SOME PROPERTIES OF MULTIDIMENSIONAL STOCHASTIC OPTIMIZATION ALGORITHM WITH A CONSTANT STEP-SIZE

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ABSTRACT

The paper studies the convergence properties of a zero-order multidimensional stochastic optimization algorithm with a constant step-size. An estimate of the finite time convergence error is given. The operation of the algorithm is illustrated by the simulation of the spacecraft spatial position adjustment within a satellite constellation, which is relevant in controlling the orientation of an interferometric mega-constellation space telescope under conditions of uncertainty.

1 INTRODUCTION

Algorithm can be used with two possible step size selection strategies: a constant step size strategy and an adaptive step size. Initially, the works considered an algorithm with a variable step size. Choosing the step size is also a major challenge. In practice, the step size affects the accuracy. The algorithm SPSA allows to solve many problems in different areas where optimization of a function of a large number of parameters is required and when it is expensive to calculate the objective function for a long time. SPSA is used for study of variational quantum algorithms Periyasamy et al. (2024), in Experimental verification of active oscillation controller for vehicle drivetrain Yonezawa et al. (2024) , The properties of the algorithm are still being researched Zago et al. (2024) and the algorithm is developing and shows its advantages when compared with others Dieleman et al. (2024) the algorithm is used in optimization of CO2 flooding for Enhanced Oil Recovery Liu et al. (2024) also for tuning for flight control system of morphing arm octorotor Kose (2024) Since the level of interference and its statistic properties is not known in advance, a constant step-size could be considered as a better option to achieve a globally optimal solution.

2 PROBLEM STATEMENT

Consider the problem of minimizing a differentiable with respect to x function.

$$f(x) := \mathbb{E}_w \left[F(w,x)\right] = \int_{\mathbb{R}^p} F(w,x)P_w(dw), \tag{1}$$

where $F(w,x)$ is some penalty function (loss function) on the set $\mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^1$, $\{w_n\}$ - uncontrolled sequence of p-dimensional random variables w_n (perturbations) with the same (maybe) unknown distribution $P_w(\cdot)$ with compact support $\mathbb{W} = supp(P_w(\cdot)) \subset \mathbb{R}^p$. Here and below $\mathbb{E}$ is a symbol of mathematical expectation. Let $x_1, x_2, \ldots, x_N$ be a sequence of measurement points (observation plan selected or controlled by the experimenter), in which at each moment of time $n = 1, 2, \ldots, N$ the value of the function $F(w_n, x_n)$ with noise:

$$y_n = F(w_n, x_n) + v_n, \tag{2}$$

where v_n is additive observation noise.

Problem: using available observations, it is necessary to construct a sequence of estimates $\hat{\theta}_n$ of the unknown vector θ that minimizes the function $f(x)$. To solve the problem, we will use an iterative algorithm with two dimensions.

Let the trial simultaneous disturbance $\Delta_n,\ n = 1, 2, \dots N$ — observable (set or user-controlled) sequence independent random vectors from $\mathbb{R}^d$ with known distribution functions $\mathrm{P}_n(\cdot)$ — and specified vector functions $\mathcal{K}_n(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, satisfy the conditions

$$\int \mathcal{K}_n(x)\mathrm{P}_n(dx) = 0, \int \mathcal{K}_n(x)x^{\mathrm{T}}\mathrm{P}_n(dx) = \mathrm{I}, \tag{3}$$

$$\sup_n \int \|\mathcal{K}_n(x)\|^2 \mathrm{P}_n(dx) < \infty, n = 1, 2, \dots N. \tag{4}$$

Let $\mathcal{F}_{n-1}$ be the σ-algebra of all probabilistic events which happened during time interval $n - 1$ before start of time interval n. Hereinafter $\mathbb{E}_{\mathcal{F}_{n-1}}$ is a symbol of the conditional mathematical expectation with respect to the σ-algebra $\mathcal{F}_{n-1}$, $\mathbb{E}$ is a symbol of the mathematical expectation. The minimum point θ of function

$$f(\theta) = \mathbb{E}_{\mathcal{F}_{n-1}} F(\cdot, \theta) \to \min_\theta$$

needs to be estimated.

More precisely, using the observations $y_1, y_2, \dots, y_n$ and inputs $\theta_1, \theta_2, \dots, \theta_n$, construct an estimate $\hat{\theta}_n$ of an unknown vector θ minimizing the time-varying mean-risk functional.

2.1 Assumptions

Let us formulate Assumptions about disturbances and functions $f(\theta), F(\cdot, \theta)$.

Assumption 1. *For $n = 1, 2, \dots N$, the successive differences $\bar{v}_n = v_n^+ - v_n^-$ of observation noise are bounded: $|\bar{v}_n| \le c_v < \infty$, or $\mathbb{E}\bar{v}_n^2 \le c_v^2$ if a sequence $\{v_n\}$ is random, where v_n^+, v_n^- are observation noises occurred during the same time interval n but at different time instants.*

Assumption 2. *Function $f(\cdot)$ have unique minimum point θ^* and $\forall\theta$ $\langle \theta - \theta^*, \mathbb{E}_{\mathcal{F}_{n-1}} \nabla F(\cdot, \theta)\rangle \ge \mu\|\theta - \theta^*\|^2$ with a constant $\mu > 0$. Here and further $\langle \cdot, \cdot \rangle$ is a scalar product of two vectors.*

Assumption 3. *The gradient ∇f is uniformly bounded in the mean-squared sense at the minimum points $\theta^* : \mathbb{E}\|\nabla f(\theta^*)\|^2 \le g^2$*

Assumption 4. *The gradient $\nabla F(\cdot, \theta)$ satisfies the Lipschitz condition: $\forall\theta', \theta''$*

$$\|\nabla F(\cdot, \theta') - \nabla F(\cdot, \theta'')\| \le M\|\theta' - \theta''\|$$

with a constant $M \ge \mu$.

Assumption 5. *$\forall n \ge 1$ random vector Δ_n does not depend on $\bar{w}_n$, random vectors $\bar{w}_n, \Delta_n$ do not depend on $\bar{w}_1, \dots, \bar{w}_{n-1}$; if $\{\bar{v}_n\}$ are random variables, then $\bar{w}_n, \Delta_n$ also do not depend on $\bar{v}_1, \dots, \bar{v}_n$.*

3 Stochastic Optimization Algorithm for Parameter Estimation

3.1 Algorithm

In case of presence of systematic error in the model a method should be chosen that gives unbiased estimates under these conditions. A stochastic approximation method, based on SPSA algorithm introduced by Spall Spall (1992), is capable of providing estimates under an unknown-but-bounded noise Granichin & Amelina (2015). In case the noise is random the knowledge of distribution and its parameters is not required, apart from upper bound of the second raw moment, to provide efficient estimates of the required variable. If the noise is not random it is only required to know an upper bound on the difference of subsequent noise values. Such variety in possible noise is caused by the usage of the randomized input Δ_n which is required to be independent of the noise. This property allows to cancel the bias induced by external disturbance when estimating the mathematical expectation of the parameter deviation. It should be noted that in theory it is not always easy to prove the required independence of the randomized input and the noise. However, in practice it is quite reasonable to assume the independence of the generated random values and external disturbances affecting the considered object.

Let's choose an arbitrary initial estimate vector $\hat{\theta}_0 \in \mathbb{R}^d$ and scalar parameters α, β for an iterative algorithm

$$\begin{cases} x_n^+ = \hat{\theta}_{n-1} + \beta\Delta_n,\ x_n^- = \hat{\theta}_{n-1} - \beta\Delta_n \\ y_n^+ = F(w_n^+, x_n^+) + v_n^+,\ y_n^- = F(w_n^-, x_n^-) + v_n^- \\ \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{\alpha}{2\beta}\mathcal{K}_n(\Delta_n)(y_n^+ - y_n^-), \end{cases} \tag{5}$$

where $\hat{\theta}_n$ is the estimate of true vector of parameters θ, θ_n^+ and θ_n^- are the disturbed estimates, y_n^+ and y_n^- are the noised measurements with noises v_n^+ and v_n^- respectively, $\Delta_n \in \mathbb{R}^d$ is the randomized vector, $K_n(\cdot)$ are vector functions specified above equation 3-equation 4, α and β are the algorithm parameters. Elements of random perturbation vectors Δ_n are chosen randomly $+1$ or -1 with equal probability.

4 Analysis of convergence of estimates

To analyze the convergence of algorithm estimates (5) the method from Polyak (1976) is used, similar to the second Lyapunov method in stability theory. For the Lyapunov function we choose

$$V(\hat{\theta}_{n-1}) = \frac{1}{2}\|\hat{\theta}_{n-1} - \theta\|^2. \tag{6}$$

Let us list the main conditions Theorems 1-3 from Polyak (1976), the fulfillment of which must be verified.

(a) The iterative process has a Markov character, i.e. the distribution of the random vector Y_n depends only on $\hat{\theta}_{n-1}$ and n:

$$Y_n = G_n(\bar{w}_n, \hat{\theta}_{n-1}). \tag{7}$$

For the algorithm (5) we have

$$\mathbb{E}\{G_n(\bar{w}_n, \hat{\theta}_{n-1})\} = \mathbb{E}\{\mathcal{K}_n(\Delta_n)\frac{y_n^+ - y_n^-}{(2\beta)}|\mathcal{F}_{n-1}\}. \tag{8}$$

(b) $V(\hat{\theta}_{n-1})$ — non-negative, $\inf V(\hat{\theta}_{n-1}) = 0$, $V(\hat{\theta}_{n-1})$ — is differentiable, and its gradient satisfies the Lipschitz condition:

$$\|\nabla V(x) - \nabla V(\theta)\| \leq L\|x - \theta\|$$

(c) Pseudogradient condition:

$$\langle \nabla V(\hat{\theta}_{n-1}), \mathbb{E}\{G_n(w, \hat{\theta}_{n-1})\}\rangle \geq \delta_n V(\hat{\theta}_{n-1}) - \gamma_n,\ \delta_n > 0,\ \gamma_n \geq 0 \tag{9}$$

For the algorithm (5), taking into account the Lyapunov function chosen in the form (6), the following expression is obtained:

$$\langle \nabla V(\hat{\theta}_{n-1}), \mathbb{E}\{G_n(w, \hat{\theta}_{n-1})\}\rangle\ = \langle \hat{\theta}_{2n-2} - \theta, \mathbb{E}\{\mathcal{K}_n(\Delta_n)\frac{y_{2n} - y_n^-}{(2\beta)}|\mathcal{F}_{n-1}\}\rangle \tag{10}$$

Considering that

$$y_n = F(w_n, x_n) + v_n \tag{11}$$

$$y_n^+ = F(w_n^+, x_n^+) + v_n^+ \tag{12}$$

$$y_n^- = F(w_n^-, x_n^-) + v_n^- \tag{13}$$

$$x_n^+ = \hat{\theta}_{n-1} + \beta\Delta_n \tag{14}$$

$$x_n^- = \hat{\theta}_{n-1} - \beta\Delta_n \tag{15}$$

$$y_n^+ = F(w_n^+, \hat{\theta}_{n-1} + \beta\Delta_n) + v_n^+ \tag{16}$$

$$y_n^- = F(w_n^-, \hat{\theta}_{n-1} - \beta\Delta_n) + v_n^- \tag{17}$$

let's rewrite $\mathbb{E}\{G_n(w, \hat{\theta}_{n-1})\}$ in the form:

$$\mathbb{E}\{G_n(w, \hat{\theta}_{n-1})\} = \mathbb{E}\{\mathcal{K}_n(\Delta_n)\frac{F(w_n^+, \hat{\theta}_{n-1} + \beta\Delta_n) + v_n^+ - F(w_n^-, \hat{\theta}_{n-1} - \beta\Delta_n) - v_n^-}{(2\beta)}|\mathcal{F}_{n-1}\} \tag{18}$$

4.1 MAIN RESULT

To provide estimates of the expected discrepancy between the system state vector and its estimates obtained via algorithm equation 5 we use Theorem 1 from Polyak (1976). To utilize the theorem, 6 propositions given inside the proof below have to be met. We show that they follow from assumptions 1–5. Let us introduce the following notation:

$$\nu = \alpha \left(0.5 - \mu - \frac{L\alpha C_\tau}{2}\right), \quad \phi = \alpha\gamma + \frac{L}{2}\alpha^2 C_\sigma, \quad \gamma = 0.5\left(C_1\beta\right)^2, \quad \psi = \frac{\phi}{\nu}$$

Here C_τ, C_σ could be chosen to satisfy inequation $\mathbb{E}\|\frac{1}{2\beta}(f(\theta+\beta\Delta_n)+v_n^+ - f(\theta-\beta\Delta_n)-v_n^-)\|^2 \leq C_\sigma + C_\tau\|\theta-\theta\|^2$ and $C_1 \geq \int \|\mathcal{K}_n(x)\|M\|x\|^2 \mathrm{P}_n(dx)$ and α chosen to satisfy following conditions:

$$0 \leq \alpha \leq \frac{4\mu - 2}{LC_\tau}, \quad \alpha \leq \frac{2\mu - 1 - \sqrt{(2\mu-1)^2 - 2LC_\tau}}{LC_\tau}$$
$$\alpha \geq \frac{2\mu - 1 + \sqrt{(2\mu-1)^2 - 2LC_\tau}}{LC_\tau}. \tag{19}$$

Theorem 1. *Let Assumptions 1–5 and conditions for kernels $\mathcal{K}$ equation 3-equation 4 and α equation 19 be satisfied. Set $\hat{\theta}_0$, choose interval size parameter k*

$$\mathbb{E}\{\|\hat{\theta}_n - \theta\|^2\} \leq \mathbb{E}\{\|\hat{\theta}_0 - \theta\|^2\}(1-\nu_i)^n + \psi(1-(1-\nu_i)^n). \tag{20}$$

Proof 1. *To analyze the convergence of the algorithm equation 5 estimates, a method from Polyak (1976) is used. Choose*

$$V(\hat{\theta}_n) = \frac{1}{2}\|\hat{\theta}_n - \theta\|^2 \tag{21}$$

as Lyapunov function. To prove the theorem equation 20 is true it is sufficient to show the following six propositions are satisfied.

Proposition 1. *The iterative process Y_n, which defines the direction of the estimate change, is a Markov process, i.e. the distribution of the random vector Y_n depends only on $\hat{\theta}_n$ and n:*

$$Y_n = \frac{1}{2\beta}\mathcal{K}_n(\Delta_n)(y_n^+ - y_n^-) \tag{22}$$

For algorithm equation 5 we have

$$\mathbb{E}\{Y_n\} = \mathbb{E}\{\mathcal{K}_n(\Delta_n)\frac{y_n^+ - y_n^-}{2\beta}|\mathcal{F}_{n-1}\}, \tag{23}$$

where the right-hand side depends only on $\hat{\theta}_n$ and n in the sense that Δ_n does not depend on any other random variables. Proposition is true.

Proposition 2. $V(\hat{\theta}_n) \geq 0$, $\inf V(\hat{\theta}_n) = 0$, $V(\hat{\theta}_n)$ *has first-order derivative, and its gradient satisfies Lipshitz condition:*

$$\|\nabla V(x) - \nabla V(\theta)\| \leq L\|x - \theta\| \quad \forall x, \theta \in \mathbb{R}^d.$$

The proposition is valid due to the choice of Lyapunov function equation 21.

Proposition 3. *Pseudo-gradient condition:*

$$\langle \nabla V(\hat{\theta}_n), \mathbb{E}\{Y_n\}\rangle \geq \delta_n V(\hat{\theta}_n) - \gamma_n,\ \delta_n > 0,\ \gamma_n \geq 0. \tag{24}$$

At first consider $\mathbb{E}\{Y_n\}$. Due to equation 5 pseudo-gradient equation 22 after using Assumption 5 becomes

$$\mathbb{E}\{Y_n\} = \mathbb{E}\{\mathcal{K}_n(\Delta_n)\frac{1}{2\beta}\left(f(\theta_n^+) - f(\theta_n^-)\right)|\mathcal{F}_{n-1}\} \tag{25}$$

Consider the expression under mathematical expectation. Using Taylor series representation it could be written as:

$$\frac{1}{2}\nabla f(\hat{\theta}_n) + \frac{1}{2\beta}\int \mathcal{K}_n(x)x^T \int_0^1 \left(\nabla_x F(\cdot, \hat{\theta}_n + t\beta x) - \nabla_x F(\cdot, \hat{\theta}_n)\right) dt P_n(dx) +$$
$$+ \frac{1}{2}\nabla f(\hat{\theta}_n) - \frac{1}{2\beta}\int \mathcal{K}_n(x)x^T \int_0^1 \left(\nabla_x F(\cdot, \hat{\theta}_n - t\beta x) - \nabla_x F(\cdot, \hat{\theta}_n)\right) dt P_n(dx)$$

Estimate absolute value of the sum of integral elements in the obtained expression. After using equation 3, Assumption 4 we get

$$|\int(\cdot)P_n(dx)| + |\int(\cdot)P_n(dx)| \leq \frac{2\beta^2}{2\beta}\int \|\mathcal{K}_n(x)\|\|x\|M\|x\|P_n(dx) \leq C_1\beta. \quad (26)$$

Substitute the estimate, elements containing gradients of F_n and equation 21 into equation 24, regard the relation $\int(\cdot)P_n(dx) \geq -|\int(\cdot)P_n(dx)|$ and Cauchy–Bunyakovsky–Schwarz inequality:

$$\langle\hat{\theta}_n - \theta, \mathbb{E}\{Y_n\}\rangle \geq \langle\hat{\theta}_n - \theta, \nabla f(\hat{\theta}_n)\rangle - \|\hat{\theta}_n - \theta\|C_1\beta.$$

Apply Assumption 2 and estimate $\|\hat{\theta}_n - \theta\|C_1\beta \leq \frac{1}{2}(\|\hat{\theta}_n - \theta\|^2 + (C_1\beta)^2)$:

$$\langle\hat{\theta}_n - \theta, \mathbb{E}\{Y_n\}\rangle \geq (2\mu - 1)\frac{1}{2}\|\hat{\theta}_n - \theta\|^2 - \frac{1}{2}C_1^2\beta^2$$

Proposition is true for $\mu > 1/2$.

Proposition 4.

$$\mathbb{E}\{\|Y_n\|^2\} \leq \sigma_n^2 + \tau V(x),\ \sigma_n \geq 0,\ \tau_n \geq 0.$$

Using equation 4, Cauchy–Bunyakovsky–Schwarz inequality, Assumptions 1 and 3 it could be shown that

$$\mathbb{E}\{\|Y_n\|^2\} \leq \frac{1}{2}\sup_x \mathcal{K}_n(x)^2\mathbb{E}\{(v_n^+)^2 + (v_n^-)^2|\mathcal{F}_{n-1}\} + \\ + \int\left(f(\hat{\theta}_n^+) - f(\hat{\theta}_n^-)\right)^2\|\mathcal{K}_n(x)\|^2\mathrm{P}_n(dx) \leq C_2\beta^2(\|\hat{\theta}_{n-1} - \theta\|^2) + C_3\beta^4 + C_4\bar{v}_n^2$$

Proposition holds true with $\tau = 2C_2\beta^2$, $\sigma_n^2 = C_3\beta^4 + C_4\bar{v}_n^2$.

Proposition 5. $\mathbb{E}V(\hat{\theta}_0) < \infty$.

Proposition is valid due to arbitrariness of initial approximation $\hat{\theta}_0$ choice and an assumption regarding final order of the external disturbance W affecting the system and thus the final order of the system state vector.

Proposition 6. $0 \leq \nu \leq 1$*;* $\sum_n \nu = \infty,\ n \to \infty$

The first inequality could be met by choice of α and the second one is true since ν is constant. Proposition is true. The fulfillment of the given propositions allow to prove the theorem using the result in Polyak (1976).

Remark 1. *After the algorithm converges the parameter estimates $\hat{\theta}_n$ continue to fluctuate around the true parameter value θ.*

5 Application for Orientation Improvement of interferometric mega-constellation space telescope

The main tasks of the telescope are: to collect radiation that falls on the mirror system, with minimal losses, and also obtain the most accurate image of the object. If the radiation is collected with significant errors, then the image will be disturbed. An important and time-consuming part of image acquisition is the precise tuning of the radio telescope antenna (or systems consisting of such antennas). The quality of the image obtained on a radio telescope directly depends on the quality of the construction of a reflecting system of mirrors that focuses the radiation coming from outside. To improve the image quality, it is necessary to focus the radiation of the device in such a way that it works as accurately as possible, especially if it is located in space. Conventional antenna tuning algorithms are sufficient. However, they lose their effectiveness under uncontrolled unpredictable external influences such as deformations of the radio telescope shields that arise due to environmental influences e.g. temperature changes, wind etc. Moreover, if we consider an autonomous system representing a grouping of spacecraft, the relevance of the approach used is obvious. In particular, the SPSA algorithm can be applied to improve the orientation of a megaconstellation interferometric space telescope such as Knapp et al. (2024)

One of the ways to solve such a problem is using of randomized stochastic approximation algorithms Granichin et al. (2021).

The main criterion for efficiency is the recording power of the desired signal and the time required for the parameters adjustment.

The antenna segments can be set to the optimal position to improve the quality of image recording. Consider an irradiator (radiation generator), a receiver and a mirror system of a radio telescope, consisting of identical plates that reflect the incoming signal $i = 1 \dots K$. Radiation is created in the irradiator, falls on the plates and is focused in the receiver. Let's consider time intervals of duration δ, n is the index of the time interval. Assume the following parameters are known:
1) the position (orientation) of each i-th plate, which is specified by the vector of parameters $(a_i, b_i, c_i)^T$, where a_i is the rotation angle of the i-th reflective element horizontally; b_i is the vertical rotation angle, c_i is the forward horizontal displacement of the i-th reflecting element. Let θ_n be a vector that contains all parameters of the mirror system in a given time interval n, $\theta_n = (a_1, a_2, ..., a_K, b_1, b_2, ..., b_K, c_1, c_2, ..., c_K)^T$,
2) radiation coming from each mirror $z_i(t)$, which depends on parameters a_i, b_i, c_i,
3) the common signal coming from all mirrors to the receiver $Z(\theta, t) = \sum_{i=1 \dots K} z_i(t)$,
4) characteristics of the signal in the generator.

The front of a signal is the sum of harmonics with different phases from a certain direction. The perfect placement of the plates brings all radiation from the objects into focus. The signal is obtained as a sum of sine waves with different phases. Different z_i arrive at different times with different phases.

Let us evaluate the difference between the signals from an ideal antenna and from a real one (with deformations). Signals reflected from ideal mirror segments will have the same phase ϕ_i ($\phi_1 = \phi_2 = \dots = \phi_K$). The signals from segments with deformations will look like this:

$$z_1 = \sin(\omega t + \phi_1), z_2 = \sin(\omega t + \phi_2), \dots z_K = \sin(\omega t + \phi_K) \tag{27}$$

The objective function of the problem ($F(\theta)$ is the signal power) is defined as follows:

$$F(\theta) = \int_0^\delta |Z(\theta, t)|^2 dt. \tag{28}$$

We maximize the objective function:

$$F(\theta) \to \max_\theta \tag{29}$$

We consider the problem of optimizing the position of mirrors in the limit over time (not over a specific time interval).

The reflective elements of the antenna are made exactly the same, so they provide equivalent observations in all directions.

At the same time, if one moves along an ideal reflective surface, its local characteristics change. Therefore, a real reflective surface composed of identical elements will repeat deviations from the ideal surface from element to element. These deviations will be greater, if the shape of the surface of the element differs from the shape of that portion of the ideal surface which this element should represent. If the size of the reflective elements increases, then the deviations naturally increase. These deviations comprise an error distributed over the reflector of a variable profile antenna, which, at large values, creates unacceptable distortions reducing the efficiency of the antenna. We will call v_n the signal power measurement errors arising due to unknown and uncontrolled deformations in the reflective elements of the antenna caused by weather, wind, and temperature changes.

After conventional procedures for adjusting the inclination angles and positions of the radio telescope mirrors, we obtain an initial approximation $\hat{\theta}_0$ of the mirror system parameters. Then there is still a necessity of tuning in a certain neighborhood of $\hat{\theta}_0$ to obtain the optimal value $\theta^\star$ corresponding to the maximum power of the received signal. For adjustment, we could use a stochastic optimization algorithm with two measurements per iteration, which allows us to reduce the negative impact of various disturbances on the power of the recorded signal.
Algorithm:

1. Select initial approximation $\hat{\theta}_0$, put $n = 0$.
2. $n \to n + 1$.
3. Generate vectors Δ_n according to Bernoulli distribution with components taking values 1 or -1 with probability 0.5.
4. Measure the power values for two positions of the antenna system: $(\hat{\theta}_n + \beta\Delta_n)$ and $(\hat{\theta}_n - \beta\Delta_n)$. Measurements are obtained with noise v_n^+ and v_n^- :

$$y_n^+ = F(\hat{\theta}_n + \beta\Delta_n) + v_n^+; \tag{30}$$

$$y_n^- = F(\hat{\theta}_n - \beta\Delta_n) + v_n^-; \tag{31}$$

5. Update estimate $\hat{\theta}_n$ according to the rule:

$$\hat{\theta}_{n+1} = \mathcal{P}_{\mathcal{T}}\left(\hat{\theta}_n + \frac{\alpha K_n(\Delta_n)}{2\beta}(y_n^+ - y_n^-)\right), \tag{32}$$

where α, β are the parameters of the algorithm, $\mathcal{P}_{\mathcal{T}}$ is the projection onto the set $\mathcal{T}$ of admissible parameter values.

The kernels $\mathcal{K}_n$ could be constructed as vectors with i.i.d. components with Bernoulli distribution, taking values ± 1 with probability 0.5.

The test disturbance is formed in such a way that $\forall n \geq 1$ random vector Δ_n does not depend on $\bar{v}_1, \ldots, \bar{v}_n$ and $\mathbb{E}\{(v_n^+ - v_n^-)^2/2\} \leq c_v^2$, $(\mathbb{E}\{\bar{v}_n^2\} \leq c_v^2)$.

The main element that reflects the incoming signal is the segment of the mirror system.

Consider a system consisting of $K = 50$ satellites with reflective mirrors. The mirrors should be placed in space in such a way that the reflected signal from the mirrors is focused in a small area where the receiver is placed. One of the possible space configurations of such mirror system is a circular paraboloid which, due to its properties, focuses the parallel beams in a single point. The satellites equipped with the sensors could adjust their relative positions and the mirrors orientation to form a paraboloid. However, due to the errors caused by measurement and the natural phenomena the mirror system needs to be adjusted for better focusing.

Stochastic optimization algorithm equation 30-32 can be applied for tuning a system which consists of a large number of satellites with reflective shields in space under conditions of interference and also when individual elements of the system are deformed.

In Fig. 1 on page 8, the dependence of the signal power on the number of iterations of the algorithm is given. The admissible adjustment angles for the mirrors lie in $[-\pi/6, \pi/6]$, with the step $\beta = \pi/36$. Step-size parameter $\alpha = 1e - 7$ is chosen manually with respect to the simulation results. The graph in Fig. 1 shows the signal power in the receiver computed according equation 28. The graph is constructed by averaging the results of 40 simulations. After converging to the maximum value the algorithm continues to fluctuate due to input randomization. The experiments show strong dependence on step-size parameter value which has to be carefully adjusted to achieve fast and monotonous convergence behavior.

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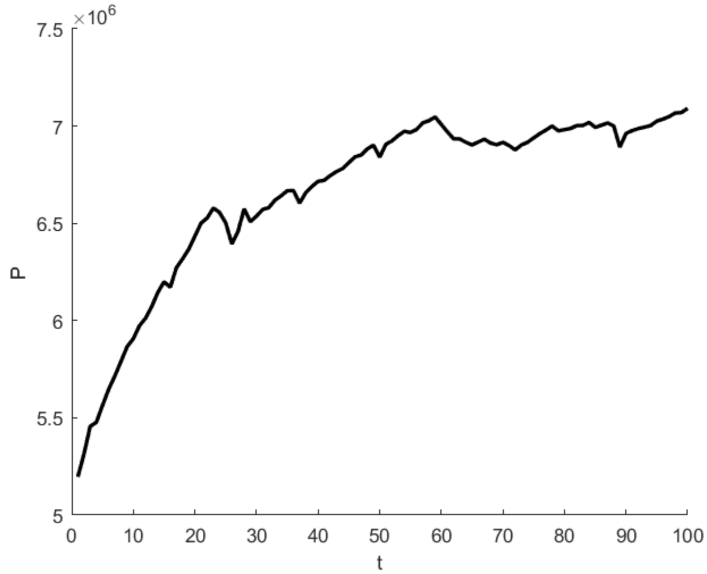

Figure 1: Convergence of the algorithm.

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