Sampling and Frame Expansions for UWB Signals

Peter Balazs
Acoustics Research Institute (ARI),
Austrian Academy of Sciences, Vienna, Austria
Email: *peter.balazs@oeaw.ac.at*

Stephen D. Casey
Department of Mathematics and Statistics,
American University, Washington, DC 20016 USA
Email: *scasey@american.edu*

***Abstract*—This article outlines sampling and frame expansions designed specifically for ultra-wideband (UWB) signals. We present two different types of expansions: ON windows with modified Gegenbauers and generalized frame expansions. They represent a contrast in the computability and flexibility of frames versus the rigid structure and precision in the sampling expansion. Both approaches rely on the multiplication of some super-windowing functions and given orthonormal sequences. We generalize this approach to the redundant case, investigating windowed families of frame sequences.**

I. Introduction

Ultra-wideband (UWB) signal processing is a technology with many features that promise potential advances in wireless communications, networking, radar, imaging, and positioning systems. A UWB communication system is a large bandwidth system based on the transmission of very short pulses with relatively low energy. UWB technology has many potential advantages, such as high data rate, low probability of interception and detection, system simplicity, low cost, reduced average power consumption, and immunity to interference.

This article outlines UWB sampling and frame expansions. The two different types of expansions are a contrast in computability and flexibility (in the frame expansions) versus the rigid structure and precision in the sampling expansion (which are given by expansion in a particular ON basis system). The paper is structured as follows. We first develop the sampling expansions, and then develop similar expansions in the frame setting.

The sampling expansions first window the signal in the analog domain, and use the Malvar-Coifman-Meyer basis folding system to construct the system with ON basis elements tailored to the signal - following the seminal works of Malvar [17] and Coifman and Meyer [9]. We refer to this as the *Projection Method*. We create a basis system, by decomposing the signal into a basis via windowing and a continuous-time inner product operation, preserving orthogonality between adjacent blocks. This allows the computation of the basis coefficients in parallel. Moreover, the windows can have variable lengths, roll-offs and smoothness. We use a modified Gegenbauer system designed specifically for UWB signals. This system minimizes the Gibbs phenomenon, giving the point values of a piecewise smooth signal with essentially the same accuracy as a smooth approximation, making it the ideal system to use for basis folding as applied to these signals. We then extend this folded basis system by developing frame systems for UWB signals. We establish the frame operators and frame bounds to give a frame extension of the folded basis system. The frame extension allows for greater flexibility and computability in implementation in comparison to the UWB sampling systems.

Mathematical definitions and computations for the paper follow those given in Benedetto [2].

II. ON Window Systems

We consider a system of 'super windows', separating a function into parts $f_k = \mathbb{W}_k \cdot f$, designed to maintain global orthogonality. For that let $[g]^\circ(t)$ denote the the W-*periodization* of g, i.e. $[g]^\circ(t) = \sum_{n=-\infty}^{\infty} g(t-nW)\, a.e.$

Definition 1 (ON Window System): An **ON Window System** is a set of functions $\{\mathbb{W}_k(t)\}$ such that for all $k \in \mathbb{Z}$,

$$\begin{aligned} (i.) &\quad \mathrm{supp}(\mathbb{W}_k(t)) \subseteq [(k-1/2)\,T - r, (k+1/2)\,T + r]\,, \\ (ii.) &\quad \mathbb{W}_k(t) \equiv 1 \text{ for } t \in [(k-1/2)\,T + r, (k+1/2)\,T - r]\,, \\ (iii.) &\quad \mathbb{W}_k \text{ is symmetric about its midpoint}\,, \\ (iv.) &\quad \sum [\mathbb{W}_k(t)]^2 \equiv 1\,, \\ (v.) &\quad \{\widehat{\mathbb{W}_k{}^\circ}[n]\} \in l^1\,. \end{aligned} \tag{1}$$

Conditions $(i.)$ and $(ii.)$ are partition properties, in that they give an exact snapshot of the input function f on $[(k-1/2)\,T + r, (k+1/2)\,T - r]$ with roll-off at the edges. Conditions $(iii.)$ and $(iv.)$ are needed to preserve orthogonality between adjacent blocks. Condition $(v.)$ is needed for the computation of Fourier coefficients. The windowing system is designed to create a lapped orthogonal transform [17], [9], covering $\mathbb{R}$ by windowing blocks that overlap neighboring windows, with the overlapping regions designed so that windowed folded basis elements preserve orthogonality.

We generate our systems by translations and dilations of a window $\mathbb{W}_I$ centered at the origin, where $\mathrm{supp}(\mathbb{W}_I) = [-\frac{T}{2} - r, \frac{T}{2} + r]$, We create the system by shifting windows by a factor of T. The window function $\mathbb{W}_I$ is m-times differentiable, has $\mathrm{supp}(\mathbb{W}_I) = [-\frac{T}{2} - r, \frac{T}{2} + r]$, and has values

$$\mathbb{W}_I = \begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \rho(t) & T/2 - r < |t| < T/2 + r\,. \end{cases} \tag{2}$$

We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{array}{ll} (a.) & \rho(T/2-r)=\rho(-T/2+r)=1\,, \\ (b.) & \rho(T/2+r)=\rho(-T/2-r)=0\,, \\ (c.) & \rho^{(n)}(T/2-r)=0\,,\ n=1,2,\ldots,m\,, \\ (d.) & \rho^{(n)}(T/2+r)=0\,,\ n=0,1,2,\ldots,m\,. \end{array}$$

with the conditions that $\rho \in C^m$ and

$$[\rho(t)]^2+[\rho(-t)]^2=1 \text{ for } t\in[\pm(\tfrac{T}{2}-r),\pm(\tfrac{T}{2}+r)]\,. \quad (3)$$

The C^m solution for ρ is given by a theorem of Schoenberg (see [20], pp. 7-8).

An explicit solution can be given by using the spline $S(t)$ with endpoints -1 and 1, i.e. $S(1)=1$, $S^{(n)}(1)=0$, $n=1,2,\ldots,m$, and $S^{(n)}(-1)=0$, $n=0,1,\ldots,m$. This is given by the integral of the function $M(t)=(-1)^m\sum_{j=0}^{m}\frac{\Psi(t-t_j)}{\phi'(t_j)}$, where Ψ is the $m+1$ convolution of characteristic functions, the knot points are $t_j=-\cos(\frac{\pi j}{m})$, $j=0,1,\ldots,m$, and $\phi(t)=\prod_{j=0}^{k}(t-t_j)$. If m is even, the midpoint occurs at the $m/2$ knot point. If m is odd, the midpoint occurs at the midpoint between the $m/2$ and $(m+1)/2$ knot points. Let $\xi=l(t)=\frac{r}{2}(t-1)$, and let $\alpha(\xi)=S\circ l(\xi)\,,\ |\xi|\le r$. Let $A=\int_{-r}^{r}\alpha(\zeta)\,d\zeta$. Now, normalize α by letting $\beta(\xi)=\frac{\pi}{2A}\alpha(\xi)\,,$ and let $\Theta(\tau)=\int_{-r}^{\tau}\beta(\xi)\,d\xi\,,\ |\tau|\le r$. Define $\rho_{\text{up}}(\tau)=\sin(\Theta(\tau))\,,\ \rho_{\text{down}}(\tau)=\cos(\Theta(\tau))$. We define our C^m window $\mathbb{W}_I(t)=\mathbb{ON}_{C^m}(t)$ as follows:

$$\begin{array}{ll} 0 & |t|\ge T/2+r\,, \\ 1 & |t|\le T/2-r\,, \\ \rho_{\text{up}}(t+(T/2+r) & -T/2-r<t<-T/2+r\,, \\ \rho_{\text{down}}(t-(T/2-r))) & T/2-r<t<T/2+r\,. \end{array} \quad (4)$$

We translate the window as needed. The resultant windowing system has variable partitioning length, variable roll-off, and variable smoothness. With each degree of smoothness, we get an additional degree of decay in frequency.

We designed the ON windows $\{\mathbb{W}_k(t)\}$ so that they preserve orthogonality of basis elements of overlapping blocks, using the techniques of Malvar, Meyer, and Coiffman. Because of the partition properties of these systems, we need only check the orthogonality of adjacent overlapping blocks. Let $\varphi\in L^2[-T/2,T/2]$. Define the folded function by

$$\widetilde{\varphi}(t) = \begin{cases} 0 & |t|\ge T/2+r \\ \varphi(t) & |t|\le T/2 \\ -\varphi(-T-t) & -T/2-r<t<-T/2 \\ \varphi(T-t) & T/2<t<T/2+r\,. \end{cases} \quad (5)$$

Then one can show the following:

Theorem 1 ([6]): Let φ_j be an orthononormal basis (ONB) for $L^2\left[-\frac{T}{2},\frac{T}{2}\right]$. Then $\{\Psi_{k,j}\}=\{\mathbb{W}_k\widetilde{\varphi_j}\}$ is an ONB for $L^2(\mathbb{R})$.

Given characteristics of the class of input signals, the choice of basis functions used can be tailored to optimal representation of the signal or a desired characteristic in the signal. A direct consequence is:

Theorem 2 ([6]): Let $\{\mathbb{W}_k(t)\}$ be ON windows, and let $\{\Psi_{k,n}\}=\{\mathbb{W}_k\widetilde{\varphi_n}\}$ be an ON basis that preserves orthogonality between adjacent windows. Let $f\in\mathbb{PW}_\Omega$ and $N=N(T,\Omega)$ be such that $\langle f,\Psi_{k,n}\rangle=0$ for all $n>N$ and all k. Then, $f(t)\approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t)=\sum_{k\in\mathbb{Z}}\left[\sum_{n=-N}^{N}\langle f,\Psi_{k,n}\rangle\Psi_{k,n}(t)\right]. \quad (6)$$

The analysis of the error generated by the method involves looking at the decay rates of the Fourier coefficients. We designed the ON windows so that the windows have decay $\mathcal{O}(1/(\omega)^{m+2})$ in frequency. This makes the error on each block summable.

Assume $\mathbb{W}_k$ is in C^m, then $\widehat{\mathbb{W}_k}(\omega)=\mathcal{O}(1/(\omega)^{m+2})$. We will analyze the error $\mathcal{E}_{k_{\mathcal{P}}}$ on a given block. Let $M=\|(f\cdot\mathbb{W}_k)\|_{L^2(\mathbb{R})}$. Then $\mathcal{E}_{k_{\mathcal{P}}}$

$$\begin{aligned} &= \sup\left|(f(t)\cdot\mathbb{W}_k)-\left[\sum_{n=-N}^{N}\langle f,\Psi_{n,k}\rangle\Psi_{n,k}(t)\right]\mathbb{W}_k(t)\right| \\ &= \sup\left[\sum_{|n|>N}\langle f,\Psi_{n,k}\rangle\Psi_{n,k}(t)\right]\mathbb{W}_k(t)\le\sum_{|n|>N}\frac{M}{n^{m+2}}\,. \end{aligned}$$

III. Modified Gegenbauer Systems

The Gegenbauer polynomials are the symmetric specialization of the Jacobi polynomials [18, Chapter 18]. They are used in a UWB communication system to construct pulses with narrow widths. The Gegenbauer waveform is used to modulate data, and has demonstrated superior performance to classic waveforms, e.g., Gaussian waveforms and the Hermite systems. We develop a modified Gegenbauer system, and use it to construct a windowed basis system for UWB signals with exponentially small error on each block.

The Gegenbauer polynomials are defined using the Gauss hypergeometric function [18, (18.5.9)] as

$$C_n^\nu(x):=\frac{(2\nu)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,2\nu+n\\ \nu+\frac{1}{2}\end{matrix};\frac{1-x}{2}\right),$$

where the Gauss hypergeometric function F_1 is defined in [18, Chapter 15], and can also be computed using three-term recurrence relations [18, Table 18.9.1] or using trigonometric [16, p. 220] series expressions. The Gegenbauer polynomials are given in terms of the more general Jacobi polynomials symmetric in parameters with [18, (18.7.1)]

$$C_n^\nu(x)=\frac{(2\nu)_n}{(\nu+\frac{1}{2})_n}P_n^{(\nu-1/2,\nu-1/2)}(x).$$

We define an ON basis for $L^2[-\frac{T}{2},\frac{T}{2}]$ using modified Gegenbauer functions, constructed from Gegenbauer polynomials. The Gegenbauer polynomials are modified so that they zero-out at the endpoints and normalized to create an ON system. This then allows UWB signals to be expanded in the folding method using the modified Gegenbauer system.

The Gegenbauer polynomials $C_n^\nu : \mathbb{C} \to \mathbb{C}$ are orthogonal over $(-1,1)$ with orthogonality relation given by [18, Table 18.3.1]

$$\int_{-1}^{1} C_n^\nu(x) C_m^\nu(x) w(x;\nu)dx = h_n^\nu \delta_{n,m}, \tag{7}$$

for $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, where

$$w(x;\nu) := (1-x^2)^{\nu-1/2}\,,\; h_n^\nu := \frac{2^{1-2\nu}\pi\Gamma(2\nu+n)}{(\nu+n)\Gamma^2(\nu)n!}\,.$$

They have a Rodrigues-type formula [18, Table 18.5.1]

$$C_n^\nu(x) := \frac{(-1)^n(2\nu)_n}{2^n(\nu+\frac{1}{2})_n n!}\frac{1}{w(x;\nu)}\frac{d^n}{dx^n}w(x;\nu+n).$$

Consider the modified Gegenbauer function $\mathcal{C}_n^\nu : [-\frac{T}{2}, \frac{T}{2}] \times (0,\infty) \to \mathbb{R}$ defined by

$$\mathcal{C}_n^\nu(t;T) := \sqrt{\frac{2\,w\left(\frac{2t}{T};\nu\right)}{Th_n^\nu}}\, C_n^\nu\left(\frac{2t}{T}\right). \tag{8}$$

It is easy to see from (7) that these functions form an ON basis for $L^2[-\frac{T}{2}, \frac{T}{2}]$ with $\nu \in (\frac{1}{2}, \infty)$, namely

$$\int_{-T/2}^{T/2} \mathcal{C}_n^\nu(t;T)\mathcal{C}_m^\nu(t;T)dt = \delta_{m,n}.$$

Note that we exclude the parameters $\nu \in (-\frac{1}{2}, \frac{1}{2}]$ in order to keep the endpoints $\pm\frac{T}{2}$ in the domain of integration. By using $w(x;\nu)$ and h_n^ν one has

$$\mathcal{C}_n^\nu(t;T) = \frac{2^{2\nu-1/2}\Gamma(\nu)}{T^\nu}\sqrt{\frac{(n+\nu)n!}{\pi\Gamma(2\nu+n)}} \times \left(\left(\tfrac{T}{2}\right)^2 - t^2\right)^{\nu/2-1/4} C_n^\nu\left(\frac{2t}{T}\right). \tag{9}$$

The modified Gegenbauer system zeros out at the endpoints, which allows us to use it to create the windowed ON basis $\{\Psi_{k,n}\} = \{\mathbb{W}_k\, \mathcal{C}_n^\nu(t;T)^\curlywedge\}$, where we window with $\mathbb{ON}_{C^m}$, and define the *folded basis elements* $\mathcal{C}_n^\nu(t;T)^\curlywedge$ by

$$\begin{array}{cl} 0 & |t| \geq T/2 + r \\ \mathcal{C}_n^\nu(t;T) & |t| \leq T/2 \\ -\mathcal{C}_n^\nu(-T-t;T) & -T/2 - r < t < -T/2 \\ \mathcal{C}_n^\nu(T-t;T) & T/2 < t < T/2 + r\,. \end{array} \tag{10}$$

We use two theorems of Gottlieb and Shu [11] to give an outline of an analytic argument showing they minimize the Gibbs phenomenon. Given an integrable function f defined on $[-\frac{T}{2}, \frac{T}{2}]$, we can compute its modified Gegenbauer coefficients $\widehat{f^\nu}(l)$ by

$$\widehat{f^\nu}(l) = \int_{-T/2}^{T/2} f(t)\mathcal{C}_l^\nu(t;T)dt\,. \tag{11}$$

This then gives the Gegenbauer expansion for the first $j+1$ terms as

$$f_j^\nu(t) = \sum_{l=0}^{j} \widehat{f^\nu}(l)\mathcal{C}_l^\nu(t;T)\,.$$

It is important to note that in order to get exponential decay in our error computation at the end of this section (15), ν must grow linearly with N. We assume $\nu = \alpha N$, for $\alpha > 0$. Now, let f be our original signal, f^m be the expansion of f into m-th degree modified Gegenbauer functions, and f_N^m be the expansion of f into m-th degree modified Gegenbauer functions truncated at N. We want to estimate $\|(f - f_N^m)\|$. By the triangle inequality,

$$\|(f - f_N^m)\| \leq \|(f - f^m)\| + \|(f^m - f_N^m)\|\,.$$

Assume $m = \beta N$, for $\beta > 0$. Let $TE(\nu,\alpha,\beta,N)$ be the truncation error in $\|(f^m - f_N^m)\|$ (Theorem 4.1, page 659 from [11]). One has

$$TE(\nu,\alpha,\beta,N) \leq C(q_T)^N\,,$$

where

$$q_T = \frac{(\beta+2\alpha)^{(\beta+2\alpha)}}{(2\alpha)^\alpha\beta^\beta}\,.$$

Note, q_T is the key to exponential decay in the truncation estimate. We have that if

$$\alpha = \beta < 2/27 \text{ then } q_T < 1\,.$$

(In fact, if $\alpha = \beta$, the function $g(x) = [(3x)^{(3x)}]/[2^x \cdot (x)^{(2x)}]$ is minimized at $x = 2/(27e)$ with value $e^{-2/(27e)}$.)

Next, we estimate the regularization error $\|(f - f^m)\|$. Let

$$RE(\nu,m) = \sup_{-\frac{T}{2}\leq t\leq\frac{T}{2}} \left| f(t) - \sum_{l=0}^{m} \widehat{f^\nu}(l)\mathcal{C}_l^\nu(t;T)\right|. \tag{12}$$

We modify Theorem 4.3, p. 662 from [11]. Here, we have to be careful, because the estimate in [11] involves $\frac{1}{C_l^\nu(1)}$ (Equation (4.21), p. 661). The modified Gegenbauer functions zero out at the endpoints. Therefore, we had to recompute the estimates. Adjusting the arguments in Gottlieb and Shu for the modified Gegenbauer functions, we arrive at $RE(\nu,m) \leq$

$$\max_{-\frac{T}{2}\leq t\leq\frac{T}{2}} |f(t)|\left[1 + \frac{2^{1-\nu}}{\sqrt{T}\Gamma(\nu+1/2)}\sum_{l=0}^{m}\sqrt{\frac{(l+\nu)\Gamma(l+2\nu)}{\Gamma(l-1)}}\right],$$

where the gamma function $\Gamma : \mathbb{C}\setminus -\mathbb{N}_0 \to \mathbb{C}$ is defined in [18, Chapter 5] for $\mathbb{N}_0 := \{0,1,2,\ldots\}$. Using Stirling's approximation formula [18, p. 141], and bounding $\sup_{-\frac{T}{2}\leq t\leq\frac{T}{2}} |f(t)|$ by M, we get that

$$RE(\nu,m) \leq \frac{M}{\sqrt{2\pi T}}\left[\frac{2^{1-\nu}}{(\nu-\frac{1}{2})^\nu}\right]\left[\frac{(m+2\nu)^{\frac{(m+2\nu)}{2}}}{m^{\frac{m}{2}}}\right]. \tag{13}$$

We can now follow the proof of Theorem 4.3, pp. 662-663 from [11]. If $\nu = \gamma m$, there exists $q_R < 1$ such that the error in this estimate satisfies, for $0 < r < 1$,

$$RE(\nu,m) \leq Cm(q_R)^m,\; q_R = \frac{(1+2\gamma)^{\frac{(1+2\gamma)}{2}}}{\gamma^{2\gamma}}r\,. \tag{14}$$

We close by computing $\mathcal{E}_{k_\mathcal{P}}$ in terms of the modified Gegenbauer system. The minimization of the Gibbs phenomenon, giving the point values of a piecewise smooth signal with essentially the same accuracy as a smooth approximation, makes this system the ideal system to use for basis folding as applied to UWB systems. Let $\sigma \in \mathbb{N}$ be the smoothness parameter, and assume $\mathbb{W}_k$ is C^σ, and so $\widehat{\mathbb{W}_k}(\omega) = \mathcal{O}(1/(\omega)^{\sigma+2})$. Now approximate the signal f with the windowed ON basis $\{\Psi_{k,n}\} = \{\mathbb{W}_k \mathcal{C}_n^\nu(t;T)^\curlywedge\}$, where we window with $\mathbb{ON}_{C^\sigma}$. Let $q = \max\{q_T, q_R\}$. Note, $q < 1$. Then, the error $\mathcal{E}_{k_\mathcal{P}}$ on a given block is

$$\sup\left|(f(t)\cdot \mathbb{W}_k) - \left[\sum_{n=-N}^{N} \langle f, \Psi_{n,k}\rangle \Psi_{n,k}(t)\right]\mathbb{W}_k(t)\right| \quad (15)$$

$$\leq \sup\left[\sum_{|n|>N}\left|\langle f, \Psi_{n,k}\rangle \Psi_{n,k}(t)\right|\right]\mathbb{W}_k(t) \leq \sum_{|n|>N} \frac{e^{\log(q)N}}{n^{\sigma+2}}.$$

Since $q < 1$, $e^{\log(q)N}$ decays exponentially as N increases. The partial sums, for $K > N+1$, are given by

$$\sum_{|n|>N}^{K} \frac{e^{\log(q)N}}{n^{\sigma+2}} = e^{\log(q)N}\left[\zeta(\sigma, N+1) - \zeta(\sigma, K+1)\right],$$

where ζ is the Riemann zeta function.

IV. Frame Expansions

The theory of frames [8], [4] gives us a more computable and flexible mathematical structure in which to expand UWB signals. Recall that a sequence of elements $\Psi = \{\psi_k\}_{k\in K}$ in a Hilbert space $\mathbb{H}$ is a *frame* in there exist constants A and B such that

$$A\|f\|^2 \leq \sum_{k\in K} |\langle f, \psi_k\rangle|^2 \leq B\|f\|^2.$$

A and B are called the *upper and lower frame bounds*, respectively. A frame is called *tight* if $A = B$, and *Parseval* if $A = B = 1$. A frame is called *exact* if it ceases to be a frame when any of its elements are removed.

By $C_\Psi : \mathbb{H} \to \ell^2(\mathbb{Z})$ we denote the *analysis operator* defined by $(C_\Psi f)_k = \langle f, \psi_k\rangle$. The adjoint of C_Ψ is the *synthesis operator* $D_\Psi(c_k) = \sum_n c_k \psi_k$. The *frame operator* $S_\Psi = D_\Psi C_\Psi$ can be written as $S_\Psi f = \sum_k \langle f, \psi_k\rangle \psi_k$. It is positive and invertible. Frames allow redundant representations, but always allow reconstruction. A frame Ψ always has a dual frame $\widetilde{\Psi} = \left(\tilde{\psi}_i\right)_{i\in I}$, such that $f = \sum \langle f, \psi_i\rangle \tilde{\psi}_i = \sum \left\langle f, \tilde{\psi}_i\right\rangle \psi_i$. There is always the canonical dual $S^{-1}\psi_i$, where S is the frame operator $Sf = \langle f, \psi_i\rangle \psi_i$.

A sequence $\{e_n\}$ is called a *Riesz basis* for $\mathbb{H}$ if it is complete and there exist $0 < A \leq B$ such that for every finite scalar sequence $\{c_n\}$

$$A\sum |c_n|^2 \leq \left\|\sum_{n\in\mathbb{Z}} c_n e_n\right\|^2 \leq B\sum |c_n|^2.$$

The constants A and B are called *Riesz bounds*. Riesz basis are exact frames, and the unique biorthogonal sequence is the canonical dual.

And ONB is a self-dual Riesz-basis. Any normalized Parseval frame is an ONB.

A. Frame Theory of Super Windows

We prove, in this section, versions of Theorem 1 for redundant systems. This generalizes similar approaches for the wavelet [19], the non-stationary Gabor [1] and the constant-Q transform [13].

We call the set of functions *admissible* if there exists numbers m, M such that

$$m \leq \sum_k [\mathbb{W}_k(t)]^2 \leq M. \quad (16)$$

We call it *uniform* if $m = M$. Let $S^{(k)} = \overline{\operatorname{supp} \mathbb{W}_k}$.

Lemma 1: Let $\Phi^{(k)} = \left(\phi_i^{(k)}\right)_{i\in I_k}$ be a frame for $L^2\left(S^{(k)}\right)$ with frame bounds $A^{(k)}$ and $B^{(k)}$, such that there are common bounds A, B. Let W_k be a family of admissible windows. Then the set

$$\Psi = \left(\psi_i^{(k)}\right)_{k\in K, i\in I_k} := \left(\mathbb{W}_k \cdot \phi_i^{(k)}\right)_{k\in K, i\in I_k}$$

is a frame for the whole space $L^2(\mathbb{R})$ with frame bounds $m\cdot A$ and $M \cdot B$.

Proof: We have that

$$\begin{aligned}\sum_{k\in K}\sum_{i\in I_k}\left|\left\langle f, \psi_i^{(k)}\right\rangle\right|^2 &= \sum_{k\in K}\sum_{i\in I_k}\left|\left\langle \mathbb{W}_k f, \phi_i^{(k)}\right\rangle\right|^2 \\ = \sum_{k\in K} B^{(k)}\|\mathbb{W}_k f\|^2 &\leq B\cdot \int_{-\infty}^{\infty}\sum_{k\in K}|\mathbb{W}_k(t)|^2|f(t)|^2\,dt \\ &\leq B\cdot M\cdot \|f\|^2. \quad (17)\end{aligned}$$

An analogous proof works for the lower bound. □

Note that this is very similar to the splitting done for fusion frames [5], [15], but the spaces $\mathfrak{W}^k = \left\{\mathbb{W}_k \cdot f \,\middle|\, f \in L^2(\mathbb{R})\right\}$ are, in general, not closed subspaces. In particular, this can never be the case in our setting, where the roll-off should be smooth.

The reason why we could call $\mathbb{W}_k$ *super windows* is that the local frames $\Phi^{(k)}$ could already involve windowing functions, like for (non-stationary) Gabor transforms [10]. See [19] how a similar approach can be used for wavelets.

An easy consequence is

Corollary 1: Let $\Phi^{(k)}$ be tight frames, i.e. $A = B$, and $\mathbb{W}_k$ be uniform. The system Ψ is a tight frame.

It is easy to show that the analysis operator is given by

$$C_\Psi f = \left(\left\langle f, \psi_i^{(k)}\right\rangle\right)_{k\in K, i\in I_k} = \left(C_{\Psi^{(k)}}\left(\mathbb{W}_k f\right)\right)_{k\in K}.$$

Its adjoint, the synthesis operator, is

$$D_\Psi(c_{k,i})_{k\in K, i\in I_k} = \sum_{k\in K, i\in I_k} c_{k,i}\psi_i^{(k)} = \sum_k \mathbb{W}_k \cdot \left(D_{\Psi^{(k)}} c_{k,\bullet}\right).$$

The frame operator is then

$$S_\Psi = \sum_k \mathbb{W}_k \cdot S_{\Psi^{(k)}} \mathbb{W}_k.$$

As a consequence we can give a reconstruction formula.

Proposition 1: Let Ψ be as Lemma 1. Then choose an admissible family of windows $\mathbb{V}_k$ with

$$\sum_k \mathbb{V}_k W_k \equiv 1.$$

Then the system $\Theta := \left(\mathbb{V}_k \cdot \tilde{\phi}_i^{(k)}\right)_{k \in K, i \in I_k}$ is a dual system of Ψ. Here, $\tilde{\phi}_i^{(k)}$ is any dual frame of $\Phi^{(k)}$. In particular, one has the reconstruction

$$f = \sum_k \sum_{i \in I_k} c_{k,i} \tilde{\theta}_i^{(k)},$$

where the coefficients are $c_{k,i} = \left\langle f, \psi_i^{(k)} \right\rangle$.

Proof: By Lemma 1, the system Θ is a frame. We just have to proof the reconstruction formula as follows –

$$\begin{aligned} \sum_k \sum_{i \in I_k} \left\langle f, \psi_i^{(k)} \right\rangle \tilde{\theta}_i^{(k)} &= \sum_k \sum_{i \in I_k} \left\langle \mathbb{W}_k f, \phi_i^{(k)} \right\rangle \mathbb{V}_k \tilde{\phi}_i^{(k)} \\ &= \sum_k \mathbb{V}_k \mathbb{W}_k f = f. \end{aligned} \tag{18}$$

□

The only thing left in this section to be able to prove Theorem 1 is that the super windows preserve (bi-)orthogonality (and dealing with the folding). This is a non-trivial task – except for the trivial case that the support of super windows do not overlap. It necessitates a careful construction, as done in Section II and III.

B. General Hilbert Spaces

One could generalize that approach to other Hilbert spaces $\mathbb{H}$. There is certainly always an isomorphism between $\mathbb{H}$ and $L^2(\mathbb{R})$. Taking the assumptions from Section IV-A, and assume that there are embeddings, i.e. $\mathfrak{T}_i : L^2\left(S^{(k)}\right) \to \mathbb{H}$ being an injective operator with closed range with pseudo inverse ${\mathfrak{T}_i}^\dagger$.

Considering the sequence $\Xi = \left(\xi_i^{(k)}\right)_{i \in I_k}$ defined by $\xi_i^{(k)} := \left(\mathfrak{T}_i^\dagger\right)^* \mathbb{W}_i \mathfrak{T}_i^* \xi_i^{(k)}$, one can apply Lemma 1 and the other results in an analogue manner.

V. Outlook

In the upcoming journal paper we will prove generalizations of Theorem 1. We will construct several combinations of windows and building blocks to achieve orthogonality. We will also go beyond the orthogonality constraints. We will work on a much more general approach of how to build such pairs. We will do the same for Riesz bases and a fitting definition of windows functions.

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