SIAM J. MATRIX ANAL. APPL.
Vol. 41, No. 1, pp. 199–220

CONDITIONING OF PARTIAL NONUNIFORM FOURIER MATRICES WITH CLUSTERED NODES*

DMITRY BATENKOV†, LAURENT DEMANET‡, GIL GOLDMAN§, AND YOSEF YOMDIN§

Abstract. We prove sharp lower bounds for the smallest singular value of a partial Fourier matrix with arbitrary "off the grid" nodes (equivalently, a rectangular Vandermonde matrix with the nodes on the unit circle) in the case when some of the nodes are separated by less than the inverse bandwidth. The bound is polynomial in the reciprocal of the so-called superresolution factor, while the exponent is controlled by the maximal number of nodes which are clustered together. As a corollary, we obtain sharp minimax bounds for the problem of sparse superresolution on a grid under the partial clustering assumptions.

Key words. Vandermonde matrix with nodes on the unit circle, partial Fourier matrix, singular values, prolate matrix, superresolution, decimation

AMS subject classifications. 15A18, 42A82, 65F22, 94A12

DOI. 10.1137/18M1212197

1. Introduction.

Vandermonde matrices and their spectral properties are of considerable interest in several fields, such as polynomial interpolation, approximation theory, numerical analysis, applied harmonic analysis, line spectrum estimation, exponential data fitting, and others (e.g., [3, 5, 10, 38, 39, 41] and references therein). Motivated by questions related to the so-called problem of superresolution (more on this in subsection 3.2 below), in this paper we study the conditioning of rectangular Vandermonde matrices $\mathbf{V}$ with irregularly spaced nodes on the unit circle, where the number of nodes s is considered to be relatively small and fixed, while the polynomial degree $N \geq s$ can be large. This question has received much attention in the literature; see, e.g., [3, 10, 31, 32, 20, 28, 7, 16]. Normalizing the matrix by $\frac{1}{\sqrt{N}}$, the magnitude of the largest singular value is $O(\sqrt{s})$, and so studying the scaling of the condition number is equivalent to estimating the smallest singular value. As long as the nodes are separated by at least $\frac{1}{N}$, the matrix $\mathbf{V}$ is known to be well-conditioned. However, as the nodes collide, the columns of $\mathbf{V}$ become increasingly correlated, and therefore the smallest singular value becomes very small, while the condition number blows up.

In this paper we show (see section 3.1) that if the nodes are separated by $\Delta \ll \frac{1}{N}$, then under certain technical conditions the smallest singular value of $\mathbf{V}$ scales with the asymptotically tight rate $\asymp (N\Delta)^{\ell-1}$, where $\ell \leq s$ is the maximal number of nodes

*Received by the editors September 18, 2018; accepted for publication (in revised form) by K. D. Usevich November 14, 2019; published electronically January 30, 2020.
https://doi.org/10.1137/18M1212197

Funding: The work of the first and second authors was partially supported by AFOSR grant FA9550-17-1-0316, National Science Foundation grant DMS-1255203, and a grant from the MIT-Skolkovo initiative. The work of the third and fourth authors was partially supported by the Minerva Foundation.

†Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, and School of Mathematical Sciences, Tel Aviv University, P.O. Box 39040, Tel Aviv 6997801, Israel (dbatenkov@tauex.tau.ac.il, https://dimabatenkov.info).

‡Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139 (ldemanet@mit.edu, https://math.mit.edu/icg).

§Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel (gil.goldman@weizmann.ac.il, yosef.yomdin@weizmann.ac.il).

which form a small "cluster" (i.e., a group of at most ℓ nodes which are separated below $\sim \frac{1}{N}$; see Definition 3.1). This improves on previous known results [16, 28] which established this scaling for the extreme case $\ell = s$ and a recent preprint [25] which deals with the special case $\ell = 2$. During the review of the present paper, the authors of [28] improved their analysis to the general case $\ell \leq s$, and we compare their results (and the follow-up work [26]) to ours in Remark 3.7 below. Kunis and Nagel [26] also provide an initial study of the multivariate case. Finally, in a very recent preprint [8] the authors derived the asymptotic scaling of *all the singular values* of $\mathbf{V}$ by completely different techniques; however, the proportionality constants in their work are not explicit.

In this work, the above-mentioned bounds on $\sigma_{\min}(\mathbf{V})$ follow from the solution of the "continuous" version of the problem, where the row index becomes a continuous "frequency" variable $\omega \in [-\Omega, \Omega]$, so that the bandwidth Ω effectively plays the role of N. In the continuous setting, we establish tight bounds for the smallest eigenvalue of the corresponding Gramian matrix $\mathbf{G}$ with irregularly spaced nodes, which generalizes well-known results due to Slepian [42] for the prolate matrix (which, in turn, plays a prominent role in the seminal study of the spectral concentration problem [43]). In fact, this continuous version is what originally appeared in the studies of the superresolution of sparse atomic measures in [17] and later in [16], and we use our results to derive minimax bounds for this problem in subsection 3.2.

The paper is organized as follows. In section 2 we provide the definitions and review known bounds for singular values of rectangular Vandermonde matrices. In section 3 we state the definition for clustered configurations and formulate the main results regarding the smallest eigenvalue of the Gramian matrix $\mathbf{G}$, the smallest singular value of the corresponding Vandermonde matrix $\mathbf{V}$, and the novel minimax bound for the problem of superresolution of point sources on the grid. In section 4 we prove the main results, and in section 5 we present numerical evidence confirming our bounds.

2. Preliminaries.

2.1. Notation.

DEFINITION 2.1. *For $N \in \mathbb{N}$ and vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_s)$ of pairwise distinct real nodes $\xi_j \in (-\pi, \pi]$, we define the rectangular $(2N+1) \times s$ Vandermonde matrix $\mathbf{V}_N(\boldsymbol{\xi})$ as*

$$\mathbf{V}_N(\boldsymbol{\xi}) := \frac{1}{\sqrt{2N}} \left[\exp(\imath k \xi_j)\right]_{k=-N,\ldots,N}^{j=1,\ldots,s}. \tag{2.1}$$

In many applications of interest, the columns of $\mathbf{V}_N$ as above arise from sampling the exponential functions $\{\exp(\imath \omega t_j)\}_{j=1}^{s}$ at equispaced points $\omega_k = \frac{k}{N}\Omega$, $|k| \leq N$, where $\Omega > 0$ is a quantity which is frequently called the bandlimit or bandwidth and the nodes $\{t_j := \frac{N\xi_j}{\Omega}\}$ represent some relevant physical parameters, such as angles of arrival, locations of point sources, etc. Therefore, in these cases it is more natural to regard $\{t_j\}$ and Ω as the primary variables instead of $\{\xi_j\}$ and N while in fact thinking about the scenario where N can be very large. According with this philosophy, we shall be primarily interested in the continuous limit $N \to \infty$.

DEFINITION 2.2. *For $N \in \mathbb{N}$, $s \in \mathbb{N}$, $\boldsymbol{x} := (t_1, \ldots, t_s)$ with $t_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ a vector of s distinct nodes, and bandwidth parameter $\Omega > 0$, denote by $\mathbf{V}_N(\boldsymbol{x}, \Omega)$ the rectangular $(2N+1) \times s$ Vandermonde matrix with complex nodes $z_{j,N} = \exp(\imath \xi_{j,N})$,*

where $\xi_{j,N} = \frac{t_j \Omega}{N}$, *i.e.,*

$$\mathbf{V}_N(\boldsymbol{x}, \Omega) := \mathbf{V}_N\left(\frac{\Omega}{N}\boldsymbol{x}\right) = \frac{1}{\sqrt{2N}}\left[\exp\left(\imath k \frac{t_j \Omega}{N}\right)\right]_{k=-N,\ldots,N}^{j=1,\ldots,s}. \tag{2.2}$$

With the above definition, the Gramian matrix $\mathbf{V}_N(\boldsymbol{x}, \Omega)^H \mathbf{V}_N(\boldsymbol{x}, \Omega)$ becomes in the limit $N \to \infty$ the kernel matrix with respect to the well-known sinc kernel.

DEFINITION 2.3. *For* $N \in \mathbb{N}$, *the Dirichlet (periodic sinc) kernel of order* N *is*

$$\mathcal{D}_N(t) := \sum_{k=-N}^{N} \exp(\imath k t) = \begin{cases} \frac{\sin\left((N+\frac{1}{2})t\right)}{\sin\frac{t}{2}}, & t \notin 2\pi\mathbb{Z}, \\ 2N+1, & \textit{else.} \end{cases}$$

DEFINITION 2.4. *For* $N \in \mathbb{N}$ *and* $\boldsymbol{x}, \Omega$ *as in Definition* 2.2, *let* $\mathbf{G}_N$ *be the* $s \times s$ *matrix*

$$\mathbf{G}_N(\boldsymbol{x}, \Omega) := \mathbf{V}_N(\boldsymbol{x}, \Omega)^H \mathbf{V}_N(\boldsymbol{x}, \Omega) = \frac{1}{2N}\left[\mathcal{D}_N\left(\frac{\Omega(t_i - t_j)}{N}\right)\right]_{i,j}.$$

DEFINITION 2.5. *Let the* sinc *function be defined by*

$$\operatorname{sinc}(t) := \frac{1}{2}\int_{-1}^{1} \exp(\imath \omega t) d\omega = \begin{cases} \frac{\sin t}{t}, & t \neq 0, \\ 1, & \textit{else.} \end{cases}$$

DEFINITION 2.6. *For* $s \in \mathbb{N}$, $\boldsymbol{x}$ *a vector of* s *distinct nodes* $\boldsymbol{x} := (t_1, \ldots, t_s)$ *with* $t_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *and bandwidth parameter* $\Omega > 0$, *let* $\mathbf{G}(\boldsymbol{x}, \Omega)$ *denote the* $s \times s$ *matrix*

$$\mathbf{G}(\boldsymbol{x}, \Omega) := \left[\operatorname{sinc}(\Omega(t_i - t_j))\right]_{1 \le i,j \le s}. \tag{2.3}$$

PROPOSITION 2.7. *For* $\boldsymbol{x}$ *a vector of pairwise distinct nodes, the matrix* $\mathbf{G}(\boldsymbol{x}, \Omega)$ *is positive definite.*

Proof. The matrix $\mathbf{G}$ is the Gramian matrix of the functions $\{\exp(\imath t_j \omega)\}_{j=1,\ldots,s}$ with the inner product $\langle f, g\rangle_\Omega := \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} f(\omega)\overline{g(\omega)}d\omega$. For any $\boldsymbol{x}$ as above and non-zero $\boldsymbol{c} = (c_1, \ldots, c_s) \in \mathbb{C}^s$, define $f_{\boldsymbol{x},\boldsymbol{c}}(\omega) := \sum_{j=1}^{s} c_j \exp(\imath t_j \omega) \not\equiv 0$; then we have $\|\mathbf{G}(\boldsymbol{x}, \Omega)\boldsymbol{c}\|_2^2 = \langle f_{\boldsymbol{x},\boldsymbol{c}}, f_{\boldsymbol{x},\boldsymbol{c}}\rangle_\Omega > 0$. ∎

For any matrix $\mathbf{G} \in \mathbb{C}^{s\times s}$ and a matrix $\mathbf{V} \in \mathbb{C}^{N\times s}$ with $N \ge s$, we denote as usual

$$\lambda_{\min}(\mathbf{G}) := \text{the minimal eigenvalue of } \mathbf{G},$$
$$\sigma_{\min}(\mathbf{V}) := \sqrt{\lambda_{\min}(\mathbf{V}^H\mathbf{V})}.$$

PROPOSITION 2.8. *With the above definitions, we have*

$$\lambda_{\min}(\mathbf{G}(\boldsymbol{x}, \Omega)) = \lim_{N\to\infty} \lambda_{\min}(\mathbf{G}_N(\boldsymbol{x}, \Omega)) = \lim_{N\to\infty} \sigma_{\min}^2(\mathbf{V}_N(\boldsymbol{x}, \Omega)). \tag{2.4}$$

Proof. Approximating the integrals by the Riemann sums, we have that

$$\operatorname{sinc}(\Omega t) = \lim_{N\to\infty} \frac{1}{2N}\sum_{k=-N}^{N} \exp\left(\imath \frac{k}{N}\Omega t\right) = \lim_{N\to\infty} \frac{1}{2N}\mathcal{D}_N\left(\frac{\Omega t}{N}\right),$$

and therefore $\mathbf{G}(\boldsymbol{x}, \Omega) = \lim_{N\to\infty} \mathbf{G}_N(\boldsymbol{x}, \Omega)$. By definition, $\mathbf{V}_N^H\mathbf{V}_N = \mathbf{G}_N$, and so, by continuity of eigenvalues [23, section 2.4.9], we conclude that (2.4) holds. ∎

The main subject of the paper is the scaling of the smallest eigenvalue of $\mathbf{G}$ and the smallest singular value of $\mathbf{V}_N$ when some of the nodes of $\boldsymbol{x}$ nearly collide (become very close to each other).

DEFINITION 2.9 (wraparound distance). *For $t \in \mathbb{R}$, we denote*

$$\|t\|_{\mathbb{T}} := \left|\operatorname{Arg}\exp(\imath t)\right| = \left|t \mod (-\pi, \pi]\right|,$$

where $\operatorname{Arg}(z)$ *is the principal value of the argument of* $z \in \mathbb{C}\backslash\{0\}$*, taking values in* $(-\pi, \pi]$.

DEFINITION 2.10 (minimal separation). *Given a vector of s distinct nodes $\boldsymbol{x} := (t_1, \dots, t_s)$ with $t_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we define the minimal separation (in the wraparound sense) as*

$$\Delta = \Delta(\boldsymbol{x}) := \min_{i \neq j} \|t_i - t_j\|_{\mathbb{T}}.$$

2.2. Known bounds. Let $\mathbf{V}_N$ be as defined in (2.2), i.e., a rectangular Vandermonde matrix with nodes $z_{j,N} = \exp(\imath \xi_{j,N})$ on the unit circle with $\xi_{j,N} = t_j \frac{\Omega}{N}$, $j = 1, \dots, s$. Denote

$$\Delta_N := \min_{i \neq j} \|\xi_{i,N} - \xi_{j,N}\|_{\mathbb{T}}.$$

Several more or less equivalent bounds on $\sigma_{\min}(\mathbf{V}_N)$ are available in the "well-separated" case $N\Delta_N > \text{const}$, using various results from analysis and number theory, such as Ingham and Hilbert inequalities, large sieve inequalities, and Selberg's majorants [24, 32, 36, 3, 33, 34, 20, 10]. The tightest bound was obtained in [3, equation (32)] (slightly improving on Moitra's bound from [32]), where it was shown that (in our notations we substitute $N \to 2N + 1$) if $2N + 1 > \frac{2\pi}{\Delta_N}$, then

$$\sigma_{\min}\left(\sqrt{2N}\mathbf{V}_N\right) \geq \sqrt{2N + 1 - \frac{2\pi}{\Delta_N}},$$

$$\sigma_{\min}(\mathbf{V}_N) \geq \sqrt{1 + \frac{1}{2N} - \frac{\pi}{N\Delta_N}}.$$

In our setting, we take N sufficiently large, and so, in particular, $N > \Omega$, implying that $\Delta_N = \Delta\frac{\Omega}{N}$. Assuming further that $\Delta\Omega \geq \pi$, we conclude that

$$\sigma_{\min}(\mathbf{V}_N) \searrow \sqrt{1 - \frac{\pi}{\Omega\Delta}}, \quad N \to \infty.$$

The case $\Delta\Omega \ll 1$, or, equivalently, $\Delta_N \ll \frac{1}{N}$, turns out to be much more difficult to analyze. All known results provide sharp bounds only in the particular case when all the nodes are clustered together or approximately equispaced.

If all the nodes t_j are equispaced, say, $t_j = t_0 + j\Delta$, $j = 1, \dots, s$, then the matrix $\mathbf{G}$ is the so-called *prolate matrix*, whose spectral properties are known exactly [45, 42]. Indeed, we have in this case

$$\mathbf{G}_{i,j} = \frac{\sin(\Omega(t_i - t_j))}{\Omega(t_i - t_j)} = \frac{\sin(\Omega\Delta(i-j))}{\Omega\Delta(i-j)} = \frac{\pi}{\Omega\Delta} \cdot \frac{\sin(2\pi W(i-j))}{\pi(i-j)}, \quad W := \frac{\Omega\Delta}{2\pi},$$

and therefore $\mathbf{G} = \frac{\pi}{\Omega\Delta}\boldsymbol{Q}(s, W)$ where $\boldsymbol{Q}(s, W)$ is the matrix defined in [42, equation (21)]. The smallest eigenvalue of $\boldsymbol{Q}(s, W)$, denoted by $\lambda_{s-1}(s, W)$ in the same paper, has the exact asymptotics for W small, given in [42, equations (64 and 65)]:

$$\lambda_{s-1}(s, W) = \frac{1}{\pi}(2\pi W)^{2s-1} C_1(s)(1 + O(W)), \quad C_1(s) := \frac{2^{2s-2}}{(2s-1)\binom{2s-2}{s-1}^3}, \tag{2.5}$$

which gives

$$\lambda_{\min}(\mathbf{G}) = C_1(s)(\Omega\Delta)^{2s-2}(1+O(\Omega\Delta)), \quad \Omega\Delta \ll 1.$$

The same scaling was shown using Szego's theory of Toeplitz forms in [16]; see also subsection 3.2. The authors showed that there exist $C > 0$ and $y^* > 0$ such that for $\Omega\Delta < y^*$,

$$\frac{C}{16}\left(\sin\frac{2\Omega\Delta}{\pi}\right)^{2s-2} \le \lambda_{\min}(\mathbf{G}) \le 16\left(\sin\frac{2\Omega\Delta}{\pi}\right)^{2s-2}.$$

To conclude the above discussion, defining the superresolution factor as

$$\mathrm{SRF} := \frac{\pi}{\Delta\Omega},$$

we have that

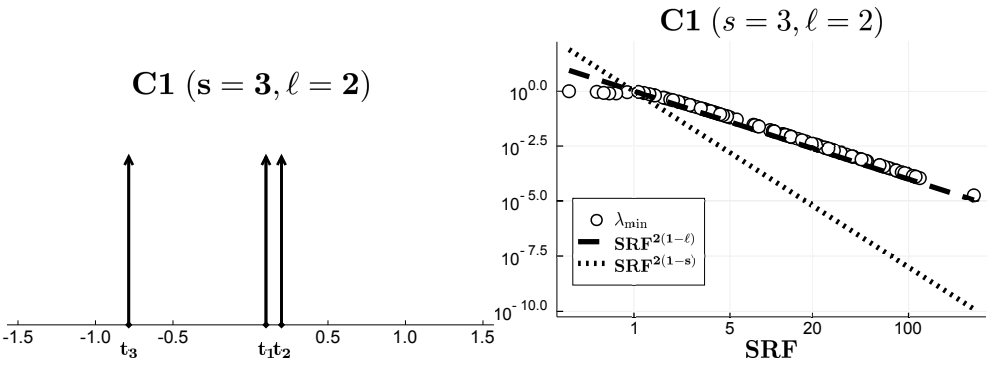
$$\lambda_{\min}(\mathbf{G}) \approx (1-\mathrm{SRF}), \quad \mathrm{SRF} \le 1, \tag{2.6}$$

$$\lambda_{\min}(\mathbf{G}) \approx \mathrm{SRF}^{-2(s-1)}, \quad \mathrm{SRF} \gg 1. \tag{2.7}$$

3. Main results.

3.1. Optimal bounds for the smallest eigenvalue. It turns out that the bound (2.7) is too pessimistic if only some of the nodes are known to be clustered. Consider, for instance, the configuration $\boldsymbol{x} = (t_1 = \Delta,\ t_2 = 2\Delta,\ t_3 = -\frac{\pi}{4})$; then, as can be seen in Figure 1, we have in fact $\lambda_{\min}(\mathbf{G}(\boldsymbol{x},\Omega)) \approx (\Delta\Omega)^2$, decaying much slower than $(\Delta\Omega)^4$, which would be the bound given by (2.7).

In this paper we bridge this theoretical gap. We consider the *partially clustered regime* where at most $2 \le \ell \le s$ neighboring nodes can form a cluster (there can be several such clusters), with two additional parameters, ρ, τ, controlling the distance between the clusters and the uniformity of the distribution of nodes within the clusters.



(a) Schematic representation of $\boldsymbol{x}$. (b) The decay of $\lambda_{\min}$.

FIG. 1. *For different values of* Δ, Ω, *we plot the quantity* $\lambda_{\min}(\mathbf{G}(\boldsymbol{x},\Omega))$ *versus the superresolution factor* $\mathrm{SRF} = \frac{\pi}{\Delta\Omega}$. (a) $\boldsymbol{x} = \left(t_1 = \Delta,\ t_2 = 2\Delta,\ t_3 = -\frac{\pi}{4}\right)$ *is a single cluster with* $s = 3$ *and* $\ell = 2$. (b) *The correct scaling is seen to be* $\lambda_{\min} \sim (\Delta\Omega)^{2(\ell-1)}$ *rather than* $\lambda_{\min} \sim (\Delta\Omega)^{2(s-1)}$. *See section* 5 *for further details regarding the experimental setup. The relationship breaks when* $\mathrm{SRF} \le O(1)$, *consistent with* (2.6).

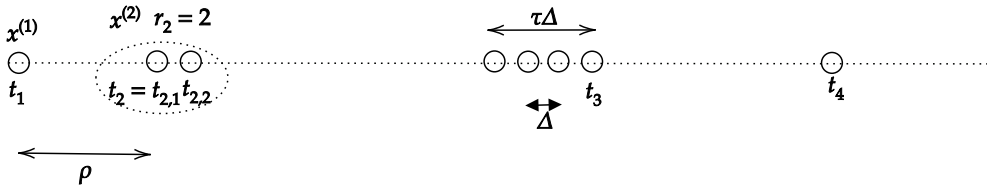


FIG. 2. *The schematic representation of a cluster configuration according to Definition* 3.1. *Here* $s = 8$ *and* $\ell = 4$. *Each node* t_j *defines its "cluster"* $\boldsymbol{x}^{(j)}$ *of size* $r_j \le \ell$. ρ *is the minimal distance from any node* t_j *to another node* y *not in* $\boldsymbol{x}^{(j)}$. *The distance between any two nodes in* $\boldsymbol{x}^{(j)}$ *is between* Δ *and* $\tau\Delta$.

DEFINITION 3.1. *The node vector* $\boldsymbol{x} = (t_1, \dots, t_s) \subset (-\frac{\pi}{2}, \frac{\pi}{2}]$ *is said to form a* $(\Delta, \rho, s, \ell, \tau)$*-clustered configuration for some* $\Delta > 0$, $2 \le \ell \le s$, $\ell - 1 \le \tau < \frac{\pi}{\Delta}$, *and* $\rho \ge 0$ *if for each* t_j *there exist at most* ℓ *distinct nodes*

$$\boldsymbol{x}^{(j)} = \{t_{j,k}\}_{k=1,\dots,r_j} \subset \boldsymbol{x},\ 1 \le r_j \le \ell, \quad t_{j,1} \equiv t_j,$$

such that the following conditions are satisfied:

1. *For any* $y \in \boldsymbol{x}^{(j)} \setminus \{t_j\}$, *we have*

$$\Delta \le \|y - t_j\|_{\mathbb{T}} \le \tau\Delta.$$

2. *For any* $y \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}$, *we have*

$$\|y - t_j\|_{\mathbb{T}} \ge \rho.$$

The different parameters are illustrated in Figure 2.

Our main result is the following generalization of (2.7) for clustered configurations.

THEOREM 3.2. *There exists a constant* $C_2 = C_2(s)$ *such that for any* $4\tau\Delta \le \rho$, *any* $\boldsymbol{x}$ *forming a* $(\Delta, \rho, s, \ell, \tau)$*-clustered configuration, and any* Ω *satisfying*

$$\frac{4\pi s}{\rho} \le \Omega \le \frac{\pi s}{\tau\Delta}, \tag{3.1}$$

we have

$$\sigma_{\min}\left(\mathbf{V}_N(\boldsymbol{x}, \Omega)\right) \ge C_2 \cdot (\Delta\Omega)^{\ell-1} \qquad \textit{whenever } N > 2s^3 \left\lceil \frac{\Omega}{4s} \right\rceil, \tag{3.2}$$

$$\lambda_{\min}\left(\mathbf{G}(\boldsymbol{x}, \Omega)\right) \ge C_2^2 \cdot (\Delta\Omega)^{2(\ell-1)}. \tag{3.3}$$

The proof of Theorem 3.2 is presented in subsection 4.3 below. It is based on the "decimation-and-blow-up" technique, previously used in the context of superresolution in [1, 2, 6, 7, 9] and references therein. In a nutshell, the main idea is to choose an appropriate "decimation" parameter $\lambda \approx \Omega$ such that the "inflated" nodes in the vector $\lambda\boldsymbol{x}$ (considered in the wraparound sense) are separated by $\lambda\Delta \approx \Omega\Delta$ from its cluster neighbors and by a constant from the other nodes. Then we fix sufficiently large N and divide the $2N+1$ rows of $\mathbf{V}_N$ into groups of s rows, separated by $\frac{\lambda N}{\Omega}$. Each of the resulting *square* Vandermonde matrices can be explicitly estimated (the inverses have well-known behavior) and has smallest singular value of the order $\frac{1}{\sqrt{N}}(\Delta\Omega)^{\ell-1}$. The main technical part is to show that such λ exists, and it is proved in Lemma 4.1 by a union bound argument, showing that the measure of all "bad" values of λ (causing a collision of at least two nodes) is small. The condition on N in (3.2) is obtained by accurate counting of how many such "bad" intervals exist.

Remark 3.3. The condition $4\tau\Delta \le \rho$ ensures that the range of admissible Ω is nonempty, and it will clearly be satisfied for all small enough Δ with all the rest of the parameters fixed.

Remark 3.4. The same node vector $\boldsymbol{x}$ can be regarded as a clustered configuration with different choices of the parameters (ℓ, ρ, τ). For example, the vector $\boldsymbol{x}$ from the beginning of this section (and also Figure 1) is both $\left(\Delta, \frac{\pi}{4} + \Delta, 3, 2, 1\right)$-clustered and $\left(\Delta, \rho, 3, 3, \frac{\pi}{4\Delta} + 2\right)$-clustered with any ρ. To obtain as tight a bound as possible, one should choose the minimal ℓ such that the condition (3.1) is satisfied for Ω within the range of interest. For instance, Ω might be too small if ρ is small enough; however, by choosing $\ell = s$ one is able to increase ρ without bound. See Figure 6 for a numerical example.

Remark 3.5. The constant C_2 is given explicitly in (4.16), and it decays in s like $\sim s^{-2s}$. It is plausible that the best possible bound would scale like $c^{-\ell}$ for some absolute constant $c > 1$; see also Remark 3.7 below.

Our next result is the analogue of (3.2) for the Vandermonde matrix $\mathbf{V}_N$ as in (2.1) albeit under an extra assumption that the nodes are restricted to the interval $\frac{1}{s^2}(-\frac{\pi}{2}, \frac{\pi}{2}]$.

COROLLARY 3.6. *There exists a constant $C_3 = C_3(s)$ such that for any $4\tau\Delta \le \min\left(\rho, \frac{1}{s^2}\right)$, any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_s) \subset \frac{1}{s^2}(-\frac{\pi}{2}, \frac{\pi}{2}]$ forming a $(\Delta, \rho, s, \ell, \tau)$-clustered configuration, and any N satisfying*

$$\max\left(\frac{4\pi s}{\rho}, 4s^3\right) \le N \le \frac{\pi s}{\tau\Delta}, \tag{3.4}$$

we have

$$\sigma_{\min}\left(\mathbf{V}_N(\boldsymbol{\xi})\right) \ge C_3 \cdot (N\Delta)^{\ell-1}. \tag{3.5}$$

Proof. Let us choose $\widetilde{\Omega} := \frac{N}{s^2}$ so that for all $j = 1, \dots, s$, we have

$$\widetilde{t}_j := \frac{N\xi_j}{\widetilde{\Omega}} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Further define $\widetilde{\Delta} := s^2\Delta$, and $\widetilde{\rho} := s^2\rho$. We immediately obtain that the vector $\widetilde{\boldsymbol{x}} := (\widetilde{t}_1, \dots, \widetilde{t}_s)$ forms a $(\widetilde{\Delta}, \widetilde{\rho}, s, \ell, \tau)$-clustered configuration according to Definition 3.1, and the rectangular Vandermonde matrix $\mathbf{V}_N(\boldsymbol{\xi})$ in (2.1) is precisely $\mathbf{V}_N(\widetilde{\boldsymbol{x}}, \widetilde{\Omega})$. Clearly, $4\tau\widetilde{\Delta} \le s^2\rho = \widetilde{\rho}$, and also

$$\widetilde{\Omega}s^2 = N \ge 4s^3 \Longrightarrow \frac{\widetilde{\Omega}}{4s} \ge 1 \Longrightarrow \frac{2\widetilde{\Omega}}{4s} > \left\lceil \frac{\widetilde{\Omega}}{4s} \right\rceil \Longrightarrow N = \widetilde{\Omega}s^2 > 2s^3 \left\lceil \frac{\widetilde{\Omega}}{4s} \right\rceil. \tag{3.6}$$

Using (3.4), we obtain precisely the conditions (3.1) with $\widetilde{\Omega}, \widetilde{\rho}$ in place of Ω, ρ, respectively. Therefore, the conditions of Theorem 3.2 are satisfied for $\widetilde{\boldsymbol{x}}, \widetilde{\Omega}, \widetilde{\rho}, \widetilde{\Delta}, \tau$, and so (3.5) follows immediately from (3.6) and (3.2) with $C_3 = C_2$. ∎

Remark 3.7. During the revision of the present paper, the authors of [28] (second version) investigated the question of bounding $\sigma_{\min}(\mathbf{V}_N)$ under assumptions on node distribution which are similar to our clustering model (they are called "sparse clumps" in [28]). They also obtain the scaling $(N\Delta)^{\ell-1}$ for the smallest singular value. Comparing their results to Corollary 3.6 (see also Remark 4 in their paper), we note the following:

1. They do not have the requirement that the vector $\boldsymbol{\xi}$ should be restricted to a small interval.
2. Their bounds hold whenever $N \geq s^2$, while we require $N \geq 4s^3$.
3. Although their model is more general, their constants are more complicated. Nevertheless, the corresponding constant C_3 scales as $\ell^{-\ell}$, which is better than our s^{-2s}. Slightly improving on their proof technique, the authors of [26] have reduced this scaling further to $C^{-\ell}$; see also discussion in section 1.3 of [8].
4. Their equation (2.5) in Theorem 2 requires the product ρN to be at least $\ell^{5/2}\frac{20s}{\sqrt{N\Delta}}$, which essentially forces a single cluster if Δ is very small (or, alternatively, prevents Δ to be too small for certain s, ℓ). In contrast, our equation (3.4) only requires $\rho N \geq 4\pi s$ and therefore does not have these restrictions (although both conditions require ρ to grow with s).

Remark 3.8. Continuing the above discussion, we would like to emphasize that Corollary 3.6 is derived by discretization of the continuous setting of Theorem 3.2, and therefore it is perhaps not surprising that the conditions for which the scaling holds are not optimal.

Returning to Theorem 3.2, it turns out that the bound (3.3) is asymptotically optimal.

THEOREM 3.9. *There exists an absolute constant $\eta \ll 1$ and a constant $C_4 = C_4(\ell)$ such that for any $2 \leq \ell \leq s$ and any Δ satisfying $\Delta < \frac{\pi}{2(\ell-1)}$, there exists a $(\Delta, \rho', s, \ell, \tau')$-clustered configuration $\boldsymbol{x}_{\min}$ with s nodes and certain ρ', τ' depending only on s, ℓ, for which*

$$\lambda_{\min}\left(\mathbf{G}\left(\boldsymbol{x}_{\min}, \Omega\right)\right) \leq C_4 \cdot (\Delta\Omega)^{2(\ell-1)}, \qquad \Delta\Omega < \eta.$$

The proof of Theorem 3.9 is presented in subsection 4.4.

We conclude by deriving the optimal scaling for the *condition number* of $\mathbf{V}_N = \mathbf{V}_N(\boldsymbol{x}, \Omega)$, which is of interest to some applications.

COROLLARY 3.10. *Fix s, ℓ, ρ, τ, and Ω. For any $(\Delta, \rho, s, \ell, \tau)$-clustered configuration $\boldsymbol{x}$, we have*

$$\kappa\left(\mathbf{V}_N\left(\boldsymbol{x}, \Omega\right)\right) := \frac{\sigma_{\max}(\mathbf{V}_N)}{\sigma_{\min}(\mathbf{V}_N)} \asymp \mathrm{SRF}^{\ell-1} \quad \text{as } \Delta \to 0,\ N \to \infty.$$

Proof. It is immediate that as $N \to \infty$, the largest singular value (the spectral norm) of $\mathbf{V}_N$ is bounded from above by a constant,

$$\sigma_{\max}(\mathbf{V}_N) = \|\mathbf{V}_N\|_2 \leq \sqrt{s\frac{2N+1}{2N}} \leq \sqrt{2s},$$

while the lower bound can be obtained by

$$\sigma_{max}(\mathbf{V}_N) = \sqrt{\lambda_{\max}(\mathbf{G}_N)} \geq \sqrt{\frac{1}{2N}\max_{t\in\mathbb{R}} \mathcal{D}_N(t)} > 1.$$

Combining this with Theorems 3.2 and 3.9 finishes the proof. ∎

3.2. Stable superresolution of point sources. The problem of (sparse) superresolution is to recover discrete, pointlike objects (or "spikes") from their noisy

and bandlimited spectral measurements. It arises in many fields, such as frequency estimation, sampling theory, array processing, astronomical imaging, seismic imaging, nonuniform FFT, statistics, radar signal detection, error correction codes, and others [4, 13, 14, 17, 11, 21, 30, 27, 37]. Our main results in section 3.1 have direct implications for the problem of superresolution under sparsity constraints in the so-called on-grid model, namely, when the spike locations are restricted to a discrete grid of step size Δ.[1] In what follows, we provide the necessary notations, formulate the main result (Theorem 3.14), and conclude with a brief discussion.

3.2.1. Notations.

DEFINITION 3.11. *For $\Delta > 0$, denote by $\mathcal{T}_\Delta$ the discrete grid*

$$\mathcal{T}_\Delta := \left\{k\Delta,\ k = -\left\lfloor \frac{\pi}{2\Delta} \right\rfloor, \ldots, \left\lfloor \frac{\pi}{2\Delta} \right\rfloor\right\} \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

DEFINITION 3.12. *For $\Delta, \rho, s, \ell, \tau$ as in Definition 3.1, let $\mathcal{R} := \mathcal{R}(\Delta, \rho, s, \ell, \tau)$ be the set of point measures of the form $\mu = \sum_{j=1}^{s} a_j \delta_{t_j}$, where $t_j \in \mathcal{T}_\Delta$ for all $j = 1, \ldots, s$, δ_t is the Dirac measure supported on $t \in \mathbb{R}$, $a_j \in \mathbb{C}$, and the node vector $(t_1, \ldots, t_s)$ forms a $(\Delta, \rho, s, \ell, \tau)$-clustered configuration according to Definition 3.1.*

Consider the problem of reconstructing $\mu \in \mathcal{R}$ from approximate spectral data $\widehat{\mu}(\omega)$ restricted to some interval $\omega \in [-\Omega, \Omega]$. Here the Fourier transform $\widehat{\mu}$ is defined as

$$\mu = \sum_{j=1}^{s} a_j \delta_{t_j} \Longrightarrow \widehat{\mu}(\omega) = \sum_{j=1}^{s} a_j \exp(\imath\omega t_j).$$

The measurement space $L_2([-\Omega, \Omega])$ contains complex-valued square-integrable functions supported on $[-\Omega, \Omega]$ with the norm

$$\|f\|_{2,\Omega}^2 := \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |f(\omega)|^2 d\omega. \tag{3.7}$$

Proceeding as in [17, 16], we define the minimax error for this problem as follows.

DEFINITION 3.13. *For $\mathcal{R}$ as above, $\varepsilon > 0$ and $\Omega > 0$, the minimax error $\mathcal{E} = \mathcal{E}(\mathcal{R}, \Omega, \varepsilon)$ is the quantity*

$$\mathcal{E} := \inf_{\widetilde{\mu}(\Phi_{\mu,e}) \in \mathcal{R}} \sup_{\mu \in \mathcal{R}} \sup_{e \in L_2([-\Omega,\Omega]),\ \|e\|_{2,\Omega} \le \varepsilon} \|\widetilde{\mu} - \mu\|_2, \tag{3.8}$$

where

- *$\Phi_{\mu,e} \in L_2([-\Omega, \Omega])$ is the measurement function given by*

$$\Phi_{\mu,e}(\omega) = \widehat{\mu}(\omega) + e(\omega); \tag{3.9}$$

- *$\widetilde{\mu}$ is any deterministic mapping from $L_2([-\Omega, \Omega])$ to $\mathcal{R}$;*
- *for $\mu = \sum_{j=1}^{s} a_j \delta_{t_j}$, the norm $\|\mu\|_2$ is the discrete ℓ_2 norm of the coefficient vector:*

$$\|\mu\|_2 := \left(\sum_{j=1}^{s} |a_j|^2\right)^{\frac{1}{2}}.$$

[1] Note that the results in the previous section are valid for an "off-grid" setting, as the nodes $\{t_j\}$ can have arbitrary real values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3.2.2. Minimax rates for on-grid superresolution. Using arguments very similar to [35, 17, 16, 28] and the novel bounds of Theorems 3.2 and 3.9, we obtain the optimal rate for the minimax error for clustered on-grid superresolution as follows (for the proof, see subsection 4.5 below).

THEOREM 3.14. *Fix* $s \geq 1,\ 2 \leq \ell \leq s,\ \varepsilon > 0$. *Put* $\mathrm{SRF} := \frac{\pi}{\Delta\Omega}$. *Then the following hold:*

1. *For any* $\rho \geq 0,\ \ell - 1 \leq \tau$, *and* $M \geq \pi$, *there exists* $\alpha \geq M$ *such that for all sufficiently small* Δ, *it holds that*

$$\mathcal{E}\left(\mathcal{R}\left(\Delta, \rho, s, \ell, \tau\right), \Omega, \varepsilon\right) \leq C_{s,\ell} \mathrm{SRF}^{2\ell-1}\varepsilon, \quad \mathrm{SRF} = \alpha, \tag{3.10}$$

   for some absolute constant $C_{s,\ell}$ *depending only on* s *and* ℓ.

2. *There exists an absolute constant* $\beta \gg 1$ *and* ρ', τ', *depending only on* s, ℓ, *such that for any* $\Delta < \frac{\pi}{2(2\ell-1)}$, *it holds that*

$$\mathcal{E}\left(\mathcal{R}\left(\Delta, \rho', s, \ell, \tau'\right), \Omega, \varepsilon\right) \geq C_{\ell} \mathrm{SRF}^{2\ell-1}\varepsilon, \quad \mathrm{SRF} > \beta, \tag{3.11}$$

   for some absolute constant C_ℓ *depending only on* ℓ.

3.2.3. Discussion. Theorem 3.14 generalizes [16, 28], where the scaling $\mathcal{E} \asymp \mathrm{SRF}^{2\ell-1}\varepsilon$ was derived for $\ell = s$. We also slightly improve on [35], where it was shown that for positive a_j and a comparable definition of ℓ as the "Rayleigh regularity," it holds that $\mathcal{E} \lessapprox \mathrm{SRF}^{2\ell}\varepsilon$ for $\ell \leq s$. A different but closely related setting was considered in the seminal paper [17], where the measure μ was assumed to have an infinite number of spikes on a grid of size Δ, with one spike per unit of time on average but whose local complexity was constrained to have not more than R spikes per any interval of length R (such R is called the "Rayleigh index"). It was shown in [17] that the minimax recovery rate for such measures scales like SRF^{α}, where $2R-1 \leq \alpha \leq 2R+1$. Our partial cluster model can be regarded as the finite-dimensional version of these "sparsely clumped" measures with finite Rayleigh index, and, while the two models cannot be compared directly, our results suggest that the correct exponent in Donoho's model should be $\alpha = 2R-1$ (i.e., our ℓ should conceptually play the role of Donoho's R).

If the grid assumption is relaxed, then it is natural to measure the accuracy of recovery $\|\widetilde{\mu} - \mu\|$ by comparing the spike locations in the recovered signal $\widetilde{\mu}$ with the true ones $\{t_j\}$. In this case, there are additional considerations which are required to derive the minimax rate, and it is possible to do so under the partial clustering assumptions. See [1, 9] for details, where we prove that $\mathcal{E} \asymp \mathrm{SRF}^{2\ell-1}\Delta\varepsilon$ in this scenario for uniform bound on the noise $\|e\|_\infty := \sup_{|\omega|\leq\Omega} |e\left(\omega\right)| \leq \varepsilon \lessapprox \mathrm{SRF}^{1-2\ell}$.

The partial clustering scenario can be considered as an intermediate case between the extremes $\ell = 1$ and $\ell = s$. While our results in this paper (and also in [9]) show that the superresolution problem may potentially be much more stable compared with the unconstrained sparse case (i.e., when $\ell \ll s$), it is an intriguing open question whether provably tractable (i.e., running in polynomial time) reconstruction algorithms exist. A recent line of work starting with [13, 12, 18, 15] (and the great number of follow-up papers) has shown that the case of well-separated spikes (i.e., clusters of size $\ell = 1$) can be effectively solved by semidefinite programming.[2] The extreme case $\ell = s$ has been treated recently in [6, 7] by polynomial homotopy methods.

[2]Note that the case of positive sources is solved in [35] by a convex program; however, this is slightly suboptimal, as discussed above.

However, the corresponding question for $1 < \ell < s$ remains open. Several candidate algorithms for sparse superresolution are well known—MUSIC, ESPRIT/matrix pencil, and variants; these have roots in parametric spectral estimation [44]. In recent years, the superresolution properties of these algorithms are a subject of ongoing interest; see, e.g., [19, 31, 40, 28, 29] and references therein. Smallest singular values of the partial Fourier matrices $\mathbf{V}_N$, for finite N, play a major role in these works, and therefore we hope that our results and techniques may be extended to investigate the optimality of these algorithms as well.

4. Proofs.

4.1. Blowup.

Here we introduce the uniform blowup of a node vector $\boldsymbol{x} = (t_1, \ldots, t_s)$ by a positive parameter λ and study the effect of such a blowup mapping on the minimal wraparound distance between the mapped nodes.

LEMMA 4.1. *Let $\boldsymbol{x}$ form a $(\Delta, \rho, s, \ell, \tau)$ cluster, and suppose that $\frac{4\pi s}{\rho} \le \Omega \le \frac{\pi s}{\tau\Delta}$. Then, for any $0 \le \xi \le 1$, there exists a set $I \subset \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right]$ of total measure $\frac{\Omega}{2s}\xi$ such that for every $\lambda \in I$, the following holds for every $t_j \in \boldsymbol{x}$:*

$$\|\lambda y - \lambda t_j\|_{\mathbb{T}} \ge \lambda\Delta \ge \frac{\Delta\Omega}{2s} \qquad \forall y \in \boldsymbol{x}^{(j)} \setminus \{t_j\}, \tag{4.1}$$

$$\|\lambda y - \lambda t_j\|_{\mathbb{T}} \ge \frac{1-\xi}{s^2}\pi \qquad \forall y \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}. \tag{4.2}$$

Furthermore, the set $I^c := \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right] \setminus I$ is a union of at most $\frac{s^2}{2}\left\lceil\frac{\Omega}{4s}\right\rceil$ intervals.

Proof. We begin with (4.1). Let $\lambda \in \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right]$; then $\lambda\tau\Delta \le \pi$, and since $\|t_j - y\|_{\mathbb{T}} \le \tau\Delta$, we immediately conclude that

$$\|\lambda t_j - \lambda y\|_{\mathbb{T}} = \lambda\|t_j - y\|_{\mathbb{T}} \ge \lambda\Delta.$$

To show (4.2), let ν be the uniform probability measure on $\left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right]$. Let $t_j \in \boldsymbol{x}$ and $y \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}$ be fixed, and put $\delta := \|y - t_j\|_{\mathbb{T}}$. For $\lambda \in \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right]$, let $\gamma(\lambda) = \gamma^{(t_j, y)}(\lambda)$ be the random variable on ν, defined by

$$\gamma^{(t_j, y)}(\lambda) := \|\lambda t_j - \lambda y\|_{\mathbb{T}}.$$

We now show that for any $0 \le \alpha \le 1$,

$$\nu\left\{\gamma(\lambda) \le \alpha\pi\right\} \le 2\alpha. \tag{4.3}$$

Since $\delta \ge \rho \ge \frac{4\pi s}{\Omega}$, we can write $\frac{\Omega}{2s} = \frac{2\pi}{\delta}(n + \zeta)$, where $n \ge 1$ is an integer and $0 \le \zeta < 1$. We break up the probability (4.3) as follows:

$$\begin{aligned}
&\nu\left\{\gamma(\lambda) \le \alpha\pi\right\} \\
&\quad = \sum_{k=1}^{n} \nu\left\{\gamma(\lambda) \le \alpha\pi \,\middle|\, \lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[k-1, k]\right\} \nu\left\{\lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[k-1, k]\right\} \\
&\qquad + \nu\left\{\gamma(\lambda) \le \alpha\pi \,\middle|\, \lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[n, n+\zeta]\right\} \nu\left\{\lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[n, n+\zeta]\right\}.
\end{aligned} \tag{4.4}$$

Now, consider the number $a = y - t_j$. As λ varies between $\frac{\Omega}{2s} + \frac{2(k-1)\pi}{\delta}$ and $\frac{\Omega}{2s} + \frac{2k\pi}{\delta}$, the number $\exp(\imath\lambda a)$ traverses the unit circle exactly once, and therefore the variable

$\gamma(\lambda)$ traverses the interval $[0, \alpha\pi]$ exactly twice. Consequently,

$$\nu\left\{\gamma(\lambda) \le \alpha\pi \middle| \lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[k-1, k]\right\} = \frac{2\alpha\pi}{2\pi} = \alpha.$$

Similarly, when λ varies between $\frac{\Omega}{2s} + \frac{2\pi n}{\delta}$ and $\frac{\Omega}{2s} + \frac{2\pi(n+\zeta)}{\delta}$, we have

$$\nu\left\{\gamma(\lambda) \le \alpha\pi \middle| \lambda - \frac{\Omega}{2s} \in \frac{2\pi}{\delta}[n, n+\zeta]\right\} \le \frac{2\alpha\pi}{2\pi\zeta} \le \frac{\alpha}{\zeta}.$$

Overall,

$$\nu\{\gamma(\lambda) \le \alpha\pi\} \le \alpha\frac{n}{n+\zeta} + \frac{\alpha}{\zeta}\frac{\zeta}{n+\zeta} = \alpha\frac{n+1}{n+\zeta} \le 2\alpha,$$

proving (4.3).

It is clear from the above that $\{\lambda : \gamma(\lambda) \le \alpha\pi\}$ is a union of intervals, each of length $2\alpha\pi$, repeating with the period of $\frac{2\pi}{\delta}$. Thus, the set $\left\{\lambda \in \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right] : \gamma(\lambda) \le \alpha\pi\right\}$ is a union of at most $\left\lceil \frac{\Omega}{2s}\frac{\delta}{2\pi} \right\rceil$ intervals. Since $\delta \le \pi$, we have $\left\lceil \frac{\Omega}{2s}\frac{\delta}{2\pi} \right\rceil \le \left\lceil \frac{\Omega}{4s} \right\rceil$, and so the set $\left\{\lambda \in \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right] : \gamma(\lambda) \le \alpha\pi\right\}$ is a union of at most $\left\lceil \frac{\Omega}{4s} \right\rceil$ intervals.

Now we put $\alpha_0 = \frac{1-\xi}{s^2}$ and apply (4.3) for every pair (t_j, y), where $j = 1, \dots, s$ and $y \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}$. Denote

$$J := \bigcup_{t_j, y \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}} \left\{\lambda \in \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right] : \gamma^{(t_j, y)}(\lambda) \le \alpha_0\pi\right\};$$

then, by the union bound, we obtain

$$\nu(J) \le \sum_{t_j, y} 2\alpha_0 = 2\binom{s}{2}\frac{1-\xi}{s^2} < 1 - \xi. \tag{4.5}$$

Fixing I as the complement of the above set, $I = \left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right] \setminus J$, we have that I is of total measure greater than or equal to $\xi\frac{\Omega}{2s}$, and for every $\lambda \in I$, the estimate (4.2) holds. Clearly, J is a union of at most $\frac{s^2}{2}\left\lceil \frac{\Omega}{4s} \right\rceil$ intervals. ∎

Fix $\xi = \frac{1}{2}$, and consider the set I given by Lemma 4.1. Let us also fix a finite and positive integer N and consider the set of $2N+1$ equispaced points in $[-\Omega, \Omega]$:

$$P_N := \left\{k\frac{\Omega}{N}\right\}_{k=-N,\dots,N}.$$

PROPOSITION 4.2. *If $N > 2s^3\left\lceil \frac{\Omega}{4s} \right\rceil$, then $P_N \cap I \neq \emptyset$.*

Proof. By Lemma 4.1, the set I^c consists of $K \le \frac{s^2}{2}\left\lceil \frac{\Omega}{4s} \right\rceil$ intervals, and by (4.5), the total length of I^c is at most $\frac{\Omega}{4s}$. Denote the lengths of those intervals by $d_1, \dots, d_K$. The distance between neighboring points in P_N is $\frac{\Omega}{N}$, and therefore each interval contains at most $\frac{d_j N}{\Omega} + 1$ points. Overall, the interval I^c contains at most

$$\sum_{j=1}^{K}\left(\frac{d_j N}{\Omega} + 1\right) \le \frac{\Omega}{4s}\frac{N}{\Omega} + K$$

points from P_N, and since the total number of points in $\left[\frac{\Omega}{2s}, \frac{\Omega}{s}\right]$ is at least $\frac{N}{2s}$, we have

$$|P_N \cap I| \ge \frac{N}{2s} - \frac{N}{4s} - K \ge \frac{N}{4s} - \frac{s^2}{2}\left\lceil \frac{\Omega}{4s} \right\rceil > 0.$$

∎

4.2. Square Vandermonde matrices. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_s)$ be a vector of s pairwise distinct complex numbers. Consider the square Vandermonde matrix

$$\mathbf{V}(\boldsymbol{\xi}) := \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \xi_1 & \xi_2 & \ldots & \xi_s \\ \xi_1^2 & \xi_2^2 & \ldots & \xi_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{s-1} & \xi_2^{s-1} & \ldots & \xi_s^{s-1} \end{bmatrix}. \tag{4.6}$$

THEOREM 4.3 ([22]). *For a matrix $A = (a_{i,j}) \in \mathbb{C}^{m\times n}$, let $\|A\|_\infty$ denote the ℓ_∞ induced matrix norm*

$$\|A\|_\infty := \max_{1\le i\le m} \sum_{1\le j\le n} |a_{i,j}|.$$

Then we have

$$\|\mathbf{V}^{-1}(\boldsymbol{\xi})\|_\infty \le \max_{1\le i\le s} \prod_{j\ne i} \frac{1+|\xi_j|}{|\xi_j - \xi_i|}. \tag{4.7}$$

PROPOSITION 4.4. *Suppose that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_s)$ is a vector of pairwise distinct complex numbers with $|\xi_j| = 1$, $j = 1, \ldots, s$, and let $r \in \mathbb{R}$ be arbitrary. Let*

$$\mathbf{V}(\boldsymbol{\xi}, r) := \begin{bmatrix} \xi_1^r & \xi_2^r & \ldots & \xi_s^r \\ \xi_1^{r+1} & \xi_2^{r+1} & \ldots & \xi_s^{r+1} \\ \xi_1^{r+2} & \xi_2^{r+2} & \ldots & \xi_s^{r+2} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{r+s-1} & \xi_2^{r+s-1} & \ldots & \xi_s^{r+s-1} \end{bmatrix}. \tag{4.8}$$

For $1 \le j < k \le s$, denote by $\delta_{j,k}$ the angular distance between ξ_j and ξ_k:

$$\delta_{j,k} := \left|\operatorname{Arg}\left(\frac{\xi_j}{\xi_k}\right)\right| = \big|\operatorname{Arg}(\xi_j) - \operatorname{Arg}(\xi_k) \mod (-\pi, \pi]\big|.$$

Then

$$\sigma_{\min}(\mathbf{V}(\boldsymbol{\xi}, r)) \ge \frac{\pi^{1-s}}{\sqrt{s}} \min_{1\le j\le s} \prod_{k\ne j} \delta_{j,k}. \tag{4.9}$$

Proof. Clearly, the matrix $\mathbf{V}(\boldsymbol{\xi}, r)$ can be factorized as

$$\mathbf{V}(\boldsymbol{\xi}, r) = \mathbf{V}(\boldsymbol{\xi}, 0) \times \operatorname{diag}\{\xi_1^r, \ldots, \xi_s^r\}.$$

Since $\mathbf{V}(\boldsymbol{\xi}, 0) = \mathbf{V}(\boldsymbol{\xi})$ as in (4.6), using (4.7) we immediately have

$$\|\mathbf{V}^{-1}(\boldsymbol{\xi}, r)\|_\infty \le 2^{s-1} \max_{1\le j\le s} \prod_{k\ne j} |\xi_j - \xi_k|^{-1}. \tag{4.10}$$

For any $|\theta| \le \frac{\pi}{2}$, we have

$$\frac{2}{\pi}|\theta| \le \sin|\theta| \le |\theta|,$$

and since for any $\xi_j \neq \xi_k$

$$|\xi_j - \xi_k| = \left|1 - \frac{\xi_j}{\xi_k}\right| = 2\sin\left|\frac{1}{2}\operatorname{Arg}\frac{\xi_j}{\xi_k}\right| = 2\sin\left|\frac{\delta_{j,k}}{2}\right|,$$

we therefore obtain

$$\frac{2}{\pi}\delta_{j,k} \leq |\xi_j - \xi_k| \leq \delta_{j,k}. \tag{4.11}$$

Plugging (4.11) into (4.10), we have

$$\sigma_{\max}\left(\mathbf{V}^{-1}(\boldsymbol{\xi}, r)\right) \leq \sqrt{s}\|\mathbf{V}^{-1}(\boldsymbol{\xi}, r)\|_\infty \leq \sqrt{s}\pi^{s-1}\max_{1\leq j\leq s}\prod_{k\neq j}\delta_{j,k}^{-1},$$

which is precisely (4.9). ∎

4.3. Proof of Theorem 3.2. We shall bound $\sigma_{\min}(\mathbf{V}_N(\boldsymbol{x}, \Omega))$ defined as in (2.2) for sufficiently large N. For any subset $R \subset \{-N, \dots, N\}$, let $\mathbf{V}_{N,R}$ be the submatrix of $\mathbf{V}_N$ containing only the rows in R. By the Rayleigh characterization of singular values, it is immediately obvious that if $\{-N, \dots, N\} = R_1 \cup \dots \cup R_P$ is any partition of the rows of $\mathbf{V}_N$, then

$$\sigma_{\min}^2(\mathbf{V}_N) \geq \sum_{n=1}^{P}\sigma_{\min}^2(\mathbf{V}_{N,R_n}). \tag{4.12}$$

Let I be the set from Lemma 4.1 for $\xi = \frac{1}{2}$. By Proposition 4.2, we have that for all $N > 2s^3\left\lceil\frac{\Omega}{4s}\right\rceil$, I will contain a rational multiple of Ω of the form $\lambda_N = \frac{\Omega}{N}m$ for some $m \in \mathbb{N}$.

Consider the "new" nodes

$$u_{j,N} := t_j\frac{\Omega}{N}m = \lambda_N t_j, \qquad j = 1, \dots, s. \tag{4.13}$$

Since $\lambda_N \in I$, we conclude by Lemma 4.1 that for every $j = 1, \dots, s$,

$$\|u_{j,N} - u_{k,N}\|_{\mathbb{T}} \geq \frac{1}{2s}(\Delta\Omega) \qquad \forall t_k \in \boldsymbol{x}^{(j)} \setminus \{t_j\}, \tag{4.14}$$

$$\|u_{j,N} - u_{k,N}\|_{\mathbb{T}} \geq \frac{\pi}{2s^2} \qquad \forall t_k \in \boldsymbol{x} \setminus \boldsymbol{x}^{(j)}. \tag{4.15}$$

Since $\lambda_N \leq \frac{\Omega}{s}$, it follows that $ms \leq N$. Now consider the particular interleaving partition of the rows $\{-N, \dots, N\}$ by blocks $R_{-m}, \dots, R_{-1}, R_0, R_1, \dots, R_m$ of s rows each, separated by $m-1$ rows between them (some rows might be left out):

$$\begin{aligned}
R_0 &= \{0, m, \dots, (s-1)m\}, \\
R_1 &= \{1, m+1, \dots, (s-1)m+1\}, \\
R_{-1} &= \{-1, -m-1, \dots, -(s-1)m-1\}, \\
&\dots \\
R_{m-1} &= \{m-1, 2m-1, \dots, sm-1\}, \\
R_{-m+1} &= \{-m+1, -2m+1, \dots, -sm+1\}.
\end{aligned}$$

For $n = -m+1, \ldots, m-1$, each $\mathbf{V}_{N,R_n}$ is a square Vandermonde-type matrix as in (4.8),

$$\mathbf{V}_{N,R_n} = \frac{1}{\sqrt{2N}} \mathbf{V}\left(\boldsymbol{\xi}, n\right),$$

with node vector

$$\boldsymbol{\xi} = \{e^{\imath u_{j,N}}\}_{j=1}^{s},$$

where $u_{j,N}$ are given by (4.13). We apply Proposition 4.4 with the crude bound obtained from (4.14) and (4.15) above,

$$\min_{1\le j\le s} \prod_{k\neq j} \delta_{j,k} \ge \frac{1}{2^{s-1} s^{2s-2}} \left(\Delta\Omega\right)^{\ell-1},$$

and obtain

$$\sigma_{\min}\left(\mathbf{V}_{N,R_n}\right) \ge \frac{C_5(s)}{\sqrt{2N}} \left(\Delta\Omega\right)^{\ell-1}, \qquad C_5(s) := \frac{1}{(2\pi)^{s-1} s^{2s-2}\sqrt{s}}.$$

Now we use (4.12) to aggregate the bounds on $\sigma_{\min}$ for each square matrix $\mathbf{V}_{N,R_n}$ and obtain

$$\lambda_{\min}\left(\mathbf{V}_N^H \mathbf{V}_N\right) = \sigma_{\min}^2\left(\mathbf{V}_N\right) \ge (2m-1)\frac{C_5^2}{2N} \left(\Delta\Omega\right)^{2(\ell-1)}.$$

Since $m = \frac{\lambda_N N}{\Omega} \ge \frac{\Omega N}{2s\Omega} = \frac{N}{2s}$ and since by assumption $N > 2s^3$, we have that $\frac{2m-1}{2N} \ge \frac{1}{4s}$, and so

$$\sigma_{\min}^2\left(\mathbf{V}_N\right) \ge \frac{C_5^2}{4s} \left(\Delta\Omega\right)^{2(\ell-1)}.$$

This proves (3.2) and (3.3) with

$$C_2(s) := \frac{1}{2(2\pi)^{s-1} s^{2s-1}}. \tag{4.16}$$

4.4. Proof of Theorem 3.9. Let ℓ, s, Δ, Ω be fixed with $\Delta\Omega < \eta$, where η will be specified during the proof below and $\Delta < \frac{\pi}{2(\ell-1)}$. We shall exhibit a $(\Delta, \rho', s, \ell, \tau')$-clustered configuration $\boldsymbol{x}_{\min}$ with certain ρ', τ' such that

$$\lambda_{\min}\left(\mathbf{G}\left(\boldsymbol{x}_{\min}, \Omega\right)\right) \le C_4 \cdot \left(\Delta\Omega\right)^{2(\ell-1)} \tag{4.17}$$

for some constant $C_4 = C_4(\ell)$.

Define $\boldsymbol{x}_{\ell,\Delta} = \{t_1, \ldots, t_\ell\}$ to be the vector of ℓ equispaced nodes separated by Δ, i.e., $t_j = j\Delta,\ j = 1, \ldots, \ell$. Let $\mathbf{G}^{(\ell,\ell)} = \mathbf{G}\left(\boldsymbol{x}_{\ell,\Delta}, \Omega\right)$ be the corresponding $\ell \times \ell$ prolate matrix.

PROPOSITION 4.5. *There exists an absolute constant* $0 < \eta_1 \ll 1$ *and* $C_6 = C_6(\ell)$ *such that whenever* $\Omega\Delta \le \eta_1$, *we have*

$$\lambda_{\min}\left(\mathbf{G}^{(\ell,\ell)}\right) \le C_6 \cdot \left(\Omega\Delta\right)^{2(\ell-1)}. \tag{4.18}$$

Proof. By Slepian's results [42] elaborated in section 2, there exists a constant $\eta' \ll 1$ for which (2.5) holds for all s, in particular for $s = \ell$, whenever $W \le \eta'$, i.e., whenever $\Omega\Delta \le \eta_1 := 2\pi\eta'$. ∎

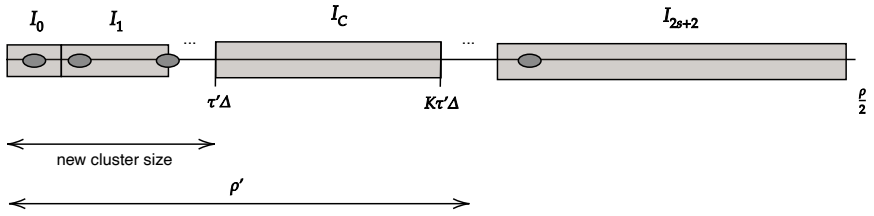


FIG. 3. *Merging μ_1 and μ_2; see Lemma 4.7. The grayed ovals represent the sets $\mathcal{S}_j$, $j = 1, \dots, s$.*

We define $\boldsymbol{x}_{\min}$ to be the extension of $\boldsymbol{x}_{\ell,\Delta}$ such that the remaining $s - \ell$ nodes are maximally equally spaced between $-\frac{\pi}{2}$ and 0, not including the endpoints. Under the assumptions on s, ℓ, Δ specified in Theorem 3.9, it is easy to check that the nodes $t_1, \dots, t_\ell$ are between 0 and $\frac{\pi}{2}$, while the remaining nodes are separated at least by

$$\rho' := \frac{\pi}{2\,(s - \ell + 1)}. \tag{4.19}$$

Therefore, $\boldsymbol{x}_{\min}$ is a particular $(\Delta, \rho', s, \ell, \tau')$-clustered configuration according to Definition 3.1 with ρ' given by (4.19) and $\tau' := \ell - 1$.

It is clear that $\mathbf{G}^{(\ell,\ell)}$ is a principal submatrix of $\mathbf{G}\,(\boldsymbol{x}_{\min}, \Omega)$, and therefore we can apply the interlacing theorem for eigenvalues of partitioned Hermitian matrices [23, Theorem 4.3.28]. Together with (4.18), this concludes the proof of (4.17) and of Theorem 3.9 with $C_4 = C_6$ and $\eta = \eta_1$.

4.5. Proof of Theorem 3.14. By the definition of the matrix $\mathbf{G}$ and (3.7), we immediately obtain the following fact.

PROPOSITION 4.6. *For $\mu = \sum_{j=1}^{s} a_j \delta_{t_j} \in \mathcal{R}$, denote $\boldsymbol{x} = \boldsymbol{x}_\mu := \operatorname{supp} \mu = (t_1, \dots, t_s) \in \mathbb{R}^s$ and $\boldsymbol{c} = \boldsymbol{c}_\mu := (a_1, \dots, a_s) \in \mathbb{C}^s$. Then we have*

$$\|\widehat{\mu}\|^2_{2,\Omega} = \boldsymbol{c}^* \mathbf{G}\,(\boldsymbol{x}, \Omega)\,\boldsymbol{c}.$$

The next result shows that for any two measures with s nodes and clusters of size ℓ, their difference has clusters of size at most 2ℓ, provided that the grid size is small enough. Note that it may happen that some nodes are in the support of both measures, in which case the difference measure will have less than $2s$ nodes, and the largest cluster may be of size strictly smaller than 2ℓ.

LEMMA 4.7. *Fix s, ℓ, ρ, τ, and let there be given $K \geq 2$. Then there exists Δ_0 such that for all $\Delta \leq \Delta_0$, the following holds: For any $\mu_1, \mu_2 \in \mathcal{R}\,(\Delta, \rho, s, \ell, \tau)$, we have*

$$\mu_1 - \mu_2 \in \mathcal{R}\,(\Delta, \rho', s', \ell', \tau')$$

for some $\ell' \leq 2\ell$, $s' \leq 2s$, some ρ', τ' satisfying $\rho' = K\tau'\Delta$ and $\tau' \geq 1$.

Proof. Let $\Delta \leq \Delta_0$ be given (with Δ_0 to be determined below), and put $\rho_0 := \tau\Delta$. Consider the intervals $I_0, I_1, \dots, I_{2s+2}$ (see Figure 3):

$$\begin{aligned} I_0 &= [0, \rho_0]\,, \\ I_1 &= [\rho_0, K\rho_0]\,, \\ I_2 &= \left[K\rho_0, K^2\rho_0\right], \\ &\dots \end{aligned}$$

such that the length a_j of each I_j is

$$a_0 = |I_0| = \rho_0,$$
$$a_j = |I_j| = (K-1)\cdot K^{j-1}\rho_0, \quad j \geq 1.$$

One can verify that

$$a_n = (K-1)\sum_{j=0}^{n-1} a_j, \quad n \geq 1,$$
$$\sum_{j=0}^{n} a_j = K^n \rho_0, \quad n \geq 0.$$

The overall length of the $2s+3$ intervals is therefore $K^{2s+2}\rho_0$. From now on we assume that this quantity is at most $\frac{\rho}{2}$, which in particular means that $\Delta \leq \Delta_0 := \frac{\rho}{2\tau K^{2s+2}}$.

For each $t_j \in \operatorname{supp}\mu_1$, consider the following sets of distances:

$$\mathcal{R}_j := \{\|y - t_j\|_{\mathbb{T}} :\ y \in \operatorname{supp}\mu_1\} \cap \left[0, \frac{\rho}{2}\right],$$
$$\mathcal{S}_j := \{\|y - t_j\|_{\mathbb{T}} :\ y \in \operatorname{supp}\mu_2\} \cap \left[0, \frac{\rho}{2}\right].$$

It is obvious that

$$\left(\bigcup_{j=1}^{s} \mathcal{R}_j\right) \cap \left(\bigcup_{j=2}^{2s+2} I_j\right) = \emptyset.$$

Now, since $\mu_2 \in \mathcal{R}$, each one of the sets $\mathcal{S}_1, \ldots, \mathcal{S}_s$ intersects at most two of the intervals $I_2, \ldots, I_{2s+2}$. Therefore, by the pigeonhole principle (Dirichlet's principle), there exists an index $C \in \{2, \ldots, 2s+2\}$ such that

$$I_C \bigcap \left(\bigcup_{j=1}^{s} \mathcal{S}_j\right) = \emptyset.$$

Clearly, we also have $I_C \bigcap \left(\bigcup_{j=1}^{s} \mathcal{R}_j\right) = \emptyset$. Put $I_C = [a, b]$. The proof is finished by taking $\tau' := \frac{a}{\Delta} \geq \frac{K\rho_0}{\Delta} = \tau \geq 1$ and $\rho' := b = Ka = K\tau'\Delta$. ∎

Proof of upper bound. Let s, ℓ, τ, ρ, and ε be fixed. Put $K := 8sM$, and let Δ_0 be as specified in Lemma 4.7. Let $\Delta \leq \Delta_0$, and put $\mathcal{R} = \mathcal{R}(\Delta, \rho, s, \ell, \tau)$ as in Definition 3.12.

Since the set $\mathcal{R}$ is finite, it is clearly possible to enumerate all its elements. To prove the upper bound for the minimax error rate $\mathcal{E}$, consider the following estimator (clearly realizable but computationally intractable) function $\widetilde{\mu}_0 = \mu_{\mathcal{R},\Omega,\varepsilon} : L_2([-\Omega, \Omega]) \to \mathcal{R}$:

$$\widetilde{\mu}_0(\Phi) := \{\text{the first } \mu \in \mathcal{R} \quad \text{subject to } \|\Phi - \widehat{\mu}\|_{2,\Omega} \leq \varepsilon\}.$$

Given $\mu \in \mathcal{R}$ and $e \in L_2([-\Omega, \Omega])$ with $\|e\|_{2,\Omega} \leq \varepsilon$, let $\widetilde{\mu}_0 = \widetilde{\mu}_0(\Phi_{\mu,e})$, where $\Phi_{\mu,e}$ is given by (3.9). Then, since $\|\Phi_{\mu,e} - \widehat{\mu}\|_{2,\Omega} = \|e\|_{2,\Omega} \leq \varepsilon$ and also by the definition of $\widetilde{\mu}_0$, we have

$$\|\widehat{\widetilde{\mu}}_0 - \widehat{\mu}\|_{2,\Omega} \leq 2\varepsilon.$$

Denote $\mu_2 := \widetilde{\mu}_0 - \mu$. By Lemma 4.7, we get that $\mu_2 \in \mathcal{R}(\Delta, \rho', s', \ell', \tau')$, where $s' \le 2s$, $\ell' \le 2\ell$, $\tau' \ge 1$, and $\rho' = 8sM\tau'\Delta$. In particular, $\rho' > 4\tau'\Delta$, and therefore, by applying Theorem 3.2, we obtain that for all Ω satisfying

$$\frac{\pi s'}{2sM\tau'\Delta} = \frac{4\pi s'}{\rho'} \le \Omega \le \frac{\pi s'}{\tau'\Delta},$$

it holds that

$$\sqrt{\lambda_{\min}\left(\mathbf{G}\left(\operatorname{supp}\mu_2, \Omega\right)\right)} \ge C_2(s')\left(\Omega\Delta\right)^{\ell'-1}.$$

In particular, for $\frac{\pi}{\Omega\Delta} := \alpha = M\tau'\frac{2s}{s'} \ge M$, we have, using the above and Proposition 4.6, that

$$\begin{aligned} 2\varepsilon \ge \|\widehat{\mu}_2\|_{2,\Omega} &\ge \sqrt{\lambda_{\min}\left(\mathbf{G}\left(\operatorname{supp}\mu_2, \Omega\right)\right)}\|\mu_2\|_2 \\ &\ge C_2(s')\left(\Omega\Delta\right)^{\ell'-1}\|\mu_2\|_2 \\ &\ge C_2(2s)\left(\Omega\Delta\right)^{2\ell-1}\|\mu_2\|_2 \end{aligned}$$

(here we also used that the constant $C_2(s)$ is decreasing with s and $\Delta\Omega < 1$). This in turn proves that $\mathcal{E} \le \frac{2}{C_2(2s)\pi^{2\ell-1}}\mathrm{SRF}^{2\ell-1}\varepsilon$. ∎

Proof of the lower bound. Let η be the constant from Theorem 3.9, and put $\beta := \frac{\pi}{\eta}$. Now suppose that $\alpha := \mathrm{SRF} > \beta$, that is, $\Omega\Delta < \eta$. Applying Theorem 3.9 with $2s, 2\ell$ we obtain ρ', τ' and the minimal configuration $\boldsymbol{x}_\alpha$. Let $\boldsymbol{c}_\alpha \in \mathbb{C}^{2s}$ denote the corresponding minimal eigenvector of $\mathbf{G}\left(\boldsymbol{x}_\alpha, \Omega\right)$ with the normalization

$$\|\boldsymbol{c}_\alpha\|_2^2 = \frac{\varepsilon^2}{C_4\alpha^{2(2\ell-1)}}. \tag{4.20}$$

Let μ be the measure defined by $\boldsymbol{x}_\alpha, \boldsymbol{c}_\alpha$. Evidently, $\mu \in \mathcal{R}(\Delta, \rho', 2s, 2\ell, \tau')$. By Proposition 4.6, (4.20) and Theorem 3.9, we have

$$\|\widehat{\mu}\|_{2,\Omega}^2 = \lambda_{\min}\left(\mathbf{G}\left(\boldsymbol{x}_\alpha, \Omega\right)\right)\|\boldsymbol{c}_\alpha\|_2^2 \le \varepsilon^2.$$

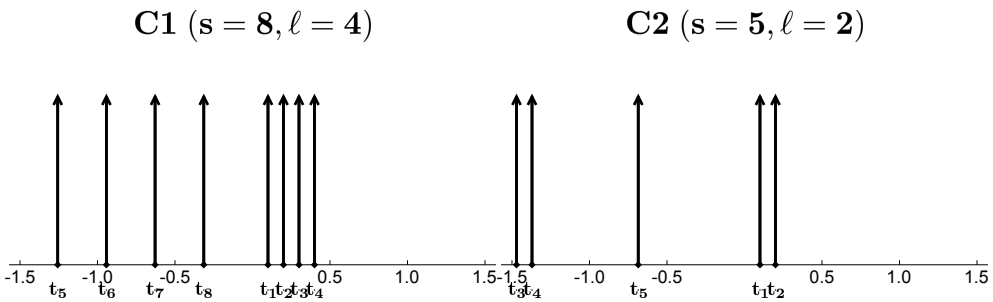
Clearly, it is possible to write $\mu = \mu_1 - \mu_2$, where $\mu_1, \mu_2 \in \mathcal{R}(\Delta, \rho', s, \ell, \tau')$. Let the measurement function Φ be such that $\Phi(\omega) := \widehat{\mu}_2(\omega)$, $\omega \in [-\Omega, \Omega]$, and let $\widetilde{\mu} = \widetilde{\mu}(\Phi) \in \mathcal{R}(\Delta, \rho', s, \ell, \tau')$. Clearly, $\Phi = \Phi_{\mu_1, -\widehat{\mu}} = \Phi_{\mu_2, 0}$ (as per (3.9)), while also

$$\begin{aligned} \frac{\varepsilon}{C_4^{\frac{1}{2}}\alpha^{2\ell-1}} &= \|\boldsymbol{c}_\alpha\|_2 = \|\mu\|_2 = \|\mu_1 - \mu_2\|_2 \\ &\le \|\mu_1 - \widetilde{\mu}\|_2 + \|\mu_2 - \widetilde{\mu}\|_2 \\ &\le 2\max\left(\|\mu_1 - \widetilde{\mu}\|_2, \|\mu_2 - \widetilde{\mu}\|_2\right), \end{aligned}$$

which shows (3.11) with $C_\ell = \frac{1}{2}C_4^{-\frac{1}{2}}$. ∎

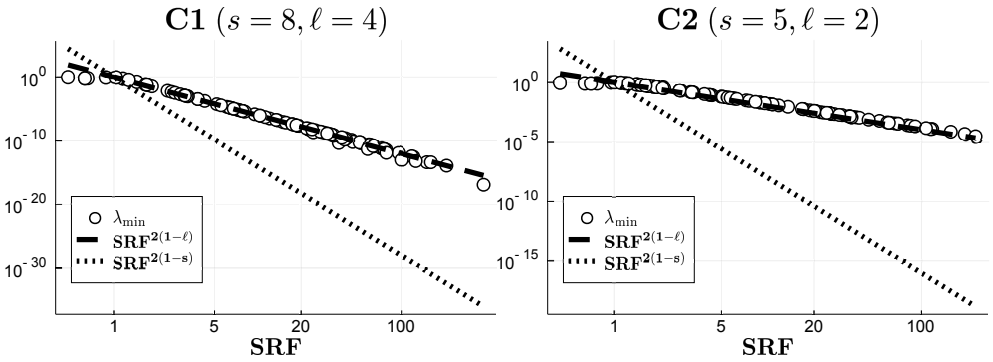
5. Numerical experiments. In order to validate the bounds of Theorems 3.2 and 3.9, we computed $\lambda_{\min}(\mathbf{G})$ for varying values of Δ, Ω, ℓ, s and the actual clustering configurations. As before, we put $\mathrm{SRF} := \frac{\pi}{\Delta\Omega}$. We checked two clustering scenarios:

C1 A single equispaced cluster of size ℓ in $[\Delta, \ell\Delta]$ with the rest of the nodes equally spaced and maximally separated in $\left(-\frac{\pi}{2}, 0\right)$. For example, in the



(a) $s = 8, \ell = 4$, 1 cluster (**C1**). (b) $s = 5, \ell = 2$, 2 clusters (**C2**).

FIG. 4. *Examples for the configurations* **C1** *and* **C2**.



(a) $s = 8, \ell = 4$, 1 cluster (configuration **C1**). (b) $s = 5, \ell = 2$, 2 clusters (configuration **C2**).

FIG. 5. *Decay rate of* $\lambda_{\min}$ *as a function of* SRF. *Results of* $n = 100$ *random experiments with randomly chosen* Δ, Ω *are plotted versus the theoretical bound* $\mathrm{SRF}^{2(1-\ell)}$. *The curve* $\mathrm{SRF}^{2(1-s)}$ *is shown for comparison. The bound stops being accurate for* $\mathrm{SRF} < O(1)$.

case $s = 8,\ \ell = 4$ (as in Figure 4a), we have $t_j = j\Delta$ for $j = 1, \ldots, 4$ and $t_j = -\frac{\pi}{2} + (j-4)\frac{\pi}{10}$ for $j = 5, \ldots, 8$.

C2 Split the s nodes into two groups, and construct two single-clustered configurations as follows:

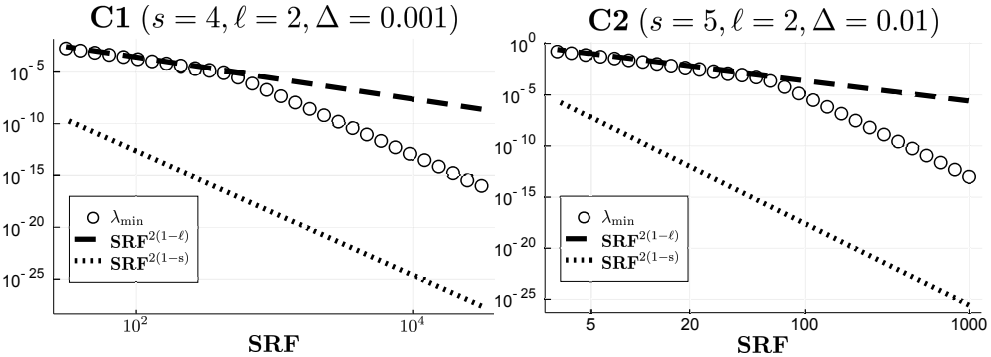
(a) $s_1 = \left\lfloor \frac{s}{2} \right\rfloor$ nodes, a single equispaced cluster of size $\ell_1 = \ell$ contained in $[\Delta, \ell\Delta]$ and the rest of the $s_1 - \ell_1$ nodes maximally separated and equally spaced in $\left(\ell\Delta, \frac{\pi}{2}\right)$;

(b) $s_2 = s - s_1$ nodes, a single equispaced cluster of size $\ell_2 = \ell$ contained in $\left[-\frac{\pi}{2} + \Delta, -\frac{\pi}{2} + \ell\Delta\right]$ and the rest of the $s_2 - \ell_2$ nodes maximally separated and equally spaced in $\left(-\frac{\pi}{2} + \ell\Delta, 0\right)$.

For example, in the case $s = 5,\ \ell = 2$ (as in Figure 4b), we have $t_1 = \Delta,\ t_2 = 2\Delta$ and $t_3 = -\frac{\pi}{2} + \Delta,\ t_4 = -\frac{\pi}{2} + 2\Delta,\ t_5 = -\frac{\pi}{4} + \Delta$.

In each experiment we fixed ℓ, s and one of the scenarios above and run $n = 100$ random tests with Δ, Ω randomly chosen within appropriate ranges for each experiment. The results are presented Figure 5.

In another experiment (Figure 6), we fixed Δ, ℓ, s and changed Ω. As expected, when Ω became small enough, the left inequality in (3.1) was violated, and indeed we can see that in this case the asymptotic decay was $\approx \mathrm{SRF}^{2(1-s)}$. See Remark 3.4 for further discussion.



(a) **C1** with $s = 4, \ell = 2$, $\Delta = 10^{-3}$. (b) **C2** with $s = 5, \ell = 2, \Delta = 10^{-2}$.

FIG. 6. *Breakdown of cluster structure. When Ω is small enough, the assumptions of Theorem 3.2 are violated for certain $\ell < s$. As a result, the decay rate of $\lambda_{\min}$ corresponds to the entire $\boldsymbol{x}$ being a single cluster of size $\ell = s$. Δ is kept fixed. See Remark 3.4.*

Acknowledgments. We would like to thank the anonymous referees whose insightful remarks and suggestions helped us immensely improve the quality of the manuscript.

REFERENCES

[1] A. AKINSHIN, D. BATENKOV, AND Y. YOMDIN, *Accuracy of spike-train Fourier reconstruction for colliding nodes*, in Proceedings of the 2015 International Conference on Sampling Theory and Applications (SampTA), Washington, DC, 2015, pp. 617–621, https://doi.org/10.1109/SAMPTA.2015.7148965.

[2] A. AKINSHIN, G. GOLDMAN, AND Y. YOMDIN, *Geometry of Error Amplification in Solving Prony System with Near-Colliding Nodes*, preprint, arXiv:1701.04058, 2017.

[3] C. AUBEL AND H. BÖLCSKEI, *Vandermonde matrices with nodes in the unit disk and the large sieve*, Appl. Comput. Harmon. Anal., (2017), https://doi.org/10.1016/j.acha.2017.07.006.

[4] J. AUTON, *Investigation of Procedures for Automatic Resonance Extraction from Noisy Transient Electromagnetics Data. Volume III. Translation of Prony's Original Paper and Bibliography of Prony's Method*, Technical report, Effects Technology Inc., Santa Barbara, CA, 1981.

[5] D. BATENKOV, *Complete algebraic reconstruction of piecewise-smooth functions from Fourier data*, Math. Comp., 84 (2015), pp. 2329–2350, https://doi.org/10.1090/S0025-5718-2015-02948-2.

[6] D. BATENKOV, *Accurate solution of near-colliding Prony systems via decimation and homotopy continuation*, Theoret. Comput. Sci., 681 (2017), pp. 27–40, https://doi.org/10.1016/j.tcs.2017.03.026.

[7] D. BATENKOV, *Stability and super-resolution of generalized spike recovery*, Appl. Comput. Harmon. Anal., 45 (2018), pp. 299–323, https://doi.org/10.1016/j.acha.2016.09.004.

[8] D. BATENKOV, B. DIEDERICHS, G. GOLDMAN, AND Y. YOMDIN, *The Spectral Properties of Vandermonde Matrices with Clustered Nodes*, preprint, https://arxiv.org/abs/1909.01927, 2019.

[9] D. BATENKOV, G. GOLDMAN, AND Y. YOMDIN, *Super-Resolution of Near-Colliding Point Sources*, preprint, https://arxiv.org/abs/1904.09186, 2019.

[10] F. BAZÁN, *Conditioning of rectangular Vandermonde matrices with nodes in the unit disk*, SIAM J. Matrix Anal. Appl., 21 (2000), p. 679.

[11] T. BLU, P.-L. DRAGOTTI, M. VETTERLI, P. MARZILIANO, AND L. COULOT, *Sparse sampling of signal innovations*, IEEE Signal Process. Mag., 25 (2008), pp. 31–40, https://doi.org/10.1109/MSP.2007.914998.

[12] E. J. CANDÈS AND C. FERNANDEZ-GRANDA, *Super-resolution from noisy data*, J. Fourier Anal. Appl., 19 (2013), pp. 1229–1254, https://doi.org/10.1007/s00041-013-9292-3.

[13] E. J. CANDÈS AND C. FERNANDEZ-GRANDA, *Towards a mathematical theory of super-resolution*, Comm. Pure Appl. Math., 67 (2014), pp. 906–956, https://doi.org/10.1002/cpa.21455.

[14] A. Cuyt, G. Labahn, A. Sidi, and W.-S. Lee, *Sparse modelling and multi-exponential analysis (Dagstuhl Seminar 15251)*, Dagstuhl Rep., 5 (2016), pp. 48–69, https://doi.org/10.4230/DagRep.5.6.48.

[15] Y. de Castro and F. Gamboa, *Exact reconstruction using Beurling minimal extrapolation*, J. Math. Anal. Appl., 395 (2012), pp. 336–354, https://doi.org/10.1016/j.jmaa.2012.05.011.

[16] L. Demanet and N. Nguyen, *The Recoverability Limit for Superresolution via Sparsity*, preprint, https://arxiv.org/abs/1502.01385, 2015.

[17] D. Donoho, *Superresolution via sparsity constraints*, SIAM J. Math. Anal., 23 (1992), pp. 1309–1331.

[18] V. Duval and G. Peyré, *Exact support recovery for sparse spikes deconvolution*, Found. Comput. Math., 15 (2014), pp. 1315–1355, https://doi.org/10.1007/s10208-014-9228-6.

[19] A. Fannjiang, *Compressive Spectral Estimation with Single-Snapshot ESPRIT: Stability and Resolution*, preprint, https://arxiv.org/abs/1607.01827, 2016.

[20] P. Ferreira, *Superresolution, the recovery of missing samples, and Vandermonde matrices on the unit circle*, in Proceedings of the 1999 Workshop on Sampling Theory and Applications; Loen, Norway, August 1999.

[21] S. Fomel, *Seismic data decomposition into spectral components using regularized nonstationary autoregression*, Geophysics, 78 (2013), pp. O69–O76, https://doi.org/10.1190/geo2013-0221.1.

[22] W. Gautschi, *On inverses of Vandermonde and confluent Vandermonde matrices*, Numer. Math., 4 (1962), pp. 117–123.

[23] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2012.

[24] A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Z., 41 (1936), pp. 367–379, https://doi.org/10.1007/BF01180426.

[25] S. Kunis and D. Nagel, *On the Condition Number of Vandermonde Matrices with Pairs of Nearly-Colliding Nodes*, https://arxiv.org/abs/1812.08645, 2018.

[26] S. Kunis and D. Nagel, *On the Smallest Singular Value of Multivariate Vandermonde Matrices with Clustered Nodes*, https://arxiv.org/abs/1907.07119, 2019.

[27] L. Li and T. P. Speed, *Parametric deconvolution of positive spike trains*, Ann. Statist., (2000), pp. 1279–1301.

[28] W. Li and W. Liao, *Stable Super-Resolution Limit and Smallest Singular Value of Restricted Fourier Matrices*, https://arxiv.org/abs/1709.03146v2, 2018.

[29] W. Li, W. Liao, and A. Fannjiang, *Super-Resolution Limit of the ESPRIT Algorithm*, preprint, https://arxiv.org/abs/1905.03782v3, 2019.

[30] Y. E. Li and L. Demanet, *Phase and amplitude tracking for seismic event separation*, Geophysics, 80 (2015), pp. WD59–WD72.

[31] W. Liao and A. Fannjiang, *MUSIC for single-snapshot spectral estimation: Stability and super-resolution*, Appl. Comput. Harmon. Anal., 40 (2016), pp. 33–67, https://doi.org/10.1016/j.acha.2014.12.003.

[32] A. Moitra, *Super-resolution: Extremal functions and the condition number of Vandermonde matrices*, in Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC '15, ACM, New York, 2015, pp. 821–830, https://doi.org/10.1145/2746539.2746561.

[33] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, American Mathematical Society, Providence, RI, 1994.

[34] H. L. Montgomery and R. C. Vaughan, *Hilbert's Inequality*, J. Lond. Math. Soc. (2), (1974), pp. 73–82, https://doi.org/10.1112/jlms/s2-8.1.73.

[35] V. I. Morgenshtern and E. J. Candès, *Super-resolution of positive sources: The discrete setup*, SIAM J. Imaging Sci., 9 (2016), pp. 412–444, https://doi.org/10.1137/15M1016552.

[36] M. Negreanu and E. Zuazua, *Discrete Ingham inequalities and applications*, SIAM J. Numer. Anal., 44 (2006), pp. 412–448, https://doi.org/10.1137/050630015.

[37] H. Pan, T. Blu, and M. Vetterli, *Towards generalized FRI sampling with an application to source resolution in radioastronomy*, IEEE Trans. Signal Process., 65 (2016), pp. 821–835.

[38] V. Pereyra and G. Scherer, *Exponential Data Fitting and Its Applications*, Bentham Science Publishers, 2010.

[39] T. Peter, D. Potts, and M. Tasche, *Nonlinear approximation by sums of exponentials and translates*, SIAM J. Sci. Comput., 33 (2011), p. 1920.

[40] D. Potts and M. Tasche, *Error estimates for the ESPRIT algorithm*, in Large Truncated Toeplitz Matrices, Toeplitz Operators, and Related Topics, D. A. Bini, T. Ehrhardt, A. Y. Karlovich, and I. Spitkovsky, eds., Vol. 259, Springer, Cham, Switzerland, 2017, pp. 621–648, https://doi.org/10.1007/978-3-319-49182-0_25.

[41] M. Reynolds, G. Beylkin, and L. Monzón, *On generalized Gaussian quadratures for bandlimited exponentials*, Appl. Comput. Harmon. Anal., 34 (2013), pp. 352–365, https://doi.org/10.1016/j.acha.2012.07.002.

[42] D. Slepian, *Prolate spheroidal wave functions, Fourier analysis, and uncertainty—V: The discrete case*, Bell Sys. Tech. J., 57 (1978), pp. 1371–1430, https://doi.org/10.1002/j.1538-7305.1978.tb02104.x.

[43] D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, SIAM Rev., 25 (1983), pp. 379–393.

[44] P. Stoica and R. Moses, *Spectral Analysis of Signals*, Pearson/Prentice Hall, Upper Saddle River, NJ, 2005.

[45] J. Varah, *The prolate matrix*, Linear Algebra Appl., 187 (1993), pp. 269–278, https://doi.org/10.1016/0024-3795(93)90142-B.