Automated Search for Conjectures on Mathematical Constants using Analysis of Integer Sequences

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Abstract

The discovery of formulas involving mathematical constants such as π and e had a great impact on various fields of science and mathematics. However, such discoveries have remained scarce, relying on the intuition of mathematicians such as Ramanujan and Gauss. Recent efforts to automate such discoveries, such as the Ramanujan Machine project, relied solely on exhaustive search and remain limited by the space of options that can be covered. Here we propose a fundamentally different method to search for conjectures on mathematical constants: through analysis of integer sequences. We introduce the Enumerated Signed-continued-fraction Massey Approve (ESMA) algorithm, which builds on the Berlekamp-Massey algorithm to identify patterns in integer sequences that represent mathematical constants. ESMA has found various known formulas and new conjectures for $e, e^2, \tan(1)$, and ratios of values of Bessel functions, many of which provide faster numerical convergence than their corresponding simple continued fractions forms. We also characterize the space of constants that ESMA can catch and quantify its algorithmic advantage in certain scenarios. Altogether, this work continues the development toward algorithm-augmented mathematical intuition, to help accelerate mathematical research.

1. Introduction

Fundamental mathematical constants like e, π and φ are ubiquitous in almost all fields of science and mathematics (Finch, 2003). The discovery of new formulas involving mathematical constants often inspired mathematical research that revealed intrinsic properties of said constants and occasionally had a great impact in seemingly unrelated fields (Andrews, 1979). In Ramanujan's 'Lost Notebook', formulas about q-series and on mock modular forms were found to have applications in physics such as calculating black hole entropy (Harvey, 2019). Furthermore, in number theory, a polynomial continued fraction (CF) formula of $\zeta(3)$ was utilized by Apery to prove it's irrationality (Apéry, 1979). However, despite their great impact, the discovery of new formulas has been a scarce occasion, relying mostly on the intuition of great mathematicians such as Ramanujan.

In the Ramanujan Machine Project (Raayoni et al., 2021), an automated approach for conjecture discovery involving fundamental constants was presented. This automation was approached as a search problem: different functions of given fundamental constants (e.g.,$(1+e)/(1-e)$) are calculated to some limited precision and stored in hash tables, which are later numerically compared with decimal values of a generated family of polynomial CFs (a CF in which the partial numerator and denominator sequences are integer-coefficient polynomials (Laughlin & Wyshinski, 2005)). Each match is further validated by a high decimal precision calculation. This algorithm obtained significant results, discovering many previously unknown formulas and conjectures for e, π, π^2, and $\zeta(3)$, along with discovering the most efficient representation of the Catalan constant G (Raayoni et al., 2021). However, the algorithm was limited in its capacities since it still relied on an exhaustive search, expensive in computational resources and limited in the space of options that it can cover.

Here we propose a fundamentally different approach. We convert each target constant into a set of integer sequences which we search for patterns utilizing the **Berlekamp-Massey** algorithm (Berlekamp, 1966; Massey, 1969). Such a pattern, if exists, may provide a formula for the target constant. We thus name the overall algorithm **Enumerated Signed-CF Massey Approve (ESMA)**.

Our approach is inspired by ideas of compression, entropy, and information theory (Shannon, 1948). Decimal representations of irrational constants such as e may seem (incorrectly) to contain an infinite amount of data with no

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discernable pattern, e.g., infinite entropy. Meaning, transmitting e may seem to require a transmission of an infinite sequence of random digits to fully describe the constant. However, since there exists a formula to calculate e to infinite accuracy, its decimal representation actually contains zero entropy. This fact can be seen in e's simple CF expansion (Olds, 1970), which shows a clear pattern $(\ldots 1, 2k, 1, 1, 2k+2, 1, 1, 2k+4, \ldots)$:

$$e - 2 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \ldots}}}}}}. \tag{1}$$

The vision behind our algorithm is to efficiently identify such patterns. We expand each target constant to a set of different CFs (with ± 1 in the numerators) with the hope that at least one of them will reveal a pattern. Such patterns can be seen as compressions of the seemingly infinite decimal representation of a constant to a zero-entropy formula.

The ESMA algorithm aims to advance mathematical discovery by automatically generating conjectures. Automatic conjecture generation has led to impressive discoveries in the past. For example, in Fajtlowicz's work on Graffiti, novel conjectures in graph and matrix theory were discovered (Fajtlowicz, 1988) and in the recent DeepMind work on Advancing mathematics by guiding human intuition with AI, new conjectures in knot theory were presented (Davies et al., 2021). Similarly, our application is to generate research directions for the math community in the form of conjectures, with the hope they provide valuable insights.

We demonstrate the potential of this algorithm with equations involving **Signed Interlaced continued fractions (SICFs)** in which the partial denominator is a sequence made of β different sub-sequences (interlaced sequence), and the partial numerator is some periodic sequence of ± 1:

$$a_0 + \mathbb{K}_1^{\infty} \frac{b_j}{a_j} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ldots}}} \tag{2}$$

where $a_j \in \mathbb{Z}$, $b_j \in \{1, -1\} \quad \forall j \in \mathbb{N}$, are the partial denominators and numerators of the CF, respectively. In these SICF's, the partial numerators $b_j = \pm 1$ determine the denominators a_j, via the Euclidean division algorithm, in which we look for patterns.

The ESMA algorithm was able to produce various known mathematical formulas, e.g.,

$$\tan(1) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \ldots}}}}}, \tag{3}$$
$$b_j = 1, \quad a_j = \begin{cases} 1 & j = 2k+1 \\ 1 + 2k & j = 2k+2 \end{cases}$$

Along with a plethora of novel conjectures, some of which converge faster than their simple CFs e.g.,

$$\frac{2+2e}{-1+3e} = 2 - \cfrac{1}{1 + \cfrac{1}{24 + \cfrac{1}{3 - \cfrac{1}{2 + \ldots}}}},$$

$$b_j = \begin{cases} -1 & j = 6k+1 \\ 1 & j = 6k+2 \\ 1 & j = 6k+3 \\ -1 & j = 6k+4 \\ 1 & j = 6k+5 \\ 1 & j = 6k+6 \end{cases}, \quad a_j = \begin{cases} 2 & j = 6k \\ 1+4k & j = 6k+1 \\ 24+64k & j = 6k+2 \\ 3+4k & j = 6k+3 \\ 2 & j = 6k+4 \\ 13+16k & j = 6k+5 \end{cases}, \quad \forall j, k \in \mathbb{N}. \tag{4}$$

$$\frac{\mathrm{J}_1(1)}{\mathrm{J}_3(1)} = 23 - \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{39 - \cfrac{1}{2 + \ldots}}}},$$

$$b_j = \begin{cases} -1 & j = 3k+1 \\ 1 & j = 3k+2 \\ 1 & j = 3k+3 \end{cases}, \quad a_j = \begin{cases} 23+16k & j = 3k \\ 1+k & j = 3k+1 \\ 1 & j = 3k+2 \end{cases}, \quad \forall j, k \in \mathbb{N}.$$

Where $\mathrm{J}_y(x)$ are the Bessel functions of the first kind of order y with argument x (see Appendix A).

Conjectures found are verified to a precision of 1000 decimal places. Therefore, obtaining a false conjecture that is merely a mathematical coincidence is highly unlikely. Specifically, every conjecture is found and verified in two steps: (1) Using the Berlekamp-Massey algorithm, a pattern is found in the a_j extracted directly from the constant. We usually test 50 elements of the sequence, though more can be tested to find longer patterns. (2) If a pattern was found, further validation is acquired by using it to calculate additional elements of the a_j sequence. We then compare the decimal representation of the CF and the constants', usually to the next 1000 digits (though more digits can be verified with ease). Consequently, the probability for a false positive is roughly 10^{-1000}. Additionally, ESMA has found various known formulas, contributing to our confidence in the algorithm. We therefore believe that our conjectures are mathematical formulas awaiting formal proof. Importantly, this estimate of likelihood of course does not substitute the need for formal proof.

Apart from the algorithm, the analysis and manipulation of CFs in our work relies on several novel results that are presented below. We present a connection between simple CFs with interlaced polynomial sequences and the SICF structure found in our results. Furthermore, we present the **Folding Transform**, which reveals a connection between polynomial CFs and CFs made of interlaced polynomial sequences or more generally polynomial matrices. These connections provide insight on the space of constants for which we can expect the ESMA algorithm to find formulas.

We summarize our contributions in this paper:

1. We present a novel use of the Berlekamp-Massey algorithm as an efficient method for pattern recognition (see Section 2.1).

2. We introduce a complementary approach to the existing exhaustive search methods (Raayoni et al., 2021) for automation of mathematical discovery. ESMA improves the efficiency of enumeration over the space of possible CFs and rational functions, increasing overall efficiency of conjectures discovery (see Section 3).

3. The ESMA algorithm proved successful in discovering various conjectures on constants that converge significantly faster than their simple CF expansions. For example, the algorithm discovered conjectures on $\mathrm{J}_1(1)/\mathrm{J}_3(1)$, $\mathrm{J}_5(1)/\mathrm{J}_3(1)$, and the Golden Ratio φ, which converge faster than their simple CF forms. A conjecture found on φ converges approximately 6 times faster than its simple CF (see Appendix A see Figure 6). In Appendix A, we plot the convergence of a sample of faster converging conjectures alongside their known formulas, presenting the significant increase in the convergence rate.

2. Preliminaries

2.1. The Berlekamp-Massey algorithm in ESMA

We present a new application of the Berlekamp-Massey algorithm (Berlekamp, 1966; Massey, 1969) in pattern recognition. Given an integer sequence, the Berlekamp-Massey algorithm finds the minimal linear recurrence with integer coefficients that can produce given the sequence, returning the coefficients of the recurrence relation (Massey, 1969).

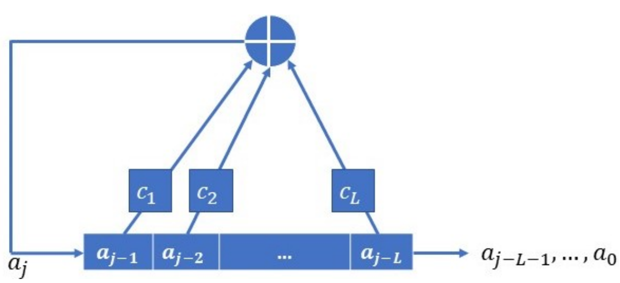

Figure 1. Linear-Feedback Shift Register containing L-cells, where a_{j-k} is the value held in the k^{th} cell, and c_k is the coefficient with which we multiply the value before linear feedback.

The Berlekamp-Massey algorithm is based on Berlekamp's decoding algorithm (Massey, 1969; Reeds & Sloane, 1985), generalized by James L. Massey to solve the task of finding the shortest Linear-Feedback Shift Register (LFSR) that outputs a given sequence over a finite field (Reeds & Sloane, 1985; Klein, 2013). In such registers, the input at every clock is a fixed linear function of the register's current state, creating a linear recursion with coefficients referred to as connection coefficients (Massey, 1969) (see Figure 1). Thus, finding the shortest integer coefficient recurrence relation of a sequence is analogous to finding the shortest linear-feedback shift register that can output the sequence. A linear-feedback shift register of length L with the initial contents of the L cells given by $a_0, a_1, \dots a_{L-1}$ (initial conditions) and connection coefficients given by $\{c_i \in \mathrm{GL}(p) \mid i \in \{1, 2, \dots, L\}\}$, for some prime number p, produces the following output sequence over GL(p) (Massey, 1969):

$$\left(a_j + \sum_{i=1}^{L} c_i a_{j-i}\right) \bmod p = 0 \quad \forall_{j \in \{L, L+1, \dots, n-1\}} \tag{5}$$

Given a sequence $a_0, \dots, a_n$, the Berlekamp-Massey algorithm finds and returns the connection coefficients c_i of the minimal linear-feedback shift register for which (5) is satisfied.

The length of the resulting recursion, which is the number of cells, further reveals to us the significance of the pattern detected. Given an input sequence of length n, if the resultant recurrence is of length $L \geq n/2$, then it is a trivial solution. In this case, we can simply take the initial conditions of the LFSR to be the first elements of the sequence and calculate the connection coefficients so they create the next $n/2$ elements. We therefore add a verification step by using the length of the register found by the algorithm to identify whether the pattern found in the sequence is significant and unique, requiring a recurrence length $L < n/2$. Even then, there is still a probability for a false positive that scales as $\sim (1/p)^{n-2L}$, and therefore we apply a second verification stage.

In ESMA, we apply the Berlekamp-Massey algorithm on a_j sequences extracted from the expansion of CFs as shown in the next sections. We use a large finite field GL (p) so that the connection parameters are found explicitly (rather than modulo the finite field). Specifically, the presented results were found with $p = 199$. One can use an even smaller value of p to make the algorithm run faster, at the price of post-processing for extracting the explicit connection parameters from their values in the finite field. For conjecture verification, we use the extracted recurrence relation to generate the sequence over $\mathbb{Z}$ to any desired length (not limiting the generated numbers to GL(p)). Altogether, ESMA enables us to identify any pattern that can be described as an integer coefficient recurrence relation.

2.2. Sign Interlaced Continued Fractions

In the most general sense, Signed Interlaced continued fractions (SICFs) are CFs in which the partial denominator is an interlaced sequence, and the partial numerator is some periodic sequence of $\pm 1, b_j \in \{-1, 1\}$, whose period, β, denotes the number of sub-sequences that are interlaced. For example, the following conjecture found by the ESMA algorithm on Bessel functions of the first kind, $\mathrm{J}_{\mathrm{y}}(x)$ (of order y at point x), is an SICF with a_j made of interlaced linear sequences:

$$\frac{\mathrm{J}_0(1)}{2\mathrm{J}_1(1)} = 1 - \cfrac{1}{8 - \cfrac{1}{3 - \cfrac{1}{16 - \dots}}}, \tag{6}$$

$$b_j = -1, \quad a_j = \begin{cases} 1 + 2k & j = 2k \\ 8 + 16k & j = 2k+1 \end{cases} \forall j, k \in \mathbb{N}$$

Where the a_j sequence can described by an integer coefficient recurrence relation: $a_j - 2a_{j-2} + a_{j-4} = 0, a_0 = 1, a_1 = 8, a_2 = 3, a_3 = 16$ In all the results so far, we obtain an a_j made of positive interlaced linear sequences ($a_j > 0 \quad \forall j \in \mathbb{N}$). Therefore, for the rest of the paper when referring to SICF, we refer specifically to those with positive polynomial sequences in the partial denominator, satisfying $a_j > 0 \quad \forall j \in \mathbb{N}$. To enable discovery of conjectures in the form of an SICF, we introduce a method to extract a sample of the a_j sequence for CFs with a signed partial numerator, $b_j = \pm 1$.

2.3. Extraction of Signed Interlaced Continued Fractions and the Euclidean algorithm

We generalize the conventional method of calculating the CF of a constant (Hardy & Wright, 1980; Chrystal, 1964; Wall, 1948; Lang, 1995) to enable sign variation in the partial numerator. This procedure enables us to expand every constant to a SICF with any sequence of $\pm$ signs in the partial numerators, allowing for a larger space of candidate formulas to be analyzed.

Simple CFs are unique, every irrational number has a single simple CF which it is equal to, and every rational number has 2 simple CF expansions (Hardy & Wright, 1980). Calculating the simple CF of some constant α is simply an application of a non-terminating Euclidean Algorithm with α and 1 (Hardy & Wright, 1980), where the respective quotients form the a_j sequence and $b_j = 1 \, \forall j \in \mathbb{N}$. When enabling sign variation in the Euclidean algorithm we enable more options in each iteration of the algorithm by allowing for negative remainders. Thus, sign variation often enables the algorithm to terminate in less iterations, meaning a more efficient CF representation is found. Take a simple example of finding the $\gcd(14, 9)$ and the resultant CF formed:

CF	Without sign variation	With sign variation	CF
$\frac{14}{9} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{4}}}$	$14 = 9\times1 + 5$ $9 = 5\times1 + 4$ $5 = 4\times1 + 1$ $1 = 1\times1 + 0$	$14 = 9\times2 - 4$ $9 = -4\times - 2 + 1$ $-4 = -1\times4 + 0$	$\frac{14}{9} = 2 - \cfrac{1}{2 + \frac{1}{4}}$

Figure 2. An illustration of a more efficient CF with a signed partial numerator

We see that enabling signed variation results in a more efficient calculation, 3 iterations rather than 4, resulting in a more efficient CF expansion of the number. More generally, sign variation ($b_j = \pm 1 \, \forall j \in \mathbb{N}$) allows for the extraction of additional CF representations for a single constant, providing us with a larger search space for conjectures, some of which may have more favorable properties.

For example, consider the conjecture found on Bessel functions of the first kind, $2/(\mathrm{J}_5(1)/\mathrm{J}_3(1) + 1)$ (see Table 3). The new and unproven conjecture converges to $\mathrm{J}_5(1)/\mathrm{J}_3(1)$ at a rate of 4.3008 digits per term (averaged over 100 terms), whilst the simple CF expansion for $\mathrm{J}_5(1)/\mathrm{J}_3(1)$ converges at a rate of 1.9046 digits per term (see Figure 6 in Appendix A). Meaning a more efficient and faster converging expansion of the constant was found.

In many cases, the SICF that results from ESMA is simpler than the simple CF of the same constant. Take for example $(2+2e)/(-1+3e)$ seen in equation 4, whose simple CF has an interlaced sequence of period 8 whereas its SICF is of period 6. In a similar manner, the conjecture on $\mathrm{J}_5(1)/\mathrm{J}_3(1)$ referred to above is of period 2 while its simple CF expansion is of period 16. Therefore, the SICF provides us with a larger search space where we can potentially detect simpler patterns. We additionally find an algorithmic advantage in utilizing SICF as more conjectures for the same search space are found. For the same search space of coefficients, coefficients between -3 and 3 and polynomial degree of 1 , there was an increase of 357% in the number of conjectures found on e when searching with SICFs rather than only simple CFs. Thus presenting the algorithmic advantage gained by signed CF extraction.

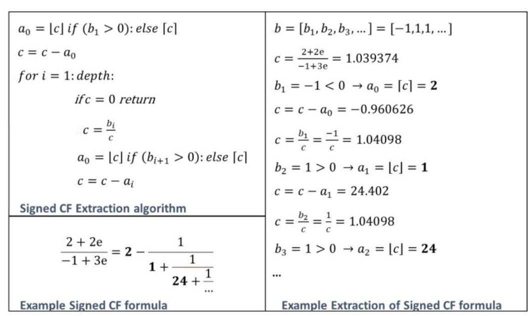

Figure 3. Extracting a signed CF from a decimal value

To extract a signed CF, we accommodate for having some $b_j = -1$ and we extract the integer part of the CF using ceil operator rather than the floor operator used for $b_j = 1$ as in the Euclidean Algorithm (see Figure 3 for an example). We utilize each given signed b_j sequence to extract a sample of the a_j integer sequence directly from the decimal representation of a constant. Consequently, we obtain an integer sequence for which we can attempt to recognize a pattern using the Berlekamp-Massey algorithm.

3. The Extract Signed-CF Massey-Approve (ESMA) Algorithm

We present a novel algorithm (Figure 4) that extracts a Signed Interlaced Continued Fraction (SICF) of a given constant c in the following form:

$$\frac{f_{m,L}(c)}{g_{m,L}(c)} = a_0 + \mathbb{K}_1^\infty \frac{b_j}{a_j} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}. \quad (7)$$

Where $f_{m,L}$ and $g_{m,L}$ are integer polynomials whose degrees are at most m with coefficients over a range of integer values $[-L, L]$. In this case, $a_0 + \mathbb{K}_1^\infty b_j/a_j$ is some SICF with partial numerator $b_j \in \{1, -1\}$ of maximal period β_b and partial denominator $a_j \in \mathbb{Z}$.

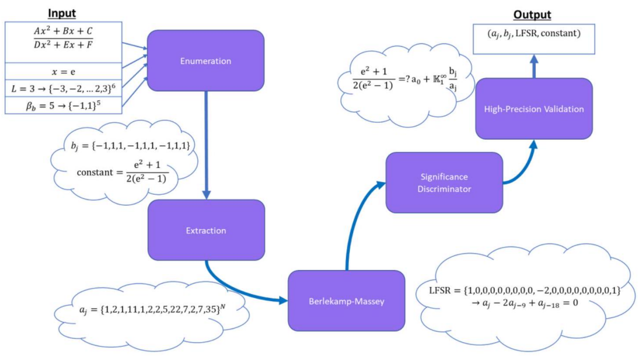

Figure 4. A graphical depiction of the stages of the ESMA algorithm, with the "clouds" presenting an example run for a single constant value and b_j pair

We begin by enumerating over all possible non-trivial rational functions $f_{m,L}(x)/g_{m,L}(x)$ (see Figure 4 for an example). For each rational function, we substitute the mathematical constant c (evaluated to 1000 decimal places in the examples shown in this work). We then enumerate over all periodic b_j sequences with periods between 1 and β_b formed of ± 1. For each b_j sequence and constant value pair, an a_j sequence is extracted up to a finite depth N, using the extraction algorithm introduced earlier. The sequence is passed to the Berlekamp-Massey algorithm in attempt to recognize a significant recurrence pattern, i.e., a minimal length linear feedback-shift register. If the resulting register length is shorter than half the length of the extracted sequence (see Section 2.1), it is considered significant and saved for verification. The resulting pair of a_j, b_j sequences represent a SICF.

We verify the result by utilizing the obtained recursion parameters to calculate a_j to a greater depth and then evaluate the SICF to compare it with the constant. The SICF is efficiently calculated by utilizing the recursive formula for numerators p_j and denominators q_j (Olds, 1963):

$$p_j = a_j p_{j-1} + b_j p_{j-2}, \quad q_j = a_j q_{j-1} + b_j q_{j-2} \quad (8)$$
$$p_{-1} = 1, \quad p_0 = a_0, \quad q_{-1} = 0, \quad q_0 = 1.$$

yielding p_j/q_j as a rational approximation of the constant given by the first j elements of the partial numerator and denominator sequences of the SICF. Each case for which the numerical values are identical for up to 1000 decimal places is considered a new conjecture. For the full implementation, refer to our github through `www.ramanujanmachine.com`.

The computational complexity of the algorithm depends on the space over which we enumerate. The space of rational functions $f_{m,L}(x)/g_{m,L}(x)$ is $\mathrm{O}\left((2L+1)^{2(m+1)}\right)$. For convenience, our system supports saving symbolic enumerations locally so we can substitute different constants into the symbolic expressions. The rational function enumerations are also simplified to reduce redundancies, ensuring no trivial cases are stored (e.g.,rational numbers, where polynomials cancel out). Later additions to the algorithm enable users to construct custom made function generators easily, not limiting the search to rational polynomial functions $f_{m,L}(c)/g_{m,L}(c)$ but rather to any family of parametric functions over a discrete parameter space.

The b_j sequence enumeration takes $\mathrm{O}\left(2^{\beta_b}\right)$, where β_b is the maximal period. In our most common searches, β_b ranges between 1 and 5 , and thus we enumerated $2^1 + 2^2 + 2^3 + 2^4 + 2^5$ combinations of ± 1 sequences. For each b_j sequence and constant value pair, we extract the a_j sequence to a depth N (the results presented in this work are found using $N = 50$). The complexity of this extraction is negligible relative to the application of the Berlekamp-Massey algorithm, whose complexity is $\mathrm{O}\left(N^2\right)$ (Reeds & Sloane, 1985; Gustavson, 1976). In case that a result was found, we verify it to 1000 decimal places by evaluating the resulting SICF and comparing with the original constant.

Given the above considerations, the time complexity of the algorithm is given by $\mathrm{O}\left((2L+1)^{2(m+1)} 2^{\beta_b} N^2\right)$. While this may seem computationally expensive, this approach provides two significant advantages over exhaustive search methods such as the first Ramanujan Machine algorithm (meet-in-the-middle regular formulas) in (Raayoni et al., 2021): (1) The enumeration over possible polynomial CFs and the expansion of each one, which are the most expensive operations, are replaced by enumerating over the signed sequences $\mathrm{O}\left(2^{\beta_b} N^2\right)$. (2) Even the enumeration over the rational functions is more efficient than in (Raayoni et al., 2021), since infinite many cases of rational functions are captured by a single run of the Berlekamp-Massey algorithm, inside the initial conditions in the register. For example, both the expressions $x, 1/x, 1/x + k, x + k$, and many other Mobius transforms of x, are covered by the same instance of running the Berlekamp-Massey algorithm, as initial sequence elements which do not follow the found pattern can be simply absorbed to the initial conditions of the recurrence relation. This applies to any value of x including the rational functions of x, presenting a more efficient enumeration.

However, despite its efficiency, the disadvantage of ESMA relative to that in (Raayoni et al., 2021) lies in the space of CFs that can be discovered. This space is analyzed in the following sections.

4. The Space of Constants Captured by the ESMA Algorithm

4.1. Example Results

The proposed ESMA algorithm has discovered many previously unknown conjectures, some having faster converging expansions for various constants (see Figure 6) relative to their known simple CFs. In Figure 5, we see a sample of conjectures and known formulas found by ESMA, all of which converge at a super-exponential rate (error decreasing as $\approx \mathrm{e}^n/n\,!$ for CF depth n (Raayoni et al., 2021)). This rate of convergence is visualized in Figure 5, where the (slower) exponential convergence of the Golden Ratio's simple CF is plotted for reference. Similar fast rates of convergence are seen in most conjectures found by ESMA (see Appendix A). To better understand our results, and the reason for their super-exponential convergence rate, we analyze the representable set of the SICF structure over which ESMA searches. Such an analysis enables us to better characterize the ESMA algorithm.

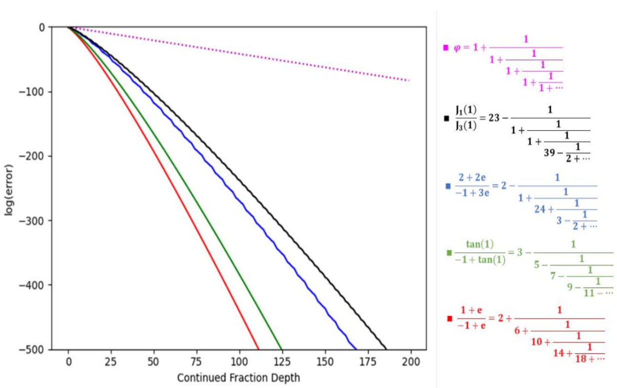

Figure 5. Convergence rates of the SICF conjectures. The figure presents the approximation error, the log of the absolute difference between the SICFs approximation at a given CF depth and the fundamental constant ($\log_{10}$(error) vs SICF depth).

4.2. Equivalent Representations of Constants

As described in the previous section, ESMA searches for CF expansions, $a_0 + \mathbb{K}_1^\infty b_j/a_j$, where the partial numerator b_j is periodic with ± 1 entries and the partial denominators a_j satisfy some recurrence relation. While a linear recurrence determines the sequence completely given "enough" initial conditions, we can sometimes decompose it into several sub-sequences and apply the recurrence to each one of them separately. For example, the linear recurrence $a_j - 2a_{j-2} + a_{j-4} = 0$ mentioned in Section 2.2 can be applied at even and odd sub-sequence indices separately obtaining an interlaced sequence. Thus, we can think of the sequence as $a_j = \begin{cases} A_1(n) & j = 2n-1 \\ A_2(n) & j = 2n \end{cases}$ (if we ignore the a_0 element). We now have a natural decomposition of the b_j and a_j sequences to β_b and β_a sub-sequences respectively, and up to taking the least common multiplier of β_b and β_a we may assume that both are equal to the same β, which we call the period of the SICF. Hence, we have functions $B_i, A_i : \mathbb{N} \to \mathbb{Z}$ for $i = 1, \ldots, \beta$ such that $b_{(n-1)\times\beta+i} = B_i(n), a_{(n-1)\times\beta+i} = A_i(n), \forall n \in \mathbb{N}\backslash 0$.

While B_i are always constant 1 or -1, in general, ESMA may find many types of A_i functions (though they must be some combination of polynomials and exponents). However, in all results so far, the functions found were linear polynomials (see Appendix A), leading us to question what sort of numbers can be represented with polynomial A_i 's and in a more general sense what constants one can expect ESMA to catch (find conjectures for). We denote this set of numbers, the representable set of the SICF structure of the form $\mathbb{K}_1^\infty \frac{b_j}{a_j}$, as $\mathfrak{R}_1$ where the following conditions are satisfied:

$$\begin{gathered} \exists\beta \text{ and } A_i, B_i \quad i = \{1, \ldots, \beta\}, \text{ s.t. } b_{(n-1)\times\beta+i} = B_i(n), \\ B_i \equiv \pm 1 \quad a_{(n-1)\times\beta+i} = A_i(n) \quad \forall n \in \mathbb{N}\backslash 0, \\ A_i \text{ is polynomial}, \mathbb{K}_1^\infty \frac{b_j}{a_j} \text{ converges} \end{gathered} \tag{9}$$

The above set refers to the set of all possible numbers that have a formula with SICF up to some Mobius transform. Similar to common standard CF representations (Cuyt et al., 2008), here too we can represent the more general CF using Mobius transforms (for details, see Appendix B.1). More specifically, given a Mobius transform:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{Z}$, we get that:

$$\mathbb{K}_1^\infty \frac{b_j}{a_j} = \lim_{N\to\infty} \prod_{j=1}^{N} \begin{pmatrix} 0 & b_j \\ 1 & a_j \end{pmatrix}(0) = \lim_{N\to\infty} \prod_{j=1}^{N+1} \begin{pmatrix} 0 & b_j \\ 1 & a_j \end{pmatrix}(\infty) \tag{10}$$

In our interlaced presentation above, the matrices come in natural batches of size β, and it is only natural to multiply each such batch together to form a single polynomial matrix, which we refer to as the collapsed matrix:

$$\prod_{j=1}^{N\beta}\begin{pmatrix} 0 & b_j \\ 1 & a_j \end{pmatrix} = \prod_{n=1}^{N}\left[\prod_{i=1}^{\beta}\begin{pmatrix} 0 & b_{(n-1)\times\beta+i} \\ 1 & a_{(n-1)\times\beta+i} \end{pmatrix}\right] \quad (11)$$

$$M_n = \prod_{i=1}^{\beta}\begin{pmatrix} 0 & b_{(n-1)\times\beta+i} \\ 1 & a_{(n-1)\times\beta+i} \end{pmatrix}, \forall n \in \mathbb{N}\backslash 0$$

We denote the collapsed matrix by M_n and we get that the entries of M_n are polynomial in n. The collapsed matrix can additionally be used to represent the more general case where both our a_j and b_j sequences are interlaced polynomial sequences, an interlaced CF. Thus, we can automatically deduce that $\Re_1$ above is contained in a much more general set,

$$\Re_2 = \left\{ \lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty) \mid \begin{array}{c}M_n \text{ is a } 2\times 2 \text{ matrix} \\ \text{with polynomial entries}\end{array} \right\}. \quad (12)$$

While the presentation as a product of a polynomial matrix can seem very general, it can almost always be presented in a much simpler manner, namely as a polynomial CF. Polynomial CFs are well known (Bowman & Laughlin, 2002; Laughlin & Wyshinski, 2005; Raayoni et al., 2021; David et al., 2021) and have been shown to enable swift proofs on the properties of CFs and the constants they can represent, thus providing information on a seemingly more general structure (e.g., Apery's proof of $\zeta(3)$'s irrationality).

Theorem 1. Let $M_n = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}$, satisfying $e_n \neq 0$, be some polynomial matrix ($c, d, e, f \in \mathbb{Z}[x]$) for which the following limit exists $\lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty)$. Then there exists some Mobius transform T with entries in $\mathbb{Z}$, and polynomials $a', b' \in \mathbb{Z}[x]$ such that:

$$\lim_{N\to\infty}\left[T\prod_{n=1}^{N} M_n\right](\infty) = \mathbb{K}_1^{\infty}\frac{b'(n)}{a'(n)}$$

Proof. See Appendix B.2

Note that since the Mobius transform T has its entries in $\mathbb{Z}$, a number α is rational if and only if $T(\alpha)$ is rational. Hence, up to Mobius equivalence we see that $\Re_2$ is contained in the set of numbers with expansions as polynomial CFs,

$$\Re_3 = \left\{\mathbb{K}_1^{\infty}\frac{b'(n)}{a'(n)} \mid a', b' \in \mathbb{Z}[x]\right\}. \quad (13)$$

As a result, we can deduce that up to an integer Mobius transformation $\Re_1 \subseteq \Re_2 \subseteq \Re_3$.

We present the Folding Transform T, which converts any polynomial matrix satisfying the theorem's conditions to a polynomial CF (see Appendix B.2). In other words, the Folding transform can convert any constant equal to an interlaced CF to a polynomial CF, up to some Mobius transform. For any $\alpha \in \mathbb{R}$ equal to a general interlaced CF where both b_j and a_j are interlaced polynomial sequences, and specifically for a SICF where $b_j = \pm 1$, we can multiply all matrices in each period to obtain a polynomial matrix, M_n. Applying the Folding transform on the resulting polynomial matrices, we obtain a polynomial CF that we denote by $T(M_n)$. We say that α and $T(M_n)$ are semi-equivalent. For example, applying the Folding transform on the simple CF of e, we obtain the following polynomial CF formula (for the full derivation see Appendix B.3):

$$e - 2 = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\dots}}}}}} \to^{T} \frac{-8e+19}{e-2} \quad (14)$$

$$= \cfrac{4\times 2^2+4\times 2-3}{8\times 2^2+16\times 2+8+\cfrac{4\times 3^2+4\times 3-3}{8\times 3^2+16\times 3+8+\cfrac{4\times 4^2+4\times 4-3}{8\times 3^2+16\times 3+8+\dots}}}$$

The Folding transform allows us to better understand the largely unexplored SICF structure by utilizing prior knowledge of polynomial CF properties, shining light on what constants SICFs can represent. We further present a novel connection showing that constants with SICF expansions can be presented in simpler forms.

4.3. Representable set of Signed Interlaced Continued Fractions (SICFs)

At first glance, the representable set of SICFs, $\Re_1$, may seem to expand the set of numbers we can represent with a simple CF by allowing for sign variation. However, we find that every SICF expansion of an irrational constant can be presented as a simple CF with a "regular" pattern up to some Mobius mapping. Rational numbers may have SICF expansions but always have a finite CF and are not of interest to us. We denote the representable set of simple CFs with interlaced polynomial sequences, also referred to as simple interlaced CFs, as:

$$\Re_4 = \left\{ \mathbb{K}_1^{\infty}\frac{1}{a_j} \middle| \begin{array}{c} \exists\beta \text{ and } A_i, B_i \quad i = 1,\dots,\beta, \text{ s.t.} \\ a_{(n-1)\times\beta+i} = A_i(n) \quad \forall n \in \mathbb{N}\backslash 0 \\ A_i \text{ polynomial} \end{array} \right\} \quad (15)$$

Unlike SICFs, simple CFs have been studied extensively (Lang, 1995; Olds, 1963; Hardy & Wright, 1980; Laughlin & Wyshinski, 2005; Bugeaud, 2012), much is known about their properties and about constants' simple CF expansions. Therefore, discovering that for irrational numbers $\Re_1 \equiv \Re_4$ up to some Mobius transform shines light on what constants we can expect ESMA to catch: those whose simple CF expansions have a pattern in their partial denominator. In other words, we expect ESMA to catch numbers who have simple interlaced CF expansions.

Theorem 2. For any signed interlaced continued fraction equal to $\alpha \in \mathbb{R}/\mathbb{Q}$, there exists some Mobius transform of α which is equal to a simple interlaced continued fraction.

Proof. See Appendix C.1

The proof of Theorem 2 is constructive, meaning we have an algorithm to convert every SICF to a simple interlaced CF expansion (see Appendix C.2 for an example).

As a result of Theorem 2, we can fully characterize the representable set of SICFs, $\Re_1$, with simple interlaced CFs, a much less general structure. This reveals a limiting property of ESMA, it can only catch constants with regular patterns or interlaced sequences in the partial denominator of their simple CF (which is unique). This set of numbers includes e, $\tan(1)$, second-degree algebraic numbers, and more. However, constants such as π which have no discernible pattern in their simple CF (Lange, 1999) probably cannot be caught by ESMA. We analyze polynomial CF expansions semi-equivalent to simple interlaced CFs to better characterize the representable set of SICFs.

We find that the representable set of simple interlaced CFs with non-constant partial denominator sequences is characterized by irrational constants whose polynomial CF expansions converge super-exponentially revealing to us why most of our results are of constants who have super-exponentially converging expansions. We do not analyze simple interlaced CFs with constant partial denominator sequences as they are trivially characterized by the 2^{nd} degree algebraic numbers (Denjoy, 1938; Balkova & Hrušková, 2013).

Theorem 3. Given a simple interlaced continued fraction satisfying $B_i(n) = 1 \quad \& \quad A_i(n) > 0$, where $A_i, B_i \in \mathbb{Z}[\mathrm{x}], i \in \{1, \dots, \beta\}, \forall n \in \mathbb{N}$ of period β, where $\exists i \in \{1, \dots, \beta\}$ s.t $\deg(A_i) > 0$, its semi-equivalent polynomial continued fraction's partial numerator b'_n and partial denominator $a'_n \quad \forall n \in \mathbb{N}$ satisfy:

$$\deg(b') = \sum_{i=1}^{\beta-1} 2 \times \deg(A_i),$$

$$\deg(a') = \left[\sum_{i=1}^{\beta-1} 2 \times \deg(A_i)\right] + \deg(A_\beta)$$

Proof. See Appendix D.1

Through the utilization of known properties of polynomial CFs (Bowman & Laughlin, 2002; Laughlin & Wyshinski, 2005; David et al., 2021) the above theorem provides us with valuable information to better understand the rate of convergence of these polynomial CF representations and the irrationality of the constants ESMA can catch.

Corollary 2. A polynomial continued fraction semi-equivalent to a simple interlaced continued fraction, satisfying $\exists i \in \{1, \dots, \beta\}$ s.t $\deg(A_i) > 0$, converges super-exponentially.

Proof. See Appendix D.2

Lemma 3. The partial denominator sequence of a polynomial continued fraction semi-equivalent to a simple interlaced continued fraction is positive: $a'_n \in \mathbb{Z}[\mathrm{x}]$, $a'_n > 0 \quad \forall n \in \mathbb{N}\backslash 0$

Proof. See Appendix D.1

Corollary 3. Any number $\alpha \in \mathbb{R}$ equal to a simple interlaced continued fraction converges to an irrational limit.

Proof. See Appendix D.3

As can be directly deduced from Theorem 3 and Lemma 1, the set of numbers representable by a simple interlaced CF, $\Re_4$, and thus SICFs is a subset of irrational numbers that have polynomial CF expansion which converge super-exponentially. Specifically, those with a positive partial denominator whose degree is greater than or equal to that of the partial numerator, $\Re_4 \subset \{\Re_3 \mid \deg(b') \leq \deg(a'), a'(n) > 0 \quad \forall n \in \mathbb{N}\backslash 0\}$. Recall, $\Re_3$ refers to the representable set of polynomial CFs. We can therefore deduce, that as $\Re_1 \equiv \Re_4 \subset \Re_3$, meaning numbers who have SICF expansions are a subset of numbers which have polynomial CF expansions.

Overall, the representable set of SICFs, which is also the space of constants we expect ESMA to catch, is characterized by irrational constants whose simple CF expansion is made of interlaced polynomial sequences (up to a Mobius transform). Such CFs with constant partial denominator sequences converge at an exponential rate to 2^{nd} degree algebraic numbers. Such CFs with non-constant sequences have polynomial CF expansions that converge at a super-exponential rate. Intriguingly, ESMA could potentially find CFs with non-polynomial subsequences. However, our extensive runs on ESMA with up to 50 different constants only found polynomial subsequences and thus further supports the conclusions of our mathematical analysis.

ESMA's main shortcoming is that it can only find conjectures that converge super-exponentially (apart from conjectures of 2^{nd} degree algebraic numbers). The limitation is even more severe - any constant which ESMA can catch must also have an interlaced polynomial pattern in its simple CF. Such a pattern is not known for mathematical constants such as $\pi, \zeta(3)$, and G, and thus we expect ESMA cannot find conjectures for them, unless such a formula could be found for a transformation of those constants.

We highlight that to potentially overcome these shortcomings and expand ESMA's search space, one could define any general b_n sequence (e.g.,$b_n = n^6$) and utilize it to extract the a_n sequence to discover conjectures as previously described in this paper.

5. Discussion and Outlook

The ESMA algorithm was designed with the hopes that the conjectures it discovers will reveal more efficient expansions of mathematical constants and more generally reveal unknown underlying patterns. Looking forward, the ESMA algorithm represents a wider effort to automate and accelerate mathematical discovery through the utilization of computational power. In particular, similar to how analysis of the algorithm's abilities led us to develop novel methods for manipulating CFs (e.g., the Folding transform), the proofs for its conjectures may require developing novel mathematical tools, further accelerating mathematical discovery.

We further note, the impact of algorithmic-assisted discoveries will be in creating large pools of results upon which generalizations can be made to provide insights on the underlying structure of constants. In this way, our algorithm is a tool to aid mathematicians in the discovery of general structures, and not only of specific formulas. For example, a large group of conjectured formulas for e in (Raayoni et al., 2021) was generalized into an infinite family which converges to e by Zeilberger (Dougherty-Bliss & Zeilberger, 2020). Such a result is much stronger than any particular formula. Therefore, we hope ESMA will be utilized in the search for the underlying structure of various fundamental constants. Moreover, one may build upon ESMA's features in attempt to adapt it to various domains.

In recent years, automation in mathematics has increased in popularity, as researchers increasingly harness the power of artificial intelligence and computational techniques to tackle complex mathematical problems. Our algorithm and its results represent one more step forward in this evolving domain, demonstrating the potential of automated methods for generating mathematical conjectures. We hope that our findings inspire and encourage more researchers to explore this promising direction, fostering further advancements at the intersection of mathematics and computer automation.

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A. Additional Results by the ESMA Algorithm

In this section we present a small sample of conjectures and known formulas found by the ESMA algorithm. For each conjecture found, we present the signed b_j sequence used for its extraction, the found LFSR or recurrence relation which models the a_j sequence, the initial conditions of the recurrence relation, the resultant a_j sequence, and the resultant CF's convergence rate. The convergence rate is a linear approximation of the number of digits obtained in the CF approximation per CF term in the log scale of approximation error. While super-exponential convergence is non-linear (in the log scale), this approximation captures the overall magnitude of convergence indicating what conjectures converge faster. Additional numerical analysis of the convergence rate is best done by plotting the approximation error with CF depth. We denote the initial conditions of the recurrence relation in list form under the relation. For example, if $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 3$ then we write the following under the recurrence relation: $[1, 1, 1, 3]$. We further denote the periodic b_j sequence using the following notation: $\{1, -1, \ldots, 1\}$, this notation describes a period of the sequence thus fully describing the sequence, $b_{(n-1)\times\beta+1} = 1, b_{(n-1)\times\beta+2} = -1, \ldots, \forall n \in \mathbb{N}\backslash 0$.

Table 1. Conjectures involving $e, e^2, \sqrt{e}, \tanh\left(\frac{1}{4}\right)$. Digits per term value refers to the number of digits in the CF approximation per CF term, averaged over 100 terms.

Novelty	Formula	b_j	LFSR	a_j	Convergence $\left[\frac{\text{digits}}{\text{term}}\right]$
known	$-1+e = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}$	$\{1,1,1\}$	$a_j - 2a_{j-3} + a_{j-6} = 0$ $[1,1,2,1,1,4]$	$a_j = \begin{cases} 1 & j=3k \\ 1 & j=3k+1 \\ 2k & j=3k+2 \end{cases}$ $k \in \{0,1,2,\ldots\}$	1.2868
new and unproven	$\frac{1+e}{4(-1+e)} = 1-\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{2-\cdots}}}$	$\{-1,1,1,-1,1,1\}$	$a_j - 2a_{j-6} + a_{j-12} = 0$ $[1,2,5,2,3,56,5,2,21,2,7,120]$	$a_j = \begin{cases} 1+4k & j=6k \\ 2 & j=6k+1 \\ 5+16k & j=6k+2 \\ 2 & j=6k+3 \\ 3+4k & j=6k+4 \\ 56+64k & j=6k+5 \end{cases}$ $k \in \{0,1,2,\ldots\}$	3.0142
known	$\frac{1+e}{-1+e} = 2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}$	$\{1\}$	$a_j - 2a_{j-1} + a_{j-2} = 0$ $[2,6]$	$a_j = 2+4j$	4.8512
new and unproven	$\frac{-5+3e}{3-e} = 12-\cfrac{1}{1+\cfrac{1}{5-\cfrac{1}{1+\cdots}}}$	$\{-1,1,-1,1\}$	$a_j - 2a_{j-4} + a_{j-8} = 0$ $[12,1,5,1,28,1,9,1]$	$a_j = \begin{cases} 12+16k & j=4k \\ 1 & j=4k+1 \\ 5+4k & j=4k+2 \\ 1 & j=4k+3 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.1508
new and unproven	$\frac{2+2e}{-1+3e} = 2-\cfrac{1}{1+\cfrac{1}{24+\cfrac{1}{3-\cdots}}}$	$\{-1,1,1,-1,1,1\}$	$a_j - 2a_{j-6} + a_{j-12} = 0$ $[2,1,24,3,2,13,2,5,88,7,2,29]$	$a_j = \begin{cases} 2 & j=6k \\ 1+4k & j=6k+1 \\ 24+64k & j=6k+2 \\ 3+4k & j=6k+3 \\ 2 & j=6k+4 \\ 13+16k & j=6k+5 \end{cases}$ $k \in \{0,1,2,\ldots\}$	3.4269
new and unproven	$\frac{-3+5e}{-6+6e} = 1+\cfrac{1}{36+\cfrac{1}{2-\cfrac{1}{4-\cdots}}}$	$\{1,1,-1,-1,1,1,-1,-1,1,1,-1,-1\}$	$a_j - 2a_{j-12} + a_{j-24} = 0$ $[2,2,18,3,3,4,3,5,2,2,6,2,2,6,54,7,3,8,3,9,2,2,10,2]$	$a_j = \begin{cases} 2 & j=12k \\ 2+4k & j=12k+1 \\ 18+36k & j=12k+2 \\ 3+4k & j=12k+3 \\ 3 & j=12k+4 \\ 4+4k & j=12k+5 \\ 3 & j=12k+6 \\ 5+4k & j=12k+7 \\ 2 & j=12k+8 \\ 2 & j=12k+9 \\ 6+4k & j=12k+10 \\ 2 & j=12k+11 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.1106
new and unproven	$\frac{1}{-2+2e^2} = 1-\cfrac{1}{1+\cfrac{1}{11+\cfrac{1}{2-\cdots}}}$	$\{-1,1,1,-1,1,-1,1,1,-1,1\}$	$a_j - 2a_{j-10} + a_{j-20} = 0$ $[1,1,11,2,1,3,2,36,3,4,4,$ $5,1,3,5,84,6,4]$	$a_j = \begin{cases} 1+3k & j=10k \\ 1 & j=10k+1 \\ 11+48k & j=10k+2 \\ 2+3k & j=10k+3 \\ 1 & j=10k+4 \\ 3 & j=10k+5 \\ 2+3k & j=10k+6 \\ 36+48k & j=10k+7 \\ 3+3k & j=10k+8 \\ 4 & j=10k+9 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.3169

known	$-\frac{1}{2}+\frac{e^2}{2}=3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cdots}}}$	$\{1\}$	$a_j-2a_{j-1}+a_{j-2}=0$ [3,5]	$a_j=3+2j$	4.2727
new and unproven	$\frac{3+3\sqrt{e}}{2}=3+\cfrac{1}{2-\cfrac{1}{1+\cfrac{1}{36-\cdots}}}$	$\{1,-1,1,-1,1,-1\}$	$a_j-2a_{j-6}+a_{j-12}=0$ [3,2,1,36,1,6,3,10,1,108,1,14]	$a_j=\begin{cases}3 & j=6k\\ 2+8k & j=6k+1\\ 1 & j=6k+2\\ 36+72k & j=6k+3\\ 1 & j=6k+4\\ 6+8k & j=6k+5\end{cases}$ $k\in\{0,1,2,\ldots\}$	2.4322
new and unproven	$\frac{2}{\tanh\left(\frac{1}{4}\right)}=8+\cfrac{1}{6+\cfrac{1}{40+\cfrac{1}{14+\cdots}}}$	$\{1,1\}$	$a_j-2a_{j-2}+a_{j-4}=0$ [8,6,40,14]	$a_j=\begin{cases}8+32k & j=2k\\ 6+8k & j=2k+1\end{cases}$ $k\in\{0,1,2,\ldots\}$	4.9003

Table 2. Conjectures for $\tan(1)$ (in radians).

Novelty	Formula	b_j	LFSR	a_j	Convergence $\left[\frac{\text{digits}}{\text{term}}\right]$
known	$\tan(1) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cdots}}}$	{1,1}	$a_j - 2a_{j-2} + a_{j-4} = 0$ [1,1,1,3]	$a_j = \begin{cases} 1 & j = 2k \\ 1 + 2k & j = 2k+1 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.8441
known	$\frac{\tan(1)}{-1+\tan(1)} = 3 - \cfrac{1}{5 - \cfrac{1}{7 - \cfrac{1}{9 - \cdots}}}$	{−1}	$a_j - 2a_{j-1} + a_{j-2} = 0$ [3,5]	$a_j = 3 + 2j$	4.2727
new and unproven	$\frac{2-\tan(1)}{-1+\tan(1)} = 1 - \cfrac{1}{4 + \cfrac{1}{2 - \cfrac{1}{1 + \cdots}}}$	{−1,1,−1,1}	$a_j - a_{j-1} + a_{j-2} - a_{j-3} - a_{j-4} + a_{j-5} - a_{j-6} + a_{j-7} = 0$ [1, 4, 2, 1, 5, 8, 2]	$a_j = \begin{cases} 1+k & j = 4k \\ 3+k & j = 4k+1 \\ 2 & j = 4k+2 \\ 1 & j = 4k+3 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.8436
new and unproven	$\frac{2}{\tan(1)} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cdots}}}$	{−1,−1,−1,−1}	$a_j - 2a_{j-4} + a_{j-8} = 0$ [2, 2, 2, 3, 14, 5, 2, 6]	$a_j = \begin{cases} 2+12k & j = 4k \\ 2+3k & j = 4k+1 \\ 2 & j = 4k+2 \\ 3+3k & j = 4k+3 \end{cases}$ $k \in \{0,1,2,\dots\}$	3.0325
new and unproven	$\frac{1}{-2+2\tan(1)} = 1 - \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{3 - \cdots}}}$	{−1,1,1,−1,1,1}	$a_j - 2a_{j-6} + a_{j-12} = 0$ [1, 9, 1, 3, 1, 1, 4, 21, 1, 6, 1, 1]	$a_j = \begin{cases} 1+3k & j = 6k \\ 9+12k & j = 6k+1 \\ 1 & j = 6k+2 \\ 3+3k & j = 6k+3 \\ 1 & j = 6k+4 \\ 1 & j = 6k+5 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.8622
new and unproven	$\frac{-2+2\tan(1)}{-3+2\tan(1)} = 10 - \cfrac{1}{4 - \cfrac{1}{2 - \cfrac{1}{5 - \cdots}}}$	{−1,−1,−1,−1}	$a_j - 2a_{j-4} + a_{j-8} = 0$ [10, 4, 2, 5, 22, 7, 2, 8]	$a_j = \begin{cases} 10+12k & j = 4k \\ 4+3k & j = 4k+1 \\ 2 & j = 4k+2 \\ 5+3k & j = 4k+3 \end{cases}$ $k \in \{0,1,2,\dots\}$	3.0555
new and unproven	$\frac{-5+4\tan(1)}{-7+5\tan(1)} = 2 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots}}}$	{−1,−1,−1, −1,−1,−1, −1,−1,−1 −1,−1,−1}	$a_j - 2a_{j-12} + a_{j-24} = 0$ [2, 3, 2, 2, 3, 27, 4, 3, 5, 3, 6, 2, 2, 7, 2, 2, 7, 63, 8, 3, 9, 3, 10, 2]	$a_j = \begin{cases} 2 & j = 12k \\ 3+4k & j = 12k+1 \\ 2 & j = 12k+2 \\ 2 & j = 12k+3 \\ 3+4k & j = 12k+4 \\ 27+36k & j = 12k+5 \\ 4+4k & j = 12k+6 \\ 3 & j = 12k+7 \\ 5+4k & j = 12k+8 \\ 3 & j = 12k+9 \\ 6+4k & j = 12k+10 \\ 2 & j = 12k+11 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.844
new and unproven	$\frac{-1+\tan(1)}{5-3\tan(1)} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$	{1,1,1, 1,1,1}	$a_j - 2a_{j-6} + a_{j-12} = 0$ [1, 1, 2, 2, 1, 16, 1, 4, 2, 5, 1, 28]	$a_j = \begin{cases} 1 & j = 6k \\ 1+3k & j = 6k+1 \\ 2 & j = 6k+2 \\ 2+3k & j = 6k+3 \\ 1 & j = 6k+4 \\ 16+12k & j = 6k+5 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.8491
new and unproven	$\frac{4-2\tan(1)}{7-4\tan(1)} = 2 - \cfrac{1}{1 + \cfrac{1}{6 - \cfrac{1}{4 - \cdots}}}$	{−1,1,−1, −1,−1,1, −1,−1}	$a_j - 2a_{j-8} + a_{j-16} = 0$ [2, 1, 6, 4, 2, 4, 2, 2, 2, 1, 18, 7, 2, 7, 2, 2]	$a_j = \begin{cases} 2 & j = 8k \\ 1 & j = 8k+1 \\ 6+12k & j = 8k+2 \\ 4+3k & j = 8k+3 \\ 2 & j = 8k+4 \\ 4+3k & j = 8k+5 \\ 2 & j = 8k+6 \\ 2 & j = 8k+7 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.3186
new and unproven	$\frac{8-5\tan(1)}{-3+2\tan(1)} = 2 - \cfrac{1}{7 - \cfrac{1}{8 + \cfrac{1}{2 - \cdots}}}$	{−1,−1,1, −1,−1,1}	$a_j - a_{j-1} + a_{j-3} - a_{j-4} - a_{j-6} + a_{j-7} - a_{j-9} + a_{j+10} = 0$ [2, 7, 8, 2, 2, 1, 8, 13, 14, 2]	$a_j = \begin{cases} 2+6k & j = 6k \\ 7+6k & j = 6k+1 \\ 8+6k & j = 6k+2 \\ 2 & j = 6k+3 \\ 1 & j = 6k+4 \\ 1 & j = 6k+5 \end{cases}$ $k \in \{0,1,2,\dots\}$	1.8851

Table 3. Conjectures of ratios of Bessel functions of different orders at $x = 1$

Novelty	Formula	b_j	LFSR	a_j	Convergence $\left[\frac{\text{digits}}{\text{term}}\right]$
known	$\frac{J_0(1)}{J_1(1)} = 2 - \cfrac{1}{4 - \cfrac{1}{6 - \cfrac{1}{8 - \cdots}}}$	$\{-1\}$	$a_j - 2a_{j-1} - a_{j-2} = 0$ [2,4]	$a_j = 2 + 2j$	4.2669
new and unproven	$-1 + \frac{J_0(1)}{J_1(1)} = 1 - \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 - \cdots}}}$	$\{-1,1,1\}$	$a_j - 2a_{j-3} + a_{j-6} = 0$ [1, 3, 1, 5, 7, 1]	$a_j = \begin{cases} 1 + 4k & j = 3k \\ 3 + 4k & j = 3k+1 \\ 1 & j = 3k+2 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.6247
new and unproven	$\frac{J_0(1)}{2J_1(1)} = 1 - \cfrac{1}{8 - \cfrac{1}{3 - \cfrac{1}{16 - \cdots}}}$	$\{-1,-1\}$	$a_j - 2a_{j-2} - a_{j-4} = 0$ [1,8,3,16]	$a_j = \begin{cases} 1 + 2k & j = 2k \\ 8 + 16k & j = 2k+1 \end{cases}$ $k \in \{0,1,2,\ldots\}$	4.2669
new and unproven	$\frac{-2J_0(1) + 4J_1(1)}{-4J_0(1) + 7J_1(1)} = 12 - \cfrac{1}{4 - \cfrac{1}{20 - \cfrac{1}{6 - \cdots}}}$	$\{-1\}$	$a_j - 2a_{j-2} - a_{j-4} = 0$ [12,4,20,6]	$a_j = \begin{cases} 12 + 8k & j = 2k \\ 4 + 2k & j = 2k+1 \end{cases}$ $k \in \{0,1,2,\ldots\}$	4.2897
new and unproven	$\frac{-12047J_1(1) + 6928J_0(1)}{8(1777J_0(1) + 3090J_1(1))} = 1 - \cfrac{1}{112 - \cfrac{1}{2 - \cfrac{1}{143 + \cdots}}}$	$\{-1,-1,-1,1,1,-1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1\}$	$a_j - 2a_{j-15} - a_{j-30} = 0$ [1, 112, 2, 143, 1, 2, 2, 45, 1, 1, 3, 2, 53, 1, 1, 3, 240, 4, 271, 1, 4, 2, 77, 1, 1, 5, 2, 85, 1, 1]	$a_j = \begin{cases} 1 + 2k & j = 15k \\ 112 + 128k & j = 15k+1 \\ 2 + 2k & j = 15k+2 \\ 143 + 128k & j = 15k+3 \\ 1 & j = 15k+4 \\ 2 + 2k & j = 15k+5 \\ 2 & j = 15k+6 \\ 45 + 32k & j = 15k+7 \\ 1 & j = 15k+8 \\ 1 & j = 15k+9 \\ 3 + 2k & j = 15k+10 \\ 2 & j = 15k+11 \\ 53 + 32k & j = 15k+12 \\ 1 & j = 15k+13 \\ 1 & j = 15k+14 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.021
known	$\frac{J_1(1)}{J_2(1)} = 4 - \cfrac{1}{6 - \cfrac{1}{8 - \cfrac{1}{10 - \cdots}}}$	$\{-1\}$	$a_j - 2a_{j-1} - a_{j-2} = 0$ [4,6]	$a_j = 4 + 2j$	4.2784
new and unproven	$\frac{J_0(1)}{J_2(1)} = 6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$	$\{1,1,1,1\}$	$a_j - 2a_{j-4} - a_{j-8} = 0$ [6,1,1,1,14,1,3,1]	$a_j = \begin{cases} 6 + 8k & j = 4k \\ 1 & j = 4k+1 \\ 1 + 2k & j = 4k+2 \\ 1 & j = 4k+3 \end{cases}$ $k \in \{0,1,2,\ldots\}$	1.8617
new and unproven	$\frac{J_1(1)}{J_3(1)} = 23 - \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{39 - \cdots}}}$	$\{-1,1,1\}$	$a_j - 2a_{j-3} - a_{j-6} = 0$ [23,1,1,39,2,1]	$a_j = \begin{cases} 23 + 16k & j = 3k \\ 1 + k & j = 3k+1 \\ 1 & j = 3k+2 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.6693
new and unproven	$\frac{J_3(1)}{J_5(1)} = 79 - \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{27 + \cdots}}}$	$\{-1,1,1,1,1,-1,1,1,1,1,-1,1,1,1,1\}$	$a_j - 2a_{j-15} + a_{j-30} = 0$ [79, 1, 2, 27, 2, 2, 1, 1, 35, 2, 2, 175, 1, 1, 1, 207, 3, 2, 59, 2, 4, 1, 1, 67, 2, 4, 303, 1, 3, 1]	$a_j = \begin{cases} 79 + 128k & j = 15k \\ 1 + 2k & j = 15k+1 \\ 2 & j = 15k+2 \\ 27 + 32k & j = 15k+3 \\ 2 & j = 15k+4 \\ 2 + 2k & j = 15k+5 \\ 1 & j = 15k+6 \\ 1 & j = 15k+7 \\ 35 + 32k & j = 15k+8 \\ 2 & j = 15k+9 \\ 2 + 2k & j = 15k+10 \\ 175 + 128k & j = 15k+11 \\ 1 & j = 15k+12 \\ 1 + 2k & j = 15k+13 \\ 1 & j = 15k+14 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.0013
new and unproven	$\frac{2}{\frac{J_5(1)}{J_3(1)} + 1} = 2 - \cfrac{1}{40 - \cfrac{1}{3 - \cfrac{1}{56 - \cdots}}}$	$\{-1,-1\}$	$a_j - 2a_{j-2} + a_{j-4} = 0$ [2,40,3,56]	$a_j = \begin{cases} 2 + k & j = 2k \\ 40 + 16k & j = 2k+1 \end{cases}$ $k \in \{0,1,2,\ldots\}$	4.3008
new and unproven	$\frac{1}{\frac{J_7(1)}{J_5(1)} + 1} = 1 - \cfrac{1}{168 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots}}}$	$\{-1,-1,-1,-1,-1,-1,-1,-1,-1\}$	$a_j - 2a_{j-9} + a_{j-18} = 0$ [1, 168, 2, 2, 2, 25, 3, 2, 264, 2, 312, 3, 2, 2, 41, 3, 3, 408]	$a_j = \begin{cases} 1 + k & j = 9k \\ 168 + 144k & j = 9k+1 \\ 2 + k & j = 9k+2 \\ 2 & j = 9k+3 \\ 2 & j = 9k+4 \\ 25 + 16k & j = 9k+5 \\ 3 & j = 9k+6 \\ 2 + k & j = 9k+7 \\ 264 + 144k & j = 9k+8 \end{cases}$ $k \in \{0,1,2,\ldots\}$	2.728

For our code please refer to our git through: `www.ramanujanmachine.com`. Notice, the code returns a sequence starting from a_0 while in our analysis we ignore a_0, moving it to the left-hand side of the equation.

As seen in the results, we solely obtained linear interlaced sequences in the partial denominator. Furthermore, all the constants have polynomial CF expansions that converge super-exponentially. ESMA can additionally trivially catch second-degree algebraic numbers whose CF expansions converge exponentially. Overall, the ESMA algorithm was run on more than 50 constants, in the following list we present a sample of constants which the algorithm did not find conjectures for: $\zeta(2), \zeta(3), \zeta(5), \pi, G, \sqrt{\varphi}, 2^{\frac{1}{3}}, 100^{\frac{1}{5}}$, and many more. We deduce that these constants could not be caught by ESMA as they are not in the representable set of the SICF structure utilized (see section 4).

A.1. Efficient Continued Fraction Expansions found by the ESMA algorithm

The ESMA algorithm found various conjectures for constants that converge significantly faster than known formulas. Even for 2nd degree algebraic numbers, whose CF can be trivially calculated, the ESMA algorithm was able to find more efficient expansions of various constants such as φ, $\sqrt{2}$, and many more. Below we denote a sample of conjectures found on $J_1(1)/J_3(1)$, $J_5(1)/J_3(1)$, and φ which converge at a faster rate, and thus more efficiently approximate the constant, than their simple CF expansions (to our knowledge).

Looking at $J_1(1)/J_3(1)$ it's simple CF expansion converges at a rate of 1.8643 digits per term whilst our conjecture on $J_1(1)/J_3(1)$ (see Table 3) converges at a rate of 2.6693 digits per term (averaged over 100 terms). In a similar manner, our conjecture on $2/((J_5(1)/J_3(1)) + 1)$ converges at a rate of 4.3008 digits per term, whilst the simple CF for $J_5(1)/J_3(1)$ converges at a rate of 1.9046, more than twice as fast. Furthermore, the simple CF of The Golden ratio, φ, which is famously given by $\varphi = 1 + 1/(1 + 1/(1 + 1/(\ldots))$ converges at a rate of 0.4096 digits per term, while the following conjecture found by converges at a rate of 2.4576 digits per term:

$$\frac{1+2\varphi}{-3+2\varphi} = 18 - \frac{1}{18 - \frac{1}{18-\ldots}}. \tag{16}$$

The formula and conjecture on φ converge at an exponential rate, and therefore the convergence rate is exactly the $\log(error)$ slope. The convergence plots for all the above conjectures and formulas are presented below:

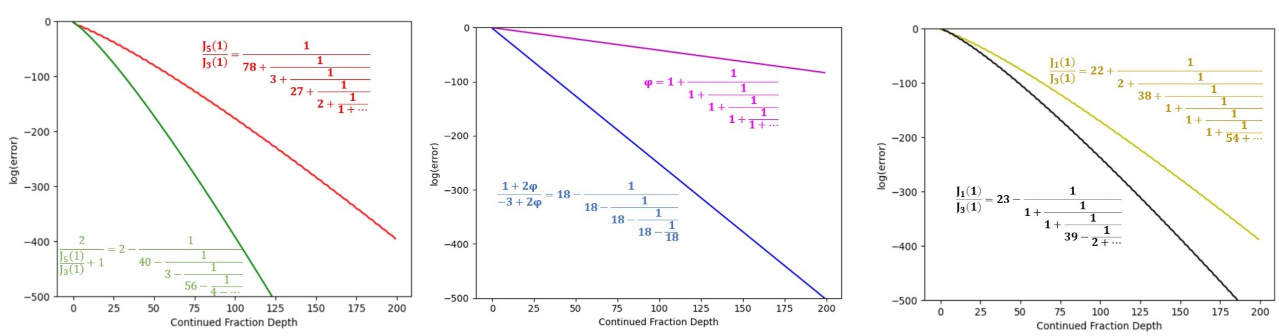

Figure 6. ESMA results that converge faster than the corresponding simple CFs. log (error) plot of a known formula and conjecture on φ, $J_1(1)/J_3(1)$, and $J_5(1)/J_3(1)$. We see the conjectures found by the ESMA algorithm (green, blue, and black) approximate the constants significantly faster.

The increased rate of convergence of our conjectures relative to their respective simple CF formulas can be clearly seen.

All the conjectures are simply a Mobius transform of the constant we attempt to approximate. In these cases, the Mobius transforms do not "bother" us as we can find an expression which converges at the same rate while isolating the constant. For example, the conjecture on φ can be written in the following way to directly approximate φ :

$$\varphi = -\frac{3}{2} + \frac{2}{17 - \frac{1}{18 - \frac{1}{18 - \frac{1}{18 - \ldots 1}}}}. \tag{17}$$

We note that some fast-converging expansions found by ESMA are known but were previously unappreciated for their fast convergence. For example, the formula found for $1/2+e^2/2$ (see Table 1), which can be easily derived from the known CF of tanh(1), converges at a rate of 4.2727 digits per term whilst its known simple CF formula converges at a rate of 2.2806 digits per term. The known formulas found further add confidence in the correctness of our conjectures.

B. Tools for Manipulating Continued Fractions

B.1. Definitions

We recall, any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a non-zero determinant, $\det(M) = ad - bc \neq 0$, can represent the following Mobius transform:

$$M(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d} \quad a,b,c,d \in \mathbb{Z}. \tag{18}$$

As such, each single fraction of a CF, e.g., $\frac{b}{a+\cdots}$, can be represented as a Mobius transform of the following form:

$$\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}(x) = \frac{b}{a+x}. \tag{19}$$

We refer to a single fraction of a CF, (19), as a layer of the CF.

As such, any CF can be represented as a composition or product of Mobius transforms which act on 0, for example:

$$\frac{b_1}{a_1 + \frac{b_2}{a_2}} = \begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & b_2 \\ 1 & a_2 \end{pmatrix}(0).$$

Therefore, we can represent a CF as a matrix product of its layers in the following form:

$$\alpha = a_0 + \lim_{N\to\infty} \prod_{j=1}^{N} \begin{pmatrix} 0 & b_j \\ 1 & a_j \end{pmatrix}(0), \quad a_j, b_j \in \mathbb{Z}. \tag{20}$$

In this paper, an interlaced continued fraction refers to a CF in which its partial numerator and denominator are interlaced sequences: a sequence formed of $\beta \geq 1$ different integer polynomial sub-sequences that alternate with a given order. An interlaced CF of a number $\alpha \in \mathbb{R}$ is a CF of the form:

$$\alpha = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\cdots + \frac{b_{\beta+1}}{a_{\beta+1}+\cdots}}}} = a_0 + \cfrac{B_1(1)}{A_1(1) + \frac{B_2(1)}{A_2(1) + \frac{B_3(1)}{\cdots + + \frac{B_1(2)}{A_1(2)+\cdots}}}} \tag{21}$$

Where $b_{(n-1)\times\beta+i} = B_i(n), a_{(n-1)\times\beta+i} = A_i(n), A_i, B_i \in \mathbb{Z}[x], \forall n \in \mathbb{N}\backslash 0$ are the partial numerator and denominator sequences, respectively, and $i \in \{1,\ldots,\beta\}$ such that $\beta \in \mathbb{N}$ represents the amount of sub-sequences that are interlaced in the partial numerator or denominator, also referred to as the period. For each i, $A_i(n)$ represents the n^{th} element of the i^{th} polynomial sub-sequence. For the rest of this paper the leading integer a_0 will be ignored as it is more comfortable to treat it as part of the constant α.

The n^{th} collapsed matrix of an interlaced CF refers to a matrix, denoted by M_n, which is the product of all matrices in the n^{th} period of the interlaced CF. Meaning given the above interlaced CF of α and of period β :

$$\alpha = \lim_{N\to\infty}\prod_{n=1}^{N}\prod_{i=1}^{\beta}\begin{pmatrix} 0 & b_{(n-1)\times\beta+i} \\ 1 & a_{(n-1)\times\beta+i}\end{pmatrix}(0) = \lim_{N\to\infty}\prod_{n=1}^{N}\prod_{i=1}^{\beta}\begin{pmatrix} 0 & B_i(n) \\ 1 & A_i(n)\end{pmatrix}(0) = \lim_{N\to\infty}\prod_{n=1}^{N} M_n(0) \tag{22}$$

$$M_n = \prod_{i=1}^{\beta}\begin{pmatrix} 0 & B_i(n) \\ 1 & A_i(n)\end{pmatrix} = \begin{pmatrix} c_n & d_n \\ e_n & f_n\end{pmatrix} \quad c,d,e,f \in \mathbb{Z}[x] \tag{23}$$

where b_j and a_j are the interlaced sequences from (21). All layers in each period are polynomial with the same n, as can be seen from the notation in (21), $b_{(n-1)\times\beta+i} = B_i(n), \quad a_{(n-1)\times\beta+i} = A_i(n) \in \mathbb{Z}[x], n \in \mathbb{N}\backslash 0$. Therefore, the elements of the collapsed matrix c, d, e, f, being a product of various integer polynomial sequences with index n, will be polynomial with index n, and the collapsed matrix is a polynomial matrix.

B.2. The Folding Transform and Proof of Theorem 1

We introduce the Folding Transform; a transform which acts on a constant's interlaced CF expansion, or in a more general sense acts on polynomial matrices. Given a constant $\alpha \in \mathbb{R}$ which has an interlaced CF expansion with a known period, we can "fold" (multiply) its layers in each period to form a polynomial matrix. Therefore, our analysis on general polynomial matrices applies to interlaced CFs in particular. The Folding transform applies a Mobius transform on α on the left-hand side of the equation while multiplying each of α 's polynomial matrices on the right-hand side. The result of the Folding transform is a polynomial CF equal to a Mobius transform of α.

Given an α which satisfies $\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty)$, with a collapsed matrix $M_n = \begin{pmatrix} c_n & d_n \\ e_n & f_n\end{pmatrix}$ where $e_n \neq 0$ $\forall n \in \mathbb{N}\backslash 0$, the Folding transform utilizes a sequence of matrices U_n such that:

$$\alpha' = (M_1 U_2)^{-1}(\alpha) = \lim_{N\to\infty}\left[\prod_{n=2}^{N} U_n^{-1} M_n U_{n+1}\right](\infty) \tag{24}$$

where $(M_1 U_2)^{-1}(\alpha)$ is a Mobius transform acting on α and $U_n^{-1} M_n U_{n+1}$ is a polynomial CF layer. Two constants or CFs that are connected by the Folding transform are said to be **semi-equivalent**. This intriguing transform reveals a novel connection between polynomial CFs and interlaced CFs. Every constant that has an interlaced CF expansion, up to some Mobius transform, must have a polynomial CF expansion regardless of the number of interlaced sequences and the seeming complexity of the interlaced CF.

It is important to note that for every converging interlaced CFs as we defined them (see Appendix B.1), $e_n \neq 0$. Assessing the collapsed matrices, if we falsely assume $e_n = 0$ we obtain a CF which does not converge. If $M_n = \begin{pmatrix} c_n & d_n \\ 0 & f_n\end{pmatrix}$ $\forall n \in \mathbb{N}\backslash 0$, the $n\beta - 1^{th}$ convergent (rational approximation of the CF) can be represented in the following form (Raayoni et al., 2021; Cuyt et al., 2008):

$$\frac{p_{n\beta-1}}{q_{n\beta-1}} = \begin{pmatrix} p_{n\beta-1} & p_{n\beta} \\ q_{n\beta-1} & q_{n\beta}\end{pmatrix}(\infty) = \begin{pmatrix} 0 & b_1 \\ 1 & a_1\end{pmatrix}\cdots\begin{pmatrix} 0 & b_{n\beta} \\ 1 & a_{n\beta}\end{pmatrix}(\infty) = M_1 M_2 \ldots M_n(\infty)$$

$$\begin{pmatrix} c_n & d_n \\ 0 & f_n\end{pmatrix}(\infty) = \infty$$

$$M_1 M_2 \ldots \begin{pmatrix} c_n & d_n \\ 0 & f_n\end{pmatrix}(\infty) = M_1 M_2 \ldots M_{n-1}(\infty) = \text{ by induction } \infty.$$

Proving the CF diverges. Therefore, any converging interlaced CF must satisfy $e_n \neq 0$ and one could apply the Folding transform on it. Note, we see here that for an interlaced CF, $\left[\prod_{n=1}^{N} M_n\right](\infty) = \left[\prod_{n=1}^{N-1} M_n\right](0)$, and therefore taking the mapping at ∞ is simply assessing a sub-sequence of the convergents the CF.

Theorem 1. Let $M_n = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}$, satisfying $e_n \neq 0$, be some polynomial matrix $(c, d, e, f \in \mathbb{Z}[x])$ for which the following limit exists $\lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty)$. Then there exists some Mobius transform T with entries in $\mathbb{Z}$, and polynomials $a', b' \in \mathbb{Z}[x]$ such that:

$$\lim_{N\to\infty}\left[T\prod_{n=1}^{N} M_n\right](\infty) = \mathbb{K}_1^{\infty}\frac{b'(n)}{a'(n)}$$

Proof:

For a given polynomial matrix M_n, satisfying $\lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty) = \alpha$, we utilize the Folding transform to obtain a polynomial CF of a Mobius transform of α which we denote α'. To prove Theorem 1 , we must prove that:

1. **The Folding transform does not change the limit of the polynomial matrix product:** $\lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty) = \alpha$

2. **The Folding transform obtains a polynomial CF.**

1. To prove that the limit is unchanged we must show that just as the original product of polynomial matrices converges to $\alpha, \lim_{\mathrm{N}\to\infty}\left[\prod_{n=1}^{\mathrm{N}} \mathrm{M_n}\right](\infty) = \alpha$, so too does the resultant polynomial CF meaning we must prove that (see equation 24):

$$\lim_{N\to\infty}\left[U_1\prod_{n=1}^{N} U_n^{-1}M_nU_{n+1}\right](\infty) = \lim_{N\to\infty} U_1U_1^{-1}M_1U_2U_2^{-1}M_2\dots M_NU_{N+1}(\infty) = \lim_{N\to\infty} M_1M_2\dots M_NU_{N+1}(\infty) = \alpha$$

Given a polynomial matrix, $M_n = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}, c, d, e, f \in \mathbb{Z}[\mathrm{x}], e_n \neq 0 \forall n \in \mathbb{N}\backslash 0$, the Folding transform utilizes the following polynomial matrices U_n :

$$U_n = \begin{pmatrix} 1 & c_n \\ 0 & e_n \end{pmatrix}, U_n^{-1} = \begin{pmatrix} e_n & -c_n \\ 0 & 1 \end{pmatrix}. \tag{25}$$

Assessing the result of mapping U_{n+1} to ∞ we obtain, $\begin{pmatrix} 1 & c_{n+1} \\ 0 & e_{n+1} \end{pmatrix}(\infty) = \infty$.

Therefore, we find that the limit is unchanged by the Folding transform as

$$\lim_{N\to\infty} M_1M_2\dots M_NU_{N+1}(\infty) = \lim_{N\to\infty} M_1M_2\dots M_N(\infty) = \lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](\infty) = \alpha$$

2. We prove that we obtain a polynomial CF. Applying the Folding transform, the following product is obtained:

$$\alpha = \lim_{N\to\infty}\left[U_1\prod_{n=1}^{N} U_n^{-1}M_nU_{n+1}\right](\infty) = \lim_{N\to\infty}\left[U_1\prod_{n=1}^{N}\begin{pmatrix} e_n & -c_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}\begin{pmatrix} 1 & c_{n+1} \\ 0 & e_{n+1} \end{pmatrix}\right](\infty)$$

$$\alpha = \lim_{N\to\infty}\left[U_1\prod_{n=1}^{N}\begin{pmatrix} 0 & e_{n+1}\left(-\det\left(M_n\right)\right) \\ e_n & e_nc_{n+1} + f_ne_{n+1} \end{pmatrix}\right](\infty).$$

We apply the equivalence transform (Lang, 1995; Cuyt et al., 2008) with the following sequence $g_n = \begin{cases} 1, n = 0 \\ e_n, n > 0 \end{cases}$ to normalize the bottom left element of each matrix in the product to obtain correct CF form (see equation 19 in Appendix B.1):

$$\alpha = \lim_{N\to\infty}\left[U_1 \prod_{n=1}^{N} \begin{pmatrix} 0 & g_{n-1}g_n e_{n+1}\left(e_n d_n - c_n f_n\right) \\ e_n & g_n\left(e_n c_{n+1} + f_n e_{n+1}\right) \end{pmatrix}\right](\infty) =$$

$$\lim_{N\to\infty}\left[U_1 \prod_{n=1}^{N} \begin{pmatrix} 0 & e_{n-1}e_{n+1}\left(e_n d_n - c_n f_n\right) \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{pmatrix}\right](\infty)$$

Note, The equivalence transform (Cuyt et al., 2008) refers to the following transform: Given a CF $\{a_j, b_j\}$ and any nonzero infinite sequence of $\{g_i\}_{i=1}^{\infty} \in \mathbb{C}$:

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \ldots}} = a_0 + \frac{g_1 b_1}{g_1 a_1 + \frac{g_1 g_2 b_2}{g_2 a_2 + \ldots}}$$

These two CFs are equivalent to any depth $n \in \mathbb{N}$.

Since $c, d, e, f \in \mathbb{Z}[\mathrm{x}]$, are all polynomial with n any product or sum of the sequences is polynomial with n. The first layer of $n = 1$ is taken out of the product as it doesn't follow the polynomial pattern of the rest of the CF, as $g_0 = 1$, and we obtain:

$$\alpha = \lim_{N\to\infty}\left[\begin{pmatrix} 1 & c_1 \\ 0 & e_1 \end{pmatrix}\begin{pmatrix} 0 & e_2\left(e_1 d_1 - c_1 f_1\right) \\ 1 & \left(e_1 c_2 + f_1 e_2\right) \end{pmatrix} \prod_{n=2}^{N} \begin{pmatrix} 0 & e_{n-1}e_{n+1}\left(e_n d_n - c_n f_n\right) \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{pmatrix}\right](\infty).$$

Where,

$$\begin{pmatrix} 1 & c_1 \\ 0 & e_1 \end{pmatrix}\begin{pmatrix} 0 & e_2\left(e_1 d_1 - c_1 f_1\right) \\ 1 & \left(e_1 c_2 + f_1 e_2\right) \end{pmatrix} = U_1 U_1^{-1} M_1 U_2 = M_1 U_2.$$

We obtain a polynomial CF:

$$\left(M_1 U_2\right)^{-1}(\alpha) = \lim_{N\to\infty}\left[\prod_{n=2}^{N} \begin{pmatrix} 0 & e_{n-1}e_{n+1}\left(e_n d_n - c_n f_n\right) \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{pmatrix}\right](\infty). \tag{26}$$

The above polynomial CF is said to be semi-equivalent to α 's interlaced CF.

We can utilize the connection revealed by the Folding transform to assess properties of interlaced CFs based on their semi-equivalent and more extensively researched polynomial CFs expansions. For example, given a polynomial CF that converges to an irrational limit $\left(M_1 U_2\right)^{-1}(\alpha)$ and that it is semi-equivalent to an interlaced CF that converges to α, we can trivially deduce that α must be irrational.

B.3. Applying the Folding Transform on the Simple CF of e

We apply the Folding transform on the simple CF of e (found in (Olds, 1970)). Looking at the simple CF of e, we can represent it in matrix form and notice that the CF is a simple interlaced CF:

$$\mathrm{e} - 2 = \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \ldots}}}}}}$$

$$\mathrm{e} - 2 = \left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\ldots\right](0)$$

The above simple interlaced CF has a period of 3 (as clarified by the highlighted colors) with $B_i(n) = 1 \quad \forall n, i \in \mathbb{N}$ and $A_i(n) = \begin{cases} 1 & i = 1 \\ 2n & i = 2 \\ 1 & i = 3 \end{cases} \forall n \in \mathbb{N}$. We find its collapsed matrices and apply the Folding transform:

$$\mathrm{e} - 2 = \lim_{N\to\infty} \left[\prod_{n=1}^{N}\prod_{i=1}^{3} \begin{pmatrix} 0 & B_i(n) \\ 1 & A_i(n) \end{pmatrix}\right](0) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 2n & 2n+1 \\ 2n+1 & 2n+2 \end{pmatrix}\right](0).$$

$M_n = \begin{pmatrix} 2n & 2n+1 \\ 2n+1 & 2n+2 \end{pmatrix}$ and therefore from equation (25), $U_n = \begin{pmatrix} 1 & 2n \\ 0 & 2n+1 \end{pmatrix}$, and the Folding transform is given by:

$$(M_1U_2)^{-1}(\mathrm{e}-2) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 2n+1 & -(2n) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2n & 2n+1 \\ 2n+1 & 2n+2 \end{pmatrix}\begin{pmatrix} 1 & 2n+2 \\ 0 & 2n+3 \end{pmatrix}\right](0),$$

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\right)^{-1}(\mathrm{e}-2) = \lim_{N\to\infty} \left[\prod_{n=2}^{N} \begin{pmatrix} 0 & 2n+3 \\ 2n+1 & 8n^2+16n+8 \end{pmatrix}\right](0).$$

We then apply equivalence theorem (see note in Appendix B.1) to reach the correct form of a CF layer and we get the following polynomial CF:

$$\frac{-8e+19}{e-2} = \lim_{N\to\infty} \left[\prod_{n=2}^{N} \begin{pmatrix} 0 & 4n^2+4n-3 \\ 1 & 8n^2+16n+8 \end{pmatrix}\right](0),$$
$$\frac{-8\mathrm{e}+19}{\mathrm{e}-2} = \frac{21}{72 + \frac{45}{128+\frac{17}{200+\cdots}}}. \tag{27}$$

B.4. Determinant Property of the Folding Transform

When applying the Folding transform one can notice the determinant property of the Folding transform. Notice that in the Folding transform's product $U_n^{-1}M_nU_{n+1}$, in the left-hand side product we obtain $U_n^{-1}M_n = \begin{pmatrix} e_n & -c_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix} = \begin{pmatrix} 0 & -\det(M_n) \\ e_n & f_n \end{pmatrix}$ where the top right element of the resultant matrix is equal to minus the determinant of the collapsed matrix: $\left|\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}\right| = e_nd_n - c_nf_n$. An interlaced CF of period β 's collapsed matrix determinant at each period is given by:

$$\left|\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & B_1(n) \\ 1 & A_1(n) \end{pmatrix}\begin{pmatrix} 0 & B_2(n) \\ 1 & A_2(n) \end{pmatrix}\cdots\begin{pmatrix} 0 & B_\beta(n) \\ 1 & A_\beta(n) \end{pmatrix}\right| = \prod_{i=1}^{\beta}(-B_i(n)) = (-1)^\beta \prod_{i=1}^{\beta} B_i(n).$$

Therefore, we can simplify the left-hand side product to:

$$\begin{pmatrix} e_n & -c_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix} = \begin{pmatrix} 0 & (-1)^\beta \prod_{i=1}^{\beta} B_i(n) \\ e_n & f_n \end{pmatrix}. \tag{28}$$

This property eases calculations when trying to predict resultant polynomial CF properties such as polynomial degree after applying the Folding transform.

C. The Simple Interlaced CF expansions of SICFs

C.1. Moving from an SICF to a Simple CF - Proof of Theorem 2

Recall, a simple CF can be represented in the following forms (Olds, 1963):

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} = [a_0; a_1, a_2, a_3, a_4 \dots] . \tag{29}$$

To represent the partial numerator, we add a sign variable to the above notation. This enables us to represent Signed Interlaced Continued Fractions (SICFs):

$$\alpha = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\dots}}}, a_j > 0 \in \mathbb{Z}, b_j \in \{1, -1\} \forall j \in \mathbb{N}, \tag{30}$$
$$\alpha = [a_0; (a_1, b_1), (a_2, b_2), (a_3, b_3), \dots]$$

For example, given the following SICF:

$$\alpha = a_0 - \frac{1}{a_1 + \frac{1}{a_2 - \frac{1}{\dots}}},$$

We can represent α in reduced notation:

$$\alpha = [a_0; (a_1, -1), (a_2, 1), (a_3, -1), \dots] .$$

For the sake of our analysis, as previously done, we will ignore a_0 and represent each SICF in the following form:

$$\alpha = [(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots] . \tag{31}$$

Theorem 2. For any signed interlaced continued fraction equal to $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, there exists some Mobius transform of α which is equal to a simple interlaced continued fraction.

Outline of Proof:

To prove Theorem 2, we address two possible cases: SICFS with constant partial denominator sequences and SICFs with non-constant partial denominator sequences.

SICFs with constant partial denominator sequences (Proof of Theorem 2)

Proof

We show that given a converging SICF of period β with constant partial denominator sequences, meaning $a_j = a_{j+\beta n}$ $\forall j \in \mathbb{N}, \forall n \in \mathbb{N}$, we converge to 2[nd] degree algebraic numbers and therefore by (Olds, 1963) there exists a Mobius transform which is equal to a simple interlaced CF (a simple CF with a pattern in its partial denominator sequence). A constant SICF of period β has a constant collapsed matrix:

$$M_n = M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z} \quad \forall n \in \mathbb{N}$$

As any simple or signed CF layers are made of unimodular matrices, and the product of unimodular matrices is a unimodular matrix, the determinant of M is given by $\det(M) = ad - bc = \pm 1$. We want to study the limit of $M^k(0)$ which is known to converge, $\alpha = \lim_{k\to\infty} p_k/q_k = \lim_{k\to\infty} M^k(0)$, therefore in the case that $\det(M) = -1$ we can instead consider the limit of the sub-sequence $M^{2k}(0)$ for which $\det\left(M^2\right) = 1$, allowing us to assume $\det(M) = 1$. We denote the eigen vectors of matrix M as $\mathbf{v_1} = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ and the eigen values as λ_1 and λ_2 which satisfy $\lambda_1\lambda_2 = \det(M) = \pm 1$. The eigen values are the roots of the second-degree monic characteristic polynomials of M and therefore are 2$^{\text{nd}}$ degree algebraic numbers.

If $\lambda_1 = \overline{\lambda_2} \in \mathbb{C}$

If the eigen values are complex, one must be the complex conjugate of the other. Complex eigen values indicate rotation and scaling and therefore the matrix M^k converges to a rational number or diverges. We show that any real 2 by 2 matrix M with complex eigen values is similar to a shift rotation matrix $R = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$.

Utilizing a single eigen vector, $\mathbf{v_1}$, we construct a matrix $C = \begin{pmatrix} \operatorname{Re}(v_{11}) & -\operatorname{Im}(v_{11}) \\ \operatorname{Re}(v_{12}) & -\operatorname{Im}(v_{12}) \end{pmatrix} = (\operatorname{Re}(\mathbf{v_1}), -\operatorname{Im}(\mathbf{v_1}))$, where Re and Im are the real and imaginary parts of a complex number respectively. Through simple matrix multiplication one can see that $M = CRC^{-1}$ by showing that $MC = CR$.

We therefore obtain a rotation matrix R which rotates vectors on the unit circle. Looking at the $k = n\beta$ convergent: $M^k(0) = CR^kC^{-1}(0)$.

For all $\theta = 0, 2\pi l \ \forall l \in \mathbb{N}$ we obtain an identity matrix and converge to rational numbers which are not of interest to us. Otherwise, M^k infinitely rotates the $C^{-1}(0)$ vector by θ and the CF does not converge.

If $|\lambda_1| \neq |\lambda_2|, \lambda_1, \lambda_2 \in \mathbb{R}$

If matrix M has two distinct eigen values, then it is diagonalizable and can be written in the following form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = PDP^{-1} = \frac{1}{v_{11}v_{22} - v_{12}v_{21}} \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_{22} & -v_{21} \\ -v_{12} & v_{11} \end{pmatrix}.$$

Note, the above matrix represents a Mobius transform and therefore, $cM = M$.

Without loss of generality, we can assume $|\lambda_1| > |\lambda_2|$. Looking at M^k :

$$M^k(0) = PD^kP^{-1}(0) = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \left(-\frac{v_{21}}{v_{11}}\right) = \begin{pmatrix} \lambda_1^k v_{11} & \lambda_2^k v_{21} \\ \lambda_1^k v_{12} & \lambda_2^k v_{22} \end{pmatrix} \left(-\frac{v_{21}}{v_{11}}\right),$$

$$M^k(0) = \frac{\lambda_1^k v_{11}\left(-\frac{v_{21}}{\lambda_{11}}\right) + \lambda_2^k v_{21}}{\lambda_{12}^k v_{12}\left(-\frac{v_{21}}{v_{11}}\right) + \lambda_2^k v_{22}} = \frac{v_{11}\left(-\frac{v_{21}}{v_{11}}\right) + \frac{\lambda_2^k}{\lambda_k^k} v_{21}}{v_{12}\left(-\frac{v_{12}}{v_{11}}\right) + \frac{\lambda_k^k}{\lambda_1^k} v_{22}} \xrightarrow{k\to\infty} = \frac{v_{11}}{v_{12}}.$$

The eigen vector must satisfy $\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$, therefore, the limit is given by: $\frac{v_{11}}{v_{12}} = \frac{\lambda_1 - d}{c}$. As such, we have shown that in this case the SICF converges to the eigen values which are 2$^{\text{nd}}$ degree algebraic numbers.

If $|\lambda_1| = |\lambda_2|, \lambda_1, \lambda_2 \in \mathbb{R}$

In this case M is either diagonalizable or non-diagonalizable. We denote $|\lambda_1| = |\lambda_2|$, and recall that $\lambda_1\lambda_2 = \det(M) = 1$ meaning the eigen values are of the same sign, $\lambda_1 = \lambda_2 = \lambda = \pm 1$. Without loss of generality we assume $\lambda = 1$ as if $\lambda = -1$ we can look at M^2. If M is diagonalizable, then M is a mapping that is equivalent to scaling and we are guaranteed to either diverge or converge to a rational number:

$$M^k = PD^kP^{-1} = P\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^k P^{-1} = P\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^k P^{-1} = PP^{-1} = I.$$

This is an identity matrix and is therefore of no interest to us in the analysis of CF of irrational numbers. If M is not diagonalizable, it must be of the following form: $P\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}P^{-1}$. This is the simple Jordan form. Where P 's columns are given by the eigen vectors of matrix M. If $M = P\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}P^{-1}$, we see that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^k(x) = x + kb$ and therefore we obtain a Mobius transform that maps any x to either infinity or to a rational number when applied repeatedly,

$$\lim_{k\to\infty} PM^kP^{-1}(0) = \lim_{k\to\infty} P\left(-\frac{v_{21}}{v_{11}} + kb\right) = \lim_{k\to\infty} \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}\left(-\frac{v_{21}}{v_{11}} + kb\right) = \lim_{k\to\infty} \frac{kb}{v_{12}kb + v_{22}}.$$

If $v_{12} \neq 0$ we will converge to a rational number otherwise we converge to infinity.

Therefore, in the cases where the SICF with constant sequences converges to an irrational number, the SICF converges to a 2nd degree algebraic number. This ensures there is a simple interlaced CF expansion for the number, where the interlaced sequences are constant sequences. Simple CFs with constant sequences (or k-periodic CFs) have a bijective correspondence to the real 2nd degree algebraic numbers as given by Minkowski's Question Mark Function (Denjoy, 1938; Balkova & Hrušková, 2013), up to initial CF layers which do not always follow the periodic pattern. Therefore, all 2nd degree algebraic numbers, up to some Mobius transform, have a mathematical formula involving a k-periodic CF (Olds, 1963). In summary, as the SICF converges to a 2nd degree algebraic number we are guaranteed that this same number will have a simple interlaced CF expansion (up to some Mobius transform).

Note, these CFs with constant sequences converge at an exponential rate (Raayoni et al., 2021). Conjectures on 2nd degree algebraic numbers can be trivially calculated (Balkova & Hrušková, 2013) and thus further analysis of periodic CFs does not provide additional insight on the algorithm.

SICFs with non-constant partial denominator sequences (Proof of Theorem 2)

Given an SICF which converges to $\alpha \in \mathbb{R}\backslash\mathbb{Q}$,

$$\alpha = \left[(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_\beta, b_\beta), \ldots\right], b_j \in \{\pm 1\},$$

We prove there exists some Mobius transform of α which equals to a simple interlaced CF. We assume $\exists i \in \{1, \beta\}$ s.t $\deg(A_i) > 0, A_i(n) = a_{(n-1)\times\beta+i}$, meaning we have non-constant sequences in the partial denominator. We address base cases and prove two lemmas to prove the above theorem in this case. We utilize the following identities to aid our proof.

Identities for proof

For a given SICF, we may find a negative partial numerator in the CF, e.g., $b_{j+1} = -1$. Our goal is to find an equivalent CF expansion with all partial numerators all equal to 1. To this end we utilize the following matrix identity:

$$(*)\ \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & a_{j+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_j - 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_{j+1} - 1 \end{pmatrix} =$$
$$\begin{pmatrix} 0 & 1 \\ 1 & a_j - 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_{j+1} - 1 \end{pmatrix}$$

We thus found a method to "get rid" of negative partial numerators but we pay for it by decreasing the denominators. In a case where all interlaced sequences only have values greater than 2 then we can simply apply (*) iteratively over all the SICF to obtain a simple interlaced CF. Therefore, we must address cases were this is not met. We introduce identities which will aid us in our proof. Most of the identities are simply a result of matrix multiplication and decomposition, with identity 1 generalizing the decomposition in (*). For example, the derivation of identity 3 is given by:

$$\begin{pmatrix} 0 & b_{j-1} \\ 1 & a_{j-1} \end{pmatrix}\begin{pmatrix} 0 & b_j \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_{j+1} \\ 1 & a_{j+1} \end{pmatrix} = \begin{pmatrix} 0 & b_{j-1}b_{j+1} \\ b_j & b_j a_{j+1} + b_{j+1}a_{j-1} \end{pmatrix} = \text{ matrix is a Mobius transform,} \quad b_j^2 = 1$$
$$\begin{pmatrix} 0 & b_{j-1}b_jb_{j+1} \\ 1 & a_{j+1} + b_jb_{j+1}a_{j-1} \end{pmatrix}.$$

Identity 1

$$\alpha = [\ldots, (a_{j-1}, b_{j-1}), (a_j, -1), \ldots] = [\ldots, (a_{j-1} - 1, b_{j-1}), (1, 1), (a_j - 1, 1), \ldots].$$

Identity 2

$$\alpha = [\ldots, (a_{j-1}, b_{j-1}), (0, b_j), (a_{j+1}, b_{j+1}), \ldots] = [\ldots, (a_{j+1} + b_{j+1} b_j a_{j-1}, b_{j-1} b_j b_{j+1}), \ldots].$$

Identity 3 - Equivalence Theorem

For any non-zero infinite sequence, g_i, the following equality is met (see note Appendix B.1)

$$\alpha = [\ldots, (a_{j-1}, b_{j-1}), (a_j, b_j), (a_{j+1}, b_{j+1}), \ldots] \rightarrow$$
$$\alpha = [\ldots, (g_{j-1} a_{j-1}, g_{j-2} g_{j-1} b_{j-1}), (g_j a_j, g_{j-1} g_j b_j), (g_{j+1} a_{j+1}, g_j g_{j+1} b_{j+1}), \ldots]$$

When applying identity 1 and 2 we assume it is applied in every period of the SICF, we will prove this does not affect convergence in the proof of lemma 1 . We further note that if we "shift" our period in an SICF, it is simply equivalent to applying a Mobius transform on our constant α. To shift our period is to look at the period in a different order, for example:

$$\alpha = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 0 & b_1 \\ 1 & A_1(n) \end{pmatrix} \cdots \begin{pmatrix} 0 & b_\beta \\ 1 & A_\beta(n) \end{pmatrix} \right] (0) \rightarrow$$
$$\left(\begin{pmatrix} 0 & b_1 \\ 1 & A_1(n+1) \end{pmatrix} \right)^{-1} (\alpha) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 0 & b_2 \\ 1 & A_2(n) \end{pmatrix} \cdots \begin{pmatrix} 0 & b_\beta \\ 1 & A_\beta(n) \end{pmatrix} \begin{pmatrix} 0 & b_1 \\ 1 & A_1(n+1) \end{pmatrix} \right] (0).$$

Proof

We prove inductively that any SICF can be represented as a simple interlaced CF up to a Mobius transform. This will follow immediately if we can prove the next lemma.

1. If all a_j's are positive, $a_j > 0$, we can decrease the number of $\boldsymbol{b_j}$'s $= \boldsymbol{-1}$ in each period at least by $\boldsymbol{1}$.

To prove the above lemma, we need to prove an additional lemma which addresses edge cases met in the first lemma:

2. Given an SICF for which there is a single $a_k = 0$ in each period ($a_k = a_{(n-1)\times\beta+k} = 0 \ \forall n \in \mathbb{N}\backslash 0$), satisfying $a_k = 0, \quad b_k = \pm 1$, while all other a_j's are positive: then there is an equivalent representation of the CF with all a_j's being positive without increasing the number of $\boldsymbol{b}_j$'s $= -1$ in each period.

Proof of Lemma 2

Given a SICF of period β,

$$\alpha = [(a_1, b_1), (a_2, b_2), \ldots, (a_\beta, b_\beta), \ldots], b_j \in \{\pm 1\},$$

for which there exists a $k \in \{1, \ldots, \beta\}$ such that $a_{(n-1)\times\beta+k} = 0 \quad \forall n \in \mathbb{N}\backslash 0$ we find an equivalent representation of the CF with all a_j's> 0 without increasing the number of b_j's$= -1$ in each period.

The proof of lemma 2 is described in the following flow chart:

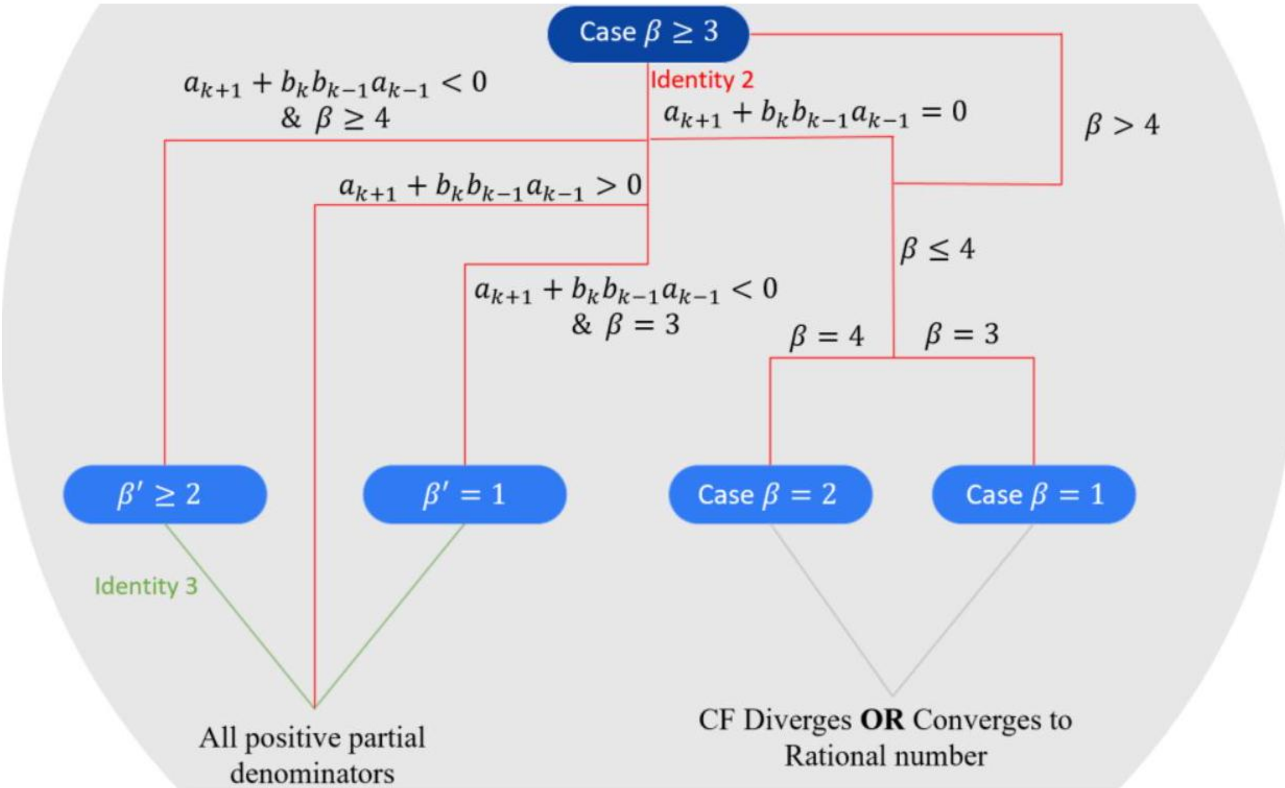

Figure 7. Flow chart proof of lemma 2: Given a converging SICF with a single $a_k = a_{(n-1)\times\beta+k} = 0 \forall n \in \mathbb{N} \backslash 0$ in each period, we show that there exists an equivalent representation (using identity 2 and 3) with all positive partial denominators. Recall, applying identity 2 decreases the period of the SICF by 2, therefore we are bound to either find a representation with all positive partial denominators or prove by contradiction that the SICF diverges or converges to a rational number.

We begin by assessing different periods.

If $\boldsymbol{\beta = 1}$:

If $\boldsymbol{\beta = 1}$, we obtain a diverging CF. We have a single layer in each period, meaning every layer has a 0 in its partial denominator and we obtain a CF which satisfies:

$$\alpha = [(0, b_1), (0, b_1), \ldots, (0, b_1), \ldots] = \frac{b_1}{0 + \frac{b_1}{0 + \cdots}}.$$

As $b_1 = \pm 1$, at each N^{th} approximation of the CF we obtain either ∞ or 0 and therefore the SICF diverges and is not relevant to our analysis. If $\boldsymbol{\beta = 2}$:

We obtain the following CF, which converges to 0,

$$\alpha = [(A_1(1), b_1), (0, b_2), \ldots] = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 0 & b_1 \\ 1 & A_1(n) \end{pmatrix} \begin{pmatrix} 0 & b_2 \\ 1 & 0 \end{pmatrix} \right] (0) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} b_1 & 0 \\ A_1(n) & b_2 \end{pmatrix} \right] (0),$$

Since $\begin{pmatrix} b_1 & 0 \\ A_1(n) & b_2 \end{pmatrix} (0) = 0$, if the limit exists it must be 0 which is rational and of no interest to us.

If $\boldsymbol{\beta \geq 3}$:

As the period is ≥ 3, we can "shift" the period such that the $a_k = 0$ element is between two matrices within the period and apply identity 2 ,

$$\alpha = [\ldots, (a_{k-1}, b_{k-1}), (0, b_k), (a_{k+1}, b_{k+1}), \ldots] =^{\text{identity 2}} [\ldots, (a_{k+1} + b_k b_{k+1} a_{k-1}, b_{k-1} b_k b_{k+1}), \ldots]$$

we see our period is reduced from β to a new period $\beta' = \beta - 2$.

If $\boldsymbol{a_{k+1} + b_k b_{k+1} a_{k-1} > 0}$, we obtain a layer with a positive partial denominator and the total number of partial numerators equal to -1 either remains the same or decreases depending on the sign of $b_{k-1} b_k b_{k+1}$, satisfying lemma 2.

If $\boldsymbol{a_{k+1} + b_k b_{k+1} a_{k-1} = 0}$, then depending on our period we treat it by induction:
If $\boldsymbol{\beta = 3}$, we obtain the case of $\beta = 1$ and therefore a CF which diverges.
If $\boldsymbol{\beta = 4}$, we obtain the case of $\beta = 2$ and therefore a CF which converges to 0.

If $\boldsymbol{a_{k+1} + b_k b_{k+1} a_{k-1} < 0}$ and $\boldsymbol{\beta = 3}$, we obtain a SICF of period 1 (as the period is reduced to $\beta - 2$) with a negative partial denominator. This is referred to as $\beta' = 1$ in the flowchart. We can apply the equivalence theorem (identity 3 with $g_j = -1 \quad \forall j \in \mathbb{N}$) to ensure that we obtain a positive partial numerator:

$$\alpha = [\ldots, (a_{k-1} - a_{k+1}, -b_{k-1}), \ldots] = [\ldots, (g_k (a_{k-1} - a_{k+1}), -g_k g_{k-1} b_{k-1}), \ldots] = [\ldots, (-(a_{k-1} - a_{j+1}), -b_{k-1}), \ldots].$$

Again, we do not increase the number of b_j's$= -1$ as $g_k g_{k-1} = 1$.

If $\boldsymbol{\beta > 4}$, we obtain the case of $\beta \geq 3$ and treat it accordingly with the new period given by $\beta - 2$.

If $\boldsymbol{a_{k+1} + b_k b_{k+1} a_{k-1} < 0}$ and $\boldsymbol{\beta \geq 4}$ then we have another layer within the period and we could use it to "absorb" the negative partial denominator, ensuring positive partial denominators. This is referred to as $\beta' \geq 2$ in the flow chart. This is done using the equivalence theorem (identity 3) with $g_{(n-1)\times\beta+k} = -1$
$\forall n \in \{1, 2, 3 \ldots\}$ and $g_j = 1 \quad \forall j \neq (n-1) \times \beta + k$.

$$\alpha = [\ldots, (a_{k-1} - a_{k+1}, -b_{k-1}), (a_{k+2}, b_{k+2}), \ldots] = [\ldots, (g_k (a_{k-1} - a_{k+1}), -g_k b_{k-1}), (a_{k+2}, g_k b_{k+2}), \ldots],$$
$$\alpha = [\ldots, (-(a_{k-1} - a_{k+1}), b_{k-1}), (a_{k+2}, -b_{k+2}), \ldots].$$

We do not increase the number the number of b_j's $= -1$ in each period. If $b_{k+2} = -1$ we decrease the total amount of negative b_j's, and if $b_{k+2} = 1$ we remain with the same amount of negative b_j's as its given that $b_k b_{k+1} = -1$ and that was canceled out when we applied identity 3 .

Thus, we ensured that all partial denominators are positive without increasing the number of b_j's $= -1$ in the period.

Furthermore, all identities were applied within a given period up to a shift which is equivalent to a Mobius transform, therefore the resultant CF with positive a_j's will have an interlaced pattern.

Proof of Lemma 1

We show that, given a SICF that converges to an irrational number, if all a_j are positive, we can always decrease the number of b_j's$= -1$ in each period at least by 1 . We first show, as base cases, that any SICF of period $\beta = 1$ or $\beta = 2$ can be represented as a simple interlaced CF up to some Mobius transform.

Base cases

The first base case is a SICF of period 1, which must be addressed as identity 1 requires at least two layers within a given period to be applied. The second base case is a SICF of period 2.

Base case: SICF of period $\beta = 1$

For any SICF of period 1, we shift the CF to some depth for which $A_1(n) > 2$ (for simplicity we keep $\mathrm{n} = 1$ as starting index regardless of shift):

$$\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} 0 & -1 \\ 1 & A_1(n) \end{pmatrix}\right](0) = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n)-1 \end{pmatrix}\right](0) =$$
$$\lim_{N\to\infty}\left[\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n)-1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\right](0) =$$
$$\lim_{N\to\infty}\left[\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n)-2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right](0).$$

We obtain a integer Mobius transform of a simple interlaced CF; Note, that we obtain a sub-sequence of the CF and as simple CFs always converge we obtain a simple interlaced expansion which converges to α.

Base case: SICF of period $\beta = 2$

There are several possibilities of SICFs of period 2, we solve the possibility which seems most complex. The other cases are solved in a similar manner and often more simply. for example,

$$\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right](0) \overset{\text{identity 1}}{=}$$
$$\lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n)-1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right](0) \overset{\text{shift, identity 2}}{=}$$
$$\lim_{N\to\infty}\left[\begin{pmatrix} 0 & 1 \\ 1 & A_1(1)-1 \end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ 1 & A_1(n+1) \end{pmatrix}\right](0).$$

A similar application of identity 1, a shift, then identity 2 can be easily applied in the case that $A_1(n)$'s layer has a negative partial numerator. We therefore analyze the more complicated case where all b_j's$= -1$:

$$\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} 0 & -1 \\ 1 & A_1(n) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right](0)$$

We notice that $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and therefore:

$$\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix} 0 & -1 \\ 1 & A_1(n) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right](0),$$
$$\alpha = \lim_{N\to\infty}\left[\begin{pmatrix} 0 & -1 \\ 1 & A_1(1) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & A_1(n+1) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right](0),$$
$$\alpha = \lim_{N\to\infty}\left[\begin{pmatrix} 0 & -1 \\ 1 & A_1(1) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix} 0 & -1 \\ 1 & A_1(n+1)-2 \end{pmatrix}\right](0).$$

We obtain a 1-periodic SICF which can be represented as a simple interlaced CF (as shown in the base case of $\beta = 1$).

General case: SICF of period $\beta \geq 3$

Given a SICF,

$$\alpha = [(a_1, b_1), (a_2, b_2), \ldots, (a_\beta, b_\beta), \ldots], \quad b_j \in \{\pm 1\}, a_j > 0$$

There are 2 cases that must be addressed:

1. $b_j = -1 \; \forall j \in \mathbb{N}$. Recall, we are discussing a SICF with non-constant partial denominator sequences, therefore $\exists i \in \{1, \beta\}$ s.t deg $(A_i) > 0$. Therefore, we can ensure $a_j > 1$ as $A_i(n) = a_{(n-1)\times\beta+k}$. As the period ≥ 3 we can "shift" the period such that this a_j is adjacent to another matrix from the left within the period. This enables us to apply identity 1 while guaranteeing that $a_j - 1 > 0$, This decreases the number of b_j's $= -1$ in each period by 1 .

$$\begin{pmatrix} 0 & b_{j-1} \\ 1 & a_{j-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_j \end{pmatrix} = \begin{pmatrix} 0 & b_{j-1} \\ 1 & a_{j-1} - 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j - 1 \end{pmatrix} = \begin{pmatrix} 0 & b_{j-1} \\ 1 & a_{j-1} - 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j - 1 \end{pmatrix}$$

We have therefore decreased the number of b_j's$= -1$ and treat potential zeros $(a_{j-1} - 1 = 0)$ as outlined in lemma 2.

We can then apply the iterative decomposition as in case 2 seen below, as we guarantee that at least one partial numerator satisfies $b_j = 1$.

2. $\exists k \in \mathbb{N} \mid b_{k-1} = 1$ & $b_k = -1$. This case can occur in any SICF that has at least a single $b_j = 1$ in each period, as then we can shift the period to match the conditions order (first positive than negative layer) and decompose the CF using identity 1, thus decreasing the total number of b_j's $= -1$ in each period.

$$\alpha = [\ldots, (a_{k-1}, 1), (a_k, -1), \ldots] \to \alpha = [\ldots, (a_{k-1} - 1, 1), (1, 1), (a_k - 1, 1), \ldots].$$

Note: this is done in every period of the SICF

We treat potential zeros on the left layer and right layer separately, enabling us to satisfy the assumption in Lemma 2: that there is only one zero partial denominator element in each period.

If $\boldsymbol{a_{k-1} - 1 = 0}$, we can simply add 1 to the previous layer, (guaranteed to have one as the period ≥ 3)

$$\alpha = [\ldots, (a_{k-2} + 1, b_{k-2}), (a_k - 1, 1), \ldots, (a_\beta, b_\beta), \ldots]$$

If $\forall \boldsymbol{n} \in \mathbb{N} \backslash \boldsymbol{0}, \boldsymbol{k} = \boldsymbol{2} + (n-1) \times \beta$ (meaning a_{k-1} is the first layer of each period), we can shift the period and add 1 to the final layer.

$$\alpha = [\ldots, (a_k - 1, 1), \ldots, (a_\beta + 1, b_\beta), \ldots]$$

If $\boldsymbol{a_k - 1 = 0}$, we utilize lemma 2 to return to a state in which all partial denominators are positive.

If we do not get any zero in the partial denominators, we increased the SICF's period from β to $\beta + 1$ and decreased the number of b_j's$= -1$, proving lemma 1.

Note, to transform the SICF to a simple interlaced CF we iteratively apply identity 1 from left to right in the period to every $b_j = -1$ we encounter. The iterative process continues until we either obtain a simple interlaced CF (reducing the number of b_j's$= -1$ at each iteration until all b_j's$= 1$) or obtain a 0 that will be dealt with as formalized in Lemma 2.

Proof of convergence

Throughout the above proof of theorem 2 we utilize 3 identities of matrix multiplication which might change the limit of the CF. Identities 2 and 3 do not change the limit as identity 2 simply collapses a sub-sequence of the CF and identity 3 does not affect convergence. Therefore, to prove the limit is unchanged we prove identity 1 does not affect the convergence of the SICF. The SICF converges therefore,

$$\alpha = [\ldots, (a_{j-1}-1, 1), (1,1), (a_j - 1, 1), \ldots].$$

We define $N(n) = j + \beta n \quad \forall n \in \mathbb{N}$ (as $N \to \infty$ means $n \to \infty$). The SICF satisfies:

$$\alpha = \lim_{N\to\infty} \frac{p_N}{q_N} = \lim_{N\to\infty} \begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_N \end{pmatrix} (0) =$$
$$\lim_{N\to\infty} \begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{N-1}-1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_N - 1 \end{pmatrix} (0).$$

We must assess whether this decomposition affects the convergence of the CF. For most subsequences we can trivially see there is no effect on the convergence, $\begin{pmatrix} 0 & 1 \\ 1 & a_{N-1}-1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (0) = \begin{pmatrix} 0 & 1 \\ 1 & a_{N-1} \end{pmatrix}$. We therefore assess a non-trivial case:

$$\begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{N-1}-1 \end{pmatrix} (0) = \begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{N-1} \end{pmatrix} (-1)$$
$$= \begin{pmatrix} p_{N-1} & p_N \\ q_{N-1} & q_N \end{pmatrix} (-1) = \frac{p_N - p_{N-1}}{q_N - q_{N-1}}$$

To prove that the above expression converges we must show that:

$$\lim_{N\to\infty} \frac{p_N - p_{N-1}}{q_N - q_{N-1}} = \lim_{N\to\infty} \frac{p_N}{q_N} = \alpha \to \lim_{N\to\infty} \frac{p_N - p_{N-1}}{q_N - q_{N-1}} - \frac{p_N}{q_N} = 0.$$

We notice that the above expression satisfies:

$$\frac{p_N - p_{N-1}}{q_N - q_{N-1}} - \frac{p_N}{q_N} = \frac{q_N (p_N - p_{N-1}) - p_N (q_N - q_{N-1})}{q_N (q_N - q_{N-1})} = \det\left(\begin{pmatrix} p_{N-1} & p_N \\ q_{N-1} & q_N \end{pmatrix}\right) = \pm 1 \frac{\pm 1}{q_N (q_N - q_{N-1}}\Bigg).$$

If the SICF converges to an irrational number, it guarantees that: $|q_N| \to_{N\to\infty} \infty$.

As if we assume that q_N is bounded, meaning $|q_N| < B \quad B \in \mathbb{Z}$, then p_N must be bounded and then we would have a sub-sequence that converges to a rational number. However, we might encounter problems for cases which $q_N = q_{N-1}$ for an infinite amount of N 's. Since, $\det\left(\begin{pmatrix} p_{N-1} & p_N \\ q_{N-1} & q_N \end{pmatrix}\right) = \pm 1$, if $q_N = q_{N-1}$ then they must be ± 1 for infinity many N 's which contradicts $|q_N| \to_{N\to\infty} \infty$. Therefore, it is not possible for $q_N = q_{N-1}$ for an infinite amount of N's and we obtain:

$$\lim_{k\to\infty} |q_N (q_N - q_{N-1})| = \infty \to \lim_{k\to\infty} \frac{\pm 1}{q_N (q_N - q_{N-1})} = 0 \to \lim_{N\to\infty} \frac{p_N - p_{N-1}}{q_N - q_{N-1}} = \lim_{N\to\infty} \frac{p_N}{q_N} = \alpha \tag{32}$$

And the resultant CF converges to the same limit.

Overall, lemma 1 guarantees that given a SICF (all a_j are positive) it can always reduce the number of b_j's$= -1$ in each period yet it may result in edge cases of 0 in the partial denominator.Lemma 2 guarantees if all a_j in each period are positive but one, $\exists k \in \mathbb{N} \mid a_k = a_{k+(n-1)\times\beta} = 0$, there is an equivalent representation with all a_j being positive without increasing the number of b_j's $= -1$ in each period. Therefore, the lemma's show that we can ensure that the number of b_j's $= -1$ is monotonically decreasing thus ensuring that for a converging SICF we can find a representation such that all b_j's$= 1$ while all a_j's remain positive. Therefore, by proving both the lemma's we prove that every constant equal to a SICF has a Mobius transform equal to a simple CF.

C.2. Example of moving from an SICF to a Simple Interlaced CF

We take the following conjecture of $\frac{2}{\tan(1)}$ found by the ESMA algorithm,

$$\frac{2}{\tan(1)} - 2 = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix}0 & -1\\ 1 & 3n-1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 3n\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 2+12n\end{pmatrix}\right](0). \tag{33}$$

We attempt to transform it to a simple interlaced CF, using the method from the constructive proof described in Appendix C.1

$$\frac{2}{\tan(1)} - 2 = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-2\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 3n\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1+12n\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 3n-3\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 3n\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1+12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 3n-3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 3n\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1+12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 3n-3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1+12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 3n-3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 3n-3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right](0),$$

$$= \lim_{N\to\infty}\left[\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n\end{pmatrix}\right](0),$$

$$-\frac{1}{\frac{2}{\tan(1)}-2} - 1 = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n-2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 12n\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 3n\end{pmatrix}\right](0).$$

D. Propeties of Simple Interlaced CFs

D.1. Proof of Theorem 3 - Predicting Degrees of Polynomial CFs semi-equivalent to Simple Interlaced CFs

Theorem 3. Given a simple interlaced continued fraction satisfying $B_i(n) = 1 \quad \& \quad A_i(n) > 0$, where $A_i,\ B_i \in \mathbb{Z}[\mathrm{x}], i \in \{1,\ldots,\beta\}, \forall n \in \mathbb{N}$ of period β, where $\exists i \in \{1,\ldots,\beta\}$ s. t $\deg(A_i) > 0$, its semi-equivalent polynomial continued fraction's partial numerator b'_n and partial denominator a'_n
$\forall n \in \mathbb{N}$ satisfy:

$$\deg(b') = \sum_{i=1}^{\beta-1} 2\times\deg(A_i), \quad \deg(a') = \left[\sum_{i=1}^{\beta-1} 2\times\deg(A_i)\right] + \deg(A_\beta).$$

Complementary theorem

As defined previously, the form of a simple interlaced CF, α, with a period of β is given by:

$$\alpha = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\prod_{i=1}^{\beta}\begin{pmatrix}0 & B_i(n)\\ 1 & A_i(n)\end{pmatrix}\right](0) = \lim_{N\to\infty}\left[\prod_{n=1}^{N} M_n\right](0) = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\begin{pmatrix}c_n & d_n\\ e_n & f_n\end{pmatrix}\right](0),$$
$$A_i \in \mathbb{Z}[x], \quad A_i(n) > 0, \forall n \in \mathbb{N}, i \in \{1,\ldots,\beta\}.$$

We prove a complementary theorem which will enable the proof of Theorem 3.

Complementary Theorem. The collapsed matrix sequences, c_n, d_n, e_n, f_n, of a simple interlaced CF's of period $\beta \geq 2$ collapsed matrix $M_n = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}$, have the following polynomial degree with n :

$$(***)\begin{cases} \deg(c) = \begin{cases} \sum_{i=2}^{\beta-1} \deg(A_i) & \text{if } \beta > 2 \\ 0 \end{cases} \\ \deg(d) = \sum_{i=2}^{\beta} \deg(A_i) \\ \deg(e) = \sum_{i=1}^{\beta-1} \deg(A_i) \\ \deg(f) = \sum_{i=1}^{\beta} \deg(A_i) \end{cases} \tag{34}$$

Proof: We prove the complementary theorem using induction.

Base case of induction $\beta = 2$:

For $\beta = 2$, we obtain the following collapsed matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & A_1(n) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & A_2(n) \end{pmatrix} = \begin{pmatrix} 1 & A_2(n) \\ A_1(n) & A_1(n)A_2(n)+1 \end{pmatrix} = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}.$$

Meaning the induction hypothesis is met.

We assume the induction hypothesis is met for all simple interlaced CFs of period $\leq \beta - 1$ and prove that it is met for all simple interlaced CF of period β. We consider the effect of multiplying some product of the first $\beta - 1$ layer to the β^{th} layer of the n^{th} collapsed matrix:

$$\begin{pmatrix} C_n & D_n \\ E_n & F_n \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & A_\beta(n) \end{pmatrix} = \begin{pmatrix} D_n & C_n + D_nA_\beta(n) \\ F_n & E_n + F_nA_\beta(n) \end{pmatrix} = \begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}$$

By our assumption, the product of the first $\beta - 1$ layers have the following polynomial degrees:

$$\deg(C) = \sum_{i=2}^{\beta-2} \deg(A_i), \deg(D) = \sum_{i=2}^{\beta-1} \deg(A_i), \deg(E) = \sum_{i=1}^{\beta-2} \deg(A_i), \deg(F) = \sum_{i=1}^{\beta-1} \deg(A_i).$$

Notice that as the elements of the collapsed matrix are a product and linear combination of positive polynomial sequences the elements are also positive polynomial sequences: $c, d, e, f \in \mathbb{Z}[x], c_n \geq 0, d_n, e_n, f_n > 0$ and therefore there can be no cancelation of polynomials. The resultant collapsed matrix satisfies:

$$\deg(c) = \sum_{i=2}^{\beta-1} \deg(A_i), \deg(d) = \sum_{i=2}^{\beta} \deg(A_i), \deg(e) = \sum_{i=1}^{\beta-1} \deg(A_i), \text{ and } \deg(f) = \sum_{i=1}^{\beta} \deg(A_i)$$

Proving our complementary theorem.

Proof of Theorem 3 (and lemma 3)

We utilize the complementary theorem to analyze the resultant polynomial CFs obtained when applying the Folding transform on the n^{th} collapsed matrix. Given a collapsed matrix $(n > 1)$ for some interlaced CF, the Folding transform on the n^{th} collapsed matrix is given by:

$$\begin{pmatrix} e_n & -c_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_n & d_n \\ e_n & f_n \end{pmatrix}\begin{pmatrix} 1 & c_{n+1} \\ 0 & e_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & e_{n+1}(e_nd_n - c_nf_n \\ e_n & e_nc_{n+1} + f_ne_{n+1} \end{pmatrix} = \text{ equivalence transform}$$
$$\begin{pmatrix} 0 & e_{n-1}e_{n+1}(e_nd_n - c_nf_n) \\ 1 & (e_nc_{n+1} + f_ne_{n+1}) \end{pmatrix}$$

Utilizing the determinant property (see Appendix B.4):

$$\left|\left(\begin{array}{cc} c_n & d_n \\ e_n & f_n \end{array}\right)\right| = (-1)^{\beta} \prod_{i=1}^{\beta} B_i.$$

In this case $B_i = 1$
$\forall i \in \mathbb{N}$, we therefore obtain:

$$-\left|\left(\begin{array}{cc} c_n & d_n \\ e_n & f_n \end{array}\right)\right| = e_n d_n - c_n f_n = (-1)^{\beta+1} = \begin{cases} 1 & \beta(\text{mod})2 = 1 \\ -1 & \beta(\text{mod})2 = 0 \end{cases},$$

And therefore, we obtain:

$$\left(\begin{array}{cc} 0 & e_{n-1}e_{n+1}\left(e_n d_n - c_n f_n\right) \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{array}\right) = \left(\begin{array}{cc} 0 & e_{n-1}e_{n+1}(-1)^{\beta+1} \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{array}\right)$$

As $c_n \geq 0, d_n, e_n, f_n > 0 \; \forall n > 1$ we can deduce that $\left(e_n c_{n+1} + f_n e_{n+1}\right) > 0$ thus proving lemma 3. Therefore, the resultant polynomial degree is the sum of polynomial degrees of the polynomial products (recalling equation (24) in Appendix B.2):

$$\left(M_1 U_2\right)^{-1}(\alpha) = \lim_{N \to \infty} \left[\prod_{n=2}^{N} \left(\begin{array}{cc} 0 & e_{n-1}e_{n+1}(-1)^{\beta+1} \\ 1 & \left(e_n c_{n+1} + f_n e_{n+1}\right) \end{array}\right)\right](0).$$

If we denote b'_n and a'_n as the partial numerator and denominator of α', respectively,

$$\deg\left(b'\right) = 2\deg(e) = \sum_{i=1}^{\beta-1} 2 * \deg\left(A_i\right), \deg\left(a'\right) = \deg(e) + \deg(f) = \sum_{i=1}^{\beta-1} 2 * \deg\left(A_i\right) + \deg\left(A_\beta\right).$$

Note, $\deg(e) + \deg(f) \geq \deg(e) + \deg(c)$ as $\deg(f) = \deg(c) + \deg\left(A_1\right) + \deg\left(A_\beta\right)$ by $(***)$.

Thus, proving Theorem 3.

D.2. Proof of Corollary 2: A Polynomial CF semi-equivalent to a simple Interlaced Continued Fraction converges super-exponentially

Corollary 2. A polynomial continued fraction semi-equivalent to a simple interlaced continued fraction, satisfying $\exists i \in \{1, \ldots, \beta\}$ s.t $\deg\left(A_i\right) > 0$, converges super-exponentially.

Proof:

In (Raayoni et al., 2021) the following condition for super-exponential convergence of a polynomial CF was proven: Given a polynomial CF with partial numerator b'_n and partial denominator a'_n the polynomial CF converges superexponentially iff $\frac{\deg\left(b'\right)}{\deg\left(a'\right)} < 2$. In the case of a simple interlaced CF, by Theorem 3 its semi-equivalent polynomial CF satisfies:

$$\frac{\deg\left(b'\right)}{\deg\left(a'\right)} = \frac{\sum_{i=1}^{\beta-1} 2 \times \deg\left(A_i\right)}{\sum_{i=1}^{\beta-1} 2 \times \deg\left(A_i\right) + \deg\left(A_\beta\right)} \leq \frac{\sum_{i=1}^{\beta-1} 2 \times \deg\left(A_i\right)}{\sum_{i=1}^{\beta-1} 2 \times \deg\left(A_i\right)} = 1 < 2.$$

Therefore, the polynomial CF converges at a super-exponential rate.

D.3. Proof of Corollary 3: Irrationality of a Simple Interlaced Continued Fraction

Corollary 3. Any number $\alpha \in \mathbb{R}$ equal to a simple interlaced continued fraction converges to an irrational limit.

Proof:

Utilizing Tietze's Criterion it can be directly proven that any simple interlaced CF is irrational. We denote α as the constant equal to the simple interlaced CF with partial numerator b_n and partial denominator a_n, and α' as its semi-equivalent constant equal to the polynomial CF. Recall Tietze's Criterion (as outlined in the second page of (Bowman & Laughlin, 2002)):

Let $\{b_j\}_{j=1}^{\infty}$ be a sequence of integers and $\{a_j\}_{j=1}^{\infty}$ a sequence of positive integers, if there exists a positive integer N_0 such that:

$$\left\{ \begin{array}{c} a_j \geq |b_j| \\ a_j \geq |b_j| + 1 \text{ for } b_{j+1} < 0 \end{array} \right. \forall j \geq N_0. \tag{35}$$

Then $\alpha' = \frac{b_1}{a_1 + \frac{b_2}{a_2 + \cdots}}$ converges and its limit is irrational.

By definition, a simple interlaced CF satisfies $a_j \geq |b_j| = 1$ (by lemma 3) and $b_j > 0 \forall j$ and therefore a simple Interlaced CF converges to an irrational limit.

E. Interesting Continued Fractions found with the Folding Transform

Representation of the Golden Ratio as a Balanced Polynomial CF

Looking at the following SICF, where $k, q, r \in \mathbb{Z}$:

$$\alpha = \lim_{N\to\infty} \left[\prod_{n=1}^{\text{N}} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & kn+q \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & kn+r \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right] (0) =$$

$$\lim_{N\to\infty} \left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k+q \end{pmatrix} \prod_{n=1}^{N} \begin{pmatrix} 0 & 1 \\ 1 & 1+k+q-r \end{pmatrix} \right] (0).$$

This converges for $(1+k+q-r)^2 > -4$ (Raayoni et al., 2021), we constrain the above expression to obtain the CF of Golden Ratio $r = 1k = 1q = 0$:

$$\varphi - 1 = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right] (0)$$

Meaning that looking at the original SICF:

$$-\frac{1}{\varphi} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} (\varphi - 1) = \lim_{N\to\infty} \left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \prod_{n=1}^{N} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right] (0),$$

However, we can also apply the Folding transform to the SICF before collapsing it to a 1-periodic CF, $r = 1, k = 1, q = 0$ so we obtain:

$$-\frac{1}{\varphi} = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & n+1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right] (0).$$

We can collapse the above layers and apply the Folding transform to obtain the following polynomial CF:

$$\begin{pmatrix} 21 & 15 \\ 3 & 1 \end{pmatrix} \left(-\frac{1}{\varphi} \right) = \lim_{N\to\infty} \left[\prod_{n=2}^{N} \begin{pmatrix} 0 & n^4 + 4n^3 + 2n^2 - 4n - 3 \\ 1 & -n^2 - 3n - 3 \end{pmatrix} \right] (0).$$

As a result, we found a balanced polynomial CF representation for the Golden Ratio:

$$\frac{21\left(-\frac{1}{\varphi}\right)+15}{3\left(-\frac{1}{\varphi}\right)+1} = \frac{-21+15\varphi}{-3+\varphi} = \lim_{N\to\infty}\left[\prod_{n=2}^{N}\begin{pmatrix} 0 & n^4+4n^3+2n^2-4n-3 \\ 1 & -n^2-3n-3 \end{pmatrix}\right](0).$$

By rationalizing the denominator, we obtain a simpler expression:

$$\frac{-21+15\varphi}{-3+\varphi} = \frac{60-48\sqrt{5}}{-30} = \frac{-30+24\sqrt{5}}{15} = \frac{-54+\frac{48(1+\sqrt{5})}{2}}{15} = \frac{-54+48\varphi}{15}$$
$$\frac{-54+48\varphi}{15} = \lim_{N\to\infty}\left[\prod_{n=2}^{N}\begin{pmatrix} 0 & n^4+4n^3+2n^2-4n-3 \\ 1 & -n^2-3n-3 \end{pmatrix}\right](0). \tag{36}$$