

Characterizing Chaotic Behavior of REM Sleep EEG Using Lyapunov Exponent

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Abstract— The Lyapunov exponent of the hippocampal EEG was calculated to examine its chaotic behavior. In particular, we study the REM sleep EEG of rats from hippocampal subfields cornu ammonis (CA1) and dentate gyrus (DG) that show prominent theta activity. Our main goal is to characterize, through nonlinear dynamics analysis, the ontogenetic variation between the granule cells of the DG and the pyramidal cells of the CA1 as the animal matures. We examined the hippocampal REM sleep EEG from rats of 15-day and 90-day of age. Our results show that the Lyapunov exponents of the hippocampal REM sleep EEG from both CA1 and DG are positive in 15-day and 90-day age groups, a strong evidence confirming their chaotic behavior. While the estimated Lyapunov exponents of hippocampal REM sleep EEG of CA1 and DG from 90-day rats are similar, the Lyapunov exponents of CA1 and DG from 15-day rats are significantly different.

Keywords- Hippocampal EEG; REM Sleep; Chaos; Lyapunov Exponent; Nonlinear Dynamics

I. INTRODUCTION

The hippocampus consists of two major interlocked subfields, the fascia dentate or *dentate gyrus* (DG) and the hippocampus proper or *cornu ammonis* (CA) [1]. Hippocampus proper or CA is in turn divided into four fields, CA1-4. The first field, CA1 is called the regio superior and consists of smaller pyramidal cells, while the dentate gyrus is made of granule cells [1]. CA1 and dentate gyrus are known as the primary generators of the hippocampal EEG pattern called rhythmical slow activity (RSA) or the theta (θ) rhythm [1]-[8]. There is no theta rhythm present in the rat brain at the time of birth [1]. Theta activity first appears in rats at 10 - 12 days of age and matures primarily during the pre-weaning period of development, i.e., birth to approximately 25 days of age [9]. Theta wave is observed in rats in a state of alertness or arousal, and also during rapid eye movement (REM) sleep [1].

Hippocampus neurons play a key role in decision making, information processing, and other behavioral and developmental functions. Even though their functions have been widely studied, their mechanisms that generate theta rhythm are still not entirely understood. Theta activity has been correlated with particular deliberate movements such as exploration, walking, and sniffing. It has been suggested that theta activity reflects hippocampal participation in cognitive functions such as reasoning, attention, memory, and transformation of newly learned information into longer term memory [10].

In the rat hippocampus, the dentate gyrus develops notably slower than CA1. More precisely, the pyramidal cells of region CA1 are already in place at the time of birth, while only 20% of the adult granule cells are present in the dentate gyrus. The dentate gyrus continues to generate new neurons well after birth and nearly 80% of the granule cells found in the dentate originate in the first three weeks of postnatal development [1]-[2]. These developmental differences in the ontogeny of the two subfields and the physiological and anatomical organization of the hippocampus make it important to examine and document the timeline of CA1 and dentate gyrus neuron excitability [8].

Hippocampal EEG has been studied to a great extent through various signal processing techniques [3]-[8]. Our previous studies conducted on the ontogeny of the hippocampal EEG showed that, as the animal matures, the theta activity appears to gradually alter its frequency features [3]-[7]. Power spectrum analyses have shown that the peak theta frequency shifts from around 4-6 Hz at birth to 7-8 Hz by 30 days of age. However, traditional linear modeling methods, such as power spectrum and autoregressive (AR) models, exclude the possibility of observing nonlinear hidden behavior in the signal. On the other hand, nonlinear signal processing algorithms adopted have addressed the non-Gaussian behavior of EEG and added new interpretation with regard to the composition of the EEG [3]-[4]. For instance, bispectral analyses were conducted to detect and quantify the presence of quadratic phase coupling (QPC) in the hippocampal EEG at subfields of region CA1 and the dentate gyrus during REM sleep. The bicoherence function—a measure describing the significance level of QPC—also has been utilized as a quantitative index to delineate the developmental changes in hippocampal subfields [3]-[4].

Information, chaos, as well as random fractal theories have introduced a breakthrough in the world of nonlinear signal analysis [11]. Having the unique ability to capture valuable information that traditional analysis tools fail to provide, these techniques have gained wide popularity in the research arena. The invention of high-speed computers further enabled researches to utilize these techniques and develop computationally efficient algorithms to gather nonlinear and chaotic data with application examples [12]. These examples include a wide variety of physical signals and have been observed in biological events, health sciences, and information systems [11]. Nonlinear time series analysis techniques have

opened a door to innovative analytical and theoretical interpretations that can deepen our knowledge of the intricate systems of interest. In literature, there are numerous application examples of nonlinear dynamics and chaos [6]-[7], [9], [11].

When analyzing real signals such as EEG, Ventricular fibrillation (VF) [13], and others, nonlinear analysis techniques are essential in capturing the bigger picture and further deepening our understanding of the signal. It is believed that certain complex properties of a nonlinear system, such as the strange attractor, can be extracted by appropriate techniques to further delineate the composition and behavior of the hippocampal EEG. The correlation dimension is an example of one such quantifiable chaotic measure that examines the geometric aspect of strange attractors with fractional dimensions [14]-[15]. The Lyapunov exponent is another measure that detects and quantifies the chaos in a nonlinear system by examining the exponential divergence or convergence of nearby orbits in phase space [11], [16]-[17].

In this paper, we quantify the chaotic dynamics of REM EEG signals using the largest Lyapunov exponent (λ_1). We used Lyapunov exponent to evaluate the developmental differences of the hippocampal EEG measured during REM sleep from CA1 and the dentate gyrus in rats of two age groups, 15 and 90 days. The Lyapunov Exponent for the system was estimated using modified Rosenstein et al.'s algorithm [19]. Our results give positive Lyapunov exponents for EEG collected from rats of both 15 days and 90 days of age, suggesting chaotic nature of EEG. More research needs to be conducted to fully understand the nonlinear chaotic dynamics of REM sleep EEG.

II. METHODS

In this study, we analyze the hippocampal REM EEG using linear and nonlinear signal processing techniques to investigate the spatial and age-dependent chaotic nature of the signal. We used linear spectra analysis to estimate the power spectra to score the dominant frequency of the EEG from CA1 and DG and furthermore, to examine the correlation between these subfields for both young and old rats using magnitude-squared coherence (MSC). In previous studies [6]-[7], we have used correlation dimension and dimension complexity as quantifiers for nonlinear dynamics analysis of the EEG. We expand our nonlinear approach in this paper by studying the divergence of close trajectories in the state-space by calculating the largest Lyapunov exponent. The hippocampal EEG was collected during REM sleep from rats of 15 and 90 days of age by chronically implanting microelectrodes in the ipsilateral dentate granule cell layer and the CA1 pyramidal cell layer. The signals recorded from these regions were passed through a band-pass (1 – 75 Hz) and a notch pass (60 Hz) filter. The REM sleep EEG segments of 8-second duration were manually chosen and a minimum of 25 segments were selected from each animal and used in the current study.

A. Power Spectra and Coherence Estimation

Power spectra and coherence function of the hippocampal REM sleep EEG measured from CA1 and DG subfields were estimated using multi-channel spectral analysis [18].

$$S_{ij}(f) = \frac{1}{L} \sum_{k=1}^L X_i^k(f) X_j^k(f)^* \quad (1)$$

Spectral density was estimated using (1). $S_{ij}(f)$ is the cross-spectrum and $S_{ii}(f)$ is the power spectrum. The calculations were carried out by first dividing the REM sleep EEG epochs $\{x_i(n)\}$ of CA1 and DG into L equal-length (N) segments where the Fourier transform of the k^{th} segment, $X_i^k(f)$, was calculated. The linear correlation between channels i and j at frequency f is quantitatively studied using magnitude-square coherence [18].

$$MSC_{i,j}(f) = \frac{|S_{i,j}(f)|^2}{S_{i,i}(f) S_{j,j}(f)} \quad (2)$$

The MSC index is a real-valued function with a theoretical value varying from 0 to 1, which provides a quantitative measure of the linear correlation level between two channels at the frequency f .

B. Lyapunov Exponent

The collection of Lyapunov exponents of a dynamical system quantifies the long-term exponential divergence and convergence of nearby trajectories on an attractor. Positive and negative values correspond to the divergence and convergence, respectively. Thus, a positive largest Lyapunov exponent (λ_1) serves as a generic indicator of chaos, and moreover, quantifies the degree of sensitivity to initial conditions [16], [19]. The Lyapunov exponents for the EEG time series were computed using a modified version of Rosenstein et al.'s algorithm [19]. For a single time series, the first step in the calculation involves reconstructing the state vectors in phase space using Takens' time-delay embedding method [20]

$$\vec{v}(i) = [x(i), x(i+L), \dots, x(i+(m-1)L)] \quad (3)$$

where L is the delay time or lag and m is the embedding dimension. We approximated the appropriate lag L for the EEG time series using fast Fourier transform. The algorithm then calculates the nearest neighbor for each point on the trajectory. For every reference point $\vec{v}(i)$, the nearest neighbor point, $\vec{v}(j)$, is located by minimizing the Euclidean distance between the two state vectors. Reconstructing the state vectors using time-delay embedding in (3) will increase the Euclidean distance between state vectors as the embedding dimension m increases. Hence, we use a modified G-P method [5] to normalize the Euclidean distance and make the estimation of Lyapunov exponent less responsive to a changing embedded dimension m . Normalization of the Euclidean distance [5] can be done by using the following equation,

$$\|\vec{v}(i) - \vec{v}(j)\| = \sqrt{\frac{\sum_{k=0}^{m-1} [x(i+kL) - x(j+kL)]^2}{m}}. \quad (4)$$

After calculating the nearest distance $D_i(0)$ for the i th pair of points, both points are propagated for a certain time t and the evolved distance $D_i(t)$ is calculated. In a chaotic system, we can expect that the nearest neighbors will diverge approximately exponentially at a rate given by the Lyapunov exponent (λ_1) which can be defined using the equation [19] as follows,

$$D_i(t) \approx D_i(0) e^{\lambda_1 t}. \quad (5)$$

The evolved distance is averaged over many reference points i and propagation time steps t . From (5), we obtain

$$\ln D_i(t) \approx \ln D_i(0) + \lambda_1 t. \quad (6)$$

The slope of the natural log plot of $D_i(t)$ in t provides an estimate of λ_1 , i.e., the largest Lyapunov exponent. We have performed the same approach to examine the chaotic behavior of the hippocampal EEG during REM sleep.

III. RESULTS AND DISCUSSION

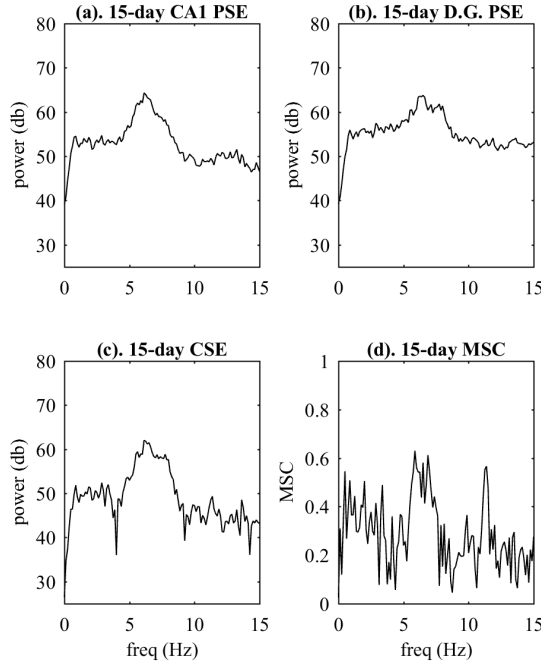


Figure 1. (a), (b) Power Spectra (c) Cross Spectra and (d) Magnitude-Squared Coherence (MSC) of REM sleep EEG of an animal of 15 days of age.

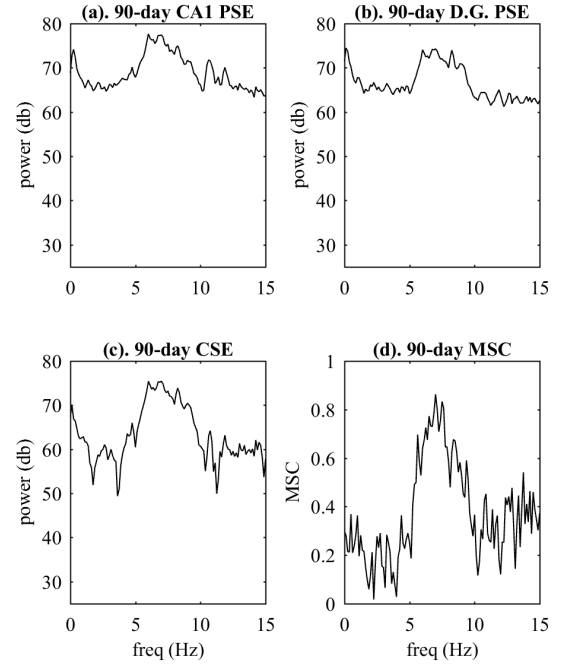


Figure 2. (a), (b) Power Spectra (c) Cross Spectra and (d) Magnitude-Squared Coherence (MSC) of REM sleep EEG of an animal of 90 days of age.

In this work, we provide a glimpse into the differential development in the cell populations of CA1 and dentate gyrus through the lens of chaotic dynamics analysis. Unlike the pyramidal cells of CA1, which are already developed at the time of birth, the granule cells of dentate gyrus grow during the postnatal period of development. This motivates us to look further into the developmental alterations in the physiological organization of the hippocampus as the animal matures.

The power spectra, cross spectra and magnitude-squared coherence (MSC) of REM sleep EEG of one 15-day old and 90-day old animal are displayed in Fig. 1 and Fig. 2 respectively. We observed that for all 15-day animals, the concentration of energy occurs in the theta frequency band around 5-6 Hertz. Higher values of MSC also appear in the same band validating the linear relationship between the two subfields CA1 and DG that generate theta activity. For older 90-day rats, we observe that the dominant theta activity shifts to a higher frequency band 7-8 Hertz with high MSC values (>0.7).

With these observations that indicate a strong correlation between the sites CA1 and DG, we examined the chaotic behavior of the REM EEG with the largest Lyapunov exponent. We present our results and examine the developmental differences between two locations (CA1 and DG) as well as two age groups of rats.

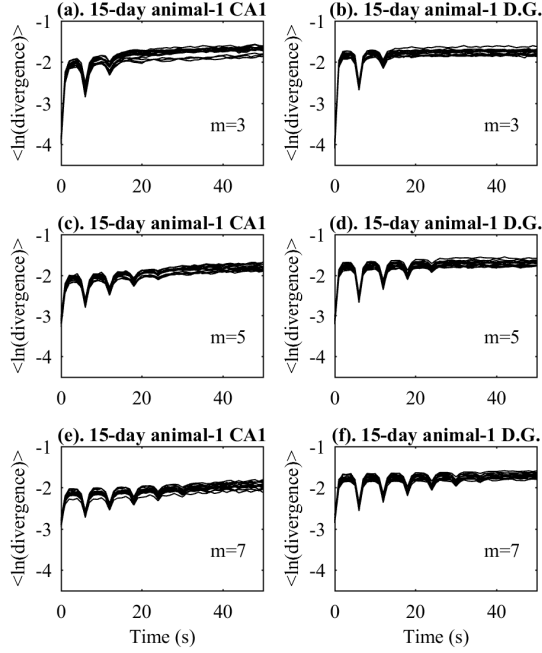


Figure 3. $\ln D_i(t)$ of hippocampal REM sleep EEG at CA1 and DG of a 15-day animal for embedding dimension $m = 3, 5$, and 7 .

A. Chaotic behavior of REM sleep EEG of 15-day animals

The average exponential divergence, given by $\langle \ln D_i(t) \rangle$ from (6) was computed and plotted versus time for individual REM sleep EEG segments of 15-day animals for both CA1 and DG sites. The results displayed in Fig. 3 and Fig. 4 represent only 15 of the total segments. It is quickly noticed from the figures that the curves are not exactly a straight line. These wiggles are artifacts that occur in continuous time systems [19]. The range of t covers until the divergence saturates, where the separation of nearby trajectories becomes comparable to the size of the attractor itself [12]. The slope of a straight line around which the system wiggles gives us the rate of the

TABLE I. LYAPUNOV EXPONENTS OF HIPPOCAMPAL REM EEG (15-DAY)

m	Animal Number	Lyapunov Exponent (λ_1)			
		CA1 $\mu \pm \sigma$	DG $\mu \pm \sigma$	t-test	p value
5	15-day-1	3.566 $\pm$ 0.236	3.705 $\pm$ 0.243	2.555	0.006
	15-day-2	3.475 $\pm$ 0.299	3.514 $\pm$ 0.254	0.579	0.282
	15-day-3	3.591 $\pm$ 0.224	3.656 $\pm$ 0.248	1.164	0.124
	15-day-4	3.636 $\pm$ 0.255	3.371 $\pm$ 0.199	7.456	<0.001
	15-day-5	3.680 $\pm$ 0.254	3.451 $\pm$ 0.179	4.416	<0.001
7	15-day-1	2.017 $\pm$ 0.221	2.301 $\pm$ 0.192	6.053	<0.001
	15-day-2	1.640 $\pm$ 0.287	1.990 $\pm$ 0.186	5.914	<0.001
	15-day-3	2.129 $\pm$ 0.215	2.228 $\pm$ 0.186	2.053	0.022
	15-day-4	2.152 $\pm$ 0.195	1.836 $\pm$ 0.203	10.240	<0.001
	15-day-5	2.218 $\pm$ 0.207	1.987 $\pm$ 0.212	4.679	<0.001

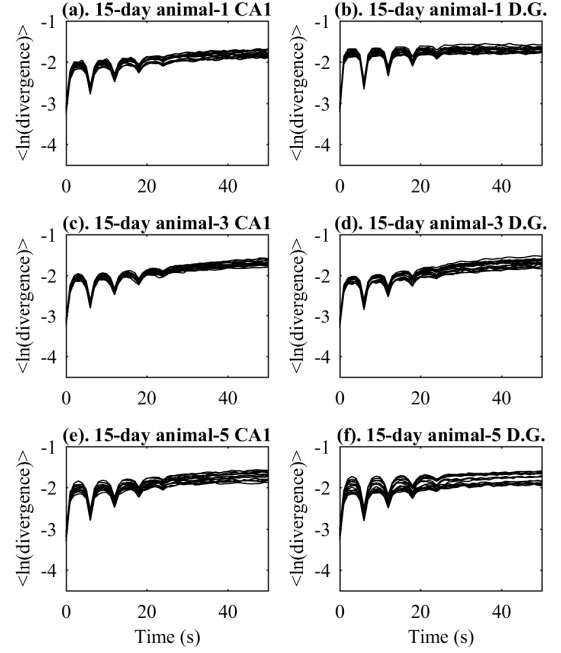


Figure 4. $\ln D_i(t)$ of hippocampal REM sleep EEG at CA1 and DG of three animals of 15 days of age for embedding dimension $m = 5$.

exponential divergence, λ_1 .

The Lyapunov exponents for both CA1 and DG sites for five 15-day animals were computed for embedding dimension $m = 3, 4, 5, 6$ and 7 . The Lyapunov exponents are positive for all animals for both CA1 and DG which indicates that the EEG is chaotic in nature. We also observed that the animals show a significant difference in the Lyapunov exponent between CA1 and DG. For lower embedding dimension $m=3$ and 4 , we observed that the Lyapunov exponent of DG is smaller than that of CA1 for three animals and the difference is significant. However, for higher m , the rate of divergence in DG gets bigger than that of CA1. For instance, the results for $m=7$, displayed in Table I., show that for animal 1, 2 and 3 the Lyapunov exponent is greater for DG than for CA1 with a significant difference. As m increases, all animals show difference between CA1 and DG with significant p values. We can tie this observation to Takens' criterion [20] of having a high enough embedding dimension to capture the chaotic behavior. Furthermore, the Lyapunov exponents also decrease for both CA1 and dentate gyrus as the embedding dimension m increases; this was an expected observation. Since more information is embedded in higher dimensions, the manifold of the system diverges at a slower rate giving us lower values of Lyapunov exponents.

B. Chaotic behavior of REM sleep EEG of 90-day animals

The average divergence is plotted against time for fifteen REM sleep EEG segments of region CA1 and DG for 90-day animals. The results are displayed in Fig. 5 and Fig. 6. The Lyapunov exponents for $m=5$ and 7 of five 90-day animals, displayed in Table II., are all positive indicating the consistent

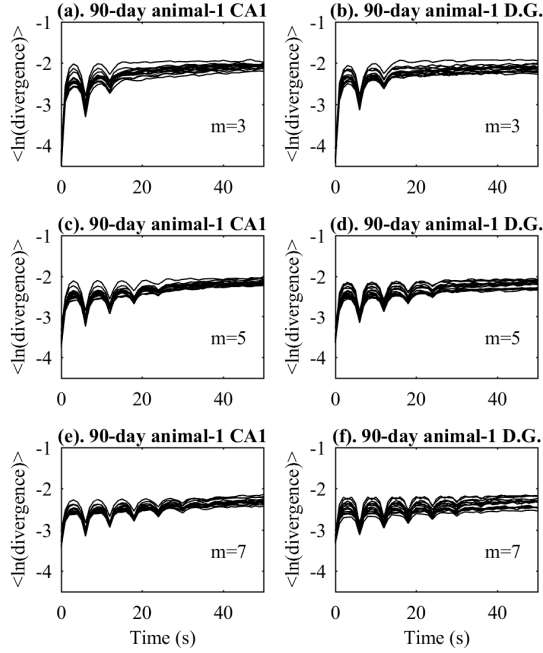


Figure 5. $\ln D_i(t)$ of hippocampal REM sleep EEG at CA1 and DG of a 90-day animal for embedding dimension $m = 3, 5$, and 7 .

chaotic behavior of the signal as the animal matures. However, three out of five animals, for both m , show that there is no significant difference in the Lyapunov exponent between CA1 and DG. Animal 2, 3 and 4 show comparable values of λ_1 in the two cell populations when increasing m from 5 to 7. However, for animal 1 and 5, the differences in Lyapunov exponent of CA1 and DG become significant at $m=7$. Furthermore, the trend of decreasing Lyapunov exponents for increasing embedding dimensions is maintained for 90-day EEG. Overall, there is no distinct pattern of significant differences in the Lyapunov exponents of CA1 and DG for older animals.

From the physiological perspective, at a young age, only 20% of the neurons in dentate gyrus are developed compared to the fully developed neurons in CA1. These differences in the excitability of the local circuit neurons at the hippocampus of a young animal could be tied to the observations from the chaotic lens of the signal. The differences in the Lyapunov exponents of CA1 and DG observed in young animals give the parallel anatomical description from the nonlinear dynamics perspective. As the dentate gyrus catches up with the developmental state of CA1 during the postnatal period, we observe that the differences between Lyapunov exponents of CA1 and DG become more comparable and less significant. Our results show no distinct pattern in the Lyapunov exponents of CA1 and DG for older animals.

Finally, age dependencies were also examined by averaging Lyapunov exponents across all animals for 15-day and 90-day animals for each of the subfields, CA1 and dentate gyrus. It was found that as embedding dimension m increases, the Lyapunov exponents of CA1 of 15-day animals get smaller than that of CA1 of 90-day animals.

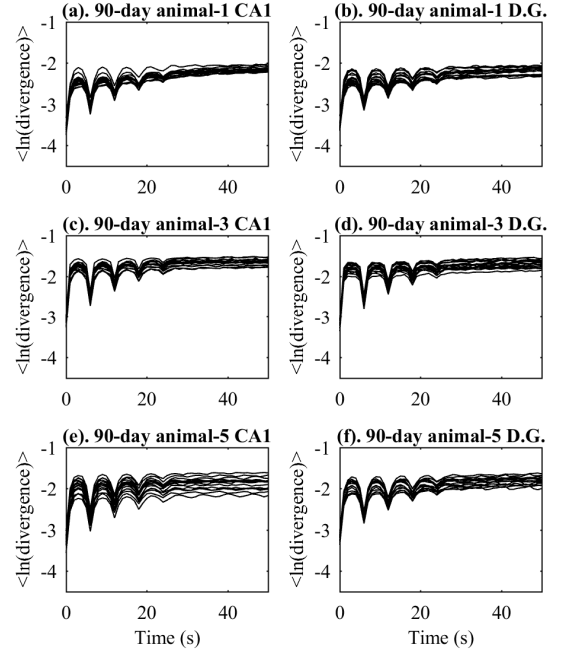


Figure 6. $\ln D_i(t)$ of hippocampal REM sleep EEG at CA1 and DG of three animals of 90 days of age for embedding dimension $m = 5$.

Dynamical systems governed by differential equations can be accurately quantified by chaotic analysis, given their fixed input parameters. However, experimental time series data, such as EEG signals, are constrained by the behavior of the animal. In this preliminary study, we have provided linear and nonlinear analysis of the hippocampal REM sleep EEG collected from CA1 and DG for animals of 15 and 90 days of age. Linear spectral techniques have confirmed a strong correlation between the theta rhythm generators, CA1 and DG. However, chaotic analysis on the EEG can only provide a glimpse of the full picture of the behavior of the signal. Lyapunov exponent is one of the several keyholes that can be

TABLE II. LYAPUNOV EXPONENTS OF HIPPOCAMPAL REM EEG (90-DAY)

m	Animal Number	Lyapunov Exponent (λ_1)			
		CA1 $\mu \pm \sigma$	DG $\mu \pm \sigma$	t -test	p value
5	90-day-1	3.440 $\pm$ 0.291	3.504 $\pm$ 0.239	1.177	0.121
	90-day-2	3.604 $\pm$ 0.220	3.612 $\pm$ 0.180	0.178	0.429
	90-day-3	3.356 $\pm$ 0.248	3.478 $\pm$ 0.202	1.964	0.025
	90-day-4	3.675 $\pm$ 0.311	3.534 $\pm$ 0.219	2.088	0.020
	90-day-5	3.400 $\pm$ 0.225	3.477 $\pm$ 0.226	1.219	0.114
7	90-day-1	1.879 $\pm$ 0.268	1.972 $\pm$ 0.215	1.855	0.033
	90-day-2	2.090 $\pm$ 0.199	2.084 $\pm$ 0.159	0.159	0.437
	90-day-3	1.919 $\pm$ 0.206	1.952 $\pm$ 0.150	0.659	0.256
	90-day-4	2.072 $\pm$ 0.326	2.097 $\pm$ 0.232	0.353	0.363
	90-day-5	1.916 $\pm$ 0.221	2.043 $\pm$ 0.194	2.201	0.016

used to look into the chaotic behavior of the signal. It serves as a touchstone to quantify strange attractors and nonlinear dynamical systems. Using this measure to underscore the developmental differences in the neuronal activities of the two subfields, we obtained positive Lyapunov exponents for both hippocampal EEG from CA1 and DG, suggesting that the underlying EEG is low dimensional chaos and deterministic in nature. However, the differences in Lyapunov exponents for varying embedding dimension and age shed further light to remind us that the EEG does not behave like a perfectly chaotic system, such as Henon or Lorenz maps. The behavior falls between a region which overlaps the qualities of a deterministic chaos and a stochastic system. We will continue our preliminary study to finely analyze the chaotic nature of hippocampal REM sleep EEG to grasp a better understanding of the differential development of the two subfields, CA1 and dentate gyrus.

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