K-theoretic Persistent Cohomology: With Applications to Graphs*

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Abstract

We develop K-theoretic persistent cohomology (KPCH): a principled extension of 1-parameter persistent (co)homology that equips the Grothendieck group of persistence modules with λ-operations arising from exterior powers. This yields new, computable persistence layers $\Lambda^i H^p$ that quantify concurrency among cohomology classes via interval intersections. We establish the core algebraic and stability results, provide an integral formula and interval-calculus for efficient computation on barcodes, as well as a concurrency detection theorem, and demonstrate empirical benefits on graph filtrations, where KPCH separates patterns that standard additive H^p summaries cannot.

Keywords: Topological Data Analysis; Persistent Cohomology; K-theory; Graph

1. Introduction

Persistent homology (PH) has matured into a central tool of topological data analysis (TDA), providing stable and computable descriptors of geometric and network data through *barcodes*, which are multisets of intervals encoding the birth and death of (co)homology classes along a filtration (Edelsbrunner et al., 2002; Edelsbrunner and Harer, 2010). However, popular scalar summaries of a single homological degree—e.g., Betti curves, total persistence, or extremal lifetimes of H^p—are *additive*: they quantify "how much" topology appears, but often miss *interactions* among simultaneously alive classes (e.g., whether two loops are present at the same time).

This paper introduces K-theoretic persistent cohomology (KPCH), a principled extension of 1-parameter persistent (co)homology that equips the Grothendieck group of persistence modules with λ-operations arising from exterior powers (Atiyah, 1967; Quillen, 1973). This yields new, computable persistence layers $\Lambda^i H^p$ that quantify concurrency among cohomology classes via interval intersections. On barcodes, these layers admit an explicit, efficient *interval-calculus*: the barcode of $\Lambda^i H^p$ consists of all i-wise *intersections* of the bars of H^p (Crawley-Boevey, 2015). In particular, $\Lambda^2 H^p$ measures *concurrency* of degree-p classes, which is precisely the information that additive H^p summaries in standard persistent homology cannot measure.

Our contributions are as follows. *(i) Foundations.* We develop KPCH as a λ-ring enhancement of K_0 for persistence modules: pointwise exterior powers define λ-operations

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$\lambda^i([M]) = [\Lambda^i M]$; a Koszul-type filtration proves additivity in K_0; and the special λ-identities hold (Atiyah, 1967; Quillen, 1973). *(ii) Interval calculus.* For 1-parameter barcodes $\mathcal{B}(M)$, we prove $\mathcal{B}(\Lambda^i M)$ consists of all i-wise intersections of intervals in $\mathcal{B}(M)$, with multiplicity (Crawley-Boevey, 2015). This makes KPCH *computable* once a standard barcode is available. *(iii) Stability.* Exterior powers are 1-Lipschitz for the interleaving distance; consequently, the diagram $\mathrm{Dgm}(\Lambda^i M)$ is bottleneck-stable (Cohen-Steiner et al., 2007; Chazal et al., 2012, 2016). *(iv) Algorithms and applications to graphs.* For weighted graphs filtered by thresholded edges and clique (flag) complexes, we compute H^1 and then $\Lambda^2 H^1$ to detect overlapping vs. sequential loop patterns (Horak et al., 2009; Petri et al., 2014; Sizemore et al., 2018). In controlled experiments on synthetic graphs, $\Lambda^2 H^1$ separates classes that H^1 totals cannot, aligning with the concurrency interpretation.[1]

2. Mathematical Preliminaries

Fix a field k and a totally ordered subset $I \subset \mathbb{R}$. A *persistence module* is a functor

$$M: \; I \longrightarrow \mathbf{Vect}_k, \qquad t \longmapsto M_t, \quad (s \le t) \longmapsto \phi^M_{s\le t} : M_s \to M_t.$$

PMod denotes the category of persistence modules and natural transformations. We write $\mathsf{PMod}_{\mathrm{pfd}} \subset \mathsf{PMod}$ for the full subcategory of *pointwise finite-dimensional (p.f.d.)* modules ($\dim_k M_t < \infty$ for all $t \in I$) (Chazal et al., 2016; Edelsbrunner and Harer, 2010). The *Betti curve* β_M of M is defined by $\beta(t) := \dim_k M_t$ for $t \in I$. A *submodule* $A \subset M$ is a subfunctor. A *filtration* of $M \in \mathsf{PMod}$ is a family of submodules $(F_\alpha)_{\alpha \in A}$ with $F_\alpha \subseteq F_\beta$ for $\alpha \le \beta$ and $\bigcup_{\alpha \in A} F_\alpha = M$. Kernels, images, cokernels in PMod are computed *pointwise*, e.g., $(\ker f)_t = \ker(f_t)$, $(\operatorname{coker} f)_t = \operatorname{coker}(f_t)$. For an interval $J \subset I$, I_J denotes the interval module.

The interval decomposition theorem states as follows (Crawley-Boevey, 2015; Chazal et al., 2016; Zomorodian and Carlsson, 2005). Let $M \in \mathsf{PMod}_{\mathrm{pfd}}$. There exists a multiset $\mathcal{B}(M)$ of intervals $J \subset I$ and a natural isomorphism $M \cong \bigoplus_{J \in \mathcal{B}(M)} I_J$ uniquely up to permutation of intervals. In particular, for each $t \in I$, only finitely many $J \in \mathcal{B}(M)$ contain t (local finiteness), and $\dim_k M_t = \#\{J \in \mathcal{B}(M) : t \in J\}$. We call $\mathcal{B}(M)$ the *barcode* of M (Ghrist, 2008). The *total persistence* of $\mathcal{B}(M)$ is

$$\mathrm{TP}\big(\mathcal{B}(M)\big) \;:=\; \sum_{L \in \mathcal{B}(M)} |L| \;=\; \sum_{L=[b,d) \in \mathcal{B}(M)} (d-b),$$

A sequence $0 \to A \to B \to C \to 0$ in PMod (or $\mathsf{PMod}_{\mathrm{pfd}}$) is *short exact* iff for every $t \in I$ the evaluated sequence $0 \to A_t \to B_t \to C_t \to 0$ is exact in $\mathbf{Vect}_k$. A functor $F : \mathsf{PMod} \to \mathsf{PMod}$ is *exact* if it sends every short exact sequence to a short exact sequence. For $\varepsilon \ge 0$, the *shift* of $M \in \mathsf{PMod}$ is the module $M(\varepsilon)$ defined by $M(\varepsilon)_t := M_{t+\varepsilon}$ and $\phi^{M(\varepsilon)}_{s\le t} := \phi^M_{s+\varepsilon \le t+\varepsilon}$. The *canonical ε-shift* is the natural transformation $\mu^\varepsilon_M : M \Rightarrow M(\varepsilon)$ with components $(\mu^\varepsilon_M)_t := \phi^M_{t \le t+\varepsilon}$ (Chazal et al., 2016; Lesnick, 2015). The *tensor product* $M \otimes N$ is defined by:

$$(M \otimes N)_t := M_t \otimes_k N_t, \qquad \phi^{M\otimes N}_{s\le t} := \phi^M_{s\le t} \otimes \phi^N_{s\le t}.$$

1. For related work, see also Maruyama (2026a,b,c); Maruyama and Yasuda (2025); Maruyama (2025a,b,c).

$\mathbf{1} \in \mathsf{PMod}$ denotes the constant module $\mathbf{1}_t := k$. Given a finite $T \subset I$, we say $M \in \mathsf{PMod}_{\mathrm{pfd}}$ is *constructible with respect to* T if for all $s \leq t$ with $(s,t] \cap T = \varnothing$, the structure map $\phi^M_{s\leq t} : M_s \to M_t$ is an isomorphism. We say M is *tame* if it is p.f.d. and constructible with respect to some finite T (called a critical set). If M is tame, then $\mathcal{B}(M)$ is finite.

A *filtration of spaces* is an increasing family $(X_\alpha)_{\alpha\in A}$ with $X_\alpha \subseteq X_\beta$ whenever $\alpha \leq \beta$. $H^p(X_\alpha; k)$ denotes the degree-p cohomology module with field coefficients.

For $n \geq 0$ and $M \in \mathsf{PMod}$, the *n-th exterior power* $\Lambda^n M$ is defined as the module

$$(\Lambda^n M)_t := \Lambda^n(M_t)$$

with structure maps $\Lambda^n(\phi^M_{s\leq t}) : \Lambda^n(M_s) \to \Lambda^n(M_t)$.[2] Functoriality of Λ^n on linear maps makes $M \mapsto \Lambda^n M$ an endofunctor on PMod preserving $\mathsf{PMod}_{\mathrm{pfd}}$.

3. K-theoretic Persistent Cohomology

Here we develop K-theoretic persistent cohomology theory. We apply the methods of Grothendieck and Quillen to persistent modules (Quillen, 1973). Note that $(\mathsf{PMod}_{\mathrm{pfd}}, \otimes, \mathbf{1})$ is a symmetric monoidal exact category; for each fixed N, the functor $M \mapsto M \otimes N$ is exact, and the same holds symmetrically in the other variable.[3]

The *Grothendieck group* $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ is generated by isomorphism classes $[X]$ with relations $[B] = [A] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathsf{PMod}_{\mathrm{pfd}}$. $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ plays the role of special λ-ring in K-theory; the following structure and properties are essential.[4]

Theorem 1 (λ-structure on $K_0(\mathsf{PMod}_{\mathrm{pfd}})$) *Equip the Grothendieck group $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ with the product operation $[M] \cdot [N] := [M \otimes N]$ and the λ-operations*

$$\lambda^n([M]) := [\Lambda^n M].$$

Then $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ satisfies the following:

$$\lambda^0(x) = 1, \qquad \lambda^1(x) = x, \qquad \lambda_t(x+y) = \lambda_t(x)\,\lambda_t(y),$$

where $\lambda_t(x) := \sum_{n\geq 0} \lambda^n(x)\, t^n \in K_0[\![t]\!]$.

Proof We first prove the well-definedness of λ^n on K_0 Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$. For each $n \geq 0$ define subfunctors $F^i \subset \Lambda^n B$ as the images of the wedge maps

$$\theta_i : \Lambda^i A \otimes \Lambda^{n-i} B \longrightarrow \Lambda^n B, \qquad (a_1 \wedge \cdots \wedge a_i) \otimes (b_{i+1} \wedge \cdots \wedge b_n) \mapsto a_1 \wedge \cdots \wedge a_i \wedge b_{i+1} \wedge \cdots \wedge b_n.$$

2. The exterior power on the right-hand side is the usual exterior power on vector spaces.
3. This can be shown as follows. Given $0 \to A \to B \to C \to 0$ in $\mathsf{PMod}_{\mathrm{pfd}}$, evaluation at $t \in I$ yields $0 \to A_t \to B_t \to C_t \to 0$ in $\mathbf{Vect}_k$. Since all k-vector spaces are flat, tensoring with N_t is exact, hence $0 \to A_t \otimes N_t \to B_t \otimes N_t \to C_t \otimes N_t \to 0$ is exact for each t. These assemble naturally to a short exact sequence $0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$, proving exactness; symmetry/associativity are inherited pointwise. Note also that PMod is abelian (limits/colimits pointwise from $\mathbf{Vect}_k$), and $\mathsf{PMod}_{\mathrm{pfd}}$ is an abelian full subcategory.
4. The other properties of special λ-ring hold as well.

Then $0 = F^{n+1} \subset F^n \subset \cdots \subset F^0 = \Lambda^n B$, and the subquotients are naturally

$$F^i/F^{i+1} \;\cong\; \Lambda^i A \;\otimes\; \Lambda^{n-i} C, \qquad i = 0, \dots, n. \tag{1}$$

Indeed, at each t, define $\phi_i : \Lambda^i A_t \otimes \Lambda^{n-i} C_t \to F_t^i/F_t^{i+1}$ by choosing lifts $b_j \in B_t$ with $\pi(b_j) = \bar{c}_j \in C_t$ and setting

$$\phi_i\big((a_1 \wedge \cdots \wedge a_i) \otimes (\bar{c}_{i+1} \wedge \cdots \wedge \bar{c}_n)\big) := \big[a_1 \wedge \cdots \wedge a_i \wedge b_{i+1} \wedge \cdots \wedge b_n\big].$$

Changing lifts by elements of A_t alters the representative by a wedge with at least $i+1$ entries from A_t, hence in F_t^{i+1}; thus ϕ_i is well-defined and natural. Surjectivity is clear by construction. Kernel identification is the classical multilinear relation showing that $\ker \phi_i = \mathrm{im}(\Lambda^{i+1} A_t \otimes \Lambda^{n-i-1} C_t \to \Lambda^i A_t \otimes \Lambda^{n-i} C_t)$. Thus (1) holds functorially in t. Passing to K_0 yields $[\Lambda^n B] = \sum_{i+j=n} [\Lambda^i A \otimes \Lambda^j C] = \sum_{i+j=n} \lambda^i([A]) \cdot \lambda^j([C])$, so λ^n descends to K_0 and respects the defining relations. Pointwise, we have $\Lambda^n(X \oplus Y) \cong \bigoplus_{i+j=n} \Lambda^i X \otimes \Lambda^j Y$; passing to K_0 gives $\lambda_t([X] + [Y]) = \lambda_t([X])\lambda_t([Y])$. Trivially, $\Lambda^0 M \cong \mathbf{1}$ (the unit) and $\Lambda^1 M \cong M$, hence $\lambda^0 = 1$ and $\lambda^1 = \mathrm{id}$. ■

For a filtration $(X_\alpha)_{\alpha \in I}$ and $p \geq 0$, consider the cohomology module $H^p(X_\alpha; k)$. Define $\kappa_p := [H^p(X_\alpha; k)] \in K_0(\mathsf{PMod}_{\mathrm{pfd}})$ and set $\lambda^i(\kappa_p) := [\Lambda^i H^p(X_\alpha; k)]$.

Theorem 2 (Interval calculus) *Let $M \in \mathsf{PMod}_{\mathrm{pfd}}$. By the interval decomposition theorem, there exist an index set R and intervals $J_r = [b_r, d_r) \subset I$ such that $M \;\cong\; \bigoplus_{r \in R} I_{J_r}$ with the property that for every $t \in I$ only finitely many $r \in R$ satisfy $t \in J_r$ (local finiteness). Then for every $i \geq 1$ there is a natural isomorphism of persistence modules*

$$\Lambda^i M \;\cong\; \bigoplus_{\{r_1 < \cdots < r_i\} \subset R} I_{\,J_{r_1} \cap \cdots \cap J_{r_i}}.$$

Equivalently,

$$\mathcal{B}(\Lambda^i M) = \Big\{ [\max_j b_{\ell_j}, \, \min_j d_{\ell_j}) \;:\; \ell_1 < \cdots < \ell_i, \; \max_j b_{\ell_j} < \min_j d_{\ell_j} \Big\},$$

i.e., the multiset of all i-wise intersections of bars of M, with multiplicities induced by the number of i-tuples producing the same intersection.

Proof For interval modules I_J and $I_{J'}$, evaluation at $t \in I$ gives $(I_J \otimes I_{J'})_t \cong (I_J)_t \otimes (I_{J'})_t$, which equals k iff $t \in J \cap J'$ and 0 otherwise; structure maps are identities along $J \cap J'$. These pointwise identifications assemble to a natural isomorphism $I_J \otimes I_{J'} \cong I_{J \cap J'}$. Iterating yields $I_{J_{r_1}} \otimes \cdots \otimes I_{J_{r_i}} \cong I_{J_{r_1} \cap \cdots \cap J_{r_i}}$. We apply this below.

Fix $t \in I$. By p.f.d. and local finiteness, only finitely many $r \in R$ satisfy $t \in J_r$, hence $M_t \cong \bigoplus_{r:\, t \in J_r} k$ is a *finite* direct sum. For finite direct sums of vector spaces in the present setting, one has the canonical decomposition $\Lambda^i(\bigoplus_r V_r) \cong \bigoplus_{r_1 < \cdots < r_i} \bigotimes_{j=1}^i V_{r_j}$. Applying this with $V_r = k$ (when $t \in J_r$) and $V_r = 0$ otherwise gives $(\Lambda^i M)_t \;\cong\; \bigoplus_{r_1 < \cdots < r_i} \bigotimes_{j=1}^i (I_{J_{r_j}})_t$. The identifications are natural in t, so they assemble to an isomorphism of persistence modules $\Lambda^i M \;\cong\; \bigoplus_{r_1 < \cdots < r_i} \big(I_{J_{r_1}} \otimes \cdots \otimes I_{J_{r_i}}\big)$. Using $I_{J_{r_1}} \otimes \cdots \otimes I_{J_{r_i}} \cong I_{J_{r_1} \cap \cdots \cap J_{r_i}}$, the right-hand side is $\bigoplus_{r_1 < \cdots < r_i} I_{J_{r_1} \cap \cdots \cap J_{r_i}}$. ■

The theorem reduces KPCH to *pure post-processing* of an existing barcode: once $\mathcal{B}(H^p)$ is computed by any standard PH routine, $\mathcal{B}(\Lambda^i H^p)$ is obtained by i-wise interval intersections; no extra boundary reductions or linear-algebra passes are required. Each interval in $\mathcal{B}(\Lambda^i H^p)$ is precisely the time window during which i *distinct* degree-p classes coexist. This provides an interpretable "concurrency timeline" that augments additive summaries (e.g., total persistence of H^p).

Theorem 3 (Stability; exterior powers are 1-Lipschitz for interleavings) *Let d_I and d_B denote the standard interleaving and bottleneck distances, respectively. For every $i \geq 1$ and p.f.d. modules M, N,*

$$d_I\big(\Lambda^i M, \Lambda^i N\big) \;\leq\; d_I(M, N).$$

Consequently, by the interleaving–bottleneck isometry in 1D p.f.d. persistence,

$$d_B\big(\mathrm{Dgm}(\Lambda^i M), \mathrm{Dgm}(\Lambda^i N)\big) \;\leq\; d_B\big(\mathrm{Dgm}(M), \mathrm{Dgm}(N)\big).$$

Proof Recall the *shift functor* S_ε: $(M(\varepsilon))_t := M_{t+\varepsilon}$ with inherited structure maps. An ε-interleaving between M and N is a pair of natural maps $\varphi : M \to N(\varepsilon)$ and $\psi : N \to M(\varepsilon)$ such that the composites $M \xrightarrow{\varphi} N(\varepsilon) \xrightarrow{\psi(\varepsilon)} M(2\varepsilon)$ and $N \xrightarrow{\psi} M(\varepsilon) \xrightarrow{\varphi(\varepsilon)} N(2\varepsilon)$ coincide with the canonical shifts $M \to M(2\varepsilon)$ and $N \to N(2\varepsilon)$, respectively.[5]

For every i and every module X there is a natural isomorphism

$$\iota^\varepsilon_X : \; \Lambda^i\big(X(\varepsilon)\big) \xrightarrow{\;\cong\;} \big(\Lambda^i X\big)(\varepsilon), \qquad \big(\iota^\varepsilon_X\big)_t = \mathrm{id}_{\Lambda^i(X_{t+\varepsilon})},$$

coming directly from the pointwise definition of Λ^i.

Given an ε-interleaving (φ, ψ) between M and N, define

$$\Phi := (\iota^\varepsilon_N) \circ \Lambda^i(\varphi) : \; \Lambda^i M \longrightarrow (\Lambda^i N)(\varepsilon), \qquad \Psi := (\iota^\varepsilon_M) \circ \Lambda^i(\psi) : \; \Lambda^i N \longrightarrow (\Lambda^i M)(\varepsilon).$$

By functoriality of Λ^i and naturality of ι^ε, we have

$$\Psi(\varepsilon) \circ \Phi \;=\; \iota^{2\varepsilon}_M \circ \Lambda^i\big(\psi(\varepsilon) \circ \varphi\big) \;=\; \iota^{2\varepsilon}_M \circ \Lambda^i\big(\mu^{2\varepsilon}_M\big) \;=\; \mu^{2\varepsilon}_{\Lambda^i M},$$

and similarly $\Phi(\varepsilon) \circ \Psi = \mu^{2\varepsilon}_{\Lambda^i N}$. Hence (Φ, Ψ) is an ε-interleaving between $\Lambda^i M$ and $\Lambda^i N$, proving $d_I(\Lambda^i M, \Lambda^i N) \leq \varepsilon$; taking the infimum over ε gives the desired inequality. The bottleneck bound follows from the $d_I = d_B$ isometry for 1D p.f.d. modules (Chazal et al., 2012, 2016). ■

Exterior powers are *nonexpansive*: any perturbation that is small for M is no larger for $\Lambda^i M$. Therefore, all KPCH layers inherit the usual bottleneck stability, making concurrency-sensitive features safe for downstream statistics and learning.

Theorem 4 (Integral formula) *Let $M \in \mathsf{PMod}$ be a tame module. For every $i \geq 1$,*

$$\mathrm{TP}\big(\mathcal{B}(\Lambda^i M)\big) \;=\; \int_I \binom{\beta_M(t)}{i}\, dt.$$

Equivalently, $\dim_k(\Lambda^i M_t) = \binom{\beta_M(t)}{i}$ for each t, and total persistence is the time integral of this pointwise dimension.

5. The concept of ε-interleaving will be used again later.

Proof Pointwise, if $M_t \cong k^{\beta_M(t)}$, then $\dim_k \Lambda^i(M_t) = \binom{\beta_M(t)}{i}$ by basic multilinear algebra. Let $\mathbf{1}_J$ denote the indicator of interval J. Since $\Lambda^i M \cong \bigoplus_{L\in\mathcal{B}(\Lambda^i M)} I_L$, we have

$$\dim(\Lambda^i M_t) \;=\; \sum_{L\in\mathcal{B}(\Lambda^i M)} \mathbf{1}_L(t).$$

Integrating and applying Fubini/Tonelli (finite sums by tameness) yields

$$\int \dim(\Lambda^i M_t)\,dt \;=\; \sum_{L\in\mathcal{B}(\Lambda^i M)} \int \mathbf{1}_L(t)\,dt \;=\; \sum_{L\in\mathcal{B}(\Lambda^i M)} |L| \;=\; \mathrm{TP}\big(\mathcal{B}(\Lambda^i M)\big).$$

The integrand equals $\binom{\beta_M(t)}{i}$ by the pointwise identity, proving the claim. ∎

The formula turns Λ^i into a *single-pass* statistic: $\mathrm{TP}(\mathcal{B}(\Lambda^i M))$ equals the area under $t \mapsto \binom{\beta_M(t)}{i}$ and can be computed by a sweep over barcode endpoints. For $i = 2$, this is precisely the "time with at least two classes alive," i.e., a quantitative measure of concurrency.

4. K-theoretic Persistent Cohomology for Graphs

We specialize KPCH to weighted graphs. Fix a finite $G = (V, E, w)$ with $w : E \to \mathbb{R}$. For a threshold α, set $G_\alpha = (V, \{e \in E : \ w(e) \le \alpha\})$ and let $X_\alpha = \mathrm{Cl}(G_\alpha)$ be the clique (flag) complex. Note that a simplex $\sigma \subseteq V$ enters at $t(\sigma) := \max\{w(e) : e \subset \sigma \text{ edge}\}$, so the filtration $\{X_\alpha\}_{\alpha\in I}$ changes only at edge weights (Edelsbrunner and Harer, 2010; Horak et al., 2009; Petri et al., 2014; Sizemore et al., 2018).

Proposition 5 (Tameness and finite barcode) *For any $p \ge 0$, $M := H^p(X_\alpha; k)$ is tame with a finite barcode. Moreover, for $p \ge 1$, all bars are finite (the filtration becomes contractible at the maximal edge-weight threshold).*

Proof Every simplex σ is supported on finitely many edges; hence $t(\sigma) := \max\{w(e) : e \subset \sigma \text{ edge}\} \in T := \{w(e) : e \in E\}$, a finite set. Between consecutive values of T, the filtered complex X_α is constant, so M is constructible with respect to T and p.f.d., yielding a finite barcode by the interval decomposition theorem. If $\alpha \ge \max T$, then G_α is complete and X_α is the full simplex on V, which is contractible; therefore $H^p(X_\alpha; k) = 0$ for all $p \ge 1$, so no bar in positive degree extends to $+\infty$. ∎

Lemma 6 (Betti–exterior identity) *Let M be tame. For $i \ge 1$ and $t \in I$, we have $\dim_k\big(\Lambda^i M_t\big) \;=\; \binom{\beta_M(t)}{i}$ with the convention $\binom{n}{i} = 0$ for $n < i$ and $\binom{n}{0} = 1$; equivalently, the pointwise Betti curve of the exterior-power layer satisfies $\beta_{\Lambda^i M}(t) \;=\; \binom{\beta_M(t)}{i}$ for all $t \in I$.*

Proof Fix $t \in I$ and write $n := \beta_M(t) = \dim_k M_t$. Choose a basis $e_1, \ldots, e_n$ of M_t; then a basis of $\Lambda^i(M_t)$ is given by the wedge products $e_{j_1} \wedge \cdots \wedge e_{j_i}$ with $1 \le j_1 < \cdots < j_i \le n$. There are precisely $\binom{n}{i}$ such i-tuples, so $\dim_k \Lambda^i(M_t) = \binom{n}{i}$. This number is independent of the chosen basis, hence canonical. ∎

Theorem 7 (Concurrency detection) *Let M be a tame module (in particular, $M = H^p(X_\alpha; k)$ from a graph filtration). Then*

(i) $\Lambda^2 M = 0 \iff$ *no two bars of M overlap in interior* $\iff \beta_M(t) \le 1$ *for all t.*

(ii) $\mathrm{TP}\big(\mathcal{B}(\Lambda^2 M)\big) > 0 \iff \{t \in I : \ \beta_M(t) \ge 2\}$ *has positive Lebesgue measure.*

Proof We use two facts established above: $\dim_k\left(\Lambda^2 M_t\right) = \binom{\beta_M(t)}{2}$ and $\mathrm{TP}(\mathcal{B}(\Lambda^2 M)) = \int_I \binom{\beta_M(t)}{2}\, dt$ (Theorem 4 with $i = 2$).

(a) $\Lambda^2 M = 0 \iff \beta_M(t) \le 1$ for all t. If $\Lambda^2 M = 0$, then every fiber $\Lambda^2 M_t$ is $\{0\}$. By the Betti–exterior identity, $0 = \dim(\Lambda^2 M_t) = \binom{\beta_M(t)}{2}$, which forces $\beta_M(t) \in \{0, 1\}$ for all t. Conversely, if $\beta_M(t) \le 1$ for all t, then $\binom{\beta_M(t)}{2} = 0$ for all t, hence every $\Lambda^2 M_t$ is zero; therefore $\Lambda^2 M$ is the zero persistence module.

(b) $\beta_M(t) \le 1 \ \forall t \iff$ no two bars overlap in interior. Assume $\beta_M(t) \le 1$ for all t. If two bars $J_1 = [b_1, d_1)$ and $J_2 = [b_2, d_2)$ had a positive-length overlap, their intersection would contain some open interval (u, v); pick $t \in (u, v)$. Then both bars are alive at t, so $\beta_M(t) \ge 2$, a contradiction. Conversely, if $\beta_M(t_0) \ge 2$ at some t_0, tameness implies that β_M is locally constant on a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon)$, hence at least two bars are simultaneously alive on that open interval; this yields a positive-length overlap.

Combining (a) and (b) proves (i).

(ii) By the integral identity, we have $\mathrm{TP}\big(\mathcal{B}(\Lambda^2 M)\big) = \int_I \binom{\beta_M(t)}{2}\, dt$. Since $\binom{n}{2} = 0$ for $n \in \{0, 1\}$ and $\binom{n}{2} \ge 1$ for $n \ge 2$, the integrand is the indicator (up to a positive integer value) of the set $S := \{t : \beta_M(t) \ge 2\}$. Hence the integral is strictly positive iff S has positive Lebesgue measure. Tameness guarantees that β_M is piecewise constant with finitely many jumps, so measure-zero events contribute nothing to the integral. ■

In summary, Proposition 5 ensures that graph-based KPCH layers in degrees $p \ge 1$ involve only finite bars, simplifying both algorithms and interpretations; Theorem 7 identifies Λ^2 as an exact detector of simultaneous p-classes, with a scalar summary given by the total persistence of $\Lambda^2 M$.

The equivalence in Theorem 7 characterizes the *absence* of pairwise concurrency purely algebraically. In practice, it distinguishes *sequential* phenomena (no overlap) from truly simultaneous ones (*overlap*). On clique filtrations of graphs (where all $p \ge 1$ bars are finite), $\mathrm{TP}(\Lambda^2 H^1)$ separates *overlapping* vs. *sequential* loop patterns that standard H^1 totals cannot detect, thus providing a drop-in diagnostic for redundancy and multi-path connectivity. We will demonstrate this below.

5. Algorithm and Experiment: detect *overlapping* vs. *sequential* loops

We present an end-to-end pipeline that turns a standard H^p barcode into KPCH outputs on graphs. The only nonstandard step is a *one-pass sweep* that computes $\mathrm{TP}(\mathcal{B}(\Lambda^i H^p))$ directly from $\mathcal{B}(H^p)$ using the integral identity (Theorem 4). The procedure is as follows:

1. **From a weighted graph to $\mathcal{B}(H^1)$.** Build the flag filtration $X_\alpha = \mathrm{Cl}(G_\alpha)$ of $G = (V, E, w)$, where a simplex enters at the maximum incident edge weight. Compute $\mathcal{B}(H^1)$ via a standard persistent cohomology routine over k (default $k = \mathbb{F}_2$) (de Silva et al., 2011; Bauer, 2021).

2. **KPCH layer (fast summary).** Compute $\mathrm{TP}(\mathcal{B}(\Lambda^i H^1)) = \int \binom{\beta_{H^1}(t)}{i}\, dt$ by a single sweep over endpoints of $\mathcal{B}(H^1)$ (see below). No pair enumeration is required.
3. **Full Λ^i barcode (optional).** If needed, form all i-wise intersections of bars in $\mathcal{B}(H^1)$ (Theorem 2) and keep the positive-length intervals.

The Sweep Algorithm for $\mathrm{TP}\big(\mathcal{B}(\Lambda^i H^p)\big)$ from $\mathcal{B}(H^p)$

Input. Barcode $\mathcal{B}(H^p) = \{[b_r, d_r)\}_{r=1}^M$ (closed–open bars), integer $i \geq 1$.
Output. $\mathrm{TP}\big(\mathcal{B}(\Lambda^i H^p)\big)$.

Procedure.

1. Form the event multiset
$$E \;\leftarrow\; \{(b_r, +1) : 1 \leq r \leq M\} \;\cup\; \{(d_r, -1) : 1 \leq r \leq M\}.$$
Here $+1$ denotes a *birth* and -1 a *death*.
2. Sort E by time; break ties by processing deaths before births:
$$(t_1, \delta_1) \prec (t_2, \delta_2) \iff \big[t_1 < t_2\big] \;\text{ or }\; \big[t_1 = t_2 \;\text{ and }\; \delta_1 = -1 < +1 = \delta_2\big].$$
3. Initialize counters:
$$c \leftarrow 0,\; \mathrm{TP} \leftarrow 0,\; \text{and if } E \neq \varnothing \text{ set } x \leftarrow \min\{t : (t, \delta) \in E\};$$
if $E = \varnothing$, return 0.
4. For each $(t, \delta) \in E$ in the sorted order, do
$$\mathrm{TP} \;\leftarrow\; \mathrm{TP} + \binom{c}{i}(t - x), \qquad c \;\leftarrow\; c + \delta, \qquad x \;\leftarrow\; t.$$
5. Return TP.

Notes.

- Correctness: follows from $\mathrm{TP}\big(\mathcal{B}(\Lambda^i H^p)\big) = \int \binom{\beta_{H^p}(t)}{i}\, dt$ (Theorem 4) and the fact that $\beta_{H^p}(t)$ is constant between consecutive event times.
- Complexity: computes total i-fold concurrency in $O(M \log M)$ time (sorting $2M$ events) and $O(M)$ memory from *endpoints only*, avoiding $O\big(\binom{M}{i}\big)$ pair/tuple enumeration.

Theorem 8 *If two filtrations are ε-interleaved, then the corresponding KPCH layers satisfy $d_B\big(\mathrm{Dgm}(\Lambda^i H^p), \mathrm{Dgm}(\Lambda^i H^{p\prime})\big) \leq \varepsilon$ for all $i \geq 1$.*

Proof Stability follows from Theorem 3 (exterior powers are 1-Lipschitz for interleavings) and the 1D isometry $d_I = d_B$ (Chazal et al., 2012, 2016). ■

The KPCH summary is *cheap* to compute and *robust* to small weight perturbations, making it suitable for large batches and noisy inputs.

Experiment: detect *overlapping* vs. *sequential* cycles in graphs

We test whether $\Lambda^2 H^1$ captures concurrency that standard H^1 totals miss.

Protocol. Generate $N = 60$ graphs (30 per class) with small jitter η to avoid ties. For each graph: (i) compute $\mathcal{B}(H^1)$; (ii) compute $\mathrm{TP}(\Lambda^2 H^1)$ via the sweep algorithm with $i = 2$; (iii) compute the baseline $\mathrm{TP}(H^1) = \sum_{[b,d)}(d - b)$. Use a median-threshold rule on each scalar to separate *overlapping* vs. *sequential* cycles classes.

Results. The results of the experiment are given in the table below. On the scalar summary $\mathrm{TP}\big(\mathcal{B}(\Lambda^2 H^1)\big)$, the two classes are cleanly separated: Sequential has median 0.00 while Overlap has median 0.62 in the same time units as edge weights. A simple threshold at the midpoint achieves accuracy 0.97 (58/60) and AUROC 0.99. By contrast, standard H^1 summaries fail to discriminate: for total persistence $\mathrm{TP}\big(\mathcal{B}(H^1)\big)$ the medians are nearly identical (1.58 vs. 1.60; AUROC 0.52, accuracy 0.55), and for max lifetime the gap is similarly negligible (AUROC 0.54, accuracy 0.53). These results align with the concurrency semantics: Sequential has $\beta_{H^1}(t) \le 1$ for all t (hence $\Lambda^2 H^1 \equiv 0$), while Overlap has a positive-measure window with $\beta_{H^1}(t) \ge 2$, yielding strictly positive $\mathrm{TP}\big(\mathcal{B}(\Lambda^2 H^1)\big)$.

Summary.

Feature	Sequential	Overlap	AUROC	Acc.
$\mathrm{TP}\big(\mathcal{B}(\Lambda^2 H^1)\big)$	0.00	0.62	0.99	0.97
$\mathrm{TP}\big(\mathcal{B}(H^1)\big)$	1.58	1.60	0.52	0.55
Max lifetime in H^1	0.86	0.88	0.54	0.53

Remarks. (i) Stability follows from the 1-Lipschitz property of exterior powers under interleavings, and finiteness of bars in $p \ge 1$ is guaranteed for clique filtrations of finite graphs (Proposition 5). (ii) No additional reductions beyond the baseline H^1 computation are required; $\mathrm{TP}\big(\mathcal{B}(\Lambda^2 H^1)\big)$ is obtained by a single sweep over endpoints.

6. Conclusion

We introduced *K-theoretic persistent cohomology* (KPCH) as a principled, computable extension of persistent (co)homology. Algebraically, we introduced the Grothendieck group $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ with λ-operations via pointwise exterior powers, so that the K-theory class $\kappa_p = [H^p(X_\alpha; k)]$ admits layers $\lambda^i(\kappa_p) = [\Lambda^i H^p(X_\alpha; k)]$. At the barcode level, we proved an *interval calculus*: $\mathcal{B}(\Lambda^i H^p)$ is obtained by i-wise intersections of bars of $\mathcal{B}(H^p)$. Analytically, exterior powers commute with shifts, hence are 1-Lipschitz for interleavings; KPCH inherits bottleneck stability (Cohen-Steiner et al., 2007; Chazal et al., 2012, 2016). Computationally, a single endpoint sweep evaluates the integral identity $\mathrm{TP}\big(\mathcal{B}(\Lambda^i H^p)\big) = \int \binom{\beta_{H^p}(t)}{i}\, dt$, giving an interpretable, low-cost summary of i-fold concurrency.

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