MIXER SEQUENCE DESIGN FOR N-PATH FILTERS

Guolong Su

Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA

Arthur J. Redfern

Texas Instruments, Inc.
12500 TI Boulevard MS 8649
Dallas, TX 75243 USA

ABSTRACT

N-path filters are a practical method for implementing high Q band pass filters in modern CMOS processes without inductors using a combination of mixers and low pass filters. This paper derives optimal practically realizable staircase and 2 level mixer sequences for maximizing the in band SNR and minimizing the out of band harmonics.

Index Terms**—** Band pass filters, N-path filters, mixers

1. INTRODUCTION

Band pass filters (BPFs) are used in many applications including radio frequency (RF) receiver (RX) paths and band pass delta sigma analog to digital converters [11]. A problem with typical BPF implementations is that they require inductors, and it's difficult to implement high quality (Q) inductors in complementary metal oxide semiconductor (CMOS) processes. Other BPF options exist, but they tend to have issues with performance, power or process technology [3].

N-path filters [5] are a type of BPF that overcome a number of these problems. The basic structure of an N-path filter is multiple paths, each path composed of a mixer, filter and mixer, summed together to form the filter output. With trends in process scaling leading to higher switching frequencies, N-path filters are a viable option for integrated BPF designs with center frequencies of interest in current communication standards. As the center frequency of the filter is decoupled from the bandwidth of the filter, high Q values are achievable.

The purpose of this paper is to look at different options for N-path filter mixer sequence design, as the mixers have a large impact on the filter performance. Optimal practically realizable mixer sequences are derived for cases where the mixer sequence is constrained to a staircase sequence and constrained to a 2 level sequence.

2. RELATION TO PRIOR WORK

N-path filters grew out of traditional RF RX chains and research into network synthesis. The first paper describing and analyzing the mixer based N-path filter structure was [5].

On the circuits side, there have been many implementations of N-path filters. While different filter structures are appropriate for different process nodes and requirements, interest seems to be increasing in N-path filters based on current trends in CMOS integration and the availability of faster transistors [1], [2], [3], [4], [6], [7], [8], [11].

With respect to mixer design, research into harmonic rejection mixers (HRMs) for RF RX chains is perhaps the closest relation to N-path filter mixer optimization. Examples of different HRM design methods can be found in [9] and [10]. These differ from the N-path filter in terms of the design criteria and the availability of the 2nd mixer sequence. A HRM is used in [4], however, there is no proof of optimality or consideration of the 2 level sequence case.

3. N-PATH FILTERS

The structure of a N-path filter is shown in Fig. 1. In each path, there are 2 real mixers, $p^{(n)}(t)$ and $q^{(n)}(t)$, with period T which determines the center frequency $\Omega_0 = 2\pi/T$ of the BPF. The mixers transform the low pass filter (LPF) shape to a pass band around Ω_0 where the double sided bandwidth of the LPF, BW, is the same as the bandwidth of the BPF.

Define $X(j\Omega)$ as the input and $Y(j\Omega)$ as the output spectrum of the N-path filter, and denote $H(j\Omega)$ as the LPF. Following the derivation in [5]:

$$Y(j\Omega) = \sum_{r=-\infty}^{\infty} H(j(\Omega - r\Omega_0)) \cdot Y_r(j\Omega), \tag{1}$$

in which the input and mixer related terms are

$$Y_r(j\Omega) = \sum_{m=-\infty}^{\infty} X(j(\Omega + (m-r)\Omega_0)) \cdot \alpha(m,r), \tag{2}$$

and

$$\alpha(m,r) = \sum_{n=1}^{N} \hat{p}_{-m}^{(n)} \cdot \hat{q}_r^{(n)} = \sum_{n=1}^{N} \left(\hat{p}_m^{(n)}\right)^* \cdot \hat{q}_r^{(n)}, \tag{3}$$

where $\hat{p}_m^{(n)}$ and $\hat{q}_m^{(n)}$ are the mth Fourier series coefficients of $p^{(n)}(t)$ and $q^{(n)}(t)$, respectively. Typically, $BW \ll \Omega_0$

and (1) implies that the output only has significant power in frequencies $\pm BW/2$ around the harmonics of Ω_0.

For simplicity, assume the power is flat in frequencies $\pm BW/2$ around a harmonic, so only the midpoint of each band (i.e., $Y(j \cdot l\Omega_0)$) is considered. When $\Omega = l\Omega_0$ in (1), only the $r = l$ term remains. Since $H(j0)$ is the same scale factor for all harmonics, let $H(j0) = 1$ to obtain

$$Y(j \cdot l\Omega_0) \approx \sum_{m=-\infty}^{\infty} X(j \cdot m\Omega_0) \cdot \alpha(m, l). \qquad (4)$$

$\alpha(m, l)$ can be viewed as the transfer coefficient from the mth harmonic in the input to the lth harmonic in the output.

Assume the stationary input signals at different harmonics are uncorrelated and both $X(j \cdot m\Omega_0)$ and $X^2(j \cdot m\Omega_0)$ have 0 mean[1]. Denoting $\mathrm{E}\{\cdot\}$ as the average operator, the output signal power spectral density (PSD) at the lth harmonic is:

$$\mathrm{E}\left\{|Y(j \cdot l\Omega_0)|^2\right\} \approx \sum_{m=-\infty}^{\infty} \mathrm{E}\left\{|X(j \cdot m\Omega_0)|^2\right\} \cdot |\alpha(m, l)|^2.$$

The in band output corresponds to $l = 1$ and has 2 components: the in band signal and the folded harmonic. The desired in band signal, which is the output of a traditional BPF, corresponds to the term $l = m = 1$ and has average power

$$P_{\text{signal}} = \mathrm{E}\left\{|X(j\Omega_0)|^2\right\} \cdot |\alpha(1, 1)|^2. \qquad (5)$$

The unwanted folded harmonics can be viewed as interference to the in band signal. They correspond to terms with $l = 1, m \neq 1$ in (4) and have average total power

$$P_{\text{folded}} = \sum_{m \neq 1} \mathrm{E}\left\{|X(j \cdot m\Omega_0)|^2\right\} \cdot |\alpha(m, 1)|^2. \qquad (6)$$

In addition to the in band output, the N-path filter typically has out of pass band outputs around the harmonics of Ω_0. For the lth harmonic ($l \neq \pm 1$), the average out of band power is

$$P_{\text{out}}(l) = \sum_{m=-\infty}^{\infty} \mathrm{E}\left\{|X(j \cdot m\Omega_0)|^2\right\} \cdot |\alpha(m, l)|^2. \qquad (7)$$

The above analysis shows that the N-path filter transforms a LPF to a BPF with 2 nonidealities: in band harmonic folding and out of band signal residue [5], [4]. A method to reduce these 2 nonideal effects is to use a loose pre and a post LPF around the N-path filter as shown in Fig. 2. With cutoff frequencies a little above Ω_0, the pre LPF attenuates signals at high harmonics to avoid folding onto the in band signal and the post LPF removes residual out of band signal power.

Alternatively, as $\alpha(m, l)$ in (5)-(7) depends on the Fourier coefficients of the mixer signals, it's possible to design mixer signals that reduce the in band harmonic folding and the out of band signal residue such that the requirements on the pre and post LPFs are reduced or eliminated. Mixer sequence design for this purpose is explored in section 4.

[1]The assumption $\mathrm{E}\{X(j \cdot m\Omega_0)\} = \mathrm{E}\left\{X^2(j \cdot m\Omega_0)\right\} = 0$ is satisfied for most digital communication systems.

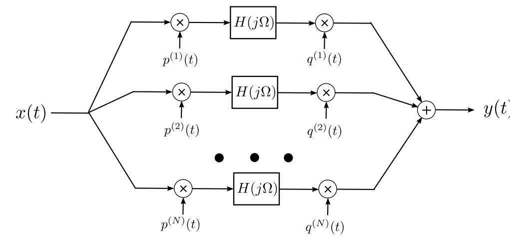

Fig. 1. N-path filter.

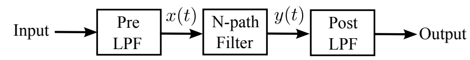

Fig. 2. Bandpass filter with N-path filter, pre and post LPF.

4. OPTIMAL MIXER DESIGN

This section determines optimal mixer sequences under different constraints. All mixers have periodic staircase sequences, i.e., each period is split into M equal time slots and each mixer sequence is a constant within each slot.

4.1. Performance Evaluation Criteria

Two criteria are considered in evaluating N-path filters:

(1) In band signal to noise ratio (SNR). If the folded harmonics are considered as in band noise, then the goal is to maximize the in band SNR = $P_{\text{signal}}/P_{\text{folded}}$.

(2) Out of band harmonic power ratio. For $l \neq \pm 1$, the goal is to minimize $R_{\text{out}}(l) = P_{\text{out}}(l)/P_{\text{signal}}$.

The following lemma and corollary, which put an upper limit on the achievable in band SNR and a lower limit on the out of band harmonic power ratio, are used in the sequence design.

Lemma: For M slot staircase mixer sequences, the harmonic power ratio for the lth harmonic is lower bounded by

$$R_{\text{out}}(l) = P_{\text{out}}(l)/P_{\text{signal}} \geq \qquad (8)$$
$$\begin{cases} \sum\limits_{b=-\infty}^{\infty} \frac{\mathrm{E}\left\{|X(j(bM+1)\Omega_0)|^2\right\}}{(l(bM+1))^2 \cdot \mathrm{E}\{|X(j\Omega_0)|^2\}}, & \text{for } l = cM+1, \\ \sum\limits_{b=-\infty}^{\infty} \frac{\mathrm{E}\left\{|X(j(bM-1)\Omega_0)|^2\right\}}{(l(bM-1))^2 \cdot \mathrm{E}\{|X(j\Omega_0)|^2\}}, & \text{for } l = cM-1, \\ 0, & \text{otherwise,} \end{cases}$$

and the lower bound is achieved if

$$\alpha(m, l) = 0, \quad \text{for } 0 \leq m < M,\ 0 \leq l < M,$$
$$\text{except for } (m, l) = (1, 1) \text{ or } (M-1, M-1). \qquad (9)$$

Short proof outline due to space limitations: The ratio $\alpha(bM+1, cM+1)/\alpha(1, 1)$ is independent of the input, where $\alpha(1, 1)$ controls P_{signal} and $\alpha(bM+1, cM+1)$ controls the power contributed from the $(bM+1)^{\text{th}}$ input harmonic to $P_{\text{out}}(cM+1)$. The bound is achieved when no other input harmonics contribute to the $(cM+1)^{\text{th}}$ output harmonic. The case $l = cM - 1$ is due to spectrum symmetry. ■

Setting $l = 1$ in the lemma leads to the corollary.

Corollary: With M slot staircase mixer sequences, the in band SNR is upper bounded by

$$SNR \leq \frac{\mathrm{E}\left\{|X(j\Omega_0)|^2\right\}}{\sum\limits_{b\neq 0} \frac{\mathrm{E}\{|X(j\cdot(bM+1)\Omega_0)|^2\}}{(bM+1)^2}}, \tag{10}$$

and the upper bound is achieved if

$$\alpha(m,1) = 0, \quad \text{for } m \neq 1, 0 \leq m < M. \tag{11}$$

4.2. Staircase Sequences

This section considers M slot staircase mixer sequences, while the signal amplitude can vary from time slot to time slot and from path to path and there is no constraint on the relationship between the mixer signals on different paths or the number of paths N. For this situation, an optimal sequence is provided by Theorem 1.

Theorem 1: The 2 path sampled quadrature filter achieves both the optimal in band SNR and the optimal harmonic power ratio at each harmonic frequency, among all N-path filters with M slot staircase mixer sequences. In the period $[0, T]$, the mixer sequences in the 2 path sampled quadrature filter have values

$$\begin{aligned} p^{(1)}(t) = q^{(1)}(t) = \cos(2\pi m/M) \\ p^{(2)}(t) = q^{(2)}(t) = \sin(2\pi m/M) \end{aligned} \tag{12}$$

where $(mT/M) \leq t < ((m+1)T/M)$ and $0 \leq m \leq M-1$.

Short proof: From the lemma and corollary in section 4.1, we only need to test the sequences in (12) satisfy (9) and (11).

Note that for $0 \leq m \leq M-1$, the only nonzero terms of Fourier series coefficients of $p^{(n)}(t)$ and $q^{(n)}(t)$ in (12) are $\hat{p}_1^{(n)}$, $\hat{p}_{M-1}^{(n)}$, $\hat{q}_1^{(n)}$ and $\hat{q}_{M-1}^{(n)}$. Thus, for $0 \leq m, l \leq M-1$, the only possible nonzero terms of $\alpha(m,l)$ are $\alpha(1,1)$, $\alpha(1, M-1)$, $\alpha(M-1, 1)$ and $\alpha(M-1, M-1)$. Direct calculation can verify that $\alpha(1, M-1) = \alpha(M-1, 1) = 0$. As such, (9) is satisfied and the minimum harmonic power ratio at each harmonic is achieved.

Constraint (11) for the optimal in band SNR is a special case of (9) with $l = 1$ which is already satisfied. Thus, the mixer signals in (12) achieve both maximum in band SNR and minimum harmonic power ratio at each harmonic. ■

Theorem 1 has a number of interesting implications. First, if the mixers are constrained to staircase sequences, then adding more than 2 paths does not provide a gain for in band SNR or out of band signal rejection. Second, the optimal sequences are independent of the input signal PSD and the location of any blockers. Third, improving the in band SNR and out of band signal rejection requires an increase of the number of time slots M in one period (i.e., the system has to run at a higher clock frequency). Fourth, the optimal sequences in (12) are also used in HRMs [4], [9], [10].

4.3. 2 Level Sequences

In this section the mixer sequences are further constrained to taking on only 2 values in each of the M slots:

$$p^{(n)}(t) \in \{1, -1\}, \quad q^{(n)}(t) \in \{A_n, -A_n\}, \tag{13}$$

where A_n is a constant gain for the nth path. Limiting the mixers to 2 levels makes them easier to implement in analog.

The cases of $N \geq M/2$ and $N < M/2$ paths are separately considered and it's assumed that M is even. Similar results can be obtained if M is odd but they are omitted due to space. Additionally, only the in band SNR criteria is used.

4.3.1. 2 Level Sequences with $N \geq M/2$ Paths

Theorem 2: Among N-path filters whose mixer sequences have M slots per period and satisfy (13), the $M/2$-path filter with the following class of mixer sequences achieves the optimal in band SNR:

$$\begin{aligned} p^{(n)}(t) &= p^{(1)}\left(t - ((n-1)T/M)\right), && (14) \\ q^{(n)}(t) &= q^{(1)}\left(t - ((n-1)T/M)\right), && (15) \\ p^{(1)}\left(t + (T/2)\right) &= -p^{(1)}\left(t\right). && (16) \end{aligned}$$

Short proof: The antisymmetric condition (16) indicates that $p^{(n)}(t)$ has no even harmonics; thus $\hat{p}_{2m}^{(n)} = 0$ and $\alpha(2m, 1) = 0$. The delay relationships in (14) and (15) result in a phase factor in the Fourier series coefficients and it can be verified that $\alpha(2m+1, 1) = 0$ for $1 \leq m < M/2$. Thus, (11) is satisfied and the optimal in band SNR is achieved. ■

In particular, the half plus half minus (HPHM) sequences

$$p^{(1)}(t) = \begin{cases} 1, & 0 \leq t < (T/2), \\ -1, & (T/2) \leq t < T, \end{cases} \tag{17}$$

$$\begin{aligned} p^{(n)}(t) &= p^{(1)}\left(t - ((n-1)T/M)\right), && (18) \\ q^{(n)}(t) &= p^{(n)}(t). && (19) \end{aligned}$$

satisfy constraints (14)-(16) and have optimal in band SNR. These sequences are particularly implementation friendly as there are only 2 level changes in one period in each path and mixer sequences in consecutive paths have a delay of one slot.

Similar to the staircase mixer in section 4.2, Theorem 2 implies that additional paths beyond $M/2$ do not provide a gain for in band SNR. However, out of band signal rejection could potentially be improved. Additionally, the optimal sequences are again independent of the input signal PSD.

4.3.2. 2 Level Sequences with $N < M/2$ Paths

This section considers 2 level M slot mixer sequences which satisfy (13) where the number of paths N is restricted to $N < M/2$ and only the in band SNR is considered. In contrast to the previous results, an input signal independent mixer sequence has not been obtained. Instead, a heuristic optimization algorithm is proposed.

Let $v_k^{(n)}$ and $w_k^{(n)}$ represent the values in the kth time slot ($1 \leq k \leq M$) of the mixer sequences $p^{(n)}(t)$ and $q^{(n)}(t)$, respectively. The in band SNR has the form of

$$SNR = \frac{\mathrm{E}\left\{|X(j\Omega_0)|^2\right\} \cdot |\alpha(1,1)|^2}{\sum_{m\neq 1} \mathrm{E}\left\{|X(j\cdot m\Omega_0)|^2\right\} \cdot |\alpha(m,1)|^2}. \quad (20)$$

Since the Fourier coefficients $\hat{p}_m^{(n)}$ and $\hat{q}_1^{(n)}$ are linear in the values of $v_k^{(n)}$ and $w_k^{(n)}$, respectively, $\alpha(m,1)$ is a bilinear form with respect to $v_k^{(n)}$ and $w_k^{(n)}$. Therefore, the powers of the signal and folded harmonic are both quadratic forms with respect to either $v_k^{(n)}$ or $w_k^{(n)}$. The total order of 4 is a challenge for optimizing (20).

To reduce the order of the objective function, an iterative 2 part heuristic algorithm is used. The first part optimizes over $v_k^{(n)}$ with $w_k^{(n)}$ held constant. The second part optimizes over $w_k^{(n)}$ with $v_k^{(n)}$ held constant.

The optimization problem in part 1 can be written as

$$\max_{\mathbf{v}\in\{-1,1\}^{MN}} SNR = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{S}\mathbf{v}}{\mathbf{v}^{\mathrm{T}}\mathbf{N}\mathbf{v}}, \quad (21)$$

where $\mathbf{S}$ and $\mathbf{N}$ are positive semidefinite matrices dependent on $N, M, w_k^{(n)}$ and the input signal PSD $\mathrm{E}\left\{|X(j\cdot m\Omega_0)|^2\right\}$. If $\mathbf{N}\mathbf{v} \neq 0$ for the all "binary" $\mathbf{v}$ vectors [2], it can be shown that the optimal objective function of (21) has the value of λ if and only if the following problem

$$\max_{\mathbf{v}\in\{-1,1\}^{MN}} \left(\mathbf{v}^{\mathrm{T}}\mathbf{S}\mathbf{v} - \lambda\cdot\mathbf{v}^{\mathrm{T}}\mathbf{N}\mathbf{v}\right) \quad (22)$$

has a maximum of 0. Equation (22) is an unconstrained binary quadratic programming problem and can be approximately solved by greedy local search. The solution of (22) never decreases SNR, which typically leads to convergence. Thus, the following iterative method is used:

Algorithm for Part 1
(1) Initialize $\mathbf{v}$ and $\mathbf{w}$ with the HPHM in (17)-(19).
(2) Compute the current $\lambda = (\mathbf{v}^{\mathrm{T}}\mathbf{S}\mathbf{v})/(\mathbf{v}^{\mathrm{T}}\mathbf{N}\mathbf{v})$.
(3) Solve (22) with the current λ.
(4) Update vector $\mathbf{v}$ with the solution to (22).
(5) If the optimal cost is 0 or the limit on iteration steps is reached, then terminate; otherwise, go to step (2).

For the second part, since the in band SNR involves only $\hat{q}_1^{(n)}$, the optimization is performed on $[\hat{q}_1^{(1)}, \ldots, \hat{q}_1^{(N)}] \in \mathbb{C}^N$. Equation (20) is the ratio of semidefinite quadratic forms with respect to $\hat{q}_1^{(n)}$, hence, its solution is available in closed form. After solving for the optimal $\hat{q}_1^{(n)}$, its phase is quantized into delays which are multiples of T/M.

The 2 parts may take multiple iterations to determine a heuristic based optimal solution for the 2 level mixer sequences in each path. There are potentially local minimums and no guarantees of global optimality are provided.

[2]The special case of $\mathbf{N}\mathbf{v} = 0$ is handled separately.

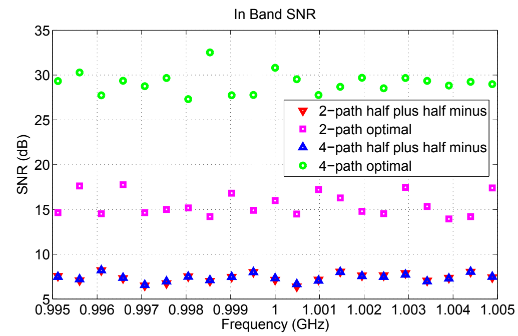

Fig. 3. N-path filter in band SNR.

In contrast to the staircase mixer sequence or 2 level sequence with $N \geq M/2$ paths, the optimal sequence in the case of $N < M/2$ paths depends on the input signal and the number of paths.

5. RESULTS

This section presents simulation results for 2 level mixer sequences with $N < M/2$. The center frequency of the passband is at 1GHz and the bandwidth is 10MHz for a quality factor of $Q = 100$. The pre LPF and post LPF in Fig. 2 are both 2nd order Chebyshev I filters with an in band ripple of 1dB and a cutoff frequency of 1.01GHz. The number of time slots is fixed at $M = 16$. The input signal in Fig. 2 has a flat PSD except for a 50dB blocker at the 9th harmonic (i.e., 9GHz).

The in band SNR of the HPHM sequences defined in (17)-(19) is compared to the optimal 2 level mixer sequences obtained from the algorithm in section 4.3.2 with various number of paths. Fig. 3 shows that the optimal sequences achieve a gain of 7dB and 22dB for in band SNR for 2 path and 4 path filters over the HPHM sequence, respectively.

6. CONCLUSIONS

This paper derived optimal mixer sequences for N-path filters under a variety of practical constraints. For staircase sequences, it was shown that a 2 path sampled quadrature filter achieves both the optimal in band SNR and the optimal out of band harmonic power ratio at each harmonic frequency. For 2 level mixer sequences with M slots per period (M even), a 2 transition per period shifted mixer sequence with $M/2$ paths achieves the optimal in band SNR. For situations where less than $M/2$ paths are available, a heuristic optimization algorithm was provided for designing 2 level mixing sequences which optimized in band SNR.

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