Stochastics and Dynamics
Vol. 22, No. 3 (2022) 2240019 (25 pages)

DOI: 10.1142/S0219493722400196

Invariant measures and boundedness in the mean for stochastic equations driven by Lévy noise

Bohdan Maslowski* and Ondrej Tybl†

Department of Probability and Mathematical Statistics, Charles University, Sokolovska 83, 186 75 Prague 8, Czech Republic
**maslow@karlin.mff.cuni.cz*
†tybl@karlin.mff.cuni.cz

Received 29 October 2021
Revised 3 February 2022
Accepted 4 February 2022
Published 14 May 2022

Existence of invariant measures and average stability in the mean are studied for stochastic differential equations driven by Lévy process. In particular, some natural conditions are found that verify stabilization of the equation (in the sense of the existence of invariant measures) by jump noise terms. These conditions are verified in several examples.

Keywords: Lévy processes; stochastic differential equations; invariant measures; stability.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

In this paper, we study the problem of existence of invariant measure for stochastic differential equations driven by Lévy process. Basically, we investigate a stability property that may be called "boundedness of solutions in probability in the mean" which together with the Feller property verifies the existence of an invariant measure by means of the well-known Krylov–Bogolyubov Theorem. In particular, we are interested in some situations when the equation is "stabilized" in the above sense by noise. This phenomenon is well understood in the case of Gaussian noise (as discussed already in the classical Khasminskii's monograph [12], for infinite-dimensional systems [15]) and we focus on contribution of the stochastic jump terms.

Stochastic differential equations driven by Lévy noise have been extensively studied in recent years. Important fundamental results in this field are presented, for instance, in the monograph [2] from which we take the basic setting of the problem.

The topic addressed in this paper is related to the problems of stability (and stabilization) of trivial solution to Lévy-driven stochastic equation which was recently studied in several papers. In [5], asymptotic a.s. stability is shown in the linear case (under conditions analogous to those in this paper if restricted to such case). In [4], asymptotic stability in probability, in the mean and the moment stability is studied. The paper [19] deals with boundedness in probability and the moment boundedness, for a time-changed Lévy noise, an analogous problem in the case of Gaussian noise is also investigated in [19]. Some related results can be also found in [8, 16] and an analogous problem for equations driven by discontinuous semimartingales are studied in [18]. Reference [17] presents a general treatise on stochastic stability.

Existence of stationary distributions has been addressed in [6, 9]. In the latter paper, the existence is shown by means of Lyapunov method but the effect of stabilization by noise is not considered (the noise has to be "small enough"). In [1], a class of invariant measures is described in general terms by means of Fokker–Planck equation. For infinite-dimensional systems, existence of invariant measure has been studied, for example, in [3] or [14]. Some other aspects of stochastic equations driven by Lévy noise have been addressed, for example, in [10, 11, 20] (filtering and control), [22] (finance mathematics), [7] (CARMA time series models) or [21] (signal processing).

A related problem of existence of random attractors for equations with two-sided Lévy noise has been treated in [23].

This paper is divided into five sections. In Sec. 1, the problem is posed and some preliminary standard results are recalled. In principle, for the most general formulation of the stability theorem we only need local boundedness of the coefficients besides existence and uniqueness of solutions. However, we also present some standard conditions (Lipschitz and linear growth conditions) which verify this basic assumption and which are also helpful in more specific cases.

In Sec. 2, a general Lyapunov-type criterion for boundedness in probability in average is proved under fairly general conditions on coefficients of the equation (the local boundedness, cf. Assumption 3.1).

The general theorem from Sec. 2 is applied to the equation containing the drift, diffusion and compensated integral terms in Sec. 3. The main result (Theorem 5.1) is formulated for locally bounded coefficients and then specified in the linear growth case (Corollary 4.1). This result allows us to discuss stabilizing roles of particular terms in the equation and their mutual influence. The section is closed by an example where such interplay of particular terms in the equation is demonstrated and also, relation to moment stability of solutions is discussed.

In Sec. 4, the general statement from Sec. 2 is applied to the equation with uncompensated integral term, drift and diffusion. Theorem 5.1 is analogous to Theorem 4.1 from the previous section. In Theorem 4.1, a different approach is adopted and a stability criterion is found which is expressed directly in terms of jumps. Section 4 is closed by two examples: In the first one, the linear equation is studied.

In the second one, the influence of the parameter dividing small and big jumps is discussed.

Section 5 summarizes consequences of the previous parts for the existence of invariant measure (stationary solution) if the equation defines a Feller Markov process, which may be viewed as the main results of the paper.

Notation. Throughout this paper, we measure the norm of a matrix $A \in \mathbb{R}^{m\times n}$ as $|A|^2 = \sum_{i=1}^m \sum_{j=1}^n A_{ij}^2$. Furthermore, we denote $B_b(\mathbb{R}^d)$ the Banach space of real, bounded, measurable functions defined on $\mathbb{R}^d$ endowed with the supremal norm $|\cdot|_\infty$, similarly we use the standard notation for the vector spaces $\mathcal{C}_b(\mathbb{R}^d)$, $\mathcal{C}^2(\mathbb{R}^d)$ and $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. When instead of real-valued functions we deal with vector- or matrix-valued functions we modify the notation in an obvious way, such as $\mathcal{C}_b(\mathbb{R}^d;\mathbb{R}^d)$, $\mathcal{C}_b(\mathbb{R}^d;\mathbb{R}^{d\times d})$, etc. without any change on the notation for the supremal norm $|\cdot|_\infty$. Finally, for $K \in (0,\infty)$ by B_K we mean the closed ball in a Euclidian space of radius K (with dimension always clear from the context) and the complements are denoted by the superscript c.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses and assume that W is an $(\mathcal{F}_t)$-Wiener process with values in $\mathbb{R}^n$ which is independent to an $(\mathcal{F}_t)$-Poisson random measure N on $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \{0\})$. The Poisson measure N has the intensity measure $dt\nu(dy)$, where ν is a Lévy measure on $\mathbb{R}^n \setminus \{0\}$, i.e. we have $\int_{\mathbb{R}^n\setminus\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty$. Let $\tilde{N}$ denote the compensator of N (cf. [2, Sec. 2.3.1]).

We study an equation driven by the pair (W, N). More specifically, assume we are given Borel measurable mappings $f : \mathbb{R}^m \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^{m\times n}$, $H, K : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and a constant $c \in (0,\infty)$ and consider the equation

$$dX_t = f(X_{t-})dt + g(X_{t-})dW_t + \int_{\{|y|<c\}} H(X_{t-}, y)\tilde{N}(dt, dy) + \int_{\{|y|\geq c\}} K(X_{t-}, y)N(dt, dy), \quad t \geq 0. \tag{2.1}$$

Recall the standard requirements on the coefficients in (2.1) that are sufficient for existence of a unique global solution for any $\mathcal{F}_0$-measurable initial condition (see [2, Sec. 6.2]).

Lipschitz condition: There exists $L_1 \in (0,\infty)$ such that

$$|f(x) - f(z)|^2 \vee |g(x) - g(z)|^2 \vee \int_{\{|y|<c\}} |H(x,y) - H(z,y)|^2 \,\nu(dy) \leq L_1 |x - z|^2, \tag{LIP}$$

for any $x, z \in \mathbb{R}^m$.

Growth condition: There exists $L_2 \in (0, \infty)$ such that

$$\int_{\{|y|<c\}} |H(x,y)|^2 \, \nu(dy) \le L_2(1 + |x|^2), \tag{GRO}$$

for any $x \in \mathbb{R}^m$.

Continuity condition: We have

$$K(\cdot, y) \in \mathcal{C}(\mathbb{R}^m; \mathbb{R}^m) \tag{CON}$$

for all $y \in \{|y| \ge c\}$.

In the following, we may proceed without (LIP), (GRO), (CON), however, under these assumptions the results can be substantially simplified.

We only assume that for any $x \in \mathbb{R}^m$ the unique solution to (2.1) denoted as X^x is given and it defines a time-homogeneous Markov process. It is convenient to assign the Markov process in the usual manner a translation semigroup of linear operators $(S_t, t \ge 0)$ acting on $B_b(\mathbb{R}^m)$ as

$$S_t f(x) = \mathbb{E}\, f(X_t^x), \quad f \in B_b(\mathbb{R}^m), \;\; x \in \mathbb{R}^m, \tag{2.2}$$

for $t \ge 0$.

For our purposes, the following concept is crucial.

Definition 2.1. We say that Markov process with translation semigroup given by (2.2) is Feller, if

$$S_t f \in \mathcal{C}_b(\mathbb{R}^m), \quad f \in \mathcal{C}_b(\mathbb{R}^m), \tag{2.3}$$

for any $t \ge 0$.

We consider two concepts of stability of Eq. (2.1). The first one is related to a particular solution and can be formulated for any stochastic process, while in the second one, Markov process induced by the equation is considered.

Definition 2.2. A measurable stochastic process Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ is *bounded in probability in the mean* if

$$\lim_{R\to\infty} \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}\,[|Y_s| \le R] ds = 1. \tag{2.4}$$

Definition 2.3. We say that Markov process induced by Eq. (2.1) possesses an invariant measure if there exists a Borel probability measure μ^* on $\mathbb{R}^m$ and a solution Y to (2.1) such that

$$\mathbb{E}\, f(Y_t) = \int_{\mathbb{R}^m} f(x) \mu^*(dx), \quad f \in B_b(\mathbb{R}^m),$$

for every $t \ge 0$.

The following statement is a particular case of Krylov–Bogolyubov Theorem.

Theorem 2.1. (Krylov–Bogolyubov) *Markov process defined by Eq.* (2.1) *possess an invariant measure, if both*

(1) *it is Feller*;
(2) *there exists* $x \in \mathbb{R}^m$ *such that* X^x *is bounded in probability in the mean.*

Proof. The proof can be found e.g., in [13]. □

3. General Lyapunov Criterion

In this section, we investigate general criterion for stability the system (2.1) in terms of boundedness in probability in the mean.

We will deal with a specific Lyapunov function that takes the following form. For $p \in (0, 1)$ denote V_p an arbitrary (but fixed in the sequel) element of $\mathcal{C}^2(\mathbb{R}^m)$ satisfying

$$DV_p \in \mathcal{C}_b(\mathbb{R}^m; \mathbb{R}^m), \quad D^2V_p \in \mathcal{C}_b(\mathbb{R}^m; \mathbb{R}^{m\times m}), \tag{V1}$$

$$V_p(x) = |x|^p, \quad |x| \geq 1, \tag{V2}$$

$$0 \leq V_p(x) \leq |x|^p, \quad |x| \leq 1. \tag{V3}$$

If follows that the derivatives of V_p take the following form:

$$DV_p(x) = p\,|x|^{p-2}\,x, \tag{3.1}$$

$$D^2V_p(x) = p(p-2)\,|x|^{p-4}\,xx^T + p\,|x|^{p-2}\,I \tag{3.2}$$

for $x \in B_1^c$, where I is the identity in $R^{m\times m}$.

Using only (V1) the Itô formula may be used to obtain the differential of $V_p(X)$, where X is a solution to (2.1).

Proposition 3.1. (Itô formula) *Let* X *be a solution to* (2.1) *and* $p \in (0, 1)$. *Then*

$$\begin{aligned} dV_p(X_t) = &\left(\langle f(X_{t-}), DV_p(X_{t-})\rangle + \frac{1}{2}\mathrm{Tr}(g(X_{t-})^T D^2V_p(X_{t-})g(X_{t-}))\right) dt \\ &+ DV_p(X_{t-})^T g(X_{t-})dW_t \\ &+ \int_{\{|y|<c\}} V_p(X_{t-} + H(X_{t-}, y)) - V_p(X_{t-})\tilde{N}(dt, dy) \\ &+ \int_{\{|y|<c\}} (V_p(X_{t-} + H(X_{t-}, y)) - V_p(X_{t-}) \\ &- \langle DV_p(X_{t-}), H(X_{t-}, y)\rangle)\nu(dy)dt \\ &+ \int_{\{|y|\geq c\}} V_p(X_{t-} + K(X_{t-}, y)) - V_p(X_{t-})N(dt, dy), \end{aligned} \tag{3.3}$$

for $t \geq 0$.

Proof. See [2, Theorem 4.4.7 and Remark below]. □

The form of Itô formula (3.3) motivates us to study the linear operator $\mathcal{L}$ that is given as follows. Denote $\mathrm{Dom}(\mathcal{L})$ the linear subspace of $\mathcal{C}^2(\mathbb{R}^m)$ of functions $V \in \mathcal{C}^2(\mathbb{R}^m)$ such that the following prescription:

$$\begin{aligned}\mathcal{L}V(x) = {} & \langle f(x), DV(x)\rangle \\ & + \frac{1}{2}\mathrm{Tr}(g(x)^T D^2 V(x) g(x)) \\ & + \int_{\{|y|<c\}} V(x + H(x,y)) - V(x) - \langle H(x,y), DV(x)\rangle \nu(dy) \\ & + \int_{\{|y|\geq c\}} V(x + K(x,y)) - V(x)\nu(dy), \quad x \in \mathbb{R}^m \end{aligned} \tag{3.4}$$

defines an element of $B_b(\mathbb{R}^m)$. Then define $\mathcal{L} : \mathrm{Dom}(\mathcal{L}) \to B_b(\mathbb{R}^m)$ by (3.4).

In fact, our aim is to rewrite (3.3) as

$$\begin{aligned} dV_p(X_t) = {} & \mathcal{L}V_p(X_{t-})dt + DV_p(X_{t-})^T g(X_{t-})dW_t \\ & + \int_{\{|y|<c\}} V_p(X_{t-} + H(X_{t-}, y)) - V_p(X_{t-})\tilde{N}(dt, dy) \\ & + \int_{\{|y|\geq c\}} V_p(X_{t-} + K(X_{t-}, y)) - V_p(X_{t-})N(dt, dy) \\ & - \int_{\{|y|\geq c\}} V_p(X_{t-} + K(X_{t-}, y)) - V_p(X_{t-})\nu(dy)dt, \end{aligned} \tag{3.5}$$

for $t \geq 0$ and any $p \in (0, 1)$.

We will prove (3.5) under some additional conditions on the coefficients.

Assumption 3.1. The following mappings:

$$x \mapsto f(x), \tag{3.6}$$

$$x \mapsto g(x), \tag{3.7}$$

$$x \mapsto \int_{\{|y|<c\}} |H(x,y)|^2 \nu(dy), \tag{3.8}$$

$$x \mapsto \int_{\{|y|\geq c\}} |K(x,y)|^p \nu(dy) \tag{3.9}$$

are locally bounded on $\mathbb{R}^m$ for $p \in (0, 1)$.

Note that the following would hold even under weaker assumption of local boundedness of (3.9) only for $p \in (0, p^*)$ for some $p^* > 0$.

Lemma 3.1. *Fix* $p \in (0, 1)$. *Under Assumption* 3.1 *we have that* $V_p \in \mathrm{Dom}(\mathcal{L})$ *and the Itô formula* (3.5) *for* V_p *holds.*

Proof. Let $p \in (0,1)$ be given. To show that the first two terms in (3.4) are well defined and locally bounded in x is straightforward. We proceed with the integral terms in more detail.

The compensated term: We use Taylor's reminder in the integral form and (V1) as follows:

$$\begin{aligned}
&\int_{\{|y|<c\}} |V_p(x+H(x,y)) - V_p(x) - \langle H(x,y), DV_p(x)\rangle|\, \nu(dy) \\
&\quad = \int_{\{|y|<c\}} \left| \int_0^1 H(x,y)^T D^2 V_p(x+\theta H(x,y)) H(x,y)(1-\theta) d\theta \right| \nu(dy) \\
&\quad \le \left| D^2 V_p \right|_\infty \int_{\{|y|<c\}} |H(x,y)|^2\, \nu(dy),
\end{aligned}$$

for $x \in \mathbb{R}^m$. By Assumption 3.1 local boundedness of the compensated term now easily follows.

The uncompensated term: We use (V2), (V3) and estimate

$$\begin{aligned}
&\int_{\{|y|\ge c\}} |V_p(x+K(x,y)) - V_p(x)|\, \nu(dy) \\
&\quad \le \int_{\{|y|\ge c\}} |V_p(x+K(x,y)) - |x+K(x,y)|^p|\, \nu(dy) \\
&\qquad + \int_{\{|y|\ge c\}} |x+K(x,y)|^p + V_p(x)\nu(dy) \\
&\quad \le \int_{\{|y|\ge c\}} 2\nu(dy) + \int_{\{|y|\ge c\}} |K(x,y)|^p + |x|^p + V_p(x)\nu(dy) \\
&\quad \le \nu(\{|y|\ge c\})\,(2+|x|^p + V_p(x)) + \int_{\{|y|\ge c\}} |K(x,y)|^p\, \nu(dy) \qquad (3.10)
\end{aligned}$$

for $x \in \mathbb{R}^m$ with the last term being locally bounded by Assumption 3.1.

The formula (3.5) is valid as it is just a different form of (3.3) provided that

$$\int_{\{|y|\ge c\}} V_p(X_{t-} + K(X_{t-}, y)) - V_p(X_{t-})\nu(dy)$$

is well defined for every $t \ge 0$ almost surely, which is the case by (3.10) and the almost sure local boundedness of the trajectories of the solution to (2.1). □

Having Lemma 3.1 we will now prove the main criterion for boundedness in probability in the mean. The proof is an adaptation of work of Khasminskii (cf. [12]) established for the special case of diffusion processes.

Theorem 3.1. *Let Assumption* 3.1 *hold. Then the solution to* (2.1) *with any deterministic initial condition is bounded in probability in the mean if there exists*

$p \in (0,1)$ such that there exists $R_0 \in (0,\infty)$ such that for all $R \in (R_0,\infty)$ there exists $A_R \in (0,\infty)$ with

$$A_R \to \infty, \quad R \to \infty \tag{3.11}$$

and

$$\mathcal{L}V_p(x) \le -A_R, \quad |x| \ge R. \tag{3.12}$$

Proof. Let $x \in \mathbb{R}^m$ and write shortly $X = X^x$ for the unique solution to (2.1) with the initial condition x. As X is a global solution, the stopping times

$$\tau_k = \inf\{t \ge 0, |X_{t-}| > k\},$$

for $k \in \mathbb{N}$, tend to infinity almost surely. Now fix $t \ge 0$, $k \in \mathbb{N}$. By Lemma 3.1, the Itô formula (3.5) implies

$$\begin{aligned}
&V_p(X_{t\wedge\tau_k}) - V(x)\\
&\quad = \int_0^{t\wedge\tau_k} \mathcal{L}V_p(X_{s-})ds + \int_0^{t\wedge\tau_k} DV_p(X_{s-})^T g(X_{s-})dW_s\\
&\qquad + \int_0^{t\wedge\tau_k}\int_{\{|y|<c\}} V_p(X_{s-} + H(X_{s-},y)) - V_p(X_{s-})\tilde{N}(ds,dy)\\
&\qquad + \int_0^{t\wedge\tau_k}\int_{\{|y|\ge c\}} V_p(X_{s-} + K(X_{s-},y)) - V_p(X_{s-})N(ds,dy)\\
&\qquad - \int_0^{t\wedge\tau_k}\int_{\{|y|\ge c\}} V_p(X_{s-} + K(X_{s-},y)) - V_p(X_{s-})\nu(dy)ds.
\end{aligned} \tag{3.13}$$

We compute expectations of all the stochastic integrals in (3.13).

We have $DV_p^T g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ and X_{s-} is bounded almost surely on $(0,\tau_k)$, therefore

$$\mathbb{E}\int_0^{t\wedge\tau_k} DV_p(X_{s-})^T g(X_{s-})dW_s = 0.$$

Similarly, the compensated integral is centered,

$$\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|<c\}} V_p(X_{s-} + H(X_{s-},y)) - V_p(X_{s-})\tilde{N}(ds,dy) = 0$$

as from Taylor's expansion and local boundedness of (3.8) we have

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|<c\}} |V_p(X_{s-} + H(X_{s-},y)) - V_p(X_{s-})|^2 \nu(dy)ds\\
&\quad = \mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|<c\}} \left|\int_0^1 \langle DV_p(X_{s-} + \theta H(X_{s-},y)), H(X_{s-},y)\rangle(1-\theta)d\theta\right|^2 \nu(dy)ds\\
&\quad \le \frac{1}{3}|DV_p|^2_\infty\,\mathbb{E}\int_0^{t\wedge\tau_k} |H(X_{s-},y)|^2\nu(dy)ds < \infty.
\end{aligned}$$

For the uncompensated term by (3.10) and local boundedness of (3.9) we have

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|\geq c\}} |V_p(X_{s-}+K(X_{s-},y))-V_p(X_{s-})|\,\nu(dy)ds \\ \leq \mathbb{E}\int_0^{t\wedge\tau_k} \nu(\{|y|\geq c\})(2+|X_{s-}|^p+V_p(X_{s-}))ds \\ +\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|\geq c\}} |K(X_{s-},y)|^p\,\nu(dy)ds<\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|\geq c\}} V_p(X_{s-}+K(X_{s-},y))-V_p(X_{s-})N(ds,dy) \\ =\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\{|y|\geq c\}} V_p(X_{s-}+K(X_{s-},y))-V_p(X_{s-})\nu(dy)ds.\end{aligned}$$

Finally, $\mathcal{L}V_p\in L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ so

$$\mathbb{E}\int_0^{t\wedge\tau_k}\mathcal{L}V_p(X_{s-})ds$$

is well defined. We have shown that

$$-V(x)\leq \mathbb{E}\,V_p(X_{t\wedge\tau_k})-V(x)=\mathbb{E}\int_0^{t\wedge\tau_k}\mathcal{L}V_p(X_{s-})ds.$$

For $R\in(R_0,\infty)$, we have from assumption (3.12) and local boundedness of $\mathcal{L}V_p$ that

$$\begin{aligned}\mathcal{L}V_p(x)\leq -A_R\mathbf{1}_{\{|x|\geq R\}}+\left(\sup_{|x|<R}\mathcal{L}V_p(x)\right)\mathbf{1}_{\{|x|<R\}} \\ \leq -A_R\mathbf{1}_{\{|x|\geq R\}}+\sup_{x\in\mathbb{R}^m}\mathcal{L}V_p(x)<\infty.\end{aligned}$$

Denote

$$\kappa=\sup_{x\in\mathbb{R}^m}\mathcal{L}V_p(x)<\infty.$$

We shall write

$$-V(x)\leq\mathbb{E}\int_0^{t\wedge\tau_k}-A_R\mathbf{1}_{\{|X_{s-}\geq R|\}}+\kappa ds.$$

Now taking the limit as $k\to\infty$. We obtain

$$-V(x)\leq -A_R\int_0^t\mathbb{P}\left[|X_{s-}|\geq R\right]ds+\kappa t$$

with $t\geq 0$ arbitrary. Finally, for $t\geq 1$

$$\frac{1}{t}\int_0^t P[|X_{s-}|\geq R]ds\leq\frac{V(x)+\kappa}{A_R}\to 0,\quad R\to\infty$$

by (3.11), which already implies (2.4). The assertion of the theorem now follows by the equality $X_{s-}=X_s$ almost surely for any $s\geq 0$. □

4. Stabilization by Compensated Jumps

Throughout this section, we assume Assumption 3.1 to hold. We investigate stabilization properties of compensated jumps. Thus, for simplicity, we put $K = 0$ in (2.1) and obtain the equation:

$$dX_t = f(X_{t-})dt + g(X_{t-})dW_t + \int_{\{|y|<c\}} H(X_{t-}, y)\tilde{N}(dt, dy), \quad t \geq 0. \tag{4.1}$$

The main result of this section is presented in Theorem 4.1 and in an important Corollary 4.1 where the conditions are simplified under specific growth assumptions. First, we prove some technical formulas.

For our purpose it is useful to denote

$$\mathcal{H}(x) = \{y \in \mathbb{R}^n : |y| < c, x + H(x, y) = 0\}, \tag{4.2}$$

for $x \in \mathbb{R}^m$.

Note that by the relations

$$\begin{aligned} |x|^2 \nu(\mathcal{H}(x)) &= \int_{\{|y|<c\}\cap\mathcal{H}(x)} |H(x, y)|^2 \nu(dy) \\ &\leq \int_{\{|y|<c\}} |H(x, y)|^2 \nu(dy) < \infty \end{aligned}$$

for $x \in \mathbb{R}^m$, which are due to Assumption 3.1, it follows that

$$\nu(\mathcal{H}(x)) < \infty, \quad x \in \mathbb{R}^m, \quad x \neq 0. \tag{4.3}$$

First, we prove a technical lemma.

Lemma 4.1. *For $p \in (0, 1)$, we have*

$$\begin{aligned} &\int_{\{|y|<c\}} V_p(x + H(x, y)) - V_p(x) - \langle H(x, y), DV_p(x)\rangle \nu(dy) \\ &\quad \leq p\,|x|^p \Bigg[\int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \log\frac{|x + H(x, y)|}{|x|} - \frac{\langle H(x, y), x\rangle}{|x|^2}\nu(dy) \\ &\qquad + \frac{p}{2}\int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \left(\log\frac{|x + H(x, y)|}{|x|}\right)^2 \nu(dy) \\ &\qquad - \left(\frac{1}{p} - 1\right)\nu(\mathcal{H}(x))\Bigg] < \infty, \end{aligned} \tag{4.4}$$

for $x \in B_1^c$.

Proof. Take $p \in (0, 1)$. Now, fix $x \in B_1^c$ and distinguish two cases.

Step I. If $y \in \mathcal{H}(x)$ then we have

$$V_p(x + H(x, y)) - V_p(x) - \langle H(x, y), DV_p(x)\rangle = |x|^p (p - 1) \tag{4.5}$$

by (3.1).

Hence, (4.3) yields

$$\int_{\mathcal{H}(x)} V_p(x + H(x,y)) - V_p(x) - \langle H(x,y), DV_p(x)\rangle \nu(dy)$$
$$\leq p\,|x|^p\,\nu\left(\mathcal{H}(x)\left(1 - \frac{1}{p}\right)\right). \tag{4.6}$$

Step II. For the second case, when $y \in \{|y| < c\} \cap \mathcal{H}(x)^c$, we remind the particular form of Taylor's expansion

$$\frac{a^p - 1}{p} = \log a + \frac{\tilde{p}}{2}(\log a)^2, \tag{4.7}$$

for $p > 0$ and some $\tilde{p} \in (0,p)$ with $a \in (0,\infty)$ fixed. We now apply (4.7) to the case

$$a = \frac{|x + H(x,y)|}{|x|}$$

and use (3.1) to compute

$$\begin{aligned}
&V_p(x + H(x,y)) - V_p(x) - \langle H(x,y), DV_p(x)\rangle \\
&\quad = |x + H(x,y)| - |x|^p - p\,|x|^p\,\frac{\langle H(x,y), x\rangle}{|x|^2} \\
&\qquad + (V_p(x + H(x,y)) - |x + H(x,y)|^p) \\
&\quad = p\,|x|^p \left(\frac{\frac{|x+H(x,y)|}{|x|} - 1}{p} - \frac{\langle H(x,y), x\rangle}{|x|^2}\right) \\
&\qquad + (V_p(x + H(x,y)) - |x + H(x,y)|^p) \\
&\quad = p\,|x|^p \left(\log\frac{|x + H(x,y)|}{|x|} + \frac{\tilde{p}}{2}\left(\log\frac{|x + H(x,y)|}{|x|}\right)^2 - \frac{\langle H(x,y), x\rangle}{|x|^2}\right) \\
&\qquad + (V_p(x + H(x,y)) - |x + H(x,y)|^p).
\end{aligned}$$

The term

$$(V_p(x + H(x,y)) - |x + H(x,y)|^p)$$

is not positive by the definition of V_p (cf. (V3)) and since $\tilde{p} < p$, we have

$$\begin{aligned}
&V_p(x + H(x,y)) - V_p(x) - \langle H(x,y), DV_p(x)\rangle \\
&\quad \leq p\,|x|^p \left(\log\frac{|x + H(x,y)|}{|x|} - \frac{\langle H(x,y), x\rangle}{|x|^2}\right) \\
&\qquad + \frac{p^2}{2}\,|x|^p \left(\log\frac{|x + H(x,y)|}{|x|}\right)^2.
\end{aligned} \tag{4.8}$$

Both the terms on the right-hand side of (4.8) are integrable over $\{|y| < c\} \cap \mathcal{H}(x)$ as follows from the fact that $\log(a) \le a - 1$ for any $a \in (0, \infty)$:

$$\begin{aligned} 0 &\le \int_{\{|y|<c\}\cap\mathcal{H}(x)} \left(\log \frac{|x + H(x,y)|}{|x|}\right)^2 \nu(dy) \\ &\le \int_{\{|y|<c\}\cap\mathcal{H}(x)} \left(\frac{|H(x,y)|}{|x|}\right)^2 \nu(dy) < \infty \end{aligned} \tag{4.9}$$

by local boundedness of (3.8). Similarly, we estimate from above

$$\begin{aligned} &\int_{\{|y|<c\}\cap\mathcal{H}(x)} \log \frac{|x + H(x,y)|}{|x|} - \frac{\langle H(x,y), x\rangle}{|x|^2} \nu(dy) \\ &\quad = \frac{1}{2} \int_{\{|y|<c\}\cap\mathcal{H}(x)} \log \frac{|x + H(x,y)|^2}{|x|^2} - 2\frac{\langle H(x,y), x\rangle}{|x|^2} \nu(dy) \\ &\quad \le \frac{1}{2} \int_{\{|y|<c\}\cap\mathcal{H}(x)} \frac{|x + H(x,y)|^2 - |x|^2}{|x|^2} - 2\frac{\langle H(x,y), x\rangle}{|x|^2} \nu(dy) \\ &\quad = \int_{\{|y|<c\}\cap\mathcal{H}(x)} \left(\frac{|H(x,y)|}{|x|}\right)^2 \nu(dy) < \infty. \end{aligned} \tag{4.10}$$

Estimation from below is not needed as the left-hand side of (4.8) is integrable. Therefore, by (4.8) we have

$$\begin{aligned} &\int_{\{|y|<c\}\cap\mathcal{H}(x)} V_p(x + H(x,y)) - V_p(x) - \langle H(x,y), x\rangle \nu(dy) \\ &\quad \le p\,|x|^p \left(\int_{\{|y|<c\}\cap\mathcal{H}(x)} \log \frac{|x + H(x,y)|}{|x|} - \frac{\langle H(x,y), x\rangle}{|x|^2} \nu(dy) \right. \\ &\qquad \left. + \frac{p}{2} \int_{\{|y|<c\}\cap\mathcal{H}(x)} \left(\log \frac{|x + H(x,y)|}{|x|}\right)^2 \nu(dy)\right). \end{aligned} \tag{4.11}$$

Now by (4.6) and (4.11) we get the desired inequality in (5.3). □

Now, we prove the main result of this section. For this purpose, we assume that there exists $b \in \mathbb{R}$ and $\underline{\sigma}, \overline{\sigma} \in (0, \infty)$ and $K > 1$ such that

$$\begin{aligned} &\langle f(x), x\rangle \le b\,|x|^2, \\ &|g(x)| \le \overline{\sigma}\,|x|, \quad \frac{|g(x)^T x|}{|x|^2} \ge \underline{\sigma} \end{aligned} \tag{4.12}$$

for $x \in B_K^c$.

Remark 4.1. An important example of coefficient $g : \mathbb{R}^m \mapsto \mathbb{R}^{m\times n}$ that satisfies (4.12) is when $n = 1$ and

$$g(x) = Gx, \quad x \in \mathbb{R}^m,$$

for some $G = (g_{ij}) \in \mathbb{R}^{m\times m}$ positive-definite. Then we may take $\overline{\sigma} = |G|$ and we have

$$\left|g(x)^T x\right| = \langle Gx, x\rangle \geq \underline{\sigma}\,|x|^2, \quad x \in \mathbb{R}^m$$

for some $\underline{\sigma} \in (0,\infty)$.

Theorem 4.1. *Assume that* f, g *satisfy* (4.12). *Let* $R_0 \in (1,\infty)$ *be such that*

$$\alpha := \sup_{x\in B^c_{R_0}} \int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \log\frac{|x+H(x,y)|}{|x|} - \frac{\langle H(x,y),x\rangle}{|x|^2}\nu(dy) < \infty \quad (4.13)$$

and

$$\sup_{x\in B^c_{R_0}} \int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \left(\log\frac{|x+H(x,y)|}{|x|}\right)^2 \nu(dy) < \infty, \quad (4.14)$$

where $\mathcal{H}(x)$ *is defined in* (4.2). *Then the solution to* (4.1) *with any deterministic initial condition is bounded in probability in the mean if*

$$b + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \alpha < 0. \quad (4.15)$$

Moreover, the condition (4.15) *need not to be satisfied if* $\nu(\mathcal{H}(x)) > 0$ *uniformly in* $x \in B^c_{R_0}$.

If we assume the growth condition (GRO), then $\alpha \leq L_2 < \infty$, where L_2 is from (GRO) and α is from (4.13) . Moreover, the condition (4.14) is satisfied. We summarize this claim in the following corollary.

Corollary 4.1. *Assume* (GRO) *and* (4.12). *Then the solution to* (4.1) *with any deterministic initial condition is bounded in probability in the mean if* (4.15) *holds, where*

$$\alpha := \sup_{x\in B^c_1} \int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \log\frac{|x+H(x,y)|}{|x|} - \frac{\langle H(x,y),x\rangle}{|x|^2}\nu(dy). \quad (4.16)$$

Proof. It easily follows by (4.9), (4.10) and Theorem 4.1. □

Proof of Theorem 4.1. We have to verify that (3.12) with A_R satisfying (3.11) holds.

Let $p \in (0,1)$ be fixed. First, observe that due to (4.12), we have $K \in (1,\infty)$ such that

$$\begin{aligned} &\langle f(x), DV_p(x)\rangle + \frac{1}{2}\mathrm{Tr}(g(x)^T D^2 V_p(x) g(x)) \\ &\quad = p\,|x|^{p-2}\,\langle f(x), x\rangle + \frac{1}{2}p(p-2)\,|x|^{p-4}\,\left|g(x)^T x\right|^2 + \frac{1}{2}p\,|x|^{p-2}\,|g(x)|^2 \\ &\quad \le p\,|x|^p \left(b + \frac{1}{2}\overline{\sigma}^2 + \frac{1}{2}(p-1)\underline{\sigma}^2\right) \end{aligned} \tag{4.17}$$

for $x \in B_K^c$.

Combining (4.17) with (5.3) we obtain

$$\begin{aligned} \mathcal{L}V_p(x) \le p\,|x|^p &\Bigg[b + \frac{1}{2}\overline{\sigma}^2 + \left(\frac{1}{2}p - 1\right)\underline{\sigma}^2 \\ &\quad + \int_{\{|y|<c\}\cap\mathcal{H}(x)^c} \log\frac{|x+H(x,y)|}{|x|} - \frac{\langle H(x,y), x\rangle}{|x|^2}\nu(dy) \\ &\quad + \frac{p}{2}\int_{\{|y|<c\}\cap\mathcal{H}(x)^c}\left(\log\frac{|x+H(x,y)|}{|x|}\right)^2\nu(dy) \\ &\quad - \left(\frac{1}{p}-1\right)\nu(\mathcal{H}(x))\Bigg] \\ \le p\,|x|^p &\Bigg[b + \frac{1}{2}\overline{\sigma}^2 + \frac{1}{2}(p-1)\underline{\sigma}^2 + \alpha \\ &\quad + \frac{p}{2}\sup_{x\in B_{R_0}}\int_{\{|y|<c\}\cap\mathcal{H}(x)^c}\left(\log\frac{|x+H(x,y)|}{|x|}\right)^2\nu(dy)\Bigg], \end{aligned} \tag{4.18}$$

for $x \in B_R^c$, where $R = R_0 \vee K$. In (4.18) note that $1/p - 1 > 0$.

Now taking $p > 0$ sufficiently small we get that there exists $\kappa > 0$ such that

$$\mathcal{L}V_p(x) \le -\kappa\,|x|^p\,, \tag{4.19}$$

for $x \in B_R^c$ if (4.15) holds.

Moreover, if $\nu(\mathcal{H}(x)) > 0$ uniformly in $B_{R_0}^c$, taking into account that

$$-\left(\frac{1}{p}-1\right)\nu(\mathcal{H}(x)) \to -\infty, \quad p \to 0+,$$

uniformly in $x \in B_R^c$, (4.18) implies (4.19) even if (4.15) does not hold.

Finally, (4.19) already guarantees (3.12) with A_R satisfying (3.11), which completes the proof. □

We can see that in the condition (4.15) the sign of α that comes from (4.13) determines if the compensated integral stabilizes the system. The sign of α is determined by the interaction of two terms

$$\log \frac{|x + H(x, y)|}{|x|} \quad \text{and} \quad -\frac{\langle H(x, y), x\rangle}{|x|^2}, \tag{4.20}$$

for $x, y \in \mathbb{R}^n$ fixed. We can interpret the coefficient value $H(x, y)$ as the (vector) jump of the solution from state point x given that the driving noise attains value of y. Moreover, both the terms in (4.20) have always opposite sign. The first one is negative if the jump $H(x, y)$ "aims towards the origin" while the second one, coming from the compensation, is negative only when $|x + H(x, y)| > |x|$. This can be seen if we rewrite it as

$$-\frac{\langle H(x, y), x\rangle}{|x|^2} = -\cos(\phi(x, y))\frac{|H(x, y)|}{|x|},$$

where $\phi(x, y)$ is the angle between $H(x, y)$ and x. The following example quantifies this interplay in the case of simple linear equation.

Example 4.1. Let $\nu(\{|y| < c\}) < \infty$, $m = n = 2$ and consider the unique solution to the equation

$$\begin{aligned} dX_t &= \int_{\{|y|<c\}} H(X_{t-}, y)\tilde{N}(dt, dy), \quad t \geq 0, \\ X_0 &= x_0 \in \mathbb{R}^2, \end{aligned} \tag{4.21}$$

where $H(x, y) = qR_\phi x$, $x, y \in \mathbb{R}^2$ for some $q > 0$, and rotation matrix R_ϕ

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

where $\phi \in [0, 2\pi)$ and $x_0 \neq 0$. In this case, we can separate jumps and the compensating drift, so (4.21) shall be written as

$$\begin{aligned} dX_t &= -q\nu(\{|y| < c\})R_\phi X_{t-}dt + qR_\phi X_{t-}dP_t, \quad t \geq 0, \\ X_0 &= \in \mathbb{R}^2, \end{aligned}$$

where $P_t = N(t, \{|y| < c\}), t \geq 0$ is a Poisson process with intensity $\nu(\{|y| < c\})$.

We can use Corollary 4.1 to assess boundedness of X in probability in the mean. If $q = 1$ then $\mathcal{H}(x) = \{|y| < c\}$ and X is bounded in probability the mean. Rewriting (4.16) we see that the same holds if

$$\alpha = \int_{\{|y|<c\}} \log \frac{|x + qR_\phi x|}{|x|} - \frac{\langle qR_\phi x, x\rangle}{|x|^2}\nu(dy) < 0$$

for $x \in \mathbb{R}^2, x \neq 0$. More specifically,

$$\int_{\{|y|<c\}} \log \frac{|x + qR_\phi x|}{|x|} - \frac{\langle qR_\phi x, x\rangle}{|x|^2}\nu(dy)$$
$$= \nu(\{|y| < c\}) \left(\frac{1}{2}\log(1 + 2q\cos\phi + q^2) - q\cos\phi\right),$$

for $x \in \mathbb{R}^2, x \neq 0$. Therefore, X is bounded in probability in the mean if

$$\log(1 + 2q\cos\phi + q^2) < 2q\cos\phi. \tag{4.22}$$

Let us inspect the condition (4.22) in more detail.

Denote by

$$\mathcal{S} = \{(q, \phi) \in (0, \infty) \times [0, 2\pi) : \log(1 + 2q\cos\phi + q^2) < 2q\cos\phi\}$$

the set of couples $(q, \phi) \in (0, \infty) \times [0, 2\pi)$ for which (4.22) holds. We briefly inspect the set $\mathcal{S}$ by considering three distinct cases.

Case 1. If $\phi \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ then if $q > 0$ is big enough, we have $(q, \phi) \in \mathcal{S}$. This corresponds to the case when the jumps point in the opposite direction to origin, therefore, the stabilization effect is driven by the compensating drift. The smaller $|\cos\phi|$ is, the smaller $q > 0$ is needed in order to have $(\phi, q) \in \mathcal{S}$ and if $\phi \in [0, \frac{\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi)$ then $(\phi, q) \in \mathcal{S}$ for any $q > 0$.

Case 2. If $\phi \in [\frac{\pi}{2}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{3\pi}{2}]$, then for any $q > 0$ we have $(\phi, q) \in \mathcal{S}^c$, where

$$\mathcal{S}^c = \{(q, \phi) \in (0, \infty) \times [0, 2\pi) : \log(1 + 2q\cos\phi + q^2) \geq 2q\cos\phi\}$$

and we do not observe the stabilization property of the compensated integral. This case also covers both the jumps and compensation being orthogonal to the direction to the origin.

Case 3. If $\phi \in (\frac{3\pi}{4}, \frac{5\pi}{4})$, then $(\phi, q) \in \mathcal{S}$ if $q > 0$ is small enough. The larger $|\cos\phi|$ is, the larger $q > 0$ we can take to keep $(\phi, q) \in \mathcal{S}$. For $\phi = \pi$, we can even take $q > 1$; then the solution jumps over the origin and still the stabilization property is obtained. However, $q = \frac{3}{2}$ is already too large and $(\pi, \frac{3}{2}) \in \mathcal{S}^c$, which means that even though the norm of the process after jump decreases, the stabilization property is lost. This is due to the compensation term, which drives the system in the direction opposite to the jumps.

Note that the solution X can be constructed directly using the interlacing procedure. Denote τ_k, $k \in \mathbb{N}$ the arrival times for P. Using the matrix exponentiation,

X_t takes the form

$$X_t = (I + R_\phi)^k e^{-\nu(\{|y|<c\})qtR_\phi} x_0,$$

if $t \in [\tau_k, \tau_{k+1})$, where $k \in \mathbb{N}_0$. Moreover, for $p > 0$, we are able to compute the moments explicitly

$$\mathbb{E}\,|X_t|^p = |x_0|^p\, e^{-\lambda t(1+pq\cos\phi - |I+qR_\phi|^p)} \tag{4.23}$$

for $t \geq 0$. Using (4.23), it can be shown that, if $p > 0$ is sufficiently small, then

$$\mathbb{E}\ |X_t|^p \to 0, \quad t \to \infty$$

if and only if $(\phi, q) \in \mathcal{S}$ and

$$\mathbb{E}\ |X_t|^p \to \infty, \quad t \to \infty$$

if and only if $(\phi, q) \in \mathcal{S}^c$.

In Corollary 4.1, we have seen that (4.1) is a sufficient condition for boundedness in probability in the mean under some assumptions. This example shows that in the case of the linear equation, (4.1) provides also if and only if condition for convergence of pth moment of the solution for sufficiently small $p > 0$ in the infinite time horizon.

5. Stabilization by Uncompensated Jumps

Throughout this section, we assume Assumption 3.1 to hold. We investigate stabilization properties of uncompensated jumps. Thus, we put $H = 0$ in (2.1) for simplicity and obtain the following equation:

$$dX_t = f(X_{t-})dt + g(X_{t-})dW_t + \int_{\{|y|\geq c\}} K(X_{t-}, y)N(dt, dy), \quad t \geq 0. \tag{5.1}$$

The main result of this section is presented in Theorem 5.1. Now, we proceed similarly as in the previous section. Set

$$\mathcal{K}(x) = \{y \in \mathbb{R}^n : |y| \geq c, x + K(x, y) = 0\}, \tag{5.2}$$

for $x \in \mathbb{R}^m$. We have the following technical lemma.

Lemma 5.1. *For $p \in (0, 1/2)$, we have*

$$\begin{aligned} &\int_{\{|y|<c\}} V_p(x + K(x, y)) - V_p(x)\nu(dy) \\ &\quad \leq p\,|x|^p \left[\int_{\{|y|<c\}\cap\mathcal{K}(x)^c} \log\frac{|x + K(x, y)|}{|x|}\nu(dy)\right. \\ &\qquad \left. + \frac{p}{2}\int_{\{|y|<c\}\cap\mathcal{K}(x)^c} \left(\log\frac{|x + K(x, y)|}{|x|}\right)^2 \nu(dy) - \left(\frac{1}{p} - 1\right)\nu(\mathcal{K}(x))\right] < \infty, \end{aligned} \tag{5.3}$$

for $x \in B_1^c$.

Proof. To prove the inequality (5.3) the same ideas as when proving the similar estimate (4.4) for the case of compensated integral can be used. This time we estimate the first integral on the right-hand side of (5.3) from above,

$$\begin{aligned}
&\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \log\frac{|x+K(x,y)|}{|x|}\nu(dy)\\
&\quad=\frac{1}{p}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \log\frac{|x+K(x,y)|^p}{|x|^p}\nu(dy)\\
&\quad\leq\frac{1}{p}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \frac{|x+K(x,y)|^p-|x|^p}{|x|^p}\nu(dy)\\
&\quad\leq\frac{1}{p}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \frac{|K(x,y)|^p}{|x|^p}\nu(dy)<\infty,
\end{aligned}$$

where we used the local boundedness of (3.9). For the second integral on the right-hand side in (5.3)

$$\begin{aligned}
0&\leq\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \left(\log\frac{|x+K(x,y)|}{|x|}\right)^2\nu(dy)\\
&=\frac{1}{p^2}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \left(\log\frac{|x+K(x,y)|^p}{|x|^p}\right)^2\nu(dy)\\
&\leq\frac{1}{p^2}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \left(\frac{|x+K(x,y)|^p-|x|^p}{|x|^p}\right)^2\nu(dy)\\
&\leq\frac{1}{p^2}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)} \frac{|K(x,y)|^{2p}}{|x|^{2p}}\nu(dy)<\infty
\end{aligned}$$

again using (3.9) (here we need $p\in(0,1/2)$). □

Having the estimate (5.3) at hand we obtain similar result for uncompensated integral as in the compensated case in Theorem 4.1.

Theorem 5.1. *Assume that f, g satisfy (4.12). Let $R_0\in(1,\infty)$ be such that*

$$\beta:=\sup_{x\in B^c_{R_0}}\int_{\{|y|\geq c\}\cap\mathcal{K}(x)^c} \log\frac{|x+K(x,y)|}{|x|}\nu(dy)<\infty \tag{5.4}$$

and

$$\sup_{x\in B^c_{R_0}}\int_{\{|y|<c\}\cap\mathcal{K}(x)^c} \left(\log\frac{|x+K(x,y)|}{|x|}\right)^2\nu(dy)<\infty, \tag{5.5}$$

where $\mathcal{K}(x)$ is defined in (5.2). Then the solution to (5.1) with any deterministic initial condition is bounded in probability in the mean if

$$b+\frac{1}{2}\overline{\sigma}^2-\underline{\sigma}^2+\beta<0. \tag{5.6}$$

Moreover, the condition (5.6) *need not to be satisfied if* $\nu(\mathcal{K}(x)) > 0$ *uniformly in* $x \in B^c_{R_0}$.

Proof. The proof is analogous to the proof of Theorem 4.1, we only here use the estimate (5.3) in place of (4.4). ☐

We see that the sign of β in (5.4) determines if the uncompensated integral stabilizes the system. Unlike in the case of compensated integral in previous section we are able to develop criterion which is determined directly by direction of the jumps. It turns out that in such case *ad hoc* computations are more efficient than using general result from Theorem 5.1. These computations depend heavily on the fact that the intensity of jumps $\nu(\{|y| \geq c\})$ of the uncompensated integral is finite.

Theorem 5.2. *Assume that* f, g *satisfy* (4.12). *Furthermore assume that there exist* $\gamma > 0$, $\alpha \in [0, 1)$, $L \in (0, \infty)$ *and* $R_0 \in (0, \infty)$ *such that*

$$|x + K(x, y)| \leq \gamma \, |x|^{1-\alpha} + L \tag{5.7}$$

for $x \in B^c_{R_0}$ *and* $y \in \{|y| \geq c\}$. *Then the solution to* (5.1) *is bounded in probability in the mean for any deterministic initial condition if either*

- $\alpha \neq 0$ *or*
- (5.6) *with* $\beta = \log \gamma$ *holds, i.e. if*

$$b + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \nu(\{|y| \geq c\}) \log \gamma < 0. \tag{5.8}$$

Proof. We verify that (2.4) holds. Taking arbitrary $p \in (0, 1)$, $n \in \mathbb{N}, n \geq R_0$ and $x \in B^c_n$, (V2), (V3) and (5.7) yield

$$\begin{aligned}
&\int_{\{|y|\geq c\}} V_p(x + K(x, y)) - V_p(x)\nu(dy) \\
&\quad= \int_{\{|y|\geq c\}} |x + K(x, y)|^p - |x|^p + (V_p(x + K(x, y)) - |x + K(x, y)|^p)\nu(dy) \\
&\quad\leq \int_{\{|y|\geq c\}} |x + K(x, y)|^p - |x|^p \, \nu(dy) + 2\nu(|y| \geq c) \\
&\quad\leq \int_{\{|y|\geq c\}} \gamma^p \, |x|^{p(1-\alpha)} - |x|^p \, \nu(dy) + 2\nu(|y| \geq c) \\
&\quad= p \, |x|^p \, \frac{(\frac{\gamma}{|x|^\alpha})^p - 1}{p} \nu(|y| \geq c) + 2\nu(|y| \geq c) \\
&\quad\leq p \, |x|^p \, \frac{(\frac{\gamma}{n^\alpha})^p - 1}{p} \nu(|y| \geq c) + 2\nu(|y| \geq c) \\
&\quad= p \, |x|^p \left(\log \frac{\gamma}{n^\alpha} + \frac{p}{2} \left(\log \frac{\gamma}{n^\alpha} \right)^2 \right) + 2\nu(\{|y| \geq c\}).
\end{aligned}$$

Taking into account (4.17) and (4.12) we have $K \in (0, \infty)$ such that

$$\mathcal{L}V_p(x) \le p\,|x|^p \left(b + \frac{1}{2}\overline{\sigma}^2 + \frac{1}{2}(p-1)\underline{\sigma}^2 + \log\frac{\gamma}{n^\alpha} + \frac{p}{2}\left(\log\frac{\gamma}{n^\alpha}\right)^2 \right) + 2\nu(\{|y| \ge c\}) \tag{5.9}$$

for $x \in B_R^c$, where $R \ge R_0 \vee K$. For the case $\alpha = 0$ we further simplify (5.9) to

$$\mathcal{L}V_p(x) \le p\,|x|^p \left(b + \frac{1}{2}\overline{\sigma}^2 + \frac{1}{2}(p-1)\underline{\sigma}^2 + \log\gamma + \frac{p}{2}(\log\gamma)^2 \right) + 2\nu(\{|y| < c\}).$$

If we take $p > 0$ sufficiently small, we see that if (5.8) holds, there exists $\kappa > 0$ and $R \in (0, \infty)$ such that

$$\mathcal{L}V_p(x) \le -\kappa\,|x|^p \tag{5.10}$$

for $x \in B_R^c$.

On the other hand, for $\alpha > 0$ the inequality (5.10) may be shown for some $\tilde{\kappa} > 0$ even without assumption (5.8) by taking $n \in \mathbb{N}$ sufficiently large. Indeed, (5.9) shall be rewritten as

$$\mathcal{L}V_p(x) \le p\,|x|^p\,(\omega - \alpha\log n) + \nu(\{|y| \ge c\})$$

for $x \in B_R^c$, where $R \ge R_0 \vee K$ and

$$\omega = b + \frac{1}{2}\overline{\sigma}^2 + \frac{1}{2}(p-1)\underline{\sigma}^2 + \left(1 + \frac{p}{2}\right)\log\gamma. \qquad \square$$

Remark 5.1. Corollary 5.2 tells us that the uncompensated term stabilizes our system if the jumps tend towards the origin and intensity of this stabilization is proportional to the intensity of the jumps. Indeed, the system may remind stable in the sense of boundedness in probability in the mean even for jumps in the direction opposite to the origin. Also, taking $\alpha \ne 0$ in Theorem 5.2 we see that if the norm of the process after jump gets small enough, then the system is stabilized even with arbitrarily small intensity of the jumps.

In the case of the linear system Corollary 5.2.

Example 5.1. Let $m = n = 1$, $\nu(\{|y| \ge c\}) > 0$ and in (5.1) we put $f(x) = bx$, $g(x) = \sigma x, K(x, y) = qx, x \in \mathbb{R}, y \in \{|y| \ge c\}$ for some $b, \sigma, q \in \mathbb{R}$, i.e. we deal with the equation

$$\begin{aligned} dX_t &= bX_{t-}dt + \sigma X_{t-}dW_t + qX_{t-}dP_t, \quad t \ge 0, \\ X_0 &= x_0 \end{aligned} \tag{5.11}$$

for some $x_0 \in \mathbb{R}$ where $P_t = \int_{\{|y|\geq c\}} N(t, \{|y| \geq c\}), t \geq 0$, is a Poisson process with intensity $\nu(\{|y| \geq c\})$. In this case, we have

$$|x + K(x, y)| = |x|\,|1 + q|$$

for $x \in \mathbb{R}, y \in \{|y| \geq c\}$. Therefore, if $q \neq -1$, we may put $\gamma = |1 + q|\,, \alpha = L = 0$ in (5.7) and obtain boundedness in probability in the mean for (5.11) provided

$$b - \frac{\sigma^2}{2} + \nu(\{|y| \geq c\}) \log |1 + q| < 0 \tag{5.12}$$

holds.

Therefore, the uncompensated term stabilizes the considered system for $q \in (-2, -1) \cup (-1, 0)$. The case $q = -1$ leads to $\mathcal{K}(x) = \{|y| \geq c\}, x \in \mathbb{R}$, where $\mathcal{K}(x)$ is defined in (5.2). Therefore, we may use Theorem 5.1 directly and obtain boundedness in probability in the mean regardless the sign in (5.12).

Now, we present an example that merges results from Secs. 3 and 4. We compare the stabilization properties of compensated and uncompensated integrals occurring together and treat the constant $c \in (0, \infty)$ as a parameter of the problem.

Example 5.2. Consider Eq. (2.1) with finite Lévy measure ν. For notational simplicity, set

$$M(x, y) = \begin{cases} H(x, y) & (x, y) \in \mathbb{R}^m \times \{|y| < c\}, \\ K(x, y) & (x, y) \in \mathbb{R}^m \times \{|y| \geq c\}. \end{cases} \tag{5.13}$$

Therefore, (2.1) can be written as

$$\begin{aligned} dX_t = f(X_{t-})dt + g(X_{t-})dW_t + \int_{\{|y|<c\}} M(X_{t-}, y)\tilde{N}(dt, dy) \\ + \int_{\{|y|\geq c\}} M(X_{t-}, y)N(dt, dy), \quad t \geq 0. \end{aligned} \tag{5.14}$$

Assume that (4.12) holds with f locally finite and in compliance with Assumption (5.7) of Theorem 5.2 let there exist $\gamma > 0$ such that

$$|x + M(x, y)| \leq \gamma\,|x|$$

for every $x \in B_1^c$ and $y \in \mathbb{R}^n \backslash \{0\}$. Furthermore, for simplicity assume that the solution X does not jump directly to the origin almost surely, i.e.

$$\nu(\{y \in \mathbb{R}^n \backslash \{0\} : x + M(x, y) = 0\}) = 0$$

for every $x \in B_1^c$.

Then Assumption 3.1 is satisfied and (5.14) can be rewritten as

$$dX_t = \left(f(X_{t-}) - \int_{\{|y|<c\}} M(X_{t-}, y)\nu(dy) \right) dt + g(X_{t-})dW_t$$
$$+ \int_{\mathbb{R}^n \setminus \{0\}} M(X_{t-}, y)N(dt, dy), \quad t \geq 0.$$

Therefore, we may expect that the parameter $c \in (0, \infty)$ influences the stability properties of our system only through the perturbation of the drift

$$-\int_{\{|y|<c\}} M(X_{t-}, y)\nu(dy).$$

We now combine proofs of Theorems 4.1 and 5.1 to assess stability in terms of boundedness in probability in the mean and obtain the condition

$$\left(b - \inf_{x \in B_1^c} \int_{\{|y|<c\}} \frac{\langle M(x, y), x\rangle}{|x|^2} \nu(dy) \right) + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \nu(\mathbb{R}^n \setminus \{0\}) \log \gamma < 0.$$

We can see that stability properties of (5.14) may depend on $c \in (0, \infty)$. In the simple case

$$M(x, y) = qx, \quad x \in \mathbb{R}^m, \ y \in \mathbb{R}^n \setminus \{0\},$$

for some $q \in \mathbb{R}, q \neq -1$, we get condition

$$b - q\nu(\{|y| < c\}) + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \nu(\mathbb{R}^n \setminus \{0\}) \log |1 + q| < 0,$$

or equivalently

$$\frac{b + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2}{\nu(\mathbb{R}^n \setminus \{0\})} + \log |1 + q| < qr(c), \tag{5.15}$$

where

$$r(c) = \frac{\nu(\{|y| < c\})}{\nu(\mathbb{R}^n \setminus \{0\})} \in [0, 1], \quad c \in (0, \infty).$$

The function $r : (0, \infty) \to [0, 1]$ is non-decreasing and by (5.15) we may conclude that if $q < 0$, which corresponds to the case when the solution exhibits jumps towards origin, the chance that (5.15) is satisfied gets smaller with increasing c. For $q > 0$, we get the opposite behavior.

6. Invariant Measure

In the final section, we formulate the main result of this paper concerning the existence of the invariant measure for Eq. (2.1). Due to Krylov–Bogolyubov Theorem (cf. Theorem 2.1), it easily follows by Theorems 4.1 and 5.1.

We treat the compensated and uncompensated term simultaneously using the notation as in (5.13) and rewrite (2.1) as

$$dX_t = f(X_{t-})dt + g(X_{t-})dW_t + \int_{\{|y|<c\}} M(X_{t-}, y)\tilde{N}(dt, dy)$$
$$+ \int_{\{|y|\geq c\}} M(X_{t-}, y)N(dt, dy), \quad t \geq 0. \tag{6.1}$$

To exclude the cases when the jumps in (6.1) aim directly into the origin, we again adopt the useful notation

$$\mathcal{M}(x) = \{y \in \mathbb{R}^n : x + M(x, y) = 0\}$$

for $x \in \mathbb{R}^m$.

Theorem 6.1. (Invariant measure) *Let Assumption* 3.1 *hold and let Eq.* (6.1) *define Markov process which is Feller. Moreover, let* $b, \underline{\sigma}, \overline{\sigma}, \gamma \in \mathbb{R}, \underline{\sigma}, \overline{\sigma} > 0$ *be such that there exists* $R_0 \in (0, \infty)$ *such that*

$$\langle f(x), x\rangle < b\,|x|^2\,,$$
$$|g(x)| < \overline{\sigma}\,|x|\,, \qquad \frac{|g(x)^T x|}{|x|^2} > \underline{\sigma},$$
$$\int_{\mathcal{M}(x)^c} \left(\log \frac{|x + M(x, y)|}{|x|} - \mathbf{1}_{\{|y|<c\}}(y) \frac{\langle M(x, y), x\rangle}{|x|^2}\right) \nu(dy) < \gamma,$$

for $x \in B^c_{R_0}$, *where* $\mathbf{1}_{\{|y|<c\}}$ *denotes the indicator function of the set* $\{|y| < c\}$, *and let*

$$\sup_{x\in B^c_{R_0}} \int_{\mathcal{M}(x)^c} \left(\log \frac{|x + M(x, y)|}{|x|}\right)^2 \nu(dy) < \infty.$$

If

$$b + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \gamma < 0$$

then Eq. (6.1) *possesses an invariant measure.*

Proof. The statement follows directly from Krylov–Bogolyubov Theorem by Theorems 4.1 and 5.1. □

Now recall the standard assumptions (LIP), (GRO) and (CON) (which translate into assumptions on M in an obvious way) under which the existence of Markov process defined by (6.1) is guaranteed. It is known (cf. [2, Secs. 6.6 and 6.7]) that this process is Feller if in place of (GRO) the following stronger condition is assumed.

Growth condition II: There exist $H_1 : \mathbb{R}^m \mapsto \mathbb{R}_+, H_2 : \mathbb{R}^n \mapsto \mathbb{R}_+$ such that

$$|H(x,y)| \leq H_1(x)H_2(y), \tag{GRO II}$$

$x \in \mathbb{R}^m, y \in \{|y| < c\}$, H_1 is Lipschitz continuous and $\int_{\{|y|<c\}} H_2(y)^2 \nu(dy) < \infty$.

Indeed, in (cf. [2, Note after Theorem 6.6.3]) it is shown that under the assumptions (LIP), (GRO II) and (CON) the (6.1) defines Markov process such that the mapping

$$x \mapsto X_t^x, \quad x \in \mathbb{R}^m$$

has an almost surely continuous modification for $t > 0$. This already implies Feller property as in (2.3) by the Dominated Convergence Theorem.

Moreover, Theorem 6.1 may be simplified as follows.

Theorem 6.2. (Invariant measure II) *Let assumption* (LIP), (GRO II) *and* (CON) *hold. Moreover, let* $b, \underline{\sigma}, \overline{\sigma}, \gamma \in \mathbb{R}$, $\underline{\sigma}, \overline{\sigma} > 0$ *be such that there exists* $R_0 \in (0, \infty)$ *such that*

$$\langle f(x), x \rangle < b\,|x|^2\,,$$

$$|g(x)| < \overline{\sigma}\,|x|\,, \quad \frac{\left|g(x)^T x\right|}{|x|^2} > \underline{\sigma},$$

$$\int_{\mathcal{M}(x)^c} \left(\log \frac{|x + M(x,y)|}{|x|} - \mathbf{1}_{\{|y|<c\}}(y) \tfrac{\langle M(x,y), x\rangle}{|x|^2} \right) \nu(dy) < \gamma,$$

for $x \in B_{R_0}^c$.

If

$$b + \frac{1}{2}\overline{\sigma}^2 - \underline{\sigma}^2 + \gamma < 0,$$

then Eq. em (6.1) *possesses an invariant measure.*

Proof. The statement follows directly from Krylov–Bogolyubov Theorem by Corollaries 4.1 and 5.2 since (6.1) defines Feller Markov process. □

Acknowledgements

This research was partially supported by Czech Science Foundation (GAČR) Grant No. 19-07140S and by the SVV Grant No. 260580.

References

1. S. Albeverio, L. Di Persio, E. Mastrogiacomo and B. Smii, A class of Lévy driven SDEs and their explicit invariant measures, *Potential Anal.* **45** (2016) 229–259.
2. D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd edn. (Cambridge Univ. Press, 2009).

3. D. Applebaum, Infinite dimensional Ornstein–Uhlenbeck processes driven by Lévy processes, *Probab. Surveys* **12** (2015) 33–54.
4. D. Applebaum and M. Siakalli, Asymptotic stability of stochastic differential equations driven by Lévy noise, *J. Appl. Probab.* **46** (2009) 1116–1129.
5. D. Applebaum and M. Siakalli, Stochastic stabilization of dynamical systems using Lévy noise, *Stoch. Dyn.* **10** (2010) 509–527.
6. C. Bhan, P. Chakraborty and V. S. Mandrekar, Invariant measure and stability of the solution of a stochastic differential equation driven by a jump Lévy process, *Int. J. Contemp. Math. Sci.* **7** (2012) 33–52.
7. J. P. Brockwell, Lévy-driven CARMA processes, *Ann. Inst. Statist. Math.* **53** (2001) 113–124.
8. M. Grigoriu, Lyapunov exponents for nonlinear systems with Poisson white noise, *Phys. Lett. A* **217** (1996) 258–262.
9. Z. J. Jurek, An integral representation of operator-self decomposable random variables, *Bull. Acad. Pol. Sci.* **30** (1982) 385–393.
10. K. Kadlec and B. Maslowski, Ergodic boundary and point control for linear stochastic PDEs driven by a cylindrical Lévy process, *Discrete Contin. Dyn. Syst.* **25** (2000) 4039–4055.
11. K. Kadlec and B. Maslowski, Ergodic control for Lévy-driven linear stochastic equations in Hilbert spaces, *Appl. Math. Optim.* **79** (2019) 547–565.
12. R. Z. Khasminskii, *Stochastic Stability of Differential Equations* (Sijthofff and Noordhoff, 1980).
13. N. Krylov and N. Bogolyubov, La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire, *Ann. Math.* **38** (1937) 65–113.
14. U. Kumar and M. Riedle, Invariant measure for the stochastic Cauchy problem driven by a cylindrical Lévy process, *J. Math. Anal. Appl.* **493** (2021) 124536.
15. G. Leha, B. Maslowski and G. Ritter, Stability of solutions to semilinear stochastic evolution equations, *Stochastic Anal. Appl.* **17** (1999) 1009–1051.
16. C. W. Li, Z. Dong and R. Situ, Almost sure stability of linear stochastic differential equations with jumps, *Probab. Theory Related Fields* **123** (2002) 121–155.
17. X. Mao, *Stochastic Differential Equations and Applications*, 2nd edn. (Elsevier, 2008).
18. X. Mao and A. E. Rodkina, Exponential stability of stochastic differential equations driven by discontinuous semimartingales, *Stoch. Int. J. Probab. Stoch. Process.* **55** (1995) 207–224.
19. E. Nane and Y. Ni, Stability of the solution of stochastic differential equation driven by time-changed Lévy noise, *Proc. Amer. Math. Soc.* **145** (2017) 3085–3104.
20. B. K. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions* (Springer, 2007).
21. A. Patel and B. Kosko, Stochastic resonance in continuous and spiking neuron models with Lévy noise, *IEEE Trans. Neural Netw.* **19** (2008) 1993–2008.
22. P. Tankov, *Financial Modelling with Jump Processes* (Chapman and Hall/CRC, 2003).
23. X. Zhang, Y. Xu, B. Schmalfuß and B. Pei, Random attractors for stochastic differential equations driven by two-sided Lévy processes, *Stochastic Anal. Appl.* **37** (2019) 1028–1041.