# Efficient Mixed-Norm Regularization: Algorithms and Safe Screening Methods

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#### Abstract

Sparse learning has recently received increasing attention in many areas including machine learning, statistics, and applied mathematics. The mixed-norm regularization based on the  $\ell_1/\ell_q$  norm with q>1is attractive in many applications of regression and classification in that it facilitates group sparsity in the model. The resulting optimization problem is, however, challenging to solve due to the inherent structure of the  $\ell_1/\ell_q$ -regularization. Existing work deals with special cases including  $q=2,\infty$ , and they can not be easily extended to the general case. In this paper, we propose an efficient algorithm based on the accelerated gradient method for solving the  $\ell_1/\ell_q$ -regularized problem, which is applicable for all values of q larger than 1, thus significantly extending existing work. One key building block of the proposed algorithm is the  $\ell_1/\ell_q$ -regularized Euclidean projection (EP<sub>1q</sub>). Our theoretical analysis reveals the key properties of  $EP_{1q}$  and illustrates why  $EP_{1q}$  for the general q is significantly more challenging to solve than the special cases. Based on our theoretical analysis, we develop an efficient algorithm for  $EP_{1q}$  by solving two zero finding problems. To further improve the efficiency of solving large dimensional  $\ell_1/\ell_q$  regularized problems, we propose an efficient and effective "screening" method which is able to quickly identify the inactive groups, i.e., groups that have 0 components in the solution. This may lead to substantial reduction in the number of groups to be entered to the optimization. An appealing feature of our screening method is that the data set needs to be scanned only once to run the screening. Compared to that of solving the  $\ell_1/\ell_q$ -regularized problems, the computational cost of our screening test is negligible. The key of the proposed screening method is an accurate sensitivity analysis of the dual optimal solution when the regularization parameter varies. Experimental results demonstrate the efficiency of the proposed algorithm.

# 1 Introduction

Regularization has played a central role in many machine learning algorithms. The  $\ell_1$ -regularization has recently received increasing attention, due to its sparsity-inducing property, convenient convexity, strong theoretical guarantees, and great empirical success in various applications. A well-known application of the  $\ell_1$ -regularization is the Lasso [37]. Recent studies in areas such as machine learning, statistics, and applied mathematics have witnessed growing interests in extending the  $\ell_1$ -regularization to the  $\ell_1/\ell_q$ regularization [2, 8, 17, 26, 32, 35, 41, 45, 46]. This leads to the following  $\ell_1/\ell_q$ -regularized minimization problem:

$$\min_{\mathbf{W}\in\mathbb{R}^{p}} f(\mathbf{W}) \equiv l(\mathbf{W}) + \lambda \varpi(\mathbf{W}), \tag{1}$$

where  $\mathbf{W} \in \mathbb{R}^p$  denotes the model parameters,  $l(\cdot)$  is a convex loss dependent on the training samples and their corresponding responses,  $\mathbf{W} = [\mathbf{w}_1^{\mathrm{T}}, \mathbf{w}_2^{\mathrm{T}}, \dots, \mathbf{w}_s^{\mathrm{T}}]^{\mathrm{T}}$  is divided into *s* non-overlapping groups,  $\mathbf{w}_i \in \mathbb{R}^{p_i}, i = 1, 2, \dots, s, \lambda > 0$  is the regularization parameter, and

$$\varpi(\mathbf{W}) = \sum_{i=1}^{s} \|\mathbf{w}_i\|_q \tag{2}$$

is the  $\ell_1/\ell_q$  norm with  $\|\cdot\|_q$  denoting the vector  $\ell_q$  norm  $(q \ge 1)$ . One of the commonly used loss function is the least square loss, i.e.,  $l(\mathbf{W})$  takes the form as

$$l(\mathbf{W}) = \frac{1}{2} \|Y - B\mathbf{W}\|_2^2 = \frac{1}{2} \left\|Y - \sum_{i=1}^s B_i \mathbf{w}_i\right\|_2^2,$$
(3)

where  $B = [B_1, B_2, \ldots, B_s] \in \mathbb{R}^{m \times p}$  is the data matrix with m samples and p features,  $B_i \in \mathbb{R}^{m \times p_i}$ ,  $i = 1, 2, \ldots, s$ , is corresponding to the  $i^{th}$  group, and  $Y \in \mathbb{R}^m$  denotes the response vector. The  $\ell_1/\ell_q$ regularization belongs to the composite absolute penalties (CAP) [46] family. When q = 1, the problem (1) reduces to the  $\ell_1$ -regularized problem. When q > 1, the  $\ell_1/\ell_q$ -regularization facilitates group sparsity in the resulting model, which is desirable in many applications of regression and classification.

The practical challenge in the use of the  $\ell_1/\ell_q$ -regularization lies in the development of efficient algorithms for solving (1), due to the non-smoothness of the  $\ell_1/\ell_q$ -regularization. According to the black-box Complexity Theory [28, 29], the optimal first-order black-box method [28, 29] for solving the class of nonsmooth convex problems converges as  $O(\frac{1}{\sqrt{k}})$  (k denotes the number of iterations), which is slow. Existing algorithms focus on solving the problem (1) or its equivalent constrained version for  $q = 2, \infty$ , and they can not be easily extended to the general case. In order to systematically study the practical performance of the  $\ell_1/\ell_q$ regularization family, it is of great importance to develop efficient algorithms for solving (1) for any q larger than 1.

### 1.1 First-Order Methods Applicable for (1)

When treating  $f(\cdot)$  as the general non-smooth convex function, we can apply the subgradient descent (SD) [6, 28, 29]:

$$\mathbf{X}_{i+1} = \mathbf{X}_i - \gamma_i \mathbf{G}_i,\tag{4}$$

where  $\mathbf{G}_i \in \partial f(\mathbf{X}_i)$  is a subgradient of  $f(\cdot)$  at  $\mathbf{X}_i$ , and  $\gamma_i$  a step size. There are several different types of step size rules, and more details can be found in [6, 28]. Subgradient descent is proven to converge, and it can yield a convergence rate of  $O(1/\sqrt{k})$  for k iterations. However, SD has the following two disadvantages: 1) SD converges slowly; and 2) the iterates of SD are very rarely at the points of non-differentiability [8], thus it might not achieve the desirable sparse solution (which is usually at the point of non-differentiability) within a limited number of iterations.

Coordinate Descent (CD) [39] and its recent extension—Coordinate Gradient Descent (CGD) can be applied for optimizing the non-differentiable composite function [40]. Coordinate descent has been applied for the  $\ell_1$ -norm regularized least squares [11],  $\ell_1/\ell_{\infty}$ -norm regularized least squares [19], and the sparse group Lasso [10]. Coordinate gradient descent has been applied for the group Lasso logistic regression [24]. Convergence results for CD and CGD have been established, when the non-differentiable part is separable [39, 40]. However, there is no global convergence rate for CD and CGD (Note, CGD is reported to have a *local* linear convergence rate under certain conditions [40, Theorem 4]). In addition, it is not clear whether CD and CGD are applicable to the problem Eq. (1) with an arbitrary  $q \geq 1$ .

Fixed Point Continuation [15, 34] was recently proposed for solving the  $\ell_1$ -norm regularized optimization (i.e.,  $\varpi(\mathbf{W}) = \|\mathbf{W}\|_1$ ). It is based on the following fixed point iteration:

$$\mathbf{X}_{i+1} = \mathcal{P}_{\lambda\tau}^{\varpi}(\mathbf{X}_i - \tau l'(\mathbf{X}_i)), \tag{5}$$

where  $\mathcal{P}_{\lambda\tau}^{\varpi}(\mathbf{W}) = \operatorname{sgn}(\mathbf{W}) \odot \max(\mathbf{W} - \lambda\tau, 0)$  is an operator and  $\tau > 0$  is the step size. The fixed point iteration Eq. (5) can be applied to solve Eq. (1) for any convex penalty  $\varpi(\mathbf{W})$ , with the operator  $\mathcal{P}_{\lambda\tau}^{\varpi}(\cdot)$  being defined as:

$$\mathcal{P}_{\lambda\tau}^{\varpi}(\mathbf{W}) = \arg\min_{\mathbf{X}} \frac{1}{2} \|\mathbf{X} - \mathbf{W}\|_{2}^{2} + \lambda\tau\varphi(\mathbf{X}).$$
(6)

The operator  $\mathcal{P}_{\lambda\tau}^{\varpi}(\cdot)$  is called the proximal operator [16, 25, 44], and is guaranteed to be non-expansive. With a properly chosen  $\tau$ , the fixed point iteration Eq. (5) can converge to the fixed point  $\mathbf{X}^*$  satisfying

$$\mathbf{X}^* = \mathcal{P}^{\varpi}_{\lambda\tau}(\mathbf{X}^* - \tau l'(\mathbf{X}^*)).$$
(7)

It follows from Eq. (6) and Eq. (7) that,

$$\mathbf{0} \in \mathbf{X}^* - (\mathbf{X}^* - \tau l'(\mathbf{X}^*)) + \lambda \tau \partial \varpi(\mathbf{X}^*), \tag{8}$$

which together with  $\tau > 0$  indicates that  $\mathbf{X}^*$  is the optimal solution to Eq. (1). In [3, 30], the gradient descent method is extended to optimize the composite function in the form of Eq. (1), and the iteration step is similar to Eq. (5). The extended gradient descent method for nonsmooth objective functions is proven to yield the convergence rate of O(1/k) for k iterations. However, as pointed out in [3, 30], the scheme in Eq. (5) can be further accelerated for solving Eq. (1).

Finally, there are various online learning algorithms that have been developed for dealing with large scale data, e.g., the truncated gradient method [18], the forward-looking subgradient [8], and the regularized dual averaging [43] (which is based on the dual averaging method proposed in [31]). When applying the aforementioned online learning methods for solving Eq. (1), a key building block is the operator  $\mathcal{P}_{\lambda\tau}^{ec}(\cdot)$ .

# 1.2 Screening Methods for (1)

Although many algorithms have been proposed to solve the mixed norm regularized problems, it remains challenging to solve especially for large-scale problems. To address this issue, "screening" has been shown to be a promising approach. The key idea of screening is to first identify the "inactive" features/groups, which have 0 coefficients in the solution. Then the inactive features/groups can be discarded from the optimization, leading to a reduced feature matrix and substantial savings in computational cost and memory size.

In [12], El Ghaoui *et al.* proposed novel screening methods, called "SAFE", to improve the efficiency for solving a class of  $\ell_1$  regularized problems, including Lasso,  $\ell_1$  regularized logistic regression and  $\ell_1$  regularized support vector machines. Inspired by SAFE, Tibshirani *et al.* [38] proposed "strong rules" for a large class of  $\ell_1$  regularized problems, including Lasso,  $\ell_1$  regularized logistic regression,  $\ell_1/\ell_q$  regularized problems and more general convex problems. Although in most of the cases strong rules are more effective in discarding features than SAFE, it is worthwhile to note that strong rules may mistakenly discard features that have non-zero coefficients in the solution. To overcome this limitation, in [42] the authors proposed the "DPP" rules for the group Lasso problem [45], that is, the  $\ell_1/\ell_q$ -regularization problem in which q = 2. The DPP rules are safe in the sense that the features/groups discarded from the optimization are guaranteed to have 0 coefficients in the solution.

The core of strong rules for the  $\ell_1/\ell_q$  problems is the assumption that the function  $\nabla l(\mathbf{X}^*(\lambda))$ , that is, the gradient of the loss function  $l(\cdot)$  at the optimal solution  $\mathbf{X}^*(\lambda)$  of problem Eq. (1), is a Lipschitz function of  $\lambda$ . However, this assumption does not always hold in practice [38]. Therefore, groups which have non-zero coefficients can be discarded mistakenly by strong rules. The key idea of DPP rules is to bound the dual optimal solution of problem Eq. (1) within a region  $\mathcal{R}$  and compute  $\max_{\theta \in \mathcal{R}} \|B_i^T \theta\|_{\bar{q}}$ ,  $i = 1, 2, \ldots, s$ , where  $1/q + 1/\bar{q} = 1$  (recall that  $B_i$  is the data matrix corresponding to  $\mathbf{w}_i$ ). The smaller the region  $\mathcal{R}$  is, the more inactive groups can be detected. In this paper, we give a more accurate estimation of the region  $\mathcal{R}$ and extend the idea to the general case with  $q \geq 1$ .

# **1.3** Main Contributions

The main contributions of this paper include the following two parts.

1. We develop an efficient algorithm for solving the  $\ell_1/\ell_q$ -regularized problem (1), for any  $q \geq 1$ . More specifically, we develop the GLEP<sub>1q</sub> algorithm<sup>1</sup>, which makes use of the accelerated gradient method [3, 30] for minimizing the composite objective functions. GLEP<sub>1q</sub> has the following two favorable properties: (1) It is applicable to any smooth convex loss  $l(\cdot)$  (e.g., the least squares loss and the logistic loss) and any  $q \geq 1$ . Existing algorithms are mainly focused on  $\ell_1/\ell_2$ -regularization and/or  $\ell_1/\ell_{\infty}$ -regularization. To the best of our knowledge, this is the first work that provides an efficient algorithm for solving (1) with any  $q \geq 1$ ; and (2) It achieves a global convergence rate of  $O(\frac{1}{L^2})$  (k

 $<sup>{}^{1}\</sup>text{GLEP}_{1q}$  stands for **G**roup Sparsity Learning via the  $\ell_1/\ell_q$ -regularized Euclidean Projection.

denotes the number of iterations) for the smooth convex loss  $l(\cdot)$ . In comparison, although the methods proposed in [1, 7, 19, 32] converge, there is no known convergence rate; and the method proposed in [24] has a *local* linear convergence rate under certain conditions [40, Theorem 4]. In addition, these methods are not applicable for an arbitrary  $q \ge 1$ .

The main technical contribution of the proposed  $\text{GLEP}_{1q}$  is the development of an efficient algorithm for computing the  $\ell_1/\ell_q$ -regularized Euclidean projection (EP<sub>1q</sub>), which is a key building block in the GLEP<sub>1q</sub> algorithm. More specifically, we analyze the key theoretical properties of the solution of EP<sub>1q</sub>, based on which we develop an efficient algorithm for EP<sub>1q</sub> by solving two zero finding problems. In addition, our theoretical analysis reveals why EP<sub>1q</sub> for the general q is significantly more challenging than the special cases such as q = 2. We have conducted experimental studies to demonstrate the efficiency of the proposed algorithm.

2. We develop novel screening methods for large-scale **mixed-n**orm (Smin) regularized problems. The proposed screening method is able to quickly identify the inactive groups, i.e., groups that have 0 components in the solution. Consequently, the inactive groups can be removed from the optimization problem and the scale of the resulting problem can be significantly reduced. Several appealing features of our screening method includes: (1) It is "safe" in the sense that the groups removed from the optimization are guaranteed to have 0 components in the solution. (2) The data set needs to be scanned only once to run the screening. (3) The computational cost of the proposed screening rule is negligible compared to that of solving the  $\ell_1/\ell_q$ -regularized problems. (4) Our screening method is independent of solvers for the  $\ell_1/\ell_q$ -regularized problems and thus it can be integrated with any existing solver to improve the efficiency. Due to the difficulty of the  $\ell_1/\ell_q$ -regularized problems, existing screening methods are limited to [38, 42]. In comparison, the method proposed in [38] is "inexact" in the sense that it may mistakenly remove groups from the optimization which have nonzero coefficients in the solution; and the method proposed in [42] is designed for the case q = 2.

The key of the proposed screening method is an accurate estimation of the region  $\mathcal{R}$  which includes the dual optimal solution of problem Eq. (1) via the "variational inequalities" [13]. After the upper bound  $\max_{\theta \in \mathcal{R}} \|B_i^T \theta\|_{\bar{q}}$  is computed, we make use of the KKT condition to determine if the  $i^{th}$  group has 0 coefficients in the solution and can be removed from the optimization. Experimental results show that the efficiency of the proposed GLEP<sub>1q</sub> can be improved by "three orders of magnitude" with the screening method, especially for the large dimensional data sets.

## 1.4 Related Work

We briefly review recent studies on  $\ell_1/\ell_q$ -regularization and the corresponding screening methods, most of which focus on  $\ell_1/\ell_2$ -regularization and/or  $\ell_1/\ell_\infty$ -regularization.

 $\ell_1/\ell_2$ -Regularization: The group Lasso was proposed in [45] to select the groups of variables for prediction in the least squares regression. In [24], the idea of group lasso was extended for classification by the logistic regression model, and an algorithm via the coordinate gradient descent [40] was developed. In [32], the authors considered joint covariate selection for grouped classification by the logistic loss, and developed a blockwise boosting Lasso algorithm with the boosted Lasso [47]. In [1], the authors proposed to learn the sparse representations shared across multiple tasks, and designed an alternating algorithm. The Spectral projected-gradient (Spg) algorithm was proposed for solving the  $\ell_1/\ell_2$ -ball constrained smooth optimization problem [4], equipped with an efficient Euclidean projection that has expected linear runtime. The  $\ell_1/\ell_2$ regularized multi-task learning was proposed in [21], and the equivalent smooth reformulations were solved by the Nesterov's method [29].

 $\ell_1/\ell_{\infty}$ -Regularization: A blockwise coordinate descent algorithm [39] was developed for the multi-task Lasso [19]. It was applied to the neural semantic basis discovery problem. In [33], the authors considered the multi-task learning via the  $\ell_1/\ell_{\infty}$ -regularization, and proposed to solve the equivalent  $\ell_1/\ell_{\infty}$ -ball constrained problem by the projected gradient descent. In [27], the authors considered the multivariate regression via the  $\ell_1/\ell_{\infty}$ -regularization, showed that the high-dimensional scaling of  $\ell_1/\ell_{\infty}$ -regularization is qualitatively similar to that of ordinary  $\ell_1$ -regularization, and revealed that, when the overlap parameter is large enough (> 2/3),  $\ell_1/\ell_{\infty}$ -regularization yields the improved statistical efficiency over  $\ell_1$ -regularization.

 $\ell_1/\ell_q$ -Regularization: In [7], the authors studied the problem of boosting with structural sparsity, and developed several boosting algorithms for regularization penalties including  $\ell_1$ ,  $\ell_{\infty}$ ,  $\ell_1/\ell_2$ , and  $\ell_1/\ell_{\infty}$ . In [46], the composite absolute penalties (CAP) family was introduced, and an algorithm called iCAP was developed. iCAP employed the least squares loss and the  $\ell_1/\ell_{\infty}$  regularization, and was implemented by the boosted Lasso [47]. The multivariate regression with the  $\ell_1/\ell_q$ -regularization was studied in [20]. In [26], a unified framework was provided for establishing consistency and convergence rates for the regularized *M*-estimators, and the results for  $\ell_1/\ell_q$  regularization was established.

 $\ell_1/\ell_q$ -Screening: To the best of our knowledge, existing screening methods for  $\ell_1/\ell_q$ -Regularization are limited to the methods proposed in [38, 42]. The methods proposed in [38], i.e., the strong rules, assume that the gradient of the loss function at the optimal solution of problem Eq. (1) is a Lipschitz function of  $\lambda$ . However, there are counterexamples showing that the assumption can be violated in practice. As a result, groups which have non-zero coefficients in the solution can be mistakenly discarded from the optimization by strong rules. In [42], the authors considered the group lasso problem [45], that is, the  $\ell_1/\ell_2$ -regularized problems and proposed the DPP rules. The key idea of DPP rules is to estimate a region  $\mathcal{R}$  which includes the dual optimal solution  $\theta^*(\lambda)$  of problem Eq. (1) by noting that  $\theta^*(\lambda)$  is nonexpansive with respect to  $\lambda$ .

### 1.5 Notation

Throughout this paper, scalars are denoted by italic letters, and vectors by bold face letters. Let  $\mathbf{X}, \mathbf{Y}, \ldots$  denote the *p*-dimensional parameters,  $\mathbf{x}_i, \mathbf{y}_i, \ldots$  the  $p_i$ -dimensional parameters of the *i*-th group, and  $x_i$  the *i*-th component of  $\mathbf{x}$ . We denote  $\bar{q} = \frac{q}{q-1}$ , and thus q and  $\bar{q}$  satisfy the following relationship:  $\frac{1}{\bar{q}} + \frac{1}{q} = 1$ . We use the following componentwise operators:  $\odot$ ,  $.^q$ ,  $|\cdot|$  and  $\operatorname{sgn}(\cdot)$ . Specifically,  $\mathbf{z} = \mathbf{x} \odot \mathbf{y}$  denotes  $z_i = x_i y_i$ ;  $\mathbf{y} = \mathbf{x}^q$  denotes  $y_i = x_i^q$ ;  $\mathbf{y} = |\mathbf{x}|$  denotes  $y_i = |x_i|$ ; and  $\mathbf{y} = \operatorname{sgn}(\mathbf{x})$  denotes  $y_i = \operatorname{sgn}(\cdot)$ , where  $\operatorname{sgn}(\cdot)$  is the signum function:  $\operatorname{sgn}(t) = 1$  if t > 0;  $\operatorname{sgn}(t) = 0$  if t = 0; and  $\operatorname{sgn}(t) = -1$  if t < 0. We use  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$  to denote the inner product of  $\mathbf{x}$  and  $\mathbf{y}$ .

# 2 The Proposed $\text{GLEP}_{1q}$ Algorithm

In this paper, we consider solving Eq. (1) in the batch learning setting, and propose to apply the accelerated gradient method [3, 30] due to its fast convergence rate. We term our proposed algorithm as "GLEP<sub>1q</sub>", which stands for **G**roup Sparsity Learning via the  $\ell_1/\ell_q$ -regularized Euclidean **P**rojection. Note that, one can develop the online learning algorithm for Eq. (1) using the aforementioned online learning algorithms, where the  $\ell_1/\ell_q$ -regularized Euclidean projection is also a key building block.

We first construct the following model for approximating the composite function  $\mathcal{M}(\cdot)$  at the point **X**:

$$\mathcal{M}_{L,\mathbf{X}}(\mathbf{Y}) = [\operatorname{loss}(\mathbf{X}) + \langle \operatorname{loss}'(\mathbf{X}), \mathbf{Y} - \mathbf{X} \rangle] + \lambda \varpi(\mathbf{Y}) + \frac{L}{2} \|\mathbf{Y} - \mathbf{X}\|_{2}^{2},$$
(9)

where L > 0. In the model  $\mathcal{M}_{L,\mathbf{X}}(\mathbf{Y})$ , we apply the first-order Taylor expansion at the point  $\mathbf{X}$  (including all terms in the square bracket) for the smooth loss function  $l(\cdot)$ , and directly put the non-smooth penalty  $\varpi(\cdot)$  into the model. The regularization term  $\frac{L}{2} \|\mathbf{Y} - \mathbf{X}\|_2^2$  prevents  $\mathbf{Y}$  from walking far away from  $\mathbf{X}$ , thus the model can be a good approximation to  $f(\mathbf{Y})$  in the neighborhood of  $\mathbf{X}$ .

The accelerated gradient method is based on two sequences  $\{\mathbf{X}_i\}$  and  $\{\mathbf{S}_i\}$  in which  $\{\mathbf{X}_i\}$  is the sequence of approximate solutions, and  $\{\mathbf{S}_i\}$  is the sequence of search points. The search point  $\mathbf{S}_i$  is the affine combination of  $\mathbf{X}_{i-1}$  and  $\mathbf{X}_i$  as

$$\mathbf{S}_i = \mathbf{X}_i + \beta_i (\mathbf{X}_i - \mathbf{X}_{i-1}), \tag{10}$$

where  $\beta_i$  is a properly chosen coefficient. The approximate solution  $\mathbf{X}_{i+1}$  is computed as the minimizer of  $\mathcal{M}_{L_i,\mathbf{S}_i}(\mathbf{Y})$ :

$$\mathbf{X}_{i+1} = \arg\min_{\mathbf{Y}} \mathcal{M}_{L_i, \mathbf{S}_i}(\mathbf{Y}), \tag{11}$$

where  $L_i$  is determined by line search, e.g., the Armijo-Goldstein rule so that  $L_i$  should be appropriate for  $\mathbf{S}_i$ .

Algorithm 1 GLEP<sub>1q</sub>: Group Sparsity Learning via the  $\ell_1/\ell_q$ -regularized Euclidean Projection

Require:  $\lambda_1 \ge 0, \lambda_2 \ge 0, L_0 > 0, \mathbf{X}_0, k$ Ensure:  $X_{k+1}$ 

1: Initialize  $\mathbf{X}_1 = \mathbf{X}_0$ ,  $\alpha_{-1} = 0$ ,  $\alpha_0 = 1$ , and  $L = L_0$ .

2: for i = 1 to k do

3:

Set  $\beta_i = \frac{\alpha_{i-2}-1}{\alpha_{i-1}}$ ,  $\mathbf{S}_i = \mathbf{X}_i + \beta_i (\mathbf{X}_i - \mathbf{X}_{i-1})$ Find the smallest  $L = L_{i-1}, 2L_{i-1}, \dots$  such that 4:

$$f(\mathbf{X}_{i+1}) \leq \mathcal{M}_{L,\mathbf{S}_i}(\mathbf{X}_{i+1}),$$

where  $\mathbf{X}_{i+1} = \arg\min_{\mathbf{Y}} \mathcal{M}_{L,\mathbf{S}_i}(\mathbf{Y})$ Set  $L_i = L$  and  $\alpha_{i+1} = \frac{1 + \sqrt{1 + 4\alpha_i^2}}{2}$ 5: 6: end for

The algorithm for solving Eq. (1) is presented in Algorithm 1.  $\text{GLEP}_{1q}$  inherits the optimal convergence rate of  $O(1/k^2)$  from the accelerated gradient method. In Algorithm 1, a key subroutine is Eq. (11), which can be computed as  $\mathbf{X}_{i+1} = \pi_{1q}(\mathbf{S}_i - l'(\mathbf{S}_i)/L_i, \lambda/L_i)$ , where  $\pi_{1q}(\cdot)$  is the  $\ell_1/\ell_q$ -regularized Euclidean projection  $(EP_{1q})$  problem:

$$\pi_{1q}(\mathbf{V},\lambda) = \arg\min_{\mathbf{X}\in\mathbb{R}^p} \frac{1}{2} \|\mathbf{X}-\mathbf{V}\|_2^2 + \lambda \sum_{i=1}^s \|\mathbf{x}_i\|_q.$$
(12)

The efficient computation of Eq. (12) for any q > 1 is the main technical contribution of this paper. Note that the s groups in Eq. (12) are independent. Thus the optimization in (12) decouples into a set of s independent  $\ell_q$ -regularized Euclidean projection problems:

$$\pi_q(\mathbf{v}) = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \left( g(\mathbf{x}) = \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|_2^2 + \lambda \|\mathbf{x}\|_q \right),\tag{13}$$

where  $n = p_i$  for the *i*-th group. Next, we study the key properties of (13).

#### Properties of the Optimal Solution to (13)2.1

The function  $q(\cdot)$  is strictly convex, and thus it has a unique minimizer, as summarized below:

**Lemma 1.** The problem (13) has a unique minimizer.

Next, we show that the optimal solution to (13) is given by zero under a certain condition, as summarized in the following theorem:

**Theorem 2.**  $\pi_q(\mathbf{v}) = \mathbf{0}$  if and only if  $\lambda \geq \|\mathbf{v}\|_{\bar{q}}$ .

*Proof.* Let us first compute the directional derivative of  $g(\mathbf{x})$  at the point **0**:

$$Dg(\mathbf{0})[\mathbf{u}] = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} [g(\alpha \mathbf{u}) - g(\mathbf{0})] = -\langle \mathbf{v}, \mathbf{u} \rangle + \lambda \|\mathbf{u}\|_q$$

where  $\mathbf{u}$  is a given direction. According to the Hölder's inequality, we have

$$|\langle \mathbf{u}, \mathbf{v} 
angle| \leq \|\mathbf{u}\|_q \|\mathbf{v}\|_{ar{q}}, orall \mathbf{u}.$$

Therefore, we have

$$Dg(\mathbf{0})[\mathbf{u}] \ge 0, \forall \mathbf{u},$$
 (14)

if and only if  $\lambda \geq \|\mathbf{v}\|_{\bar{q}}$ . The result follows, since (14) is the necessary and sufficient condition for **0** to be the optimal solution of (13). 



Figure 1: Illustration of the failure of the fixed point iteration  $\mathbf{x} = \mathbf{v} - \lambda \|\mathbf{x}\|_q^{1-q} \mathbf{x}^{(q-1)}$  for solving (13). We set  $\mathbf{v} = [1,3]^T$  and the starting point  $\mathbf{x} = [1,3]^T$ . The vertical axis denotes the values of  $x_1$  during the iterations.

Next, we focus on solving (13) for  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ . We first consider solving (13) in the case of  $1 < q < \infty$ , which is the main technical contribution of this paper. We begin with a lemma that summarizes the key properties of the optimal solution to the problem (13):

**Lemma 3.** Let  $1 < q < \infty$  and  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ . Then,  $\mathbf{x}^*$  is the optimal solution to the problem (13) if and if only it satisfies:

$$\mathbf{x}^* + \lambda \|\mathbf{x}^*\|_q^{1-q} \mathbf{x}^{*(q-1)} = \mathbf{v},$$
(15)

where  $\mathbf{y} \equiv \mathbf{x}^{(q-1)}$  is defined component-wisely as:  $y_i = \operatorname{sgn}(x_i)|x_i|^{q-1}$ . Moreover, we have

$$\pi_q(\mathbf{v}) = \operatorname{sgn}(\mathbf{v}) \odot \pi_q(|\mathbf{v}|),\tag{16}$$

$$\operatorname{sgn}(\mathbf{x}^*) = \operatorname{sgn}(\mathbf{v}),\tag{17}$$

$$0 < |x_i^*| < |v_i|, \forall i \in \{i | v_i \neq 0\}.$$
(18)

*Proof.* Since  $\lambda < \|\mathbf{v}\|_{\bar{q}}$ , it follows from Theorem 2 that the optimal solution  $\mathbf{x}^* \neq \mathbf{0}$ .  $\|\mathbf{x}\|_q$  is differentiable when  $\mathbf{x} \neq \mathbf{0}$ , so is  $g(\mathbf{x})$ . Therefore, the sufficient and necessary condition for  $\mathbf{x}^*$  to be the solution of (13) is  $g'(\mathbf{x}^*) = 0$ , i.e., (15). Denote  $c^* \equiv \lambda \|\mathbf{x}^*\|_q^{1-q} > 0$ . It follows from (15) that (16) holds, and

$$\operatorname{sgn}(x_i^*)\left(|x_i^*| + c^* |x_i^*|^{q-1}\right) = v_i,\tag{19}$$

from which we can verify (17) and (18).

It follows from Lemma 3 that i) if  $v_i = 0$  then  $x_i^* = 0$ ; and ii)  $\pi_q(\mathbf{v})$  can be easily obtained from  $\pi_q(|\mathbf{v}|)$ . Thus, we can restrict our following discussion to  $\mathbf{v} > \mathbf{0}$ , i.e.,  $v_i > 0, \forall i$ . It is clear that, the analysis can be easily extended to the general  $\mathbf{v}$ . The optimality condition in (15) indicates that  $\mathbf{x}^*$  might be solved via the fixed point iteration

$$\mathbf{x} = \eta(\mathbf{x}) \equiv \mathbf{v} - \lambda \|\mathbf{x}\|_q^{1-q} \mathbf{x}^{(q-1)},$$

which is, however, not guaranteed to converge (see Figure 1 for examples), as  $\eta(\cdot)$  is not necessarily a contraction mapping [17, Proposition 3]. In addition,  $\mathbf{x}^*$  cannot be trivially solved by firstly guessing  $c = \|\mathbf{x}\|_q^{1-q}$  and then finding the root of  $\mathbf{x} + \lambda c \mathbf{x}^{(q-1)} = \mathbf{v}$ , as when c increases, the values of  $\mathbf{x}$  obtained from  $\mathbf{x} + \lambda c \mathbf{x}^{(q-1)} = \mathbf{v}$  decrease, so that  $c = \|\mathbf{x}\|_q^{1-q}$  increases as well (note, 1 - q < 0).

# 2.2 Computing the Optimal Solution x<sup>\*</sup> by Zero Finding

In the following, we show that  $\mathbf{x}^*$  can be obtained by solving two zero finding problems. Below, we construct our first auxiliary function  $h_c^v(\cdot)$  and reveal its properties:

**Definition 4** (Auxiliary Function  $h_c^v(\cdot)$ ). Let c > 0,  $1 < q < \infty$ , and v > 0. We define the auxiliary function  $h_c^v(\cdot)$  as follows:

$$h_c^v(x) = x + cx^{q-1} - v, 0 \le x \le v.$$
(20)

**Lemma 5.** Let c > 0,  $1 < q < \infty$ , and v > 0. Then,  $h_c^v(\cdot)$  has a unique root in the interval (0, v).

*Proof.* It is clear that  $h_c^v(\cdot)$  is continuous and strictly increasing in the interval [0, v],  $h_c^v(0) = -v < 0$ , and  $h_c^v(v) = cv^{q-1} > 0$ . According to the Intermediate Value Theorem,  $h_c^v(\cdot)$  has a unique root lying in the interval (0, v). This concludes the proof.

**Corollary 6.** Let  $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$ , c > 0,  $1 , and <math>\mathbf{v} > \mathbf{0}$ . Then, the function

$$\varphi_c^{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + c\mathbf{x}^{(q-1)} - \mathbf{v}, \mathbf{0} < \mathbf{x} < \mathbf{v}$$
(21)

has a unique root.

Let  $\mathbf{x}^*$  be the optimal solution satisfying (15). Denote  $c^* = \lambda \|\mathbf{x}^*\|_q^{1-q}$ . It follows from Lemma 3 and Corollary 6 that  $\mathbf{x}^*$  is the unique root of  $\varphi_{c^*}^{\mathbf{v}}(\cdot)$  defined in (21), provided that the optimal  $c^*$  is known. Our methodology for computing  $\mathbf{x}^*$  is to first compute the optimal  $c^*$  and then compute  $\mathbf{x}^*$  by computing the root of  $\varphi_{c^*}^{\mathbf{v}}(\cdot)$ . Next, we show how to compute the optimal  $c^*$  by solving a single variable zero finding problem. We need our second auxiliary function  $\omega(\cdot)$  defined as follows:

**Definition 7** (Auxiliary Function  $\omega(\cdot)$ ). Let  $1 < q < \infty$  and v > 0. We define the auxiliary function  $\omega(\cdot)$  as follows:

$$c = \omega(x) = (v - x)/x^{q-1}, 0 < x \le v.$$
(22)

**Lemma 8.** In the interval (0, v],  $c = \omega(x)$  is i) continuously differentiable, ii) strictly decreasing, and iii) invertible. Moreover, in the domain  $[0, \infty)$ , the inverse function  $x = \omega^{-1}(c)$  is continuously differentiable and strictly decreasing.

*Proof.* It is easy to verify that, in the interval (0, v],  $c = \omega(x)$  is continuously differentiable with a non-positive gradient, i.e.,  $\omega'(x) < 0$ . Therefore, the results follow from the Inverse Function Theorem.

It follows from Lemma 8 that given the optimal  $c^*$  and  $\mathbf{v}$ , the optimal  $\mathbf{x}^*$  can be computed via the inverse function  $\omega^{-1}(\cdot)$ , i.e., we can represent  $\mathbf{x}^*$  as a function of  $c^*$ . Since  $\lambda \|\mathbf{x}^*\|_q^{1-q} - c^* = 0$  by the definition of  $c^*$ , the optimal  $c^*$  is a root of our third auxiliary function  $\phi(\cdot)$  defined as follows:

**Definition 9** (Auxiliary Function  $\phi(\cdot)$ ). Let  $1 < q < \infty$ ,  $0 < \lambda < \|\mathbf{v}\|_{\bar{q}}$ , and  $\mathbf{v} > \mathbf{0}$ . We define the auxiliary function  $\phi(\cdot)$  as follows:

$$\phi(c) = \lambda \psi(c) - c, c \ge 0, \tag{23}$$

where

$$\psi(c) = \left(\sum_{i=1}^{n} (\omega_i^{-1}(c))^q\right)^{\frac{1-q}{q}},$$
(24)

and  $\omega_i^{-1}(c)$  is the inverse function of

$$\omega_i(x) = (v_i - x)/x^{q-1}, 0 < x \le v_i.$$
(25)

Recall that we assume  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$  (otherwise the optimal solution is given by zero from Theorem 2). The following lemma summarizes the key properties of the auxiliary function  $\phi(\cdot)$ :

Lemma 10. Let  $1 < q < \infty$ ,  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ ,  $\mathbf{v} > \mathbf{0}$ , and

$$\epsilon = (\|\mathbf{v}\|_{\bar{q}} - \lambda) / \|\mathbf{v}\|_{\bar{q}}.$$
(26)

Then,  $\phi(\cdot)$  is continuously differentiable in the interval  $[0,\infty)$ . Moreover, we have

$$\phi(0) = \lambda \|\mathbf{v}\|_q^{1-q} > 0, \phi(\overline{c}) \le 0.$$

where

$$\overline{c} = \max_{i} c_i, \tag{27}$$

$$c_i = \omega_i(v_i\epsilon), i = 1, 2, \dots, n.$$

$$(28)$$

*Proof.* From Lemma 8, the function  $\omega_i^{-1}(c)$  is continuously differentiable in  $[0,\infty)$ . It is easy to verify that  $\omega_i^{-1}(c) > 0, \forall c \in [0, \infty).$  Thus,  $\phi(\cdot)$  in (23) is continuously differentiable in  $[0, \infty)$ . It is clear that  $\phi(0) = \lambda \|\mathbf{v}\|_q^{1-q} > 0$ . Next, we show  $\phi(\overline{c}) \leq 0$ . Since  $0 < \lambda < \|\mathbf{v}\|_{\overline{q}}$ , we have

$$0 < \epsilon < 1. \tag{29}$$

It follows from (25), (27), (28) and (29) that  $0 < c_i \leq \overline{c}, \forall i$ . Let  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  be the root of  $\varphi_{\overline{c}}^{\mathbf{v}}(\cdot)$  (see Corollary 6). Then,  $x_i = \omega_i^{-1}(\overline{c})$ . Since  $\omega_i^{-1}(\cdot)$  is strictly decreasing (see Lemma 8),  $c_i \leq \overline{c}, v_i \epsilon = \omega_i^{-1}(c_i)$ , and  $x_i = \omega_i^{-1}(\overline{c})$ , we have

$$x_i \le v_i \epsilon. \tag{30}$$

Combining (25), (30), and  $\overline{c} = \omega_i(x_i)$ , we have  $\overline{c} \ge v_i(1-\epsilon)/x_i^{q-1}$ , since  $\omega_i(\cdot)$  is strictly decreasing. It follows that  $x_i \ge \left(\frac{v_i(1-\epsilon)}{\overline{c}}\right)^{\frac{1}{q-1}}$ . Thus, the following holds:

$$\psi(\overline{c}) = \left(\sum_{i=1}^{n} (\omega_i^{-1}(\overline{c}))^q\right)^{\frac{1-q}{q}} = \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1-q}{q}} \le \frac{\overline{c}}{\|\mathbf{v}\|_{\bar{q}}(1-\epsilon)},$$

which leads to

$$\phi(\overline{c}) = \lambda \psi(\overline{c}) - \overline{c} \le \overline{c} \left( \frac{\lambda}{\|\mathbf{v}\|_{\bar{q}}(1-\epsilon)} - 1 \right) = 0,$$

where the last equality follows from (26).

**Corollary 11.** Let  $1 < q < \infty$ ,  $0 < \lambda < \|\mathbf{v}\|_{\bar{q}}$ ,  $\mathbf{v} > \mathbf{0}$ , and  $\underline{c} = \min_i c_i$ , where  $c_i$ 's are defined in (28). We have  $0 < c < \overline{c}$  and  $\phi(c) > 0$ .

Following Lemma 10 and Corollary 11, we can find at least one root of  $\phi(\cdot)$  in the interval  $[c, \overline{c}]$ . In the following theorem, we show that  $\phi(\cdot)$  has a unique root:

**Theorem 12.** Let  $1 < q < \infty$ ,  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ , and  $\mathbf{v} > \mathbf{0}$ . Then, in  $[\underline{c}, \overline{c}]$ ,  $\phi(\cdot)$  has a unique root, denoted by  $c^*$ , and the root of  $\varphi_{c^*}^{\mathbf{v}}(\cdot)$  is the optimal solution to (13).

*Proof.* From Lemma 10 and Corollary 11, we have  $\phi(\overline{c}) \leq 0$  and  $\phi(\underline{c}) \geq 0$ . If either  $\phi(\overline{c}) = 0$  or  $\phi(\underline{c}) = 0$ ,  $\overline{c}$ or c is a root of  $\phi(\cdot)$ . Otherwise, we have  $\phi(c)\phi(\overline{c}) < 0$ . As  $\phi(\cdot)$  is continuous in  $[0,\infty)$ , we conclude that  $\phi(\cdot)$  has a root in  $(c, \overline{c})$  according to the Intermediate Value Theorem.

Next, we show that  $\phi(\cdot)$  has a unique root in the interval  $[0,\infty)$ . We prove this by contradiction. Assume that  $\phi(\cdot)$  has two roots:  $0 < c_1 < c_2$ . From Corollary 6,  $\varphi_{c_1}^{\mathbf{v}}(\cdot)$  and  $\varphi_{c_2}^{\mathbf{v}}(\cdot)$  have unique roots. Denote  $\mathbf{x}^1 = [x_1^1, x_2^1, \dots, x_n^1]^T$  and  $\mathbf{x}^2 = [x_1^2, x_2^2, \dots, x_n^2]^T$  as the roots of  $\varphi_{c_1}^{\mathbf{v}}(\cdot)$  and  $\varphi_{c_2}^{\mathbf{v}}(\cdot)$ , respectively. We have  $0 < x_1^i, x_i^2 < v_i, \forall i$ . It follows from (23-25) that

$$\begin{split} \mathbf{x}^{1} + \lambda \|\mathbf{x}^{1}\|_{q}^{1-q} \mathbf{x}^{1(q-1)} - \mathbf{v} &= \mathbf{0}, \\ \mathbf{x}^{2} + \lambda \|\mathbf{x}^{2}\|_{q}^{1-q} \mathbf{x}^{2(q-1)} - \mathbf{v} &= \mathbf{0}. \end{split}$$

According to Lemma 3,  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are the optimal solution of (13). From Lemma 1, we have  $\mathbf{x}^1 = \mathbf{x}^2$ . However, since  $x_i^1 = \omega_i^{-1}(c_1)$ ,  $x_i^2 = \omega_i^{-1}(c_2)$ ,  $\omega_i^{-1}(\cdot)$  is a strictly decreasing function in  $[0, \infty)$  by Lemma 8, and  $c_1 < c_2$ , we have  $x_i^1 > x_i^2$ ,  $\forall i$ . This leads to a contradiction. Therefore, we conclude that  $\phi(\cdot)$  has a unique root in  $[\underline{c}, \overline{c}]$ .

From the above arguments, it is clear that, the root of  $\varphi_{c^*}^{\mathbf{v}}(\cdot)$  is the optimal solution to (13).

**Remark 13.** When q = 2, we have  $\underline{c} = \overline{c} = \frac{\lambda}{\|\mathbf{v}\|_2 - \lambda}$ . It is easy to verify that  $\phi(\underline{c}) = \phi(\overline{c}) = 0$  and

$$\pi_2(\mathbf{v}) = \frac{\|\mathbf{v}\|_2 - \lambda}{\|\mathbf{v}\|_2} \mathbf{v}.$$
(31)

Therefore, when q = 2, we obtain a closed-form solution.

# 2.3 Solving the Zero Finding Problem by Bisection

Let  $1 < q < \infty$ ,  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ ,  $\mathbf{v} > \mathbf{0}$ ,  $\bar{v} = \max_i v_i$ ,  $\underline{v} = \min_i v_i$ , and  $\delta > 0$  be a small constant (e.g.,  $\delta = 10^{-8}$  in our experiments). When q > 2, we have

$$\underline{c} = \frac{1-\epsilon}{\epsilon^{q-1}\overline{v}^{q-2}}$$
 and  $\overline{c} = \frac{1-\epsilon}{\epsilon^{q-1}\underline{v}^{q-2}}$ 

When 1 < q < 2, we have

$$\underline{c} = \frac{1-\epsilon}{\epsilon^{q-1}\underline{v}^{q-2}}$$
 and  $\overline{c} = \frac{1-\epsilon}{\epsilon^{q-1}\overline{v}^{q-2}}$ .

If either  $\phi(\overline{c}) = 0$  or  $\phi(\underline{c}) = 0$ ,  $\overline{c}$  or  $\underline{c}$  is the unique root of  $\phi(\cdot)$ . Otherwise, we can find the unique root of  $\phi(\cdot)$  by bisection in the interval  $(\underline{c}, \overline{c})$ , which costs at most

$$N = \log_2 \frac{(1-\epsilon)|\overline{v}^{q-2} - \underline{v}^{q-2}|}{\epsilon^{q-1}\overline{v}^{q-2}\underline{v}^{q-2}\delta}$$

iterations for achieving an accuracy of  $\delta$ . Let  $[c_1, c_2]$  be the current interval of uncertainty, and we have computed  $\omega_i^{-1}(c_1)$  and  $\omega_i^{-1}(c_2)$  in the previous bisection iterations. Setting  $c = \frac{c_1+c_2}{2}$ , we need to evaluate  $\phi(c)$  by computing  $\omega_i^{-1}(c)$ , i = 1, 2, ..., n. It is easy to verify that  $\omega_i^{-1}(c)$  is the root of  $h_c^{v_i}(\cdot)$  in the interval  $(0, v_i)$ . Since  $\omega_i^{-1}(\cdot)$  is a strictly decreasing function (see Lemma 8), the following holds:

$$\omega_i^{-1}(c_2) < \omega_i^{-1}(c) < \omega_i^{-1}(c_1),$$

and thus  $\omega_i^{-1}(c)$  can be solved by bisection using at most

$$\log_2 \frac{\omega_i^{-1}(c_2) - \omega_i^{-1}(c_1)}{\delta} < \log_2 \frac{v_i}{\delta} \le \log_2 \frac{\overline{v}}{\delta}$$

iterations for achieving an accuracy of  $\delta$ . For given  $\mathbf{v}, \lambda$ , and  $\delta$ , N and  $\overline{v}$  are constant, and thus it costs O(n) for finding the root of  $\phi(\cdot)$ . Once  $c^*$ , the root of  $\phi(\cdot)$  is found, it costs O(n) flops to compute  $\mathbf{x}^*$  as the unique root of  $\varphi_{c^*}^{\mathbf{v}}(\cdot)$ . Therefore, the overall time complexity for solving (13) is O(n).

We have shown how to solve (13) for  $1 < q < \infty$ . For q = 1, the problem (13) is reduced to the one used in the standard Lasso, and it has the following closed-form solution [3]:

$$\pi_1(\mathbf{v}) = \operatorname{sgn}(\mathbf{v}) \odot \max(|\mathbf{v}| - \lambda, 0).$$
(32)

For  $q = \infty$ , the problem (13) can computed via Eq. (32), as summarized in the following theorem:

**Theorem 14.** Let  $q = \infty$ ,  $\bar{q} = 1$ , and  $0 < \lambda < ||\mathbf{v}||_{\bar{q}}$ . Then we have

$$\pi_{\infty}(\mathbf{v}) = \operatorname{sgn}(\mathbf{v}) \odot \min(|\mathbf{v}|, t^*), \tag{33}$$

where  $t^*$  is the unique root of

$$h(t) = \sum_{i=1}^{n} \max(|v_i| - t, 0) - \lambda.$$
(34)

*Proof.* Making use of the property that  $\|\mathbf{x}\|_{\infty} = \max_{\|\mathbf{y}\|_1 \leq 1} \langle \mathbf{y}, \mathbf{x} \rangle$ , we can rewrite (13) in the case of  $q = \infty$  as

$$\min_{\mathbf{x}} \max_{\mathbf{y}: \|\mathbf{y}\|_1 \le \lambda} s(\mathbf{x}, \mathbf{y}) \equiv \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|_2^2 + \langle \mathbf{y}, \mathbf{x} \rangle.$$
(35)

The function  $s(\mathbf{x}, \mathbf{y})$  is continuously differentiable in both  $\mathbf{x}$  and  $\mathbf{y}$ , convex in  $\mathbf{x}$  and concave in  $\mathbf{y}$ , and the feasible domains are solids. According to the well-known von Neumann Lemma [28], the min-max problem (35) has a saddle point, and thus the minimization and maximization can be exchanged. Setting the derivative of  $s(\mathbf{x}, \mathbf{y})$  with respect to  $\mathbf{x}$  to zero, we have

$$\mathbf{x} = \mathbf{v} - \mathbf{y}.\tag{36}$$

Thus we obtain the following problem:

$$\min_{\mathbf{y}:\|\mathbf{y}\|_1 \le \lambda} \frac{1}{2} \|\mathbf{y} - \mathbf{v}\|_2^2, \tag{37}$$

which is the problem of the Euclidean projection onto the  $\ell_1$  ball [4, 7, 23]. It has been shown that the optimal solution  $\mathbf{y}^*$  to (37) for  $\lambda < \|\mathbf{v}\|_1$  can be obtained by first computing  $t^*$  as the unique root of (34) in linear time, and then computing  $\mathbf{y}^*$  as

$$\mathbf{y}^* = \operatorname{sgn}(\mathbf{v}) \odot \max(|\mathbf{v}| - t^*, 0).$$
(38)

It follows from (36) and (38) that (33) holds.

We conclude this section by summarizing the results for solving the  $\ell_q$ -regularized Euclidean projection in Algorithm 2.

Algorithm 2  $Ep_q$ :  $\ell_q$ -regularized Euclidean projection

```
Require: \lambda > 0, q \ge 1, \mathbf{v} \in \mathbb{R}^n
Ensure: \mathbf{x}^* = \pi_q(\mathbf{v}) = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|_2^2 + \lambda \|\mathbf{x}\|_q
  1: Compute \bar{q} = \frac{q}{q-1}
  2: if \|\mathbf{v}\|_{\bar{q}} \leq \lambda then
           Set \mathbf{x}^* = \mathbf{0}, return
  3:
  4: end if
  5: if q = 1 then
           Set \mathbf{x}^* = \operatorname{sgn}(\mathbf{v}) \odot \max(|\mathbf{v}| - \lambda, 0)
  6:
  7: else if q = 2 then
           Set \mathbf{x}^* = \frac{\|\mathbf{v}\|_2 - \lambda}{\|\mathbf{v}\|_2} \mathbf{v}
  8:
  9: else if q = \infty then
           Obtain t^*, the unique root of h(t), via the improved bisection method [23]
 10:
           Set \mathbf{x}^* = \operatorname{sgn}(\mathbf{v}) \odot \min(|\mathbf{v}|, t^*)
 11:
 12:
       else
           Compute c^*, the unique root of \phi(c), via bisection in the interval [\underline{c}, \overline{c}]
13:
           Obtain \mathbf{x}^* as the unique root of \varphi_{c^*}^{\mathbf{v}}(\cdot)
14:
15: end if
```

# 3 The Proposed Screening Method (Smin) for the $\ell_1/\ell_q$ -regularized Problems

In this section, we assume the loss function  $\ell(\cdot)$  is the least square loss, i.e., we consider the following problem:

$$\min_{\mathbf{W}\in\mathbb{R}^p} f(\mathbf{W}) = \frac{1}{2} \left\| Y - \sum_{i=1}^s B_i \mathbf{w}_i \right\|_2^2 + \lambda \sum_{i=1}^s \|\mathbf{w}_i\|_q.$$
(39)

The dual problem of (39) takes the form as:

$$\max_{\theta} \frac{1}{2} \|Y\|_{2}^{2} - \frac{\lambda^{2}}{2} \left\|\theta - \frac{Y}{\lambda}\right\|_{2}^{2},$$
s.t.  $\|B_{i}^{\mathrm{T}}\theta\|_{\bar{q}} \leq 1, i = 1, 2, \dots, s.$ 
(40)

Let  $\mathbf{X}^*(\lambda)$  and  $\theta^*(\lambda)$  be the optimal solutions of problems (39) and (40) respectively. The KKT conditions read as:

$$Y = \sum_{i=1}^{s} B_i \mathbf{X}_i^*(\lambda) + \lambda \theta^*(\lambda), \tag{41}$$

$$B_i^{\mathrm{T}}\theta^*(\lambda) \in \begin{cases} \frac{\mathbf{X}_i^*(\lambda)}{\|\mathbf{X}_i^*(\lambda)\|_{\bar{q}}}, & \text{if } \mathbf{X}_i^*(\lambda) \neq 0, \\ \mathbf{U}_i, \ \mathbf{U}_i \in \mathbb{R}^{p_i}, \|\mathbf{U}_i\|_{\bar{q}} \leq 1, & \text{if } \mathbf{X}_i^*(\lambda) = 0, \end{cases} \quad i = 1, 2, \dots, s.$$

$$(42)$$

In view of Eq. (42), we can see that

$$\|B_i^{\mathrm{T}}\theta^*(\lambda)\|_{\bar{q}} < 1 \Rightarrow \mathbf{X}_i^*(\lambda) = 0.$$
(R)

In other words, if  $\|B_i^{\mathrm{T}}\theta^*(\lambda)\|_{\bar{q}} < 1$ , then the KKT conditions imply that the coefficients of  $B_i$  in the solution  $\mathbf{X}^*(\lambda)$ , that is,  $\mathbf{X}_i^*(\lambda)$ , are 0 and thus the  $i^{th}$  group can be safely removed from the optimization of problem (39). However, since  $\theta^*(\lambda)$  is in general unknown, (R) is not very helpful to discard inactive groups. To this end, we will estimate a region  $\mathcal{R}$  which contains  $\theta^*(\lambda)$ . Therefore, if  $\max_{\theta \in \mathcal{R}} \|B_i^{\mathrm{T}}\theta\|_{\bar{q}} < 1$ , we can also conclude that  $\mathbf{X}_i^*(\lambda) = 0$  by (R). As a result, the rule in (R) can be relaxed as

$$\varphi(\theta^*(\lambda), B_i) := \max_{\theta \in \mathcal{R}} \|B_i^{\mathrm{T}}\theta\|_{\bar{q}} < 1 \Rightarrow \mathbf{X}_i^*(\lambda) = 0.$$
 (R')

In this paper, (R') serves as the cornerstone for constructing the proposed screening rules. From (R'), we can see that screening rules with smaller  $\varphi(\theta^*(\lambda), B_i)$  are more effective in identifying inactive groups. To give a tight estimation of  $\varphi(\theta^*(\lambda), B_i)$ , we need to restrict the region  $\mathcal{R}$  containing  $\theta^*(\lambda)$  as small as possible.

In Section 3.1, we give an accurate estimation of the possible region of  $\theta^*(\lambda)$  via the variational inequality. We then derive the upper bound  $\varphi(\theta^*(\lambda), B_i)$  in Section 3.2 and construct the proposed screening rules, that is, Smin, in Section 3.3 based on (R').

# **3.1** Estimating the Possible Region for $\theta^*(\lambda)$

In this section, we briefly discuss the geometric properties of the optimal solution  $\theta^*(\lambda)$  of problem (40), and then give an accurate estimation of the possible region of  $\theta^*(\lambda)$  via the variational inequality.

Consider problem

$$\min_{\theta} g(\theta) := \frac{1}{2} \left\| \theta - \frac{Y}{\lambda} \right\|_2^2,$$
s.t.  $\|B_i^{\mathrm{T}} \theta\|_{\bar{q}} \le 1, i = 1, 2, \dots, s.$ 

$$(43)$$

It is easy to see that problems (40) and (43) have the same optimal solution. Let  $\mathcal{F} = \{\theta : \|B_i^{\mathrm{T}}\theta\|_{\bar{q}} \leq 1, i = 1, 2, \ldots, s.\}$  denote the feasible set of problem (43). We can see that the optimal solution  $\theta^*(\lambda)$  of problem (43) is the projection of  $\frac{Y}{\lambda}$  onto the feasible set  $\mathcal{F}$ . The following theorem shows that problem (39) and its dual (40) admit closed form solutions when  $\lambda$  is large enough.

**Theorem 15.** Let  $\mathbf{X}^*(\lambda)$  and  $\theta^*(\lambda)$  be the optimal solutions of problem (39) and its dual (40). Then if  $\lambda \geq \lambda_{max} := \max_i \|B_i^{\mathrm{T}}Y\|_{\bar{q}}$ , we have

$$\mathbf{X}^*(\lambda) = 0 \quad and \quad \theta^*(\lambda) = \frac{Y}{\lambda}.$$
(44)

*Proof.* We first consider the cases in which  $\lambda > \lambda_{max}$ .

When  $\lambda > \lambda_{max}$ , we can see that  $\|B_i^T \frac{Y}{\lambda}\|_{\bar{q}} < 1$  for all i = 1, 2, ..., s, i.e.,  $\frac{Y}{\lambda}$  is itself an interior point of  $\mathcal{F}$ . As a result, we have  $\theta^*(\lambda) = \frac{Y}{\lambda}$ . Therefore, the rule in (R) implies that  $\mathbf{X}_i^*(\lambda) = 0, i = 1, 2, ..., s$ , i.e., the optimal solution  $\mathbf{X}^*(\lambda)$  of problem (39) is 0.

Next let us consider the case in which  $\lambda = \lambda_{max}$ . Because  $\mathcal{F}$  is convex and  $\theta^*(\lambda)$  is the projection of  $\frac{Y}{\lambda}$  onto  $\mathcal{F}$ , we can see that  $\theta^*(\lambda)$  is nonexpansive with respect to  $\lambda$  [5] and thus  $\theta^*(\lambda)$  is a continuous function of  $\lambda$ . Therefore, it is easy to see that

$$\theta^*(\lambda_{max}) = \lim_{\lambda \downarrow \lambda_{max}} \theta^*(\lambda) = \frac{Y}{\lambda_{max}}$$

Let  $h(\lambda) := B\mathbf{X}^*(\lambda) = \sum_{i=1}^s B_i \mathbf{X}_i^*(\lambda)$ . By Eq. (41), we have

$$h(\lambda) = Y - \lambda \theta^*(\lambda),$$

which implies that  $h(\lambda)$  is also continuous with respect to  $\lambda$ . Therefore, we have

$$h(\lambda_{max}) = \lim_{\lambda \downarrow \lambda_{max}} h(\lambda) = 0$$

Clearly, we can set  $\overline{\mathbf{X}} = 0$  such that  $h(\lambda_{max}) = B\overline{X} = 0$  can be satisfied. Therefore, both of the KKT conditions in Eq. (41) and Eq. (42) are satisfied by  $\overline{X}$  and  $\theta^*(\lambda_{max}) = \frac{Y}{\lambda_{max}}$ . Because problem (39) is a convex optimization problem, the satisfaction of the KKT conditions implies that  $\overline{\mathbf{X}} = 0$  is an optimal solution of (39). Therefore, we can choose 0 for  $\mathbf{X}^*(\lambda_{max})$ . Moreover, we can see that  $\mathbf{X}^*(\lambda_{max})$  must be zero because otherwise  $f(\mathbf{X}^*(\lambda_{max})) = \frac{1}{2} \|Y\|_2^2 + \lambda \sum_{i=1}^s \|\mathbf{X}^*_i(\lambda_{max})\|_{\bar{q}} > \frac{1}{2} \|Y\|_2^2 = f(0)$ . Therefore, we have  $\mathbf{X}^*(\lambda_{max}) = 0$  which completes the proof.

Suppose we are given two distinct parameters  $\lambda'$  and  $\lambda''$  and the corresponding optimal solutions of (43) are  $\theta^*(\lambda')$  and  $\theta^*(\lambda'')$  respectively. Without loss of generality, let us assume  $\lambda_{max} \geq \lambda' > \lambda'' > 0$ . Then the variational inequalities [13] can be written as

$$\langle \theta^*(\lambda') - \theta^*(\lambda''), \nabla g(\theta^*(\lambda'')) \rangle \ge 0, \tag{45}$$

$$\langle \theta^*(\lambda'') - \theta^*(\lambda'), \nabla g(\theta^*(\lambda')) \rangle \ge 0.$$
 (46)

Because  $\nabla g(\theta) = \theta - \frac{Y}{\lambda}$ , the variational inequalities in Eq. (45) and Eq. (46) can be rewritten as:

$$\left\langle \theta^*(\lambda') - \theta^*(\lambda''), \theta^*(\lambda'') - \frac{Y}{\lambda''} \right\rangle \ge 0, \tag{47}$$

$$\left\langle \theta^*(\lambda'') - \theta^*(\lambda'), \theta^*(\lambda') - \frac{Y}{\lambda'} \right\rangle \ge 0.$$
(48)

When  $\lambda' = \lambda_{max}$ , Theorem 15 tells that  $\theta^*(\lambda') = \theta^*(\lambda_{max}) = \frac{Y}{\lambda_{max}}$  and thus the inequality in Eq. (48) is trivial. Let  $B_* := \operatorname{argmax}_i \|B_i^{\mathrm{T}} \theta^*(\lambda_{max})\|_{\bar{q}}$  and  $\phi(\theta) := \|B_*^{\mathrm{T}} \theta\|_{\bar{q}}$  where  $\theta \in \mathbb{R}^m$ . Then the subdifferential of  $\phi(\cdot)$  can be found as:

$$\partial \phi(\theta) := \{ B_* \mathbf{d} : \| \mathbf{d} \|_q \le 1, \langle \mathbf{d}, B_*^{\mathsf{T}} \theta \rangle = \| B_*^{\mathsf{T}} \theta \|_{\bar{q}} \}$$

To simplify notations, let  $\theta_{max} := \theta^*(\lambda_{max})$ . Consider  $\partial \phi(\theta_{max})$ . It is easy to see that  $B_*^{\mathrm{T}} \theta_{max} \neq 0$  since  $\|B_*^{\mathrm{T}} \theta_{max}\|_{\bar{q}} = 1$ . Let

$$\mathbf{d}_{max} = \operatorname{sgn}(B_*^{\mathrm{T}}\theta_{max}) \odot |B_*^{\mathrm{T}}\theta_{max}|^{\bar{q}/q}.$$
(49)

It is easy to check that  $\|\mathbf{d}_{max}\|_q = 1$  and  $\langle \mathbf{d}_{max}, B_*^{\mathrm{T}} \theta_{max} \rangle = \|B_*^{\mathrm{T}} \theta_{max}\|_{\bar{q}} = 1$ , which implies that  $B_* \mathbf{d}_{max} \in \partial \phi(\theta_{max})$ . As a result, the hyperplane

$$\mathcal{H} := \{ \theta \in \mathbb{R}^m : \langle B_* \mathbf{d}_{max}, \theta \rangle = \| B_*^{\mathrm{T}} \theta \|_{\bar{q}} = 1 \}$$
(50)

is a supporting hyperplane [16] to the set

$$\mathcal{C} := \{ \theta \in \mathbb{R}^m : \|B_*^{\mathrm{T}}\theta\|_{\bar{q}} \le 1 \}.$$

As a result,  $\mathcal{H}$  is also a supporting hyperplane to the set  $\mathcal{F}$ . [Recall that  $\mathcal{F}$  is the feasible set of problem (43) and  $\mathcal{C} \subseteq \mathcal{F}$ .] Therefore, when  $\lambda' = \lambda_{max}$ , we have

$$\langle \theta - \theta^*(\lambda'), -B_* \mathbf{d}_{max} \rangle \ge 0, \ \forall \theta \in \mathcal{F}.$$
 (51)

Suppose  $\theta^*(\lambda')$  is known, for notational convenience, let

$$\mathbf{a}(\lambda'',\lambda') = \frac{1}{2} \left[ \frac{Y}{\lambda''} - \theta^*(\lambda') \right],\tag{52}$$

$$\mathbf{b}(\lambda') = \begin{cases} \frac{Y}{\lambda'} - \theta^*(\lambda'), & \text{if } \lambda' \in (0, \lambda_{max}) \\ B_* \mathbf{d}_{max}, & \text{if } \lambda' = \lambda_{max}, \end{cases}$$
(53)

$$\mathbf{v}(\lambda'',\lambda') = \mathbf{a}(\lambda'',\lambda') - \frac{\langle \mathbf{a}(\lambda'',\lambda'), \mathbf{b}(\lambda') \rangle}{\|\mathbf{b}(\lambda')\|_2^2} \mathbf{b}(\lambda'), \tag{54}$$

and

$$\mathbf{o}(\lambda'',\lambda') = \theta^*(\lambda') + \mathbf{v}(\lambda'',\lambda').$$
(55)

The next lemma shows that the inner product between  $\mathbf{a}(\lambda'', \lambda')$  and  $\mathbf{b}(\lambda')$  is nonnegative.

**Lemma 16.** Suppose  $\theta^*(\lambda')$  and  $\theta^*(\lambda'')$  are the optimal solutions of problem (43) for two distinct parameters  $\lambda_{max} \geq \lambda' > \lambda'' > 0.$  Then

$$\langle \mathbf{a}(\lambda'',\lambda'),\mathbf{b}(\lambda')\rangle \ge 0.$$
 (56)

*Proof.* We first show that the statement holds when  $\lambda' = \lambda_{max}$ . By Theorem 15, we can see that  $\theta^*(\lambda_{max}) = \frac{Y}{\lambda_{max}}$  and thus

$$\mathbf{a}(\lambda'',\lambda_{max}) = \frac{1}{2} \left(\frac{1}{\lambda''} - \frac{1}{\lambda_{max}}\right) Y.$$

If Y = 0, the statement is trivial. Let us assume  $Y \neq 0$ . Then

$$\langle \mathbf{a}(\lambda'', \lambda_{max}), \mathbf{b}(\lambda_{max}) \rangle = \frac{1}{2} \left( \frac{1}{\lambda''} - \frac{1}{\lambda_{max}} \right) \langle Y, B_* \mathbf{d}_{max} \rangle.$$
(57)

On the other hand, because the zero point is also a feasible point of problem (43), i.e.,  $0 \in \mathcal{F}$ , we can see that

$$\langle 0 - \theta^*(\lambda_{max}), -B_* \mathbf{d}_{max} \rangle = \langle -\frac{Y}{\lambda_{max}}, -B_* \mathbf{d}_{max} \rangle \ge 0.$$
 (58)

In view of Eq. (57) and Eq. (58), we have  $\langle \mathbf{a}(\lambda'', \lambda_{max}), \mathbf{b}(\lambda_{max}) \rangle \geq 0$ .

Next, let us consider the case with  $0 < \lambda' < \lambda_{max}$ . In fact, we can see that

$$\begin{split} \langle 2\mathbf{a}(\lambda'',\lambda'),\mathbf{b}(\lambda')\rangle &= \left\langle \frac{Y}{\lambda''} - \theta^*(\lambda'), \frac{Y}{\lambda'} - \theta^*(\lambda') \right\rangle \\ &= \left(\frac{\lambda'}{\lambda''} - 1\right) \left\langle \frac{Y}{\lambda'}, \frac{Y}{\lambda'} - \theta^*(\lambda') \right\rangle + \left\| \frac{Y}{\lambda'} - \theta^*(\lambda') \right\|_2^2 \\ &\geq \left(\frac{\lambda'}{\lambda''} - 1\right) \left\langle \theta^*(\lambda'), \frac{Y}{\lambda'} - \theta^*(\lambda') \right\rangle. \end{split}$$

Because  $0 \in \mathcal{F}$ , the variational inequality leads to

$$\langle 0 - \theta^*(\lambda'), \nabla g(\theta^*(\lambda')) \rangle = \left\langle -\theta^*(\lambda'), \theta^*(\lambda') - \frac{Y}{\lambda'} \right\rangle \ge 0$$

Therefore, it is easy to see that

$$\langle 2\mathbf{a}(\lambda'',\lambda'),\mathbf{b}(\lambda')\rangle \geq 0,$$

which completes the proof.

Next we show how to bound  $\theta^*(\lambda'')$  inside a ball via the above variational inequalities.

**Theorem 17.** Suppose  $\theta^*(\lambda')$  and  $\theta^*(\lambda'')$  are the optimal solutions of problem (43) for two distinct parameters  $\lambda_{max} \geq \lambda' > \lambda''$  and  $\theta^*(\lambda')$  is known. Then

$$\|\theta^*(\lambda'') - \mathbf{o}(\lambda'',\lambda')\|_2^2 \le \|\mathbf{v}(\lambda'',\lambda')\|_2^2.$$
(59)

*Proof.* To simplify notations, let  $c = \frac{\langle \mathbf{a}(\lambda'',\lambda'), \mathbf{b}(\lambda') \rangle}{\|\mathbf{b}(\lambda')\|_2^2}$ . Then, Lemma 16 implies that  $c \ge 0$ . Because  $\theta^*(\lambda'') \in \mathcal{F}$ , the inequalities in Eq. (48) and Eq. (51) lead to

$$\langle \theta^*(\lambda'') - \theta^*(\lambda'), \mathbf{b}(\lambda') \rangle \le 0, \tag{60}$$

and thus

$$\langle \theta^*(\lambda'') - \theta^*(\lambda'), 2c\mathbf{b}(\lambda') \rangle \le 0.$$
 (61)

By adding the inequalities in Eq. (47) and Eq. (61), we obtain

$$\left\langle \theta^*(\lambda'') - \theta^*(\lambda'), \theta^*(\lambda'') - \frac{Y}{\lambda''} + 2c\mathbf{b}(\lambda') \right\rangle \le 0.$$
(62)

By noting that

$$\begin{aligned} \theta^*(\lambda'') - \frac{Y}{\lambda''} + c\mathbf{b}(\lambda') &= \theta^*(\lambda'') - \theta^*(\lambda') - \left(\frac{Y}{\lambda''} - \theta^*(\lambda') - 2c\mathbf{b}(\lambda')\right) \\ &= \theta^*(\lambda'') - \theta^*(\lambda') - 2\left(\mathbf{a}(\lambda'',\lambda') - c\mathbf{b}(\lambda')\right) \\ &= \theta^*(\lambda'') - \theta^*(\lambda') - 2\mathbf{v}(\lambda'',\lambda'), \end{aligned}$$

the inequality in Eq. (62) becomes

$$\langle \theta^*(\lambda'') - \theta^*(\lambda'), \theta^*(\lambda'') - \theta^*(\lambda') - 2\mathbf{v}(\lambda'', \lambda') \rangle \le 0$$

which is equivalent to Eq. (59).

Theorem 17 bounds  $\theta^*(\lambda'')$  inside a ball. For notational convenience, let us denote

$$\mathcal{R}(\lambda'',\lambda') = \{\theta : \|\theta - \mathbf{o}(\lambda'',\lambda')\|_2 \le \|\mathbf{v}(\lambda'',\lambda')\|_2\}.$$
(63)

# 3.2 Estimating the Upper Bound

Given two distinct parameters  $\lambda'$  and  $\lambda''$ , and assume the knowledge of  $\theta^*(\lambda')$ , we estimate a possible region  $\mathcal{R}(\lambda'', \lambda')$  for  $\theta^*(\lambda'')$  in Section 3.1. To apply the screening rule in (R') to identify the inactive groups, we need to estimate an upper bound of  $\varphi(\theta^*(\lambda''), B_i) = \max_{\theta \in \mathcal{R}(\lambda'', \lambda')} \|B_i^{\mathrm{T}}\theta\|_{\bar{q}}$ . If  $\varphi(\theta^*(\lambda''), B_i) < 1$ , then  $\mathbf{X}_i^*(\lambda'') = 0$  and the *i*<sup>th</sup> group can be safely removed from the optimization of problem (39).

We need the following technical lemma for the estimation of an upper bound.

**Lemma 18.** Let  $A \in \mathbb{R}^{n \times t}$  be a matrix and  $\mathbf{u} \in \mathbb{R}^{t}$ , and  $q \in [1, \infty]$ . Then the following holds:

$$\|A\mathbf{u}\|_q \le T_A^q \|\mathbf{u}\|_2,\tag{64}$$

where

$$T_{A}^{q} = \begin{cases} \left(\sum_{k=1}^{n} \left(\sum_{j=1}^{t} a_{k,j}^{2}\right)^{q/2}\right)^{1/q}, & \text{if } q \in [1,\infty), \\ \max_{k \in \{1,\dots,n\}} \left(\sum_{j=1}^{t} a_{k,j}^{2}\right)^{1/2}, & \text{if } q = \infty, \end{cases}$$
(65)

and  $a_{k,j}$  is the  $(k,j)^{th}$  entry of A.

*Proof.* Let  $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]^{\mathrm{T}}$ , i.e.,  $\mathbf{a}_k$  is the  $k^{th}$  row of A. When q > 1, we can see that

$$\|A\mathbf{u}\|_{q} = \left(\sum_{k=1}^{n} |\langle \mathbf{a}_{k}, \mathbf{u} \rangle|^{q}\right)^{1/q} \leq \left(\sum_{k=1}^{n} \|\mathbf{a}_{k}\|_{2}^{q} \|\mathbf{u}\|_{2}^{q}\right)^{1/q} = \left(\sum_{k=1}^{n} \|\mathbf{a}_{k}\|_{2}^{q}\right)^{1/q} \|\mathbf{u}\|_{2}.$$
 (66) egument, we can prove the statement with  $q = \infty$ .

By a similar argument, we can prove the statement with  $q = \infty$ .

The following theorem gives an upper bound of  $\varphi(\theta^*(\lambda''), B_i)$ .

**Theorem 19.** Given two distinct parameters  $\lambda_{max} \geq \lambda' > \lambda'' > 0$ . Let  $\mathcal{R}(\lambda'', \lambda')$  be defined as in Eq. (63), then

$$\varphi(\theta^*(\lambda''), B_i) = \max_{\theta \in \mathcal{R}(\lambda'', \lambda')} \|B_i^{\mathrm{T}}\theta\|_{\bar{q}} \le T_{B_i^{\mathrm{T}}}^{\bar{q}} \|\mathbf{v}(\lambda'', \lambda')\|_2 + \|B_i^{\mathrm{T}}\mathbf{o}(\lambda'', \lambda')\|_{\bar{q}},\tag{67}$$

where  $\mathbf{o}(\lambda'', \lambda')$  and  $\mathbf{v}(\lambda'', \lambda')$  are defined in Eq. (55) and Eq. (54) respectively.

*Proof.* Recall that

$$\mathcal{R}(\lambda'',\lambda') = \{\theta : \|\theta - \mathbf{o}(\lambda'',\lambda')\|_2 \le \|\mathbf{v}(\lambda'',\lambda')\|_2\}.$$

Then for  $\theta \in \mathcal{R}(\lambda'', \lambda')$ , we have

$$|B_{i}^{\mathrm{T}}\theta||_{\bar{q}} \leq ||B_{i}^{\mathrm{T}}(\theta - \mathbf{o}(\lambda'', \lambda'))||_{\bar{q}} + ||B_{i}^{\mathrm{T}}\mathbf{o}(\lambda'', \lambda')||_{\bar{q}}$$

$$\leq T_{B_{i}^{\mathrm{T}}}^{\bar{q}}||\theta - \mathbf{o}(\lambda'', \lambda')||_{2} + ||B_{i}^{\mathrm{T}}\mathbf{o}(\lambda'', \lambda')||_{\bar{q}}$$

$$\leq T_{B_{T}}^{\bar{q}}||\mathbf{v}(\lambda'', \lambda')||_{2} + ||B_{i}^{\mathrm{T}}\mathbf{o}(\lambda'', \lambda')||_{\bar{q}}.$$
(68)

The second inequality in Eq. (68) follows from Lemma 18 and the proof is completed.

# 

#### 3.3The Proposed Screening Method (Smin) for $\ell_1/\ell_q$ -Regularized Problems

Using (R'), we are now ready to construct screening rules for the  $\ell_1/\ell_q$ -regularized problems.

**Theorem 20.** For problem (39), assume  $\theta^*(\lambda')$  is known for a specific parameter  $\lambda' \in (0, \lambda_{max}]$ . Let  $\lambda'' \in (0, \lambda')$ . Then  $\mathbf{X}_i^*(\lambda'') = 0$  if

$$\|B_{i}^{\mathrm{T}}\mathbf{o}(\lambda'',\lambda')\|_{\bar{q}} < 1 - T_{B_{i}^{\bar{q}}}^{\bar{q}} \|\mathbf{v}(\lambda'',\lambda')\|_{2}.$$
(69)

*Proof.* From  $(\mathbf{R}')$ , we know that

$$\varphi(\theta^*(\lambda''), B_i) < 1 \Rightarrow \mathbf{X}_i^*(\lambda'') = 0.$$
(70)

By Theorem 19, we have

$$\varphi(\theta^*(\lambda''), B_i) \le T_{B_i^{\mathrm{T}}}^{\bar{q}} \| \mathbf{v}(\lambda'', \lambda') \|_2 + \| B_i^{\mathrm{T}} \mathbf{o}(\lambda'', \lambda') \|_{\bar{q}}.$$
(71)

In view of Eq. (69), Eq. (71) results in

$$\varphi(\theta^*(\lambda''), B_i) < 1,$$

which completes the proof.

By setting  $\lambda' = \lambda_{max}$  in Theorem 20, we immediately obtain the following basic screening rule.

**Corollary 21.** (Smin<sub>b</sub>) Consider problem (39), let  $\lambda_{max} = \max_i ||B_i^T Y||_{\bar{q}}$ . If  $\lambda \ge \lambda_{max}$ ,  $\mathbf{X}_i^*(\lambda) = 0$  for all  $i = 1, \ldots, s$ . Otherwise, we have  $\mathbf{X}_i^*(\lambda) = 0$  if the following holds:

$$\|B_i^{\mathrm{T}}\mathbf{o}(\lambda,\lambda_{max})\|_{\bar{q}} < 1 - T_{B_i^{\mathrm{T}}}^{\bar{q}} \|\mathbf{v}(\lambda'',\lambda_{max})\|_2,$$

$$(72)$$

where  $\theta^*(\lambda_{max}) = \frac{Y}{\lambda_{max}}$ ,  $\mathbf{o}(\lambda'', \lambda_{max})$  and  $\mathbf{v}(\lambda'', \lambda_{max})$  are defined in Eq. (55) and Eq. (54) respectively.

In practical applications, the optimal parameter value of  $\lambda$  is unknown and needs to be estimated. Commonly used approaches such as cross validation and stability selection involve solving the  $\ell_1/\ell_q$ -regularized problems over a grid of tuning parameters  $\lambda_1 > \lambda_2 > \ldots > \lambda_K$  to determine an appropriate value for  $\lambda$ . As a result, the computation is very time consuming. To address this challenge, we propose the sequential version of the proposed Smin.

**Corollary 22.** (Smin<sub>s</sub>) For the problem in (39), suppose we are given a sequence of parameter values  $\lambda_{max} = \lambda_0 > \lambda_1 > \ldots > \lambda_{\mathcal{K}}$ . For any integer  $0 \le k < \mathcal{K}$ , we have  $\mathbf{X}_i^*(\lambda_{k+1}) = 0$  if  $\mathbf{X}^*(\lambda_k)$  is known and the following holds:

$$\|B_{i}^{\mathrm{T}}\mathbf{o}(\lambda_{k+1},\lambda_{k})\|_{\bar{q}} < 1 - T_{B_{i}^{\mathrm{T}}}^{\bar{q}}\|\mathbf{v}(\lambda_{k+1},\lambda_{k})\|_{2},$$
(73)

where  $\theta^*(\lambda_k) = \frac{Y - \sum_{i=1}^{s} B_i \mathbf{X}_i^*(\lambda_k)}{\lambda_k}$ ,  $\mathbf{o}(\lambda_{k+1}, \lambda_k)$  and  $\mathbf{v}(\lambda_{k+1}, \lambda_k)$  are defined in Eq. (55) and Eq. (54) respectively.

*Proof.* The statement easily follows by setting  $\lambda'' = \lambda_{k+1}$  and  $\lambda' = \lambda_k$  and applying Eq. (41) and Theorem 20.

# 4 Experiments

In Sections 4.1 and 4.2, we conduct experiments to evaluate the efficiency of the proposed algorithm, that is, GLEP<sub>1</sub>q, using both synthetic and real-world data sets. We evaluate the proposed Smin<sub>s</sub> on large-scale data sets and compare the performance of Smin<sub>s</sub> with DPP and strong rules which achieve state-of-theart performance for the  $\ell_1/\ell_q$ -regularized problems in Section 4.3. We set the regularization parameter as  $\lambda = r \times \lambda_{\text{max}}$ , where  $0 < r \leq 1$  is the ratio, and  $\lambda_{\text{max}}$  is the maximal value above which the  $\ell_1/\ell_q$ norm regularized problem (1) obtains a zero solution (see Theorem 15). We try the following values for  $q: 1, 1.25, 1.5, 1.75, 2, 2.33, 3, 5, \text{ and } \infty$ . The source codes, included in the SLEP package [22], are available online<sup>2</sup>.

# 4.1 Simulation Studies

We use the synthetic data to study the effectiveness of the  $\ell_1/\ell_q$ -norm regularization for reconstructing the jointly sparse matrix under different values of q > 1. Let  $A \in \mathbb{R}^{m \times d}$  be a measurement matrix with entries being generated randomly from the standard normal distribution,  $X^* \in \mathbb{R}^{d \times k}$  be the jointly sparse matrix with the first  $\tilde{d} < d$  rows being nonzero and the remaining rows exactly zero,  $Y = AX^* + Z$  be the response matrix, and  $Z \in \mathbb{R}^{m \times k}$  be the noise matrix whose entries are drawn randomly from the normal distribution with mean zero and standard deviation  $\sigma = 0.1$ . We treat each row of  $X^*$  as a group, and estimate  $X^*$  from A and Y by solving the following  $\ell_1/\ell_q$ -norm regularized problem:

$$X = \arg\min_{W} \frac{1}{2} \|AW - Y\|_{F}^{2} + \lambda \sum_{i=1}^{d} \|W^{i}\|_{q},$$

where  $W^i$  denotes the *i*-th row of W. We set m = 100, d = 200, and  $\tilde{d} = k = 50$ . We try two different settings for  $X^*$ , by drawing its nonzero entries randomly from 1) the uniform distribution in the interval [0, 1] and 2) the standard normal distribution.

<sup>&</sup>lt;sup>2</sup>http://www.public.asu.edu/~jye02/Software/SLEP/



Figure 2: Performance of the  $\ell_1/\ell_q$ -norm regularization for reconstructing the jointly sparse  $X^*$ . The nonzero entries of  $X^*$  are drawn randomly from the uniform distribution for the plots in the first column, and from the normal distribution for the plots in the second column. Plots in the first two rows show  $||X - X^*||_F$ , the Frobenius norm difference between the solution and the truth; and plots in the third row show the  $\ell_2$ -norm of each row of the solution X.

We compute the solutions corresponding to a sequence of decreasing values of  $\lambda = r \times \lambda_{\max}$ , where  $r = 0.9^{i-1}$ , for i = 1, 2, ..., 100. In addition, we use the solution corresponding to the  $0.9^i \times \lambda_{\max}$  as the "warm" start for  $0.9^{i+1} \times \lambda_{\max}$ . We report the results in Figure 2, from which we can observe: 1) the distance between the solution X and the truth X\* usually decreases with decreasing values of  $\lambda$ ; 2) for the uniform distribution (see the plots in the first row), q = 1.5 performs the best; 3) for the normal distribution (see the plots in the second row), q = 1.5, 1.75, 2 and 3 achieve comparable performance and perform better than q = 1.25, 5 and  $\infty$ ; 4) with a properly chosen threshold, the support of X\* can be exactly recovered by the  $\ell_1/\ell_q$ -norm regularization with an appropriate value of q, e.g., q = 1.5 for the uniform distribution, and q = 2 for the normal distribution; and 5) the recovery of X\* with nonzero entries drawn from the normal distribution.

The existing theoretical results [20, 26] can not tell which q is the best; and we believe that the optimal q depends on the distribution of  $X^*$ , as indicated from the above results. Therefore, it is necessary to conduct the distribution-specific theoretical studies (note that the previous studies usually make no assumption on  $X^*$ ). The proposed GLEP<sub>1q</sub> algorithm shall help verify the theoretical results to be established.

#### 4.2 Performance on the Letter Data Set

We apply the proposed GLEP<sub>1q</sub> algorithm for multi-task learning on the Letter data set [32], which consists of 45,679 samples from 8 default tasks of two-class classification problems for the handwritten letters: c/e, g/y, m/n, a/g, i/j, a/o, f/t, h/n. The writings were collected from over 180 different writers, with the letters being represented by  $8 \times 16$  binary pixel images. We use the least squares loss for  $l(\cdot)$ .



Figure 3: Computational time (seconds) comparison between  $\text{GLEP}_{1q}$  (q = 2) and Spg under different values of  $\lambda = r \times \lambda_{\text{max}}$  and m.

#### 4.2.1 Efficiency Comparison with Spg

We compare  $\text{GLEP}_{1q}$  with the Spg algorithm proposed in [4]. Spg is a specialized solver for the  $\ell_1/\ell_2$ -ball constrained optimization problem, and has been shown to outperform existing algorithms based on blockwise



Figure 4: Computation time (seconds) of  $\text{GLEP}_{1q}$  under different values of m, q and r.

coordinate descent and projected gradient. In Figure 3, we report the computational time under different values of m (the number of samples) and  $\lambda = r \times \lambda_{\max}$  (q = 2). It is clear from the plots that  $\text{GLEP}_{1q}$  is much more efficient than Spg, which may attribute to: 1)  $\text{GLEP}_{1q}$  has a better convergence rate than Spg; and 2) when q = 2, the  $\text{EP}_{1q}$  in  $\text{GLEP}_{1q}$  can be computed analytically (see Remark 13), while this is not the case in Spg.

#### 4.2.2 Efficiency under Different Values of q

We report the computational time (seconds) of  $\text{GLEP}_{1q}$  under different values of q,  $\lambda = r \times \lambda_{\max}$  and m (the number of samples) in Figure 4. We can observe from this figure that the computational time of  $\text{GLEP}_{1q}$  under different values of q (for fixed r and m) is comparable. Together with the result on the comparison with Spg for q = 2, this experiment shows the promise of  $\text{GLEP}_{1q}$  for solving large-scale problems for any  $q \ge 1$ .

#### 4.2.3 Performance under Different Values of q

We randomly divide the Letter data into three non-overlapping sets: training, validation, and testing. We train the model using the training set, and tune the regularization parameter  $\lambda = r \times \lambda_{\text{max}}$  on the validation set, where r is chosen from  $\{10^{-1}, 5 \times 10^{-2}, 2 \times 10^{-2}, 1 \times 10^{-2}, 5 \times 10^{-3}, 2 \times 10^{-3}, 1 \times 10^{-3}\}$ . On the testing set, we compute the balanced error rate [14]. We report the results averaged over 10 runs in Figure 5. The title of each plot indicates the percentages of samples used for training, validation, and testing. The results show that, on this data set, a smaller value of q achieves better performance.

## 4.3 Performance of The Proposed Screening Method

In this section, we evaluate the proposed screening method (Smin) for solving problem (39) with different values of q,  $n_g$  and p, which correspond to the mixed-norm, the number of groups and the feature dimension,



Figure 5: The balanced error rate achieved by the  $\ell_1/\ell_q$  regularization under different values of q. The title of each plot indicates the percentages of samples used for training, validation, and testing.

respectively. We compare the performance of  $\text{Smin}_s$ , the sequential DPP rule, and the sequential strong rule in identifying inactive groups of problem (39) along a sequence of 91 tuning parameter values equally spaced on the scale of  $r = \frac{\lambda}{\lambda_{max}}$  from 1.0 to 0.1. To measure the performance of the screening methods, we report the rejection ratio, i.e., the ratio between the number of groups discarded by the screening methods and the number of groups with 0 coefficients in the true solution. It is worthwhile to mention that  $\text{Smin}_s$  and DPP are "safe", that is, the discarded groups are guaranteed to have 0 coefficients in the solution. However, strong rules may mistakenly discard groups which have non-zero coefficients in the solution (notice that, DPP can be easily extended to the general  $\ell_1/\ell_q$ -regularized problems with  $q \neq 2$  via the techniques developed in this paper).

More importantly, we show that the efficiency of the proposed  $\text{GLEP}_{1q}$  can be improved by "three orders of magnitude" with Smin<sub>s</sub>. Specifically, in each experiment, we first apply  $\text{GLEP}_{1q}$  with warm-start to solve problem (39) along the given sequence of 91 parameters and report the total running time. Then, for each parameter, we first use the screening methods to discard the inactive groups and then apply  $\text{GLEP}_{1q}$  to solve problem (39) with the reduced data matrix. We repeat this process until all of the problems have been solved. The total running time (including screening) is reported to demonstrate the effectiveness of the proposed Smin<sub>s</sub> in accelerating the computation. Finally, we also report the total running time of the screening methods in discarding the inactive groups for problem (39) with the given sequence of parameters. The entries of the response vector Y and the data matrix B are generated i.i.d. from a standard Gaussian distribution, and the correlation between Y and the columns of B is uniformly distributed on the interval [-0.8, 0.8]. For each experiment, we repeat the computation 20 times and report the average results.

### 4.3.1 Efficiency with Different Values of q

In this experiment, we evaluate the performance of the proposed  $\text{Smin}_s$  and demonstrate its effectiveness in accelerating the computation of  $\text{GLEP}_{1q}$  with different values of q. The size of the data matrix B is fixed to be  $1000 \times 10000$ , and we randomly divide B into 1000 non-overlapping groups.

Figure 6 presents the rejection ratios of DPP, strong rule and  $\text{Smin}_s$  with 9 different values of q. We can see that the performance of  $\text{Smin}_s$  is very robust to the variation of q. The rejection ratio of  $\text{Smin}_s$  is almost 100% along the sequence of 91 parameters for all different q, which implies that almost all of the inactive groups are discarded by  $\text{Smin}_s$ . As a result, the number of groups that need to be entered into the optimization is significantly reduced. The performance of the strong rule is less robust to different values of q than  $\text{Smin}_s$ , and DPP is more sensitive than the other two. Figure 6 indicates that the performance of  $\text{Smin}_s$  significantly outperforms that of DPP and the strong rule.

Table 1 shows the total running time for solving problem (39) along the sequence of 91 values of  $r = \lambda/\lambda_{max}$  by  $\text{GLEP}_{1q}$  and  $\text{GLEP}_{1q}$  with screening methods. We also report the running time for the screening methods. We can see that the running times of  $\text{GLEP}_{1q}$  combined with  $\text{Smin}_s$  are only about 1% of the running times of  $\text{GLEP}_{1q}$  without screening when  $q = 1, 1.25, 1.5, 1.75, 2, \infty$ . In other words,  $\text{Smin}_s$  improves

q	$GLEP_{1q}$	$GLEP_{1q}+DPP$	$GLEP_{1q} + SR$	$GLEP_{1q} + Smin_s$	DPP	Strong Rule	$Smin_s$
1	284.89	3.99	3.63	2.35	1.07	1.98	1.02
1.25	338.76	5.50	5.18	3.65	1.05	2.03	1.06
1.5	380.02	11.58	5.58	4.19	1.12	2.04	1.08
1.75	502.07	50.27	7.53	5.83	1.21	1.97	1.08
2	484.11	121.90	10.64	4.94	1.54	1.88	1.01
2.33	809.42	266.60	89.52	47.72	1.72	2.33	1.08
3	959.83	220.81	105.61	92.14	1.52	2.05	1.12
5	1157.69	185.55	131.85	128.77	1.24	2.04	1.06
$\infty$	221.83	9.35	3.62	2.69	1.01	1.81	0.99

Table 1: Running time (in seconds) for solving the mixed-norm regularized problems along a sequence of 91 tuning parameter values equally spaced on the scale of  $r = \frac{\lambda}{\lambda_{max}}$  from 1.0 to 0.1 by (a): GLEP<sub>1q</sub> (reported in the second column) without screening; (b): GLEP<sub>1q</sub> combined with different screening methods (reported in the third to the fifth columns). The last three columns report the total running time (in seconds) for the screening methods. The data matrix B is of size 1000 × 10000 and q = 2.

$n_g$	$\mathrm{GLEP}_{1q}$	$GLEP_{1q}+DPP$	$GLEP_{1q} + SR$	$GLEP_{1q} + Smin_s$	DPP	Strong Rule	$Smin_s$
100	421.62	275.94	140.52	41.91	1.07	2.39	1.10
400	364.61	168.04	34.14	6.12	1.55	1.98	1.01
800	410.65	128.33	9.20	5.03	1.44	2.02	1.03
1200	627.85	136.68	12.33	6.84	1.56	1.99	1.13
1600	741.25	115.77	10.69	7.10	1.42	1.91	1.01
2000	786.89	130.74	12.54	7.24	1.57	2.19	1.09

Table 2: Running time (in seconds) for solving the mixed-norm regularized problems along a sequence of 91 tuning parameter values equally spaced on the scale of  $r = \frac{\lambda}{\lambda_{max}}$  from 1.0 to 0.1 by (a): GLEP<sub>1q</sub> (reported in the second column) without screening; (b): GLEP<sub>1q</sub> combined with different screening methods (reported in the third to the fifth columns). The last three columns report the total running time (in seconds) for the screening methods. The data matrix B is of size 1000 × 10000 and q = 2.

the efficiency of  $\text{GLEP}_{1q}$  by about 100 times. When q = 2.33, 3, 5, the efficiency of  $\text{GLEP}_{1q}$  is improved by more than 10 times. The last three columns of Table 1 also indicate that  $\text{Smin}_s$  is the most efficient screening method among DPP and the strong rule in terms of the running time. Moreover, we can see that the running time of the strong rule is about twice as long as the other two methods. The reason is because the strong rule needs to check its screening results [38] by verifying the KKT conditions each time.

#### 4.3.2 Efficiency with Different Numbers of Groups

In this experiment, we evaluate the performance of  $\text{Smin}_s$  with different numbers of groups. The data matrix B is fixed to be  $1000 \times 10000$  and we set q = 2. Recall that  $n_g$  denotes the number of groups. Therefore, let  $s_g$  be the average group size. For example, if  $n_g$  is 100, then  $s_g = p/n_g = 100$ .

From Figure 7, we can see that the screening methods, i.e.,  $\text{Smin}_s$ , DPP and strong rule, are able to discard more inactive groups when the number of groups  $n_g$  increases. The reason is due to the fact that the estimation of the region  $\mathcal{R}$  [please refer to Eq. (63) and Section 3.1] is more accurate with smaller group size. Notice that, a large  $n_g$  implies a small average group size. Figure 7 implies that compared with DPP and the strong rule, Smin<sub>s</sub> is able to discard more inactive groups and more robust with respect to different values of  $n_g$ .

Table 2 further demonstrates the effectiveness of  $\text{Smin}_s$  in improving the efficiency of  $\text{GLEP}_{1q}$ . When  $n_g = 100, 400$ , the efficiency of  $\text{GLEP}_{1q}$  is improved by 10 and 50 times respectively. For the other cases, the efficiency of  $\text{GLEP}_{1q}$  is boosted by about 100 times with  $\text{Smin}_s$ .

p	$GLEP_{1q}$	$GLEP_{1q}+DPP$	$GLEP_{1q} + SR$	$GLEP_{1q} + Smin_s$	DPP	Strong Rule	$Smin_s$
20000	1103.52	218.49	12.54	7.42	2.75	3.66	2.01
50000	3440.17	508.61	23.07	11.74	7.21	9.24	4.93
80000	6676.28	789.35	33.26	17.17	12.31	15.06	7.97
100000	8525.73	983.03	37.49	18.88	16.68	18.82	9.73
150000	14752.20	1472.79	55.28	26.88	26.16	29.00	15.28
200000	19959.09	1970.02	69.32	33.51	38.04	38.88	21.04

Table 3: Running time (in seconds) for solving the mixed-norm regularized problems along a sequence of 91 tuning parameter values equally spaced on the scale of  $r = \frac{\lambda}{\lambda_{max}}$  from 1.0 to 0.1 by (a): GLEP<sub>1q</sub> (reported in the second column) without screening; (b): GLEP<sub>1q</sub> combined with different screening methods (reported in the third to the fifth columns). The last three columns report the total running time (in seconds) for the screening methods. The size of the data matrix B is  $1000 \times p$ ,  $n_g = p/10$  and q = 2.

#### 4.3.3 Efficiency with Different Dimensions

In this experiment, we apply GLEP<sub>1q</sub> with screening methods to large dimensional data sets. We generate the data sets  $B \in \mathbb{R}^{1000 \times p}$ , where p = 20000, 50000, 80000, 100000, 150000, 200000. Notice that, the data matrix B is a dense matrix. We set q = 2 and the average group size  $s_q = 10$ .

As shown in Figure 8, the performance of all three screening methods, i.e.,  $Smin_s$ , DPP and the strong rule, are robust to different dimensions. We can observe from the figure that  $Smin_s$  is again the most effective screening method in discarding inactive groups.

The computational savings gained by the proposed  $\text{Smin}_s$  method are more significant than those reported in Sections 4.3.1 and 4.3.2. From Table 3, when p = 200000, the efficiency of  $\text{GLEP}_{1q}$  is improved by more than 6000 times with  $\text{Smin}_{1q}$ , which demonstrates that  $\text{Smin}_{1q}$  is a powerful tool to accelerate the optimization of the mixed-norm regularized problems especially for large dimensional data.

# 5 Conclusion

In this paper, we propose the GLEP<sub>1q</sub> algorithm and the corresponding screening method, that is, Smin, for solving the  $\ell_1/\ell_q$ -norm regularized problem with any  $q \ge 1$ . The main technical contributions of this paper include two parts.

First, we develop an efficient algorithm for the  $\ell_1/\ell_q$ -norm regularized Euclidean projection (EP<sub>1q</sub>), which is a key building block of GLEP<sub>1q</sub>. Specifically, we analyze the key theoretical properties of the solution of EP<sub>1q</sub>, based on which we develop an efficient algorithm for EP<sub>1q</sub> by solving two zero finding problems. Our analysis also reveals why EP<sub>1q</sub> for the general q is significantly more challenging than the special cases such as q = 2.

Second, we develop a novel screening method (Smin) for large dimensional mixed-norm regularized problems, which is based on an accurate estimation of the possible region of the dual optimal solution. Our method is safe and can be integrated with any existing solvers. Our extensive experiments demonstrate that the proposed Smin is very powerful in discarding inactive groups, resulting in huge computational savings (up to three orders of magnitude) especially for large dimensional problems.

In this paper, we focus on developing efficient algorithms for solving the  $\ell_1/\ell_q$ -regularized problem. We plan to study the effectiveness of the  $\ell_1/\ell_q$  regularization under different values of q for real-world applications in computer vision and bioinformatics, e.g., imaging genetics [36]. We also plan to conduct the distribution-specific [9] theoretical studies for different values of q.

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Figure 6: Comparison of  $\text{Smin}_s$ , DPP and strong rules with different values of q:  $B \in \mathbb{R}^{1000 \times 10000}$  and  $n_g = 1000$ .



Figure 7: Comparison of  $\text{Smin}_s$ , DPP and strong rules with different numbers of groups:  $B \in \mathbb{R}^{1000 \times 10000}$ and q = 2.



Figure 8: Comparison of Smin<sub>s</sub>, DPP and strong rules with different dimensions. We set  $n_g = p/10$  and q = 2.