Spectral properties of the Gram matrix for Gabor systems generated by B-splines

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***Abstract*—We investigate the structure and spectrum of the Gram matrix corresponding to time-frequency shifts of the second-order B-spline. In particular we show that under a specific finite sampling of the time-frequency lattice, the Gram matrix has Toeplitz block structure and is per-Hermitian, making spectral asymptotics amenable to classical Szego-type limit theorems. We also study the relationship between the spectra of the finite-dimensional Gram matrices as well as their constituent blocks and the spectrum of the infinite-dimensional Gram operator. A complete characterization of the spectrum of the Toeplitz blocks within the Gram matrix is provided as well as explicit descriptions of the asymptotics of its eigenvalues and estimates on the corresponding frame bounds. We expect that the spectral analysis of these Gram matrices will shed new light on the frame set of higher order B-splines.**

I. Introduction

Introduced by Duffin and Schaeffer in their seminal paper [10], frame theory centers around the study of overcomplete representations of functions via fundamental building blocks called frame vectors. Whereas a basis provides a unique representation of a function, frames and the corresponding frame vectors offer a flexible alternative and non-unique representations of a function. This allows for sparsity, redundancy, and stability of the representation that draws on rich mathematical theory at the intersection harmonic analysis, numerical analysis, and sampling theory underpinning applications in signal processing, quantum information theory, and machine learning [12], [22] , [23], [24], [6].

We consider a specific family of frames, *Gabor frames*, generated by a window $g \in L^2(\mathbb{R})$ and time-frequency parameters $a, b > 0$. The problem of characterizing the pairs (a, b) such that the system of time-frequency shifts

$$\mathcal{G}(g, a, b) := \{M_{lb}T_{ka}g = e^{2\pi i l b \cdot} g(\cdot - ka) : (l, k) \in \mathbb{Z}^2\}$$

is a frame for $L^2(\mathbb{R})$ is referred to as the *frame set* problem and is in general an unsolved and difficult problem in Gabor analysis [14], [18]. Celebrated results in the 90's characterize the frame set when the window is a Gaussian or in related function classes [15], [25], [9], [8]. Since then progress has only been made in a handful of classes including the first-order B-spline, where the frame set has been completely characterized [7], [21], [14]. Here, we zero-in on the case where the window g is a higher order B-spline and take its time-frequency shifts along the lattice

$$\Lambda := a\mathbb{Z} \times b\mathbb{Z}.$$

This class of B-splines has many advantages as a choice of window including compact support, partitions of unity, linear independence, and regularity. Furthermore, finite subsets of Gabor systems generated by functions with compact support are a basis for their linear span, a solved special case of the HRT conjecture [19]. In applications, B-spline wavelet frame systems are central to finite element methods and successfully applied in molecular dynamics simulations, motion smoothing, and recurrent neural networks [3], [16], [26], [20].

Our approach investigates the structure and spectrum of the infinite Gram matrix (operator) $G(a, b) : \ell^2(\mathbb{Z}^2_+) \to \ell^2(\mathbb{Z}^2_+)$ with entries given by the inner products $\langle g_\lambda, g_{\lambda'}\rangle_{L^2(\mathbb{R})}$, $\lambda, \lambda' \in \Lambda$, and the finite submatrices corresponding to a $n^2 \times n^2$ set of lattice points centered at the origin. Each of these finite submatrices is viewed as a finite Gram matrix $G(a, b, n) \in \mathbb{C}^{n^2\times n^2}$. Ultimately, we expect that the spectral analysis of these Gram matrices will shed new light on the frame set of higher order B-splines. This is due in part to (1) the frame operator and Gram operator share a spectrum except possibly at zero: $\{0\} \subset \mathrm{Spec}\big(G(a, b)\big) \subset \{0\} \cup [A, B]$ and (2) the lower frame bound corresponds to the (reciprocal) operator norm of the pseudo-inverse $||G^\dagger(a, b)||$ [17] [1]. We summarize our contributions as follows:

1) In Section II, under a specific sampling scheme of the time-frequency lattice Λ, the Gram matrix $G(a, b, n)$ has remarkable structure amenable to spectral analysis such as having Toeplitz blocks.
2) In Section III, via the Toeplitz structure and Szego-type limit theorems, we provide an asymptotic analysis and complete description of the spectrum of the blocks within $G(a, b, n)$ which correspond to samplings of lines on the lattice Λ.

A system $\{g_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R})$ forms a *frame* for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$,

$$A||f||^2 \le \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \le B||f||^2, \ \forall f \in L^2(\mathbb{R}).$$

The constants A, B are called *frame bounds,* and estimating the optimal frame bounds is generally an unresolved problem whose solution can impact the numerical accuracy and speed of reconstructing and representing a function through a given frame. The frame inequality above implies the existence of *dual frames* $\{h_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R})$ such that each $f \in L^2(\mathbb{R})$ has the following representation or reconstruction:

$$f = \sum_{\lambda\in\Lambda} \langle f, h_\lambda\rangle g_\lambda = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle h_\lambda$$

Given a *window* $g \in L^2(\mathbb{R})$ and real numbers $a, b > 0$, the set of time-frequency shifts $\mathcal{G}(g, a, b)$ is called a *Gabor frame* if it forms a frame for $L^2(\mathbb{R})$. The parameters (a, b) control the resolution of the time-frequency lattice that we sample with the hopes of reconstructing a function from its time-frequency shifts of the window g. The set

$$\mathcal{F}(g) := \{(a, b) \in \mathbb{R}^2_+ : \mathcal{G}(g, a, b) \text{ is a frame for } L^2(\mathbb{R})\}$$

is referred to as the *frame set* of g and the problem of characterizing the frame set is an open problem for many classes of window functions g. In the cases where g is a member of

1) $\{e^{-\pi x^2}, \frac{1}{\cosh x}, \chi_{[0,\infty)}e^{-x}, e^{-|x|}\}$
2) a totally positive function of finite type
3) a totally positive function of exponential type

then the frame set has a simple characterization as the region under the hyperbola $ab < 1$, $\mathcal{F}(g) = \{(a,b) \in \mathbb{R}^2_+,\ ab < 1\}$. Furthermore, if the window is the indicator function $g = \chi_{[0,c]}$, the frame set is also known (referred to as Janssen's tie [21]) and completely described by Dai and Sun [7].

In this paper we study the case where the window is the second-order B-spline,

$$g \coloneqq \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]} = \begin{cases} 1-|x|, & x \in [-1,1] \\ 0, & x \notin [-1,1]. \end{cases}$$

Complete knowledge of the regions under the hyperbola $ab < 1$ that belong to the frame set $\mathcal{G}(g,a,b)$ is a fundamental open problem in Gabor analysis. It is known for example that $\mathcal{G}(g,a,b)$ is an open set in $\mathbb{R}^2$ [11]. Recent work by Atindehou, Frederick, Kouagou, and Okoudjou [2] discovered new regions by explicitly finding a dual window for the second order B-spline with these new time-frequency parameters (a,b). There are few previous works that investigate connection between the spectrum of the Gram matrix of Gabor systems and the corresponding frame set problem. In [1], Adcock and Huybrechs provide a numerical analysis of regularized SVD decomposition of the finite Gram matrices $G(a,b,n)$ to get estimates on the frame bounds. In [17], Harrison characterizes the null space of the Gram operator for exponential frames and provides a theoretical and a numerical analysis of its spectrum.

II. Lattice Sampling and Gram Matrix Structure

Fix $a, b > 0$ and consider the time-frequency shifts of the B-spline,

$$\{g_{\lambda_{ij}}\}_{i,j=1}^n = \{g(x-y_j)e^{2\pi i \omega_i x}\}_{i,j=1}^n$$

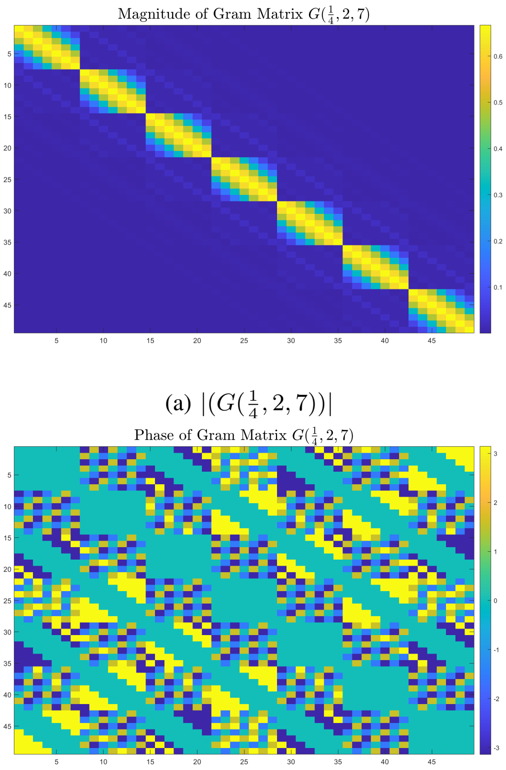
in which the time-frequency shifts belong to a set $\Lambda_{n^2} = \{\lambda_{ij} = (y_j, \omega_i)\}_{i,j=1}^n$ of enumerated elements centered at the origin, where

$$y_j = a(\lfloor n/2 \rfloor + j - n),\ j = 1,2,\dots,n$$
$$\omega_i = b(\lfloor n/2 \rfloor + i - n),\ i = 1,2,\dots,n.$$

This sampling creates a finite square sub-lattice centered at the origin of the $y-\omega$ time-frequency plane. With this sampling scheme, natural block structure forms in the Gram matrix $G(a,b,n) \in \mathbb{C}^{n^2 \times n^2}$, due to a fixed phase factor $e^{-2\pi i(\omega_i - \omega_{i'})x}$ that appears in the inner product seen in equation (1). The blocks are notated $G^{ii'}(a,b,n)$ for $i, i' \in \{1 \dots n\}$ and to index into each block we notate for $j, j' \in \{1, \dots, n\}$, so that each entry $G^{ii'}_{jj'} = \langle g_{\lambda_{ij}}, g_{\lambda_{i'j'}} \rangle_{L^2(\mathbb{R})}$ is the integral:

$$\int_{\mathbb{R}} g(x-y_j)g(x-y_{j'})e^{-2\pi i(\omega_i-\omega_{i'})x}dx. \qquad (1)$$

The ordering of $(\lambda_{ij})_{i,j=1}^n$ can be done without loss of generality when considering the spectrum of $G(a,b,n)$. This is because other orderings will be permutations of the rows and columns of $G(a,b,n)$ that are simple similarity relations that leave the spectrum unchanged. The blocks $G^{ii'}(a,b,n)$ have the phase factor $e^{-2\pi i(\omega_i - \omega_{i'})x}$ in the integrand and the entries $G^{ii'}_{jj'}$ are a function of the shifting parameters y_j and $y_{j'}$, see Figure 1. We first justify the choice of centering the finite sub-lattice at the origin by showing that if one were to consider a square sub-lattice centered at another location in the time-frequency plane, the entries of $G(a,b,n)$ change in a predictable way by a potential phase factor. The proof



(a) $|(G(\frac{1}{4},2,7))|$

(b) $\mathrm{Arg}\big(G(\frac{1}{4},2,7)\big)$

Fig. 1: The magnitude and phase of the finite Gram matrix $G(\frac{1}{4},2,7) \in \mathbb{C}^{49\times 49}$ with blocks $G^{ii'}(\frac{1}{4},2,7) \in \mathbb{C}^{7\times 7}$ for $i, i' \in \{1,2\dots,7\}$.

is a simple change-of-variables in the inner product seen in equation (1).

Proposition II.1. *Suppose we have a square subset of the lattice $\Lambda_M \subset \Lambda_{n^2} \subset a\mathbb{Z} \times b\mathbb{Z}$ where $|\Lambda_M| = M$. Let*

$$\tilde{\lambda}_{ij} = (y_j + sa, \omega_i + tb) = \lambda_{ij} + (sa, tb)$$
$$\tilde{\lambda}_{i'j'} = (y_{j'} + sa, \omega_{i'} + tb) = \lambda_{i'j'} + (sa, tb)$$

for $j, j' \in \{1,2,\dots,n\}$ and $i, i' \in \{1,2,\dots,n\}$ and $s, t \in \mathbb{Z}$. This shift $\Lambda_M + (sa,tb)$ introduces entry-wise phase factors into $G(a,b,n)$. The shifting factor s and the parameter b can be chosen so that $G(a,b,n)$ is unchanged. Furthermore, the Gershgorin discs remain unchanged independent of (a,b) and (s,t).

In the Section III we study the asymptotics of the eigenvalues of $G(a,b,n)$ and the spectrum of $G(a,b)$ that builds on Toeplitz structure seen below in Proposition II.2.

Proposition II.2. *Each of the blocks $G^{ii}(a,b,n) \in \mathbb{C}^{n\times n}$ for $i \in \{1,2,\dots,n\}$ within the finite Gram matrix $G(a,b,n) \in \mathbb{C}^{n^2\times n^2}$ are identical, real-valued, banded, and Toeplitz. Further, $G(a,b,n)$ itself is block-Toeplitz and its magnitude-squared $|G(a,b,n)|^2$ is Toeplitz-block-Toeplitz.*

The Toeplitz structure extends to the infinite blocks $G^{ii}(a,b)$ in the infinite Gram matrix $G(a,b)$. This allows us to associate a finite Fourier series constructed from the bands of $G^{ii}(a,b)$ which provides complete information about the spectrum.

III. Spectral Asymptotics

Due to the Toeplitz structure in the finite Gram matrices $G(a,b,n)$ seen in Section II, we can understand the asymptotic behavior of its eigenvalues and spectrum. This is due to a Fourier series one can construct from the Toeplitz bands and a class of *asymptotically equivalent* circulant matrices whose eigenvalues we explicitly provide. In this section we focus on the spectrum of the finite and infinite blocks $G^{ii}(a,b,n)$ and $G^{ii}(a,b)$ for

$i \in \{1, 2, \ldots\}$ to make conclusions about the spectra of the finite and infinite Gram matrices $G(a, b, n)$ and $G(a, b)$, respectively. For example, since the $G^{ii}(a, b, n)$ are principle sub-matrices of $G(a, b, n)$, the eigenvalues interlace and we can obtain upper and lower estimates on the minimum and maximum eigenvalues $\gamma_1(a, b, n)$ and $\gamma_n(a, b, n)$ of $G(a, b, n)$. Consequently, these provide upper and lower estimates on the frame bounds associated with the finite Gabor frame $\mathcal{G}(g, a, b, n)$,

$$\mathcal{G}(g, a, b, n) = \{M_{lb}T_{ka}g = e^{2\pi i l b \cdot} g(\cdot - ka) : (lb, ka) \in \Lambda_{n^2}\}.$$

Let $\mathbf{T} := \{t \in \mathbb{C} : |t| = 1\}$ be the complex unit circle. The *Wiener algebra* W is the set of functions $f : \mathbf{T} \to \mathbb{C}$ with absolutely convergent Fourier series. Thus, each function $f \in W$ has the representation and norm:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}_n t^n, \;\; ||f||_W = \sum_{n=-\infty}^{\infty} |\hat{f}_n| < \infty.$$

The Fourier coefficients $\{\hat{f}_n\}_{n=-\infty}^{\infty}$ are of course computed as:

$$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta,$$

where the identification $t \sim e^{i\theta}, \;\theta \in [0, 2\pi)$ is made. We can associate $f \in W$ with an infinite Toeplitz matrix built from the coefficients $\{f_n\}_{n=-\infty}^{\infty}$,

$$T(f) = \begin{bmatrix} \hat{f}_0 & \hat{f}_{-1} & \hat{f}_{-2} & \cdots \\ \hat{f}_1 & \hat{f}_0 & \hat{f}_{-1} & \cdots \\ \hat{f}_2 & \hat{f}_1 & \hat{f}_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

It is known that $T(f)$ is a bounded linear operator, $T(f) \in \mathcal{B}\big(l^p(\mathbb{Z}_+)\big)$, with estimate $||T||_p \leq ||f||_W$ [5]. Furthermore, a classic result [4] in analysis of Toeplitz operators is that the spectrum of $T(f)$ is completely and beautifully characterized by the range of the function $f \in W$,

$$\text{sp } T(f) = f(\mathbf{T}) \cup \{\lambda \in \mathbb{C} - f(\mathbf{T}) : \text{wind}(f - \lambda) \neq 0\}.$$

As seen in Theorem II.2, the infinite blocks $G^{ii}(a, b)$ for $i \in \{1, 2, \ldots\}$ within $G(a, b)$ are not just Toeplitz but real-valued and banded with only finitely many non-zero off-diagonal entries. Therefore, the corresponding Fourier series has only finitely many non-zero coefficients. These banded Toeplitz matrices correspond to *Laurent polynomials* $g \in W$, i.e. the functions represented by a finite Fourier series,

$$g(t) = \sum_{j=-r}^{s} g_j t^j, \;\; t \in \mathbf{T}.$$

When $T(g)$ is banded and real-valued, we get a further refinement on its spectrum: it has no eigenvalues and the spectrum is the line segment $[\min g, \max g]$ [4]. This leads to Theorem III.1.

Theorem III.1. *The infinite blocks $G^{ii}(a, b) \in \mathcal{B}\big(l^2(\mathbb{Z}_+)\big)$ for $i \in \mathbb{Z}_+$ on the diagonal of $G(a, b)$ have no eigenvalues and their spectrum is the interval*

$$Spec\big(G^{ii}(a, b)\big) = [\inf_x g^{ii}(x), \sup_x g^{ii}(x)]$$

and $g^{ii} \in W$ is

$$g^{ii}(x) = 2/3 + \sum_{k=1}^{\lfloor 1/a \rfloor} g_k \cos(2\pi k x) + \frac{1}{3}\sum_{k=\lfloor 1/a \rfloor + 1}^{\lfloor 2/a \rfloor} g_k \cos(2\pi k x)$$

where the coefficients g_k are the inner products $G^{ii}_{jj'} = G^{ii}_{|j-j'|}\langle g_{\lambda_{ij}}, g_{\lambda_{ij'}}\rangle_{L^2(\mathbb{R})}$ on the Toeplitz bands of $G^{ii}(a, b)$ indexed by $k = |j - j'|$. These coefficients have the form,

$$g_k = \begin{cases} (ak)^3 - 2(ak)^2 + \frac{4}{3}, & k \leq \lfloor \frac{1}{a} \rfloor \\ (2 - ak)^3, & \lfloor \frac{1}{a} \rfloor < k \leq \lfloor \frac{2}{a} \rfloor \\ 0, & k > \lfloor \frac{2}{a} \rfloor. \end{cases}$$

For example, if we take $a = 1/4$ and $b = 2$, then $g^{ii}(x)$ (Figure 2) has the form:

$$g^{ii}(x) = 2/3 + \sum_{k=1}^{4}(\frac{k^3}{64} - \frac{k^2}{8} + \frac{4}{3})\cos(2\pi k x) + \frac{1}{3}\sum_{k=5}^{8}(2 - \frac{k}{4})^3\cos(2\pi k x).$$

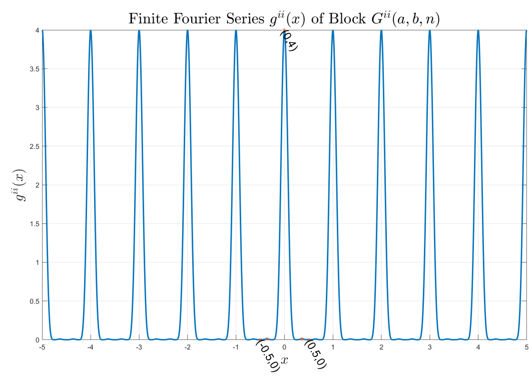


Fig. 2: The Laurent polynomial $g^{ii}(x) \in W$ associated to the Toeplitz block $G^{ii}(a, b) \in \mathcal{B}\big(l^2(\mathbb{Z}_+)\big)$. The spectrum of $G^{ii}(a, b)$ is the interval $[\inf g^{ii}(x), \sup g^{ii}(x)]$, numerically and analytically computed to be $[0, 4]$ in the case where $a = 1/4$.

As all of the coefficients g_k are positive, the $g^{ii}(x)$ attains its maximum when $\cos(2\pi k x) = 1$. Using a standard counting argument we can for certain cases explicitly compute the maximum and minimum of $g^{ii}(x)$.

Corollary III.1.1. *When $\lfloor \frac{1}{a} \rfloor = \frac{1}{a}$, the Laurent polynomial $g^{ii}(x)$ associated to the Toeplitz $G^{ii}(a, b, n)$ has a maximum at $x = k$ for $k \in \mathbb{Z}$,*

$$||G^{ii}(a, b)|| = \max_x g^{ii}(x) = g^{ii}(k) = \lfloor \frac{1}{a} \rfloor.$$

The upper frame bound for $\mathcal{G}(g, a, b)$ is therefore $B(a, b) = \frac{1}{a}$. Furthermore, the $g^{ii}(x)$ has a minimum at $x = k/2$ for $k \in \mathbb{Z}$ odd,

$$\min_x g^{ii}(x) = g^{ii}(\frac{k}{2}) = 0.$$

The spectrum of the $G^{ii}(a, b)$ is therefore the interval

$$Spec\big(G^{ii}(a, b)\big) = [g(\frac{k}{2}), g(k)] = [0, \lfloor \frac{1}{a} \rfloor].$$

Analysis of the Laurent polynomial $g^{ii}(x)$ including its maximum and minimum provides information on the spectrum of the infinite Gram matrix $G(a, b)$ and its finite counterpart $G(a, b, n)$, and hence information on their frame bounds. This is summarized in Corollary III.1.2 and is due to:

1) classic Szego-type limit theorems [13] that state the minimum and maximum eigenvalues, $\gamma_1^{ii}(a, b, n)$ and $\gamma_n^{ii}(a, b, n)$, of the Toeplitz $G^{ii}(a, b, n)$ belong to the interval $[\inf_x g^{ii}(x), \sup_x g^{ii}(x)]$ with

$$\lim_{n\to\infty} \gamma_1^{ii}(a, b, n) = \inf_x g^{ii}(x)$$
$$\lim_{n\to\infty} \gamma_n^{ii}(a, b, n) = \sup_x g^{ii}(x)$$

2) the blocks $G^{ii}(a,b,n)$ are principle submatrices and hence the eigenvalues interlace with $G(a,b,n)$,

$$\gamma_1(a,b,n) \leq \gamma_1^{ii}(a,b,n) \leq \ldots$$
$$\ldots \leq \gamma_n^{ii}(a,b,n) \leq \gamma_n(a,b,n).$$

Corollary III.1.2. *Let $A(a,b,n) = \gamma_1(a,b,n) \in \mathbb{R}$ and $B(a,b,n) = \gamma_n(a,b,n) \in \mathbb{R}$ be the lower and upper frame bounds associated to the finite Gabor frame $\mathcal{G}(g,a,b,n)$. There exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, we have the upper and lower estimates on $A(a,b,n)$ and $B(a,b,n)$:*

$$A(a,b,n) \leq \inf_x g^{ii}(x) \leq \gamma_1^{ii}(a,b,n)$$
$$\gamma_n^{ii}(a,b,n) \leq \sup_x g^{ii}(x) \leq B(a,b,n).$$

This analysis can be further refined with the introduction of a family of circulant matrices that are asymptotically equivalent to the Toeplitz block $G^{ii}(a,b)$ and allow estimates on the eigenvalues $\gamma_1(a,b,n)$ and $\gamma_n(a,b,n)$ and hence on the frame bounds $A(a,b,n), B(a,b,n)$. Two sequences of matrices $\{G_n\}_{n=1}^{\infty}, \{H_n\}_{n=1}^{\infty}$ are said to be *asymptotically equivalent*, $G_n \sim H_n$, if:

1) G_n and H_n are uniformly bounded in the $||\cdot||_2$ norm, $||G_n||_2, ||H_n||_2 \leq M < \infty$
2) $|G_n - H_n| \to 0$, where $|\cdot|$ is the scaled Hilbert-Schmidt norm $|G_n| = (\frac{1}{n}\sum_{i,j=1}^{n}|g_{ij}|^2)^{1/2}$

Asymptotically equivalent sequences of matrices have eigenvalues that behave similarly for large n. Indeed, for a continuous function $f(x)$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} |f(\gamma_m) - f(\psi_m)| = 0$$

and the eigenvalues are said to be *asymptotically absolutely equally distributed*; the average difference in the eigenvalues of asymptotically equivalent matrices tends to zero [13]. In the special case where $\{G_n\}_{n=1}^{\infty}$ is a sequence of banded Toeplitz matrices and $\{H_n\}_{n=1}^{\infty}$ is a sequence of circulant matrices constructed from the Toeplitz $\{H_n\}_{n=1}^{\infty}$, we have that the eigenvalues individually converge [27]. These circulant matrices are constructed by padding the rows of the banded Toeplitz matrix appropriately [13]. The advantage of working with circulant matrices to study the spectrum of a banded Toeplitz matrix is that their eigenvalues $\{\psi_m\}_{m=1}^{n}$ can be computed explicitly as the discrete Fourier transform of a row of the H_n,

$$\psi_m = \sum_{k=0}^{n-1} h_k e^{-2\pi imk/n}, \ m \in \{1,2,\ldots,n\}.$$

Let $\mathrm{Comb}(x) = \sum_{l=-\infty}^{\infty}\delta(x-l)$ be the Dirac comb. We use the following relation between the discrete Fourier transform (DFT) of a sampled function $\mathbf{f} := \big(f\chi\mathrm{Comb}\big)(m)$ for $m \in \{0,1,\ldots,n-1\}$ and samples of the Fourier transform $\mathcal{F}\{f\}$ of f:

$$\begin{aligned}\mathrm{DFT}\{\mathbf{f}\}(m) &= \big(\mathcal{F}\{f\} * \mathrm{sinc} * \mathrm{Comb}\big)(\frac{m}{n}) \\ &= \sum_{l=-\infty}^{\infty}\big(\mathcal{F}\{f\} * \mathrm{sinc}\big)(\frac{m}{n} - l).\end{aligned}$$

Theorem III.2. *Let $H^{ii}(a,b,n) \in \mathbb{C}^{n\times n}$ be the circulant matrix constructed from the Toeplitz block $G^{ii}(a,b,n)$. Then, the eigenvalues $\{\psi_m^{ii}(a,b,n)\}_{m=1}^{n}$ of $H^{ii}(a,b,n)$ have the form*

$$\psi_m^{ii}(a,b,n) = \frac{1}{a}\Big(\sum_{l=-\infty}^{\infty}\mathrm{sinc}^2\big(\frac{1}{a}(\frac{m}{n} - l)\big)\Big)^2$$

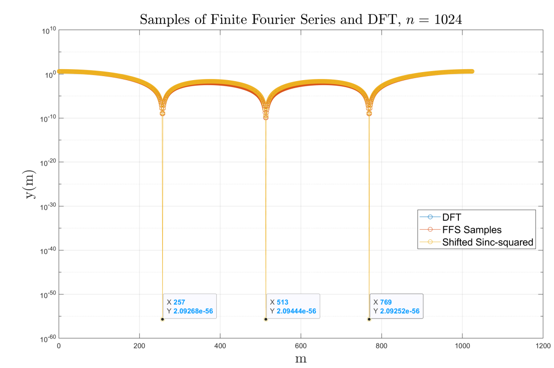
and are the square of the discrete Fourier transform of an a-dilated B-spline and correspond to samples of the Laurent polynomial $g^{ii}(x)$ from Theorem III.1.

Because the eigenvalues of the finite blocks $G^{ii}(a,b,n)$ interlace those of $G(a,b,n)$, it must be the case that if the minimum eigenvalue of $G^{ii}(a,b,n)$ tends to zero, so does that of $G(a,b,n)$. The minimum eigenvalue of $G^{ii}(a,b,n)$ tends to zero when $H^{ii}(a,b,n)$ has an eigenvalue of zero and corresponds to when $g^{ii}(x)$ has a zero, see Figure 3. We see that, for example, if $a = \frac{1}{4}$ and $n = 1024$, then

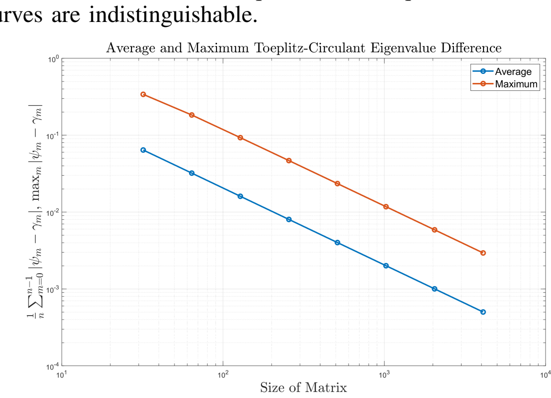
$$\psi_m^{ii}(a,b,n) = \frac{1}{4}\sum_{l=-\infty}^{\infty}\mathrm{sinc}^2(\frac{m}{256} - 4l) \tag{2}$$

has zeros at $m = 256, 512$, and 768. Hence, when $a = \frac{1}{4}$ it must be that the eigenvalue and frame bound $\gamma_1(a,b,n) = A(a,b,n) \to 0$ as $n \to \infty$. It is known that for finite Gabor frames, the lower frame bound must tend to zero [1]. Thus, Corollary III.2.1 can be viewed as generalizing this result to Gabor systems in general and tying them to the eigenvalues of their Gram matrices.

Corollary III.2.1. *If the minimum eigenvalue $\gamma_1^{ii}(a,b,n)$ of the finite block $G^{ii}(a,b,n) \in \mathbb{C}^{n\times n}$ tends to zero as $n \to \infty$, then (1) $a \in \mathbb{Q}$ and (2) the minimum eigenvalue $\gamma_1(a,b,n)$ of the finite Gram matrices $G(a,b,n)$ tend to zero as $n \to \infty$.*



(a) Samples of the Laurent polynomial $g^{ii}(x)$ (red), samples $\frac{1}{\sqrt{a}}\sum_{l=-\infty}^{\infty}\mathrm{sinc}^2(\frac{1}{a}(\frac{m}{n} - l))$ (orange), and the DFT of an a-dilated B-spline $\sqrt{a}g(ax)$ (blue) for $a = 1/4$, $b = 2$. Theorem III.2 states that these three quantities are equal; hence the three curves are indistinguishable.



(b) Average and maximum difference in eigenvalues between the Toeplitz block $G^{ii}(a,b,n)$ and circulant $H^{ii}(a,b,n)$ tends to zero as the size n of the matrices grow. Since $\psi_m^{ii}(a,b,n)$ can be explicitly computed (Equation (2)), we can study the frame bounds of the Gabor system $\mathcal{G}(a,b,n)$.

Fig. 3

Acknowledgment

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References

[1] B. Adcock and D. Huybrechs. Frames and numerical approximation. *Siam Review*, 61(3):443–473, 2019.

[2] A. G. D. Atindehou, C. Frederick, Y. B. Kouagou, and K. A. Okoudjou. On the frame set of the second-order b-spline. *Applied and Computational Harmonic Analysis*, 62:237–250, 2023.

[3] P. Bornemann and F. Cirak. A subdivision-based implementation of the hierarchical b-spline finite element method. *Computer Methods in Applied Mechanics and Engineering*, 253:584–598, 2013.

[4] A. Böttcher and S. M. Grudsky. *Toeplitz matrices, asymptotic linear algebra and functional analysis*, volume 67. Springer, 2000.

[5] A. Böttcher and S. M. Grudsky. *Spectral properties of banded Toeplitz matrices*. SIAM, 2005.

[6] G. Chen and D. Needell. Compressed sensing and dictionary learning. *Finite Frame Theory: A Complete Introduction to Overcompleteness*, 73:201, 2016.

[7] X.-R. Dai and Q. Sun. *The abc-problem for Gabor systems*, volume 244. American Mathematical Society, 2016.

[8] X.-R. Dai and M. Zhu. Frame set for gabor systems with haar window. *Applied and Computational Harmonic Analysis*, 71:101655, 2024.

[9] I. Daubechies and A. Grossmann. Frames in the bargmann space of entire functions. *Communications on Pure and Applied Mathematics*, 41(2):151–164, 1988.

[10] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic fourier series. *Transactions of the American Mathematical Society*, 72(2):341–366, 1952.

[11] H. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of gabor frames. *Transactions of the American Mathematical Society*, 356(5):2001–2023, 2004.

[12] V. K. Goyal, J. Kovačević, and J. A. Kelner. Quantized frame expansions with erasures. *Applied and Computational Harmonic Analysis*, 10(3):203–233, 2001.

[13] R. M. Gray. Toeplitz and circulant matrices: A review. *Foundations and Trends® in Communications and Information Theory*, 2(3):155–239, 2006.

[14] K. Gröchenig. The mystery of gabor frames. *Journal of Fourier Analysis and Applications*, 20(4):865–895, 2014.

[15] K. Gröchenig, J. L. Romero, and J. Stöckler. Sampling theorems for shift-invariant spaces, gabor frames, and totally positive functions. *Inventiones mathematicae*, 211:1119–1148, 2018.

[16] Z. Hajiabotorabi, A. Kazemi, F. F. Samavati, and F. M. M. Ghaini. Improving dwt-rnn model via b-spline wavelet multiresolution to forecast a high-frequency time series. *Expert Systems with Applications*, 138:112842, 2019.

[17] M. L. Harrison. *Frames and irregular sampling from a computational perspective*. University of Maryland, College Park, 1998.

[18] C. Heil. History and evolution of the density theorem for gabor frames. *Journal of Fourier Analysis and Applications*, 13:113–166, 2007.

[19] C. Heil, J. Ramanathan, and P. Topiwala. Linear independence of time-frequency translates. *Proceedings of the American Mathematical Society*, 124(9):2787–2795, 1996.

[20] C.-C. Hsieh. B-spline wavelet-based motion smoothing. *Computers & industrial engineering*, 41(1):59–76, 2001.

[21] A. J. Janssen. Zak transforms with few zeros and the tie. *Advances in Gabor analysis*, pages 31–70, 2003.

[22] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part i). *IEEE Signal Processing Magazine*, 24(4):86–104, 2007.

[23] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part ii). *IEEE Signal Processing Magazine*, 24(5):115–125, 2007.

[24] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45(6):2171–2180, 2004.

[25] K. Seip. Density theorems for sampling and interpolation in the bargmann-fock space i. 1992.

[26] J. Yang, G. Zhu, D. Tong, L. Lu, and Z. Shen. B-spline tight frame based force matching method. *Journal of Computational Physics*, 362:208–219, 2018.

[27] Z. Zhu and M. B. Wakin. On the asymptotic equivalence of circulant and toeplitz matrices. *IEEE Transactions on Information Theory*, 63(5):2975–2992, 2017.