Cooperating on networks: inequality and social structure

Extended Abstract

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1 Abstract

This paper analyses how inequality in endowments and social structure jointly affect individuals' ability to cooperate. Individuals repeatedly invest in a local public good ("cooperation") in an environment that is described by a distribution of endowments and a network of beneficiaries. We measure the cooperativeness of an environment by the minimum discount factor needed to sustain (any) cooperation in equilibrium. We characterise the endowment distribution that maximises cooperativeness for any given network and the corresponding minimum discount factor. The latter is shown to be inversely proportional to the maximal index of the graph describing the network. The corresponding Perron eigenvalue of the adjacency matrix characterises the most cooperative income distribution. Moreover, we show that if an environment maximises cooperativeness (over all income distributions and networks of a certain size), then the network is described by a *nested split graphs*. We further show that this is the same class of graphs that maximise welfare for any given discount factor, and yet, the most cooperative graph need not be equal to the most efficient.

2 Overview

Cooperation in social dilemmas - situations where efficiency requires individuals to overcome an incentive to free-ride - often relies on the threat of mutual defection (so-called 'trigger strategies') (Ellison, 1994; Wolitzky, 2013). It is well-understood that for this to be effective, the value placed on future interactions needs to be sufficiently high (Fudenberg and Maskin, 1986). However, asymmetries in who benefits from an individual's cooperation, and inequality in the resources individuals can contribute, influence the effectiveness of such punishment. The focus of this paper is to analyse how asymmetries in social structure and resources interact. We analyse an indefinitely repeated public goods game played on a network. Individuals can invest their endowments in the production of a local public good that benefits their neighbours. Individuals (potentially) differ in network position and endowments. We deliberately exclude other possible sources of inequality to get a tractable model of the interaction between social structure and endowment inequality. In particular, individuals have access to the same production and monitoring technology.[1] We focus on two aspects: (i) the cooperativeness of an environment, which we

[1] Kinateder and Merlino (2017) examine heterogeneity in production technology in endogenous networks and Wolitzky (2013) analyses (heterogeneous) network monitoring.

measure by the minium discount factor (or continuation probability) needed to sustain positive contributions to the public good in a subgame perfect equilibrium. And (ii), the efficiency, meaning the maximum utilitarian welfare that can be achieved in equilibrium (for a given discount factor).

To gain some intuition, suppose all individuals have an identical number of neighbours, meaning the network is described by a regular graph. Then individuals are homogeneous in their network position. We might reasonably conjecture that an endowment distribution that maximises cooperativeness and/or efficiency must be equally homogeneous. If, however, a network is heterogeneous, cooperativeness and efficiency might be higher if endowments somehow reflect this heterogeneity. We characterise exactly how these forms of heterogeneity relate at the optimum: cooperativeness is maximised if endowments (or contributions) correspond to the eigenvector centrality of players. Even though the benefits from cooperation are only conferred upon neighbours, this result implies that the properties of the entire network determine cooperativeness, not just direct relations. A player can have more neighbours than another, and yet lower centrality. To maximise cooperativeness, players with higher centrality, not (just) higher degree, should have larger endowments. We use this result to show that both the most cooperative as well as the most efficient graphs belong to the class of of nested split graphs.[2] These graphs have distinct hierarchy levels with the neighbourhoods N_i and N_j of any two vertices i, j ordered by set inclusion in the sense that $N_i \setminus \{j\} \subset N_j \setminus \{i\}$, or $N_i \setminus \{j\} \supset N_j \setminus \{i\}$ (or both). This has important implications for how inequality in network structure and endowments interact. We show that inequality on both dimensions can be strictly beneficial for both cooperativeness and welfare.

Model basics. There are n individuals, described by the set N, who interact repeatedly. In any given round, an individual i receives an endowment $e_i \geq 0$ and decides what fraction $x_i \in [0, 1]$ of the endowment to invest in the production of a local public good ('level of cooperation'), and what fraction to consume. The vector $\mathbf{e}$ describes the endowments of all $i \in N$. Individuals interact on an undirected network which characterises who benefits from each investment. It is described by an adjacency matrix $G = (g_{ij})_{1 \leq i,j \leq n}$, which is used synonymously with the graph it represents. The production technology of the local public good is linear and identical across individuals. Each individual receives a (constant) marginal benefit b from their own investment, as well as all the investments of their neighbours. An individual i's payoff in a given round is

$$\pi_i(\mathbf{x}, \boldsymbol{e}, G) = b\big(x_i e_i + \sum_{j \in N} g_{ij} x_j e_j\big) + (1 - x_i) e_i.$$

The interactions recur indefinitely, with a probability $\delta \in (0, 1)$ that interactions continue after each round. The network and the endowments are constant across time.

A history of play in a round t, denoted by $h(t) = \{\mathbf{x}(1), \mathbf{x}(2), ..., \mathbf{x}(t-1)\}$, contains the investment choices of all individuals in the network prior to t. We initially assume that this is common knowledge among players, meaning monitoring is public and global. Let $\mathcal{H}(t)$ denote the set of possible histories in a round t, and $\mathcal{H}$ the set of all possible histories of any length. A (pure) strategy of player i specifies an investment choice for every element of $\mathcal{H}$. Such a strategy is a (subgame perfect) **Nash equilibrium** if for every player i and history $h \in \mathcal{H}$, the continuation strategy for player i is optimal given the strategies of all other players. Given the global monitoring assumption and the structure of the game, it is straightforward to show that all individuals are willing to contribute their entire endowment in equilibrium for

[2] Nested split graphs have, for instance, been shown to emerge in settings where networks form endogenously (Kinateder and Merlino, 2017; König et al., 2014) and in games on networks with local complementarities (Belhaj et al., 2016).

at least some strategy if and only if:

$$\delta \geq \frac{(1-b)e_i}{b\sum_{j\neq i} g_{ij}e_j}, \quad \forall i \in N. \tag{1}$$

As we are interested in social dilemmas, we restrict attention to cases where individuals have an incentive to free-ride, yet investing in the public good is utility maximising between any two individuals, i.e., we assume $1/2 < b < 1$. We define the minimum discount factor needed for (1) to hold for an individual i in an environment $(\mathbf{e}, G)$ as $\delta_i^{\min}(\mathbf{e}, G) = \frac{(1-b)e_i}{b\sum_{j\neq i} g_{ij}e_j}$. The minimum discount factor for which all individuals are willing to contribute their entire endowment in at least some equilibrium is

$$\delta^{\min}(\boldsymbol{e}, G) = \max_{i\in N} \delta_i^{\min}(\mathbf{e}, G) = \max_{i\in N} \frac{(1-b)e_i}{b\sum_{j\neq i} g_{ij}e_j}. \tag{2}$$

The smallest $\delta^{\min}$ across all possible endowment distributions is

$$\delta^{\min}(G) = \min_{\boldsymbol{e}\in \mathrm{int}(\Delta^{n-1})} \delta^{\min}(\boldsymbol{e}, G) = \min_{\boldsymbol{e}\in \mathrm{int}(\Delta^{n-1})} \max_{i\in N} \frac{(1-b)e_i}{b\sum_{j\neq i} g_{ij}e_j}, \tag{3}$$

where $\mathrm{int}(\Delta^{n-1})$ is the interior of the $(n-1)$–simplex. We refer to $\delta^{\min}(\boldsymbol{e}, G)$ as the **cooperativeness** of an *environment* $(\boldsymbol{e}, G)$. If $\delta^{\min}(\boldsymbol{e}, G) < \delta^{\min}(\boldsymbol{e}', G')$, then $(\boldsymbol{e}, G)$ allows for full contributions in equilibrium for a strictly larger range of discount factors. We say $(\boldsymbol{e}, G)$ is *more cooperative* than $(\boldsymbol{e}', G')$.

Welfare is defined as the sum of individual utilities. For a particular round t, it can be expressed as a function of contributions, endowments, and the network:

$$w(\mathbf{x}(t), \boldsymbol{e}, G) = \sum_i \pi_i(\mathbf{x}(t), \boldsymbol{e}, G) = \sum_i e_i\big(1 + x_i(t)(bd_i - 1)\big).$$

The (discounted) welfare across all periods is

$$W\big(\{\mathbf{x}(t)\}_{t=1}^{\infty}, \boldsymbol{e}, G, \delta\big) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} w(\mathbf{x}(t), \boldsymbol{e}, G).$$

We call an equilibrium **efficient** if its equilibrium (on-path) contribution sequence $\{\mathbf{x}^*(t)\}_{t=1}^{\infty}$ maximises discounted welfare, i.e., $W(\{\mathbf{x}^*(t)\}_{t=1}^{\infty}, \mathbf{e}, G, \delta) = \sup\{W(\{\mathbf{x}(t)\}_{t=1}^{\infty}, \mathbf{e}, G, \delta) : \{\mathbf{x}(t)\}_{t=1}^{\infty} \in \mathcal{X}(\mathbf{e}, G, \delta)\}$, with $\mathcal{X}(\mathbf{e}, G, \delta)$ the set of all (on-path) equilibrium contribution sequences given endowments $\boldsymbol{e}$, network G, and discount factor δ.

Key results.

- The **most cooperative endowment distribution** $\mathbf{e}^*$ for a network G is such that for every individual i, their endowment e_i^* is **proportional to their eigenvector centrality**. The vector $\mathbf{e}^*$ is equal to the Perron eigenvector and the minimum discount factor is $\delta^{\min}(G) = \frac{1-b}{b}\frac{1}{\lambda^{\max}}$, where $\lambda^{\max}$ is the Perron eigenvalue of G.

- We show that the **most cooperative and most efficient graph** among all graphs with a given number of nodes and links is a **nested split graph**. For a given δ, the most efficient graph might differ from the most cooperative one.

(G) (G')

Figure 1: Graphs with 4 vertices and 4 edges. G' is a nested split graph. It is most cooperative ($\delta^{\min}(G') \approx 0.23 > \delta^{\min}(G) = 0.25$, and most efficient for all $\delta \geq \delta^{\min}(G')$.

- We derive an **upper bound on the cooperativeness for any network** of a given size. For any δ below this threshold, no positive cooperation is possible for any graph of that size.
- We show that **efficiency demands income transfers from the poorest to the richest** individuals relative to the most cooperative environment.

Example. Suppose there are 4 individuals (labelled A-D) and they have a total of 4 connections among them. Figure 1 shows two different graphs representing such networks. G is regular (the circle), where each individual has the same degree of 2. The adjacency matrix has a maximal eigenvalue equal to the degree of each individual. This means the most cooperative endowments are also equal, with $e_i^* = 1/4$, and $\delta^{\min}(G) = \frac{1-b}{(b)}\frac{1}{2}$. G' has the same number of nodes and links, but there is heterogeneity (A has degree 3, while D has degree 1). The most cooperative endowments are such that $e_A^* > e_b^* = e_C^* > e_D^*$. However, G' is more cooperative that G, meaning $\delta^{\min}(G') < \delta^{\min}(G)$. For instance, if $b = 2/3$, then $\delta^{\min}(G') \approx 0.23 < \delta^{\min}(G) = 0.25$. While the set of graphs with 4 edges and 4 vertices includes a regular graph, and even though equal endowments are most cooperative given a regular graph, there exists a more heterogenous network and unequal endowments distribution that is more cooperative. There is a positive interaction between network and endowment inequality in terms of cooperativeness. As can be shown, G' is not just more cooperative, also achieves (strictly) higher welfare for all $\delta \geq \delta^{\min}(G')$. G' is also the most efficient network among all networks with 4 nodes and 4 vertices. Efficiency for $\delta > \delta^{\min}(G')$ demands even higher endowment inequality: if $\mathbf{e}^\dagger$ is efficient for some $\delta > \delta^{\min}(G')$, then $e_A^\dagger > e_A^* > e_D^* > e_D^\dagger$. Network and endowment inequality also positively interact in terms of welfare.
Since both the most cooperative and the most efficient graph are nested split graphs, the question arises if the same graph maximises both. This is, however, not generally the case. To see this, suppose now there are 5 individuals (A-E) and they have a total of 7 connections among them. Figure 2 shows two different corresponding nested split graphs. G'' is similar to G': it has three different (degree) hierarchy levels. A,B,C, and D form a clique, but A has an additional link with E, who has no other neighbours. G''' has two hierarchy levels, with A and B are linked to all other individuals, while C,D, and E are not linked to each other. We can establish that $\delta^{\min}(G'') = 0.162 < \delta^{\min}(G''') = 0.167$. However, for $\delta > 1/4$, G''' is more efficient. While the most cooperative and the most efficient graph must be nested split graphs, for a given δ, the exact type of graph might differ.

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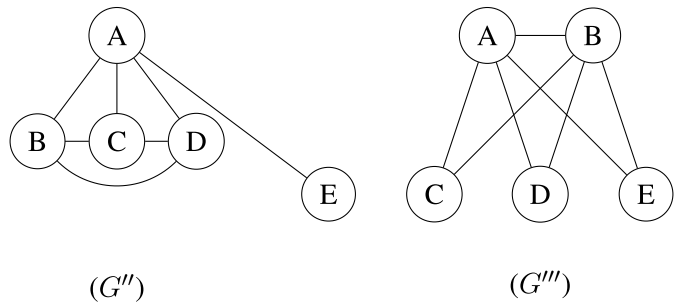

Figure 2: Two graphs with 5 vertices and 7 edges. G'' is the most cooperative graph, while G''' is more efficient for $\delta > 0.25$.

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