$$\left\| \frac{M_1 + M_1'}{2} \right\|_{op} \le \|M_2\|_{op} + \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{op} \le \left(1 + 2^{-29} \right) 2d^{1/4} \sigma_1^{1/2}(\Sigma), \tag{294}$$

$$||M_{1}'M_{2}M_{1} - M_{1}M_{2}M_{1}'||_{op} \le ||M_{1} - M_{1}'|| ||M_{2}|| ||M_{1} + M_{1}'||$$

$$\le (1 + 2^{-29}) 2^{-25} c_{1}^{-9} c_{2}^{-3} d^{-109/8} \epsilon^{3} \sigma_{1}^{3/4}(\Sigma).$$
(295)

By combining all results above, for $t \in [T_1, T_1 + T_2 - 1]$ such that $||W_1 - W_1'||_F \le 3c_1 d\epsilon$ holds,

$$\begin{aligned} &3652\\ &3653\\ &\|W_{1}(t+1)-W_{1}'(t+1)\|_{F}^{2}-\|W_{1}(t)-W_{1}'(t)\|_{F}^{2}\\ &\leq -2\eta\sigma_{1}(\Sigma)\sigma_{min}(W_{2})^{2}\,\|W_{1}(t)-W_{1}'(t)\|_{F}^{2}\\ &3656\\ &+\eta\|M_{2}(t)\|_{F}\,\|M_{1}'(t)-M_{1}(t)\|_{op}\,\|M_{2}(t)\|_{op}\,\left(\|W_{1}'(t)\|_{op}+\|W_{1}(t)\|_{op}\right)\,\|W_{1}(t)-W_{1}'(t)\|_{F}\\ &3657\\ &+2\eta\,\|-M_{1}'(t)M_{2}(t)M_{1}(t)+M_{1}(t)M_{2}(t)M_{1}'(t)\|_{op}\,\|W_{1}'(t)\|_{F}\,\|W_{1}(t)-W_{1}'(t)\|_{F}\\ &3659\\ &+2\eta\,\max_{j}\|W_{j}(t)\|_{op}^{3}\|\Sigma-W(t)\|_{F}\left(\|R(t)^{H}R(t)-I\|_{op}+\|I-R(t)R(t)^{H}\|_{op}\right)\|W_{1}(t)-W_{1}'(t)\|_{F}\\ &3661\\ &+2\eta ae_{\Delta}(t)\,\|W_{1}(t)-W_{1}'(t)\|_{F}^{2}\\ &+4\eta ae_{\Delta}(t)\,\|W_{2}(t)^{-1}\|_{op}\,\|W_{2}(t)\|_{F}\,\|W_{1}'(t)\|_{op}\,\|W_{1}(t)-W_{1}'(t)\|_{F}\\ &3664\\ &+\eta^{2}O\left(\left[\max_{j\in[1,4]\cap\mathbb{N}^{*}}\|W_{j}(t)\|_{op}^{5}\,\|Y_{2}(t+1)^{-1}\|_{op}\right)\right.\\ &\left.\left.\left.\left.\left(\max_{j\in[1,4]\cap\mathbb{N}^{*}}\|W_{j}(t)\|_{op}^{5}\,\|W_{2}(t+1)^{-1}\|_{op}\right)\right.\right.\right)\\ &\leq &-2\eta\sigma_{1}(\Sigma)\sigma_{min}(W_{2})^{2}\,\|W_{1}(t)-W_{1}'(t)\|_{F}^{2}+2^{-17}\eta c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma). \end{aligned}$$

From Theorem 49, at $t = T_1$,

(296)

3673
3674
$$||W_1(T_1) - W_1'(T_1)||_F \le ||W_1(T_1)||_F + ||W_1'(T_1)||_F \le ||W_1(T_1)||_F + ||W_4(T_1)||_F ||R(T_1)||_{op}$$
3675
$$\le (1 + 2^{-20}) 2c_1 d\epsilon.$$

(297)

Thus $\|W_1(t) - W_1'(t)\|_F^2 \le \sqrt{\left[\left(1 + 2^{-20}\right)2c_1d\epsilon\right]^2 + 2^{-17}c_1^{-8}c_2^{-3}d^{-25/2}\epsilon^4\sigma_1(\Sigma)\eta(t - T_1)}$, when both $t \in [T_1, T_1 + T_2]$ and $||W_1(t) - W_1'(t)||_F^2 \le 3c_1 d\epsilon$ hold. Then

$$||W_{1}(T_{1}+T_{2})-W_{1}'(T_{1}+T_{2})||_{F}^{2} \leq \sqrt{\left[\left(1+2^{-20}\right)2c_{1}d\epsilon\right]^{2}+2^{-17}c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma)\eta T_{2}}$$

$$\leq \sqrt{\left[\left(1+2^{-20}\right)2c_{1}d\epsilon\right]^{2}+2^{-12}c_{1}^{-2}c_{2}^{-1}d^{-7/2}\epsilon^{2}} < 3c_{1}d\epsilon,$$
(298)

which completes the proof.

Proof for Lemma 10:

Proof. We analyze the dynamics of $\lambda_{min} \left(\left(W_1 + W_1' \right)^H \left(W_1 + W_1' \right) \right) = \sigma_{min}^2$.

$$\begin{array}{lll} \text{From} & \left\| M_2 - \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq \\ & \left(1 + 2^{-29} \right) 2 d^{1/4} \sigma_1^{1/2}(\Sigma), \text{ define} & \leq & 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-3} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} d^{-113/8} \epsilon^3 \sigma_1^{-1/4}(\Sigma) & \text{and} & \left\| \frac{M_1 + M_1'}{2} \right\|_{op} & \leq & 2^{-28} c_1^{-9} c_2^{-9} c$$

$$E(t) := \sigma_1(\Sigma) \left(M_2(t) - \frac{M_1(t) + M_1'(t)}{2} \right) - \left(M_2(t) \left(\frac{M_1(t) + M_1'(t)}{2} \right) M_2(t) - \left(\frac{M_1(t) + M_1'(t)}{2} \right)^3 \right).$$
(299)

Then

$$||E(t)||_{op} \le 2^{-28}c_1^{-9}c_2^{-3}d^{-113/8}\epsilon^3\sigma_1^{3/4}(\Sigma) + (1+2^{-28})2^{-24}c_1^{-9}c_2^{-3}d^{-109/8}\epsilon^3\sigma_1^{3/4}(\Sigma)$$

$$\le (1+2^{-4}+2^{-28})2^{-24}c_1^{-9}c_2^{-3}d^{-109/8}\epsilon^3\sigma_1^{3/4}(\Sigma).$$
(300)

By Lemma 9, $\|W_1 - W_1'\|_{op} \le \|W_1 - W_1'\|_F \le 3c_1 d\epsilon$, and under $\sigma_{min}(t) \ge \frac{\epsilon}{2c_1^3c_2d^{9/2}}$,

$$\sigma_{min}(t+1)^2 \ge \lambda_{min} \left(W_{new}(t)^H W_{new}(t) \right) - 2^{-18} \sigma_1(\Sigma) \sigma_{min}(t)^4, \tag{301}$$

where

$$W_{new}(t) = \left(I + \eta \left[\sigma_1(\Sigma) \left(\frac{M_1(t) + M_1'(t)}{2}\right) - \left(\frac{M_1(t) + M_1'(t)}{2}\right)^3 + E(t)\right]\right) (W_1(t) + W_1'(t)).$$
(302)

Denote $P=\frac{W_1+W_1'}{2}$, $Q=\frac{W_1-W_1'}{2}$. Notice that $PP^H+QQ^H=\frac{M_1+M_1'}{2}$. Then by invoking Lemma 29 (omit t here) the first term becomes

$$\begin{array}{ll} 3728 & \lambda_{min}\left(W_{new}^{H}W_{new}\right) = \lambda_{min}\left(W_{new}W_{new}^{H}\right) \\ & = 4\lambda_{min}\left(\left(I + \eta\left[\sigma_{1}(\Sigma)\left(PP^{H} + QQ^{H}\right) - \left(PP^{H} + QQ^{H}\right)^{3} + E\right]\right)PP^{H} \\ & \cdot \left(I + \eta\left[\sigma_{1}(\Sigma)\left(PP^{H} + QQ^{H}\right) - \left(PP^{H} + QQ^{H}\right)^{3} + E\right]\right)\right) \\ & 3733 \\ & > \sigma_{min}^{2} + 8\eta\left(\sigma_{1}(\Sigma) - 2\|Q\|_{op}^{2}\|\frac{M_{1} + M_{1}'}{2}\|_{op}\right)\left(\frac{\sigma_{min}^{2}}{4}\right)^{2} \\ & - 8\eta\left\|\frac{M_{1} + M_{1}'}{2}\|_{op}\left(\frac{\sigma_{min}^{2}}{4}\right)^{3} \\ & - 8\eta\left(\|E\|_{op} + \|Q\|_{op}^{4}\|\frac{M_{1} + M_{1}'}{2}\|_{op}\right)\left(\frac{\sigma_{min}^{2}}{4}\right) \\ & - 8\eta\left(\|E\|_{op} + \|Q\|_{op}^{4}\|\frac{M_{1} + M_{1}'}{2}\|_{op}\right)\left(\frac{\sigma_{min}^{2}}{4}\right) \\ & + \eta^{2}O\left(\left(\sigma_{1}(\Sigma)^{2}\|\frac{M_{1} + M_{1}'}{2}\|_{op}^{2} + \|\frac{M_{1} + M_{1}'}{2}\|_{op}^{6} + \|E\|_{op}^{2}\right)\|\frac{M_{1} + M_{1}'}{2}\|_{op}\right). \end{array}$$

Notice $||Q||_{op} = \frac{1}{2} ||W_1 - W_1'||_F \le \frac{3}{2} c_1 d\epsilon \le \sigma_k \cdot 3c_1^4 c_2 d^{11/2}, \epsilon \le \frac{1}{32c_1^5 c_2 d^{53/8}} \sigma_1^{1/4}(\Sigma)$, then under $\sigma_{min}(t) \geq \frac{\epsilon}{2c_1^3c_2d^{9/2}},$

$$\sigma_{min}(t+1)^{2} \ge \sigma_{min}(t)^{2} + (2^{-1} - 81(1+2^{-4})2^{-10})\eta\sigma_{1}(\Sigma)\sigma_{min}(t)^{4} - \frac{1}{32}\eta\sigma_{min}(t)^{8}.$$
(304)

Notice that $\sigma_{min}(t)$ is bounded by $O\left(d^{1/8}\sigma_1^{1/4}(\Sigma)\right)$. By taking reciprocal,

$$\frac{1}{\sigma_{min}(t+1)^{2}} \leq \frac{1}{\sigma_{min}(t)^{2}} + \frac{(2^{-1} - 81(1+2^{-4})2^{-10})\eta\sigma_{1}(\Sigma)\sigma_{min}(t)^{4} - \frac{1}{32}\eta\sigma_{min}(t)^{8}}{\sigma_{min}(t)^{4} + (2^{-1} - 81(1+2^{-4})2^{-10})\eta\sigma_{1}(\Sigma)\sigma_{min}(t)^{6} - \frac{1}{32}\eta\sigma_{min}(t)^{10}} \\
\leq \frac{1}{\sigma_{min}(t)^{2}} + \frac{3}{8}\eta\sigma_{1}(\Sigma) - \frac{1}{32}\eta\sigma_{min}(t)^{4}.$$
(305)

This indicates that $\sigma_{min}(t)$ takes at most time $\Delta t' = \frac{1}{\frac{1}{8}\eta\sigma_1(\Sigma)} \left| \frac{1}{\sigma_{min}(t=0)^2} - \frac{1}{\left(2^{3/4}\sigma_1^{1/4}(\Sigma)\right)^2} \right| < T_2$

to increase to $2^{3/4}\sigma_1^{1/4}(\Sigma)$, and never decrease to less than $2^{3/4}\sigma_1^{1/4}(\Sigma)$ afterwards (in $t \in [T_1 + T_1]$ $\delta t', T_2$).

STAGE 3: LOCAL CONVERGENCE STAGE

In this stage, we analysis the time to reach δ -convergence, that is

$$T(\delta) = \inf_{t} \{ \mathcal{L}(t) \le \delta \}. \tag{306}$$

Theorem 57. Local convergence.

For
$$t \in [T_1 + T_2, +\infty)$$
,

3781
$$\mathcal{L}_{ori}(t) \leq \mathcal{L}_{ori}(T_1 + T_2) \exp\left(-\eta \sigma_1^{3/2}(\Sigma)(t - T_1 - T_2)\right)$$
3783
$$\mathcal{L}_{reg}(t) \leq l_{reg} \exp\left(-\eta \sigma_1^{3/2}(\Sigma)(t - T_1 - T_2)\right)$$
3784
$$\sigma_{min}\left(W_1(t) + W_1'(t)\right) \geq 2^{3/4} \sigma_1^{1/4}(\Sigma)$$
3786
$$\|W_1(t) - W_1'(t)\|_F \leq 3c_1 d\epsilon,$$
3787

where
$$\mathcal{L}_{ori}(T_1 + T_2) = \frac{1.01^2}{2} \cdot d\sigma_1^2(\Sigma)$$
, and $l_{reg} = \max\left(\frac{a}{4}\left(1.01 \cdot 2^{-44}c_1^{-21}c_2^{-7}d^{-269/8}\epsilon^7\sigma_1^{-5/4}(\Sigma)\right)\right)^2, \frac{1}{4a}\left(1.01 \cdot 2^{-30}c_1^{-15}c_2^{-5}d^{-187/8}\epsilon^5\sigma_1^{1/4}(\Sigma)\right)^2\right)$.

Proof. Prove by induction.

At $t = T_2$ these properties holds.

Suppose at some time $t \in [T_2, \infty)$ they holds, then follow the same arguments in Lemma 55, $\max_i \|W_i(t)\|_{op} \leq \sqrt{2}d^{1/8}\sigma_1^{1/4}(\Sigma)$.

To address the bound of $\|W_2^{-1}\|_{op}$

$$\left\| \frac{M_{1}(t) - M_{1}'(t)}{2} \right\|_{op} \leq \|W_{1}(t) - W_{1}'(t)\|_{op} \left\| \frac{W_{1}(t) + W_{1}'(t)}{2} \right\|_{op} \leq 8c_{1}d^{9/8}\sigma_{1}^{1/4}(\Sigma)\epsilon$$

$$\left\| M_{2}(t) - \frac{M_{1}(t) + M_{1}'(t)}{2} \right\|_{op} \leq \|\Delta_{12}(t)\|_{op} + \left\| \frac{M_{1}(t) - M_{1}'(t)}{2} \right\|_{op} \leq 16c_{1}d^{9/8}\sigma_{1}^{1/4}(\Sigma)\epsilon$$

$$\sigma_{min}(W_{2}(t)) = \sqrt{\lambda_{min}(M_{2}(t))} \geq \sqrt{\lambda_{min}\left(\frac{M_{1}(t) + M_{1}'(t)}{2}\right) - 16c_{1}d^{9/8}\sigma_{1}^{1/4}(\Sigma)\epsilon}$$

$$\geq \sqrt{\sigma_{min}^{2}\left(\frac{W_{1}(t) + W_{1}'(t)}{2}\right) - 16c_{1}d^{9/8}\sigma_{1}^{1/4}(\Sigma)\epsilon} \geq \frac{1}{2^{3/8}}\sigma_{1}^{1/4}(\Sigma).$$
(308)

Similarly, $\min_{j,k}(\sigma_k(W_j(t))) \geq \frac{1}{2^{3/8}}\sigma_1^{1/4}(\Sigma)$.

Then following the derivations in Lemma 9 and 10,

$$||W_{1}(t+1) - W_{1}'(t+1)||_{F}^{2} \leq \left(1 - 2\eta\sigma_{1}(\Sigma)\sigma_{min}(W_{2})^{2}\right) ||W_{1}(t) - W_{1}'(t)||_{F}^{2} + 2^{-17}\eta c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma)$$

$$\leq \left(1 - \eta\sigma_{1}^{3/2}(\Sigma)\right) ||W_{1}(t) - W_{1}'(t)||_{F}^{2} + 2^{-17}\eta c_{1}^{-8}c_{2}^{-3}d^{-25/2}\epsilon^{4}\sigma_{1}(\Sigma) \leq 3c_{1}d\epsilon$$

$$\frac{1}{\sigma_{min}\left(W_{1}(t+1) + W_{1}'(t+1)\right)^{2}} \leq \frac{1}{\sigma_{min}(t)^{2}} + \frac{3}{8}\eta\sigma_{1}(\Sigma) - \frac{1}{32}\eta\sigma_{min}(t)^{4} < \frac{1}{\left(2^{3/4}\sigma_{1}^{1/4}(\Sigma)\right)^{2}}.$$
(309)

Then by Theorem 54 and 30,

$$\mathcal{L}_{ori}(t+1) \leq \mathcal{L}_{ori}(t) - 2^{3/4} \eta \sigma_{1}^{3/2}(\Sigma) \mathcal{L}_{ori}(t)
+ \eta^{2} O\left(\max_{j} \|W_{j}(t)\|_{op}^{8} \left(\max_{j} \|W_{j}(t)\|_{op}^{4} + \sqrt{\mathcal{L}_{ori}(t)}\right) \mathcal{L}_{ori}(t) + a \max_{j} \|W_{j}(t)\|_{op}^{4} \sqrt{\mathcal{L}_{ori}(t)} \mathcal{L}_{reg}(t)\right)
+ \eta^{4} O\left(\max_{j} \|W_{j}(t)\|_{op}^{16} \mathcal{L}_{ori}(t)^{2} + a^{2} \max_{j} \|W_{j}(t)\|_{op}^{8} \mathcal{L}_{reg}(t)^{2}\right)
\leq \left(1 - \eta \sigma_{1}^{3/2}(\Sigma)\right) \mathcal{L}_{ori}(t),$$
(310)

$$\mathcal{L}_{reg}(t+1) \leq \left(1 - \frac{1}{3}\eta a d^{-1/4} \sigma_{1}^{1/2}(\Sigma)\right) \cdot \mathcal{L}_{reg}(t) + \eta^{2} O\left(a^{2} M^{4} \mathcal{L}_{reg}(t) + \sqrt{a \mathcal{L}_{reg}(t)} M^{6} \mathcal{L}_{ori}(t)\right)
+ \eta^{4} O\left(a M^{12} \mathcal{L}_{ori}(t)^{2} + a^{3} M^{4} \mathcal{L}_{reg}(t)^{2}\right)
\leq \left(1 - \frac{1}{4}\eta a d^{-1/4} \sigma_{1}^{1/2}(\Sigma)\right) \cdot \mathcal{L}_{reg}(t) \leq \left(1 - \eta a d^{-1/4} \sigma_{1}^{3/2}(\Sigma)\right) \cdot \mathcal{L}_{reg}(t).$$
(311)

This completes the proof.

By Combining the three-stage results, the global convergence guarantee of Theorem 48 is proved.

I NUMERICAL SIMULATIONS

Through out this section, we consider numerical simulations under four-layer matrix factorization on square matrices with dimension of 5.

I.1 SADDLE AVOIDANCE DYNAMICS UNDER BALANCE INITIALIZATION

This section presents numerical simulations of the saddle avoidance stage under balanced initialization. In this experiment, $\epsilon = 0.05$, $\eta = 0.1$, $\Sigma_w(0) = \epsilon \cdot \text{diag}(1, 0.8, 0.6, 0.5, 0.9)$.

We set the target matrix to $\Sigma = I$ in Figure 1 and to $\Sigma = \text{diag}(2.00, 1.55, 1.10, 0.65, 0.20)$ in Figure 2. Each pair of solid and dashed lines of the same color represents the logarithms of the k-th singular value of Σ_W and that of $\frac{1}{2}(U+V)\Sigma_W$, respectively.

These figures clearly exhibit the following properties:

- $\sigma_k\left(\frac{1}{2}(U+V)\Sigma_W\right)$ provides a tight lower bound for $\sigma_k\left(\Sigma_W\right)$, verifying the conclusion of Lemma 20.
- The eigen-gap of the target matrix introduces non-smoothness and non-monotonicity into
 the original lower bound for singular values of the product matrix, leading to segmented
 rather than global smoothness and monotonicity. This explains why the dynamics are easier
 to analyze when the target matrix is the identity.
- The 1/2 probability of converging to a saddle point under real balanced initialization is a general phenomenon, even if the target matrix is not identity. However, in the setting of Figure 2, initializations with $\det(U^{\top}V)=1$ fail to converge, which contrasts with the identity target case.

I.2 ALIGNMENT DYNAMICS UNDER BALANCE REGULARIZATION TERM

This section exhibits the dynamics of weight matrices under regularization term. The original square loss \mathcal{L}_{ori} is omitted. Here $a=1,\,\epsilon=1,\,\eta=0.001$.

Figure 3 illustrates the conclusion of Theorem 27 and 31. Clearly the maximum among all the singular values are non-increasing while the minimum is non-decreasing.

Figure 4 illustrates the dynamics of main term $\sigma_{min}(W_1 + W_2^{-1}W_3^HW_4^H)$. For real initialization with $\det(W(0)) < 0$, $\sigma_{min}(W_1 + W_2^{-1}W_3^HW_4^H)$ decays to 0 at a linear rate, while for $\det(W(0)) > 0$ and complex initialization it stays at a small value after some oscillation.

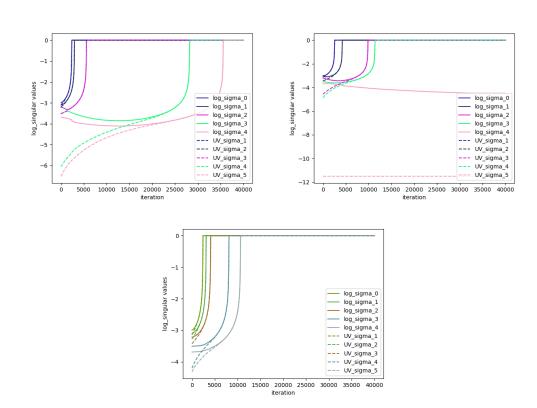


Figure 1: Dynamics of singular values (log scale) for an identity target matrix. From left to right, up to down: real initialization with $\det(U^{\top}V) = 1$, $\det(U^{\top}V) = -1$, and complex initialization.

J LLM USAGE DECLARATION

In the preparation of this paper, large language models (LLMs) served only as an auxiliary tool for enhancing writing clarity, checking grammar, and assisting in the drafting and debugging of simulation code. These tasks were performed under the authors' complete oversight. The central scientific ideas, theoretical results, and research contributions are entirely the work of the authors.

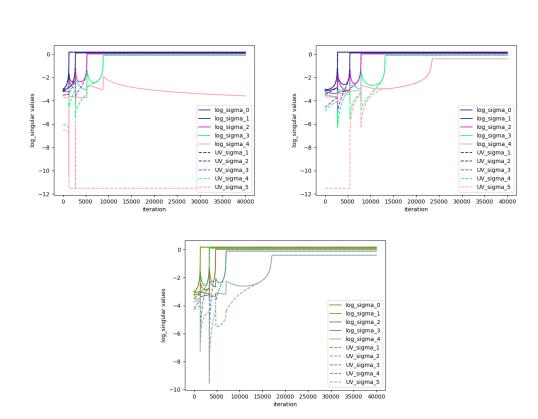


Figure 2: Dynamics of singular values (log scale) for a non-identity target matrix. From left to right, up to down: real initialization with $\det(U^\top V) = 1$, $\det(U^\top V) = -1$, and complex initialization.

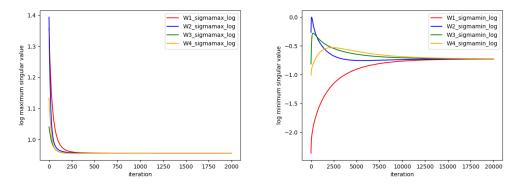


Figure 3: Dynamics of extreme singular values (log scale) for four weight matrices.

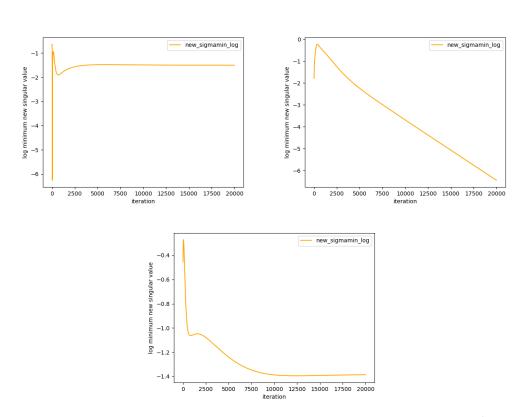


Figure 4: Dynamics of the minimum singular value of hermitian main term $W_1 + W_2^{-1}W_3^HW_4^H$ (log scale). From left to right, up to down: real initialization with $\det(W) > 0$, $\det(W) < 0$, and complex initialization.